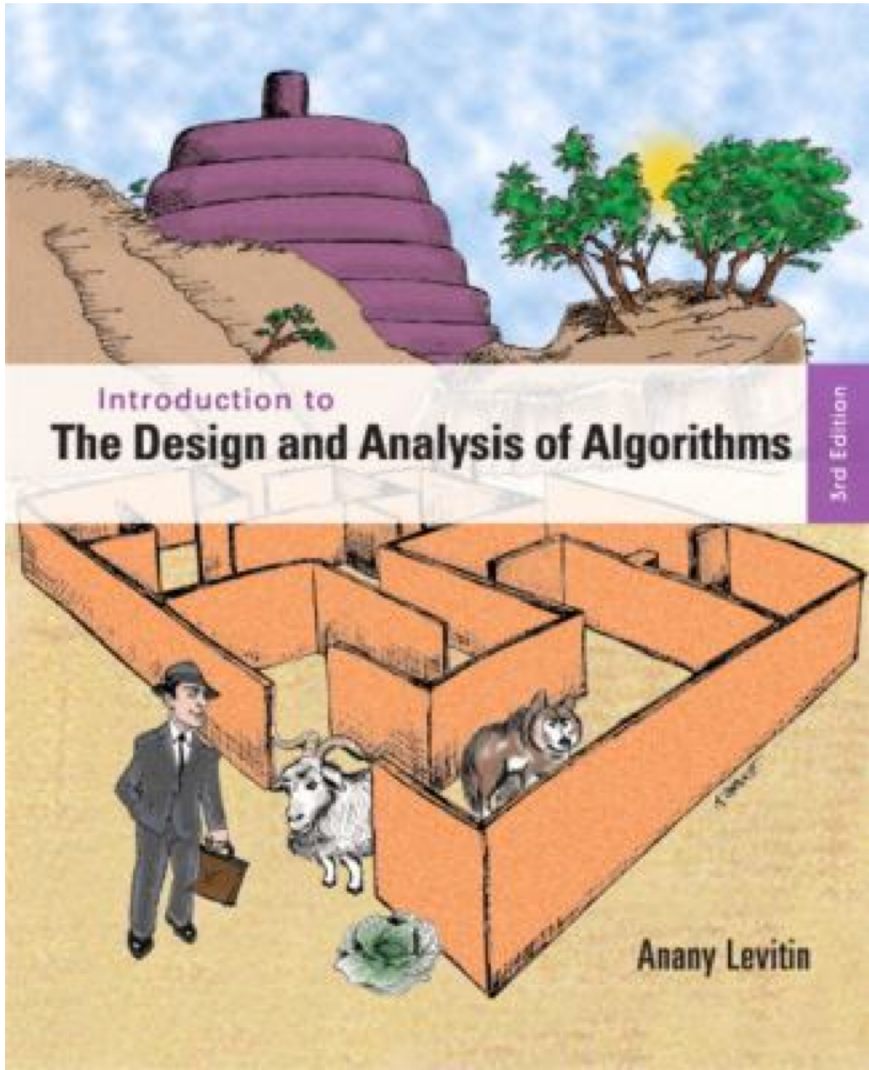
Chapter 9

Greedy Technique



Greedy Technique

- Constructs a solution to an *optimization problem* by making locally optimal choice at each stage with the hope of finding a global optimum

- *Example:* Prim's algorithm for finding a minimum spanning tree for a weighted undirected graph.

- For some problems, yields an optimal solution for every instance. For most, does not but can be useful for fast approximations.

Applications of the Greedy Strategy

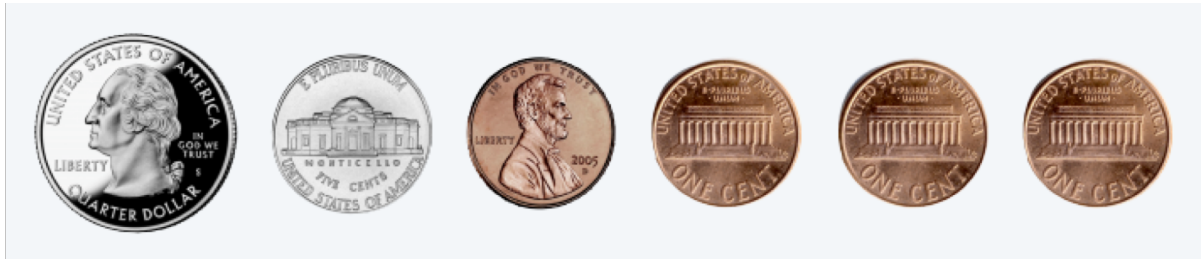
- Optimal solutions:
  - change making for “normal” coin denominations
  - minimum spanning tree (MST)
  - single-source shortest paths
  - simple scheduling problems
  - Huffman codes

- Approximations:
  - traveling salesman problem (TSP)
  - knapsack problem
  - other combinatorial optimization problems

Change-Making Problem

Given unlimited amounts of coins of denominations $d_1 < \ldots < d_m$, give change for amount n with the least number of coins

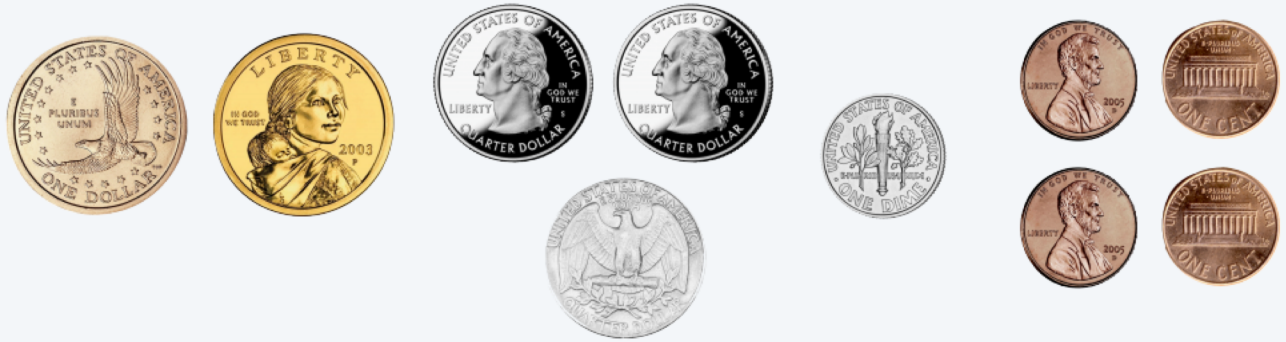
Example: $d_1 = 1c$, $d_2 = 5c$, $d_3 = 10c$, $d_4 = 25c$ and $n = 34c$



Cashier's algorithm

- At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Ex. $2.89.



Cashier's algorithm

- At each iteration, add coin of the largest value that does not take us past the amount to be paid.

CASHIERS-ALGORITHM $(x, c_1, c_2, \ldots, c_n)$

SORT n coin denominations so that $c_1 < c_2 < \ldots < c_n$

$S \leftarrow \phi$ ⟵ set of coins selected

WHILE $x > 0$

  $k \leftarrow$ largest coin denomination c_k such that $c_k \leq x$

  IF no such k, RETURN "no solution"

  ELSE

    $x \leftarrow x - c_k$

    $S \leftarrow S \cup \{k\}$

RETURN S

- Question: Is cashier's algorithm optimal?

Cashier's algorithm for other denominations

- Is cashier's algorithm for any set of denominations?
- Answer: ?
- Consider U.S. postage: 1, 10, 21, 34, 70, 100, 350.
  - Cashier's algorithm: 140¢.
  - Optimal: 140¢

Cashier's algorithm for other denominations

- Is cashier's algorithm for any set of denominations?
- Answer: No.
- Consider U.S. postage: 1, 10, 21, 34, 70, 100, 350.
  - Cashier's algorithm: 140¢ = 100 + 34 + 1 + 1 + 1 + 1 + 1 + 1.
  - Optimal: 140¢ = 70 + 70

Algorithm Comparison

- Time / Space
  - Greedy: O(1) space, O(n) time
  - Dynamic programming: O(n) space and O(m x n) time
- Applicability
  - Greedy: special sets of denominations, e.g. US coins
  - Dynamic programming: any denominations

Minimum Spanning Tree (MST)

- *Spanning tree* of a connected graph *G*: a connected acyclic subgraph of *G* that includes all of *G*'s vertices

- *Minimum spanning tree* of a weighted, connected graph *G*: a spanning tree of *G* of minimum total weight

Example:

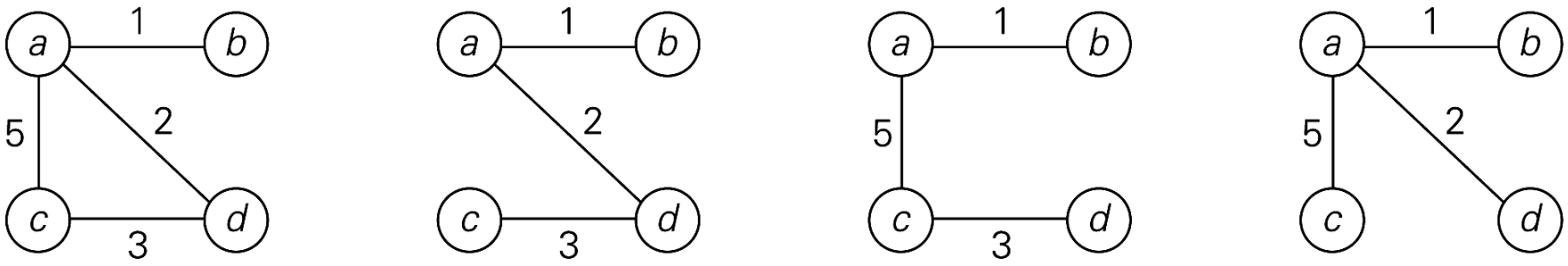


FIGURE 9.1 Graph and its spanning trees; T_1 is the minimum spanning tree

Prim's MST Algorithm

- Start with tree T_1 consisting of one (any) vertex and "grow" tree one vertex at a time to produce MST through a series of expanding subtrees $T_1, T_2, \ldots, T_n$

- On each iteration, construct T_{i+1} from T_i by adding vertex not in T_i that is closest to those already in T_i (this is a "greedy" step!)

- Stop when all vertices are included

Refining Prim‘s Algorithm

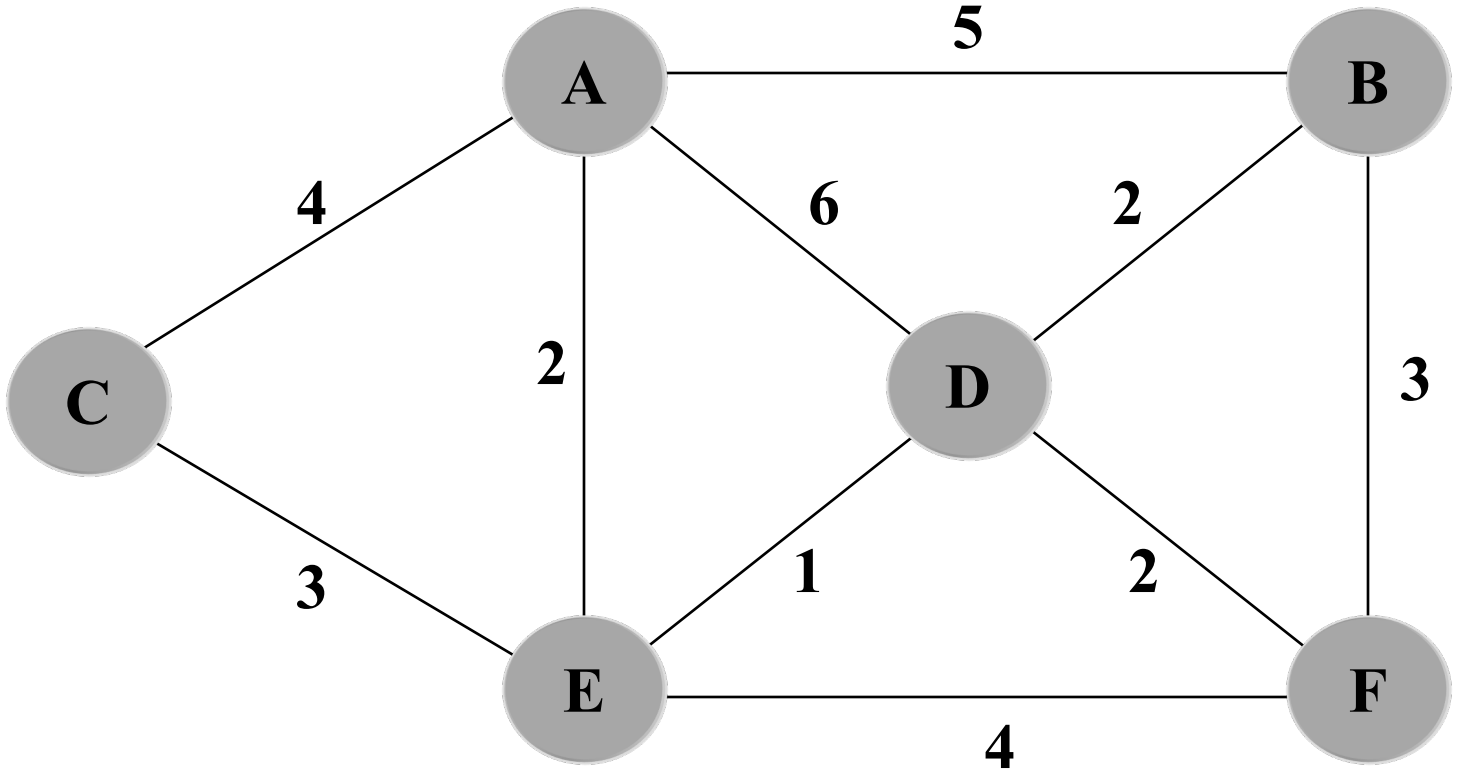
1. All vertices are marked as not visited

2. Any vertex *v* you like is chosen as starting vertex and is marked as visited (define a cluster *C*)

3. The smallest-weighted edge $e = (v, u)$, which connects one vertex *v* inside the cluster *C* with another vertex *u* outside of *C*, is chosen and is added to the MST.

4. The process is repeated until a spanning tree is formed

Illustration of Prim's Algorithm



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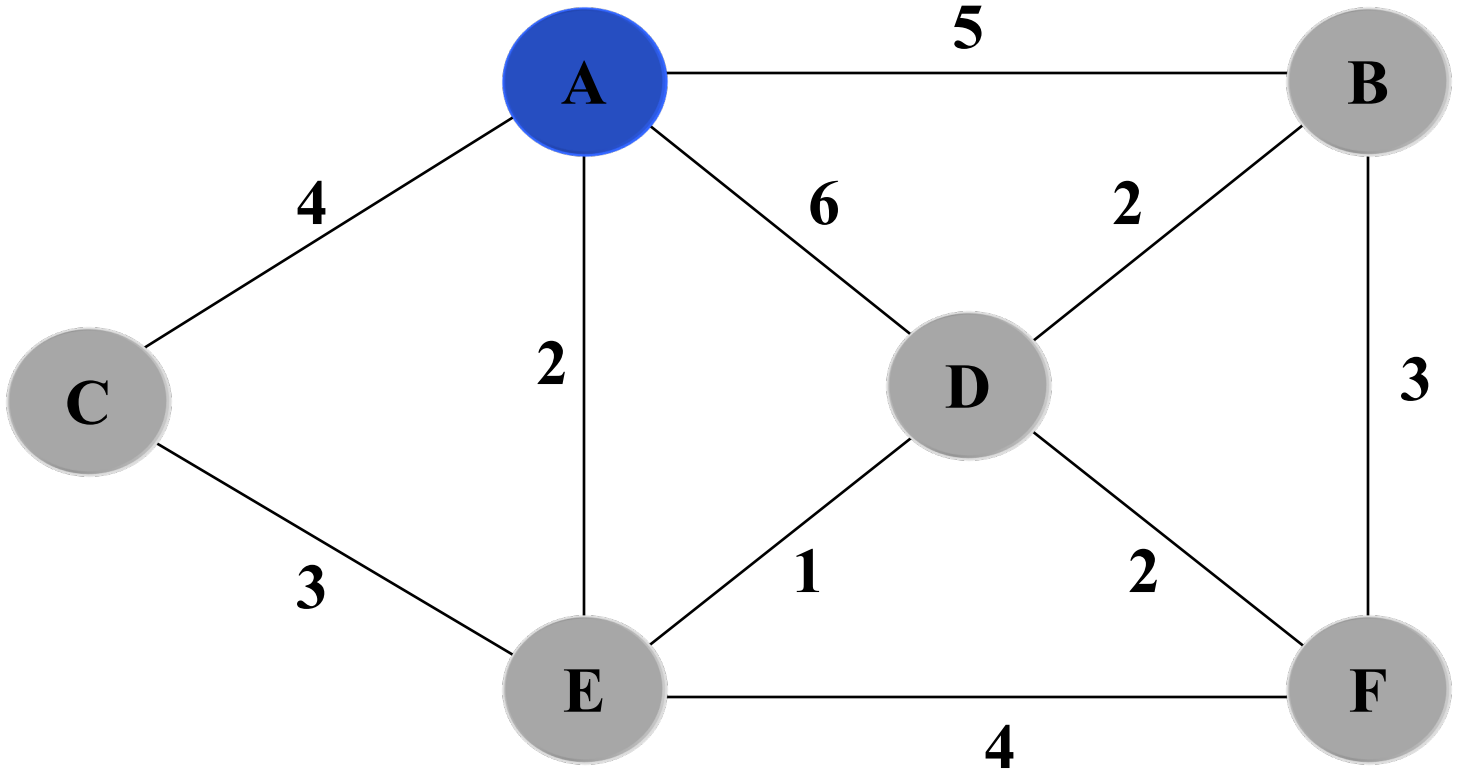

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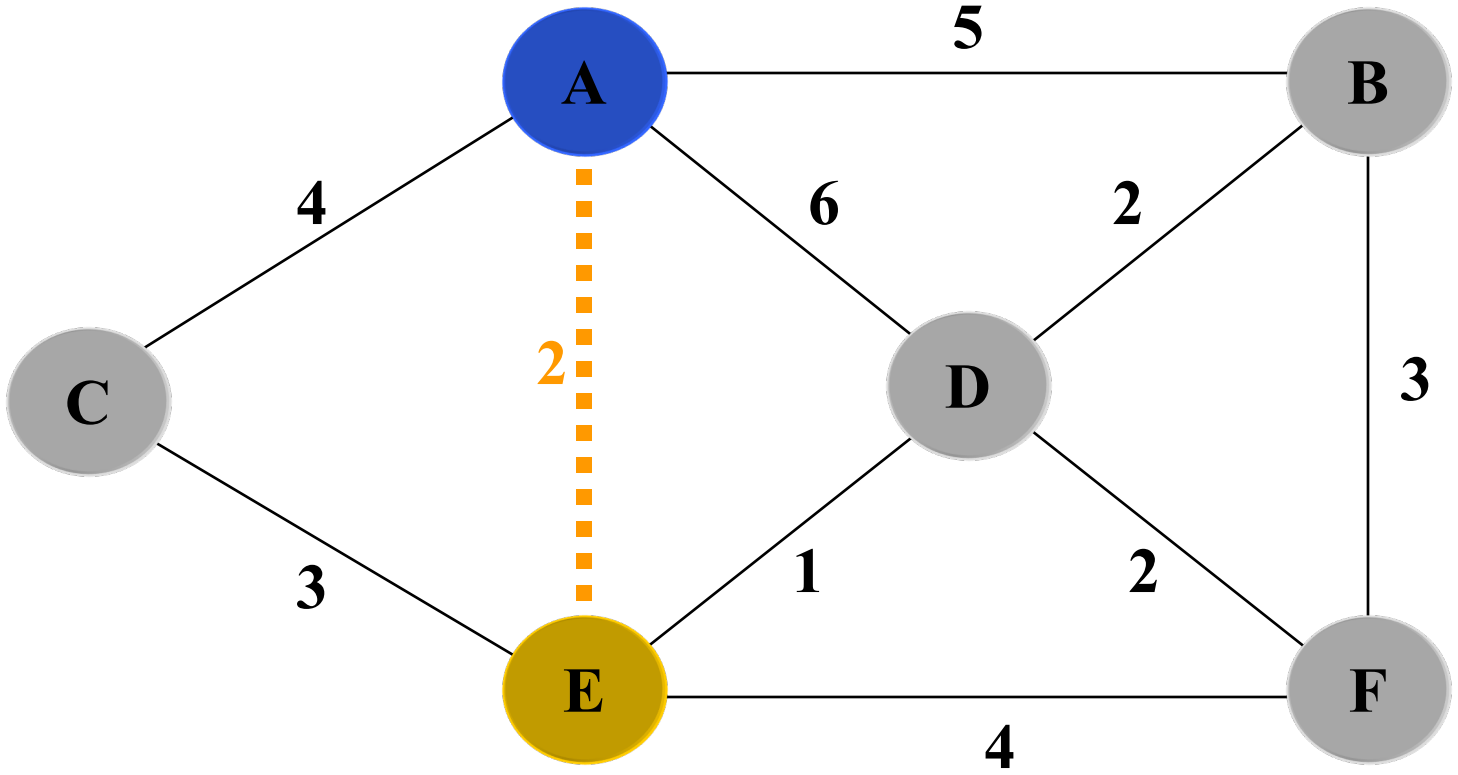

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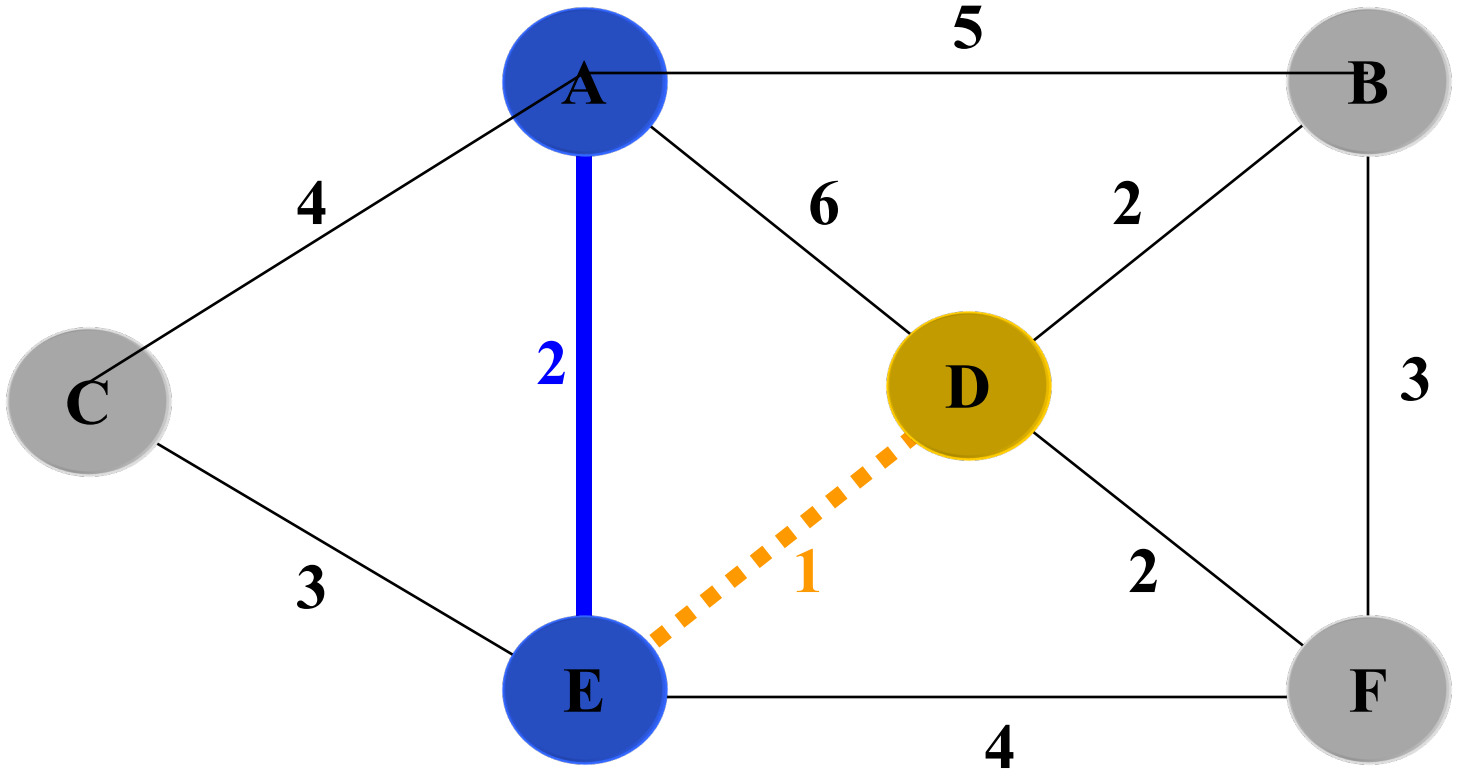

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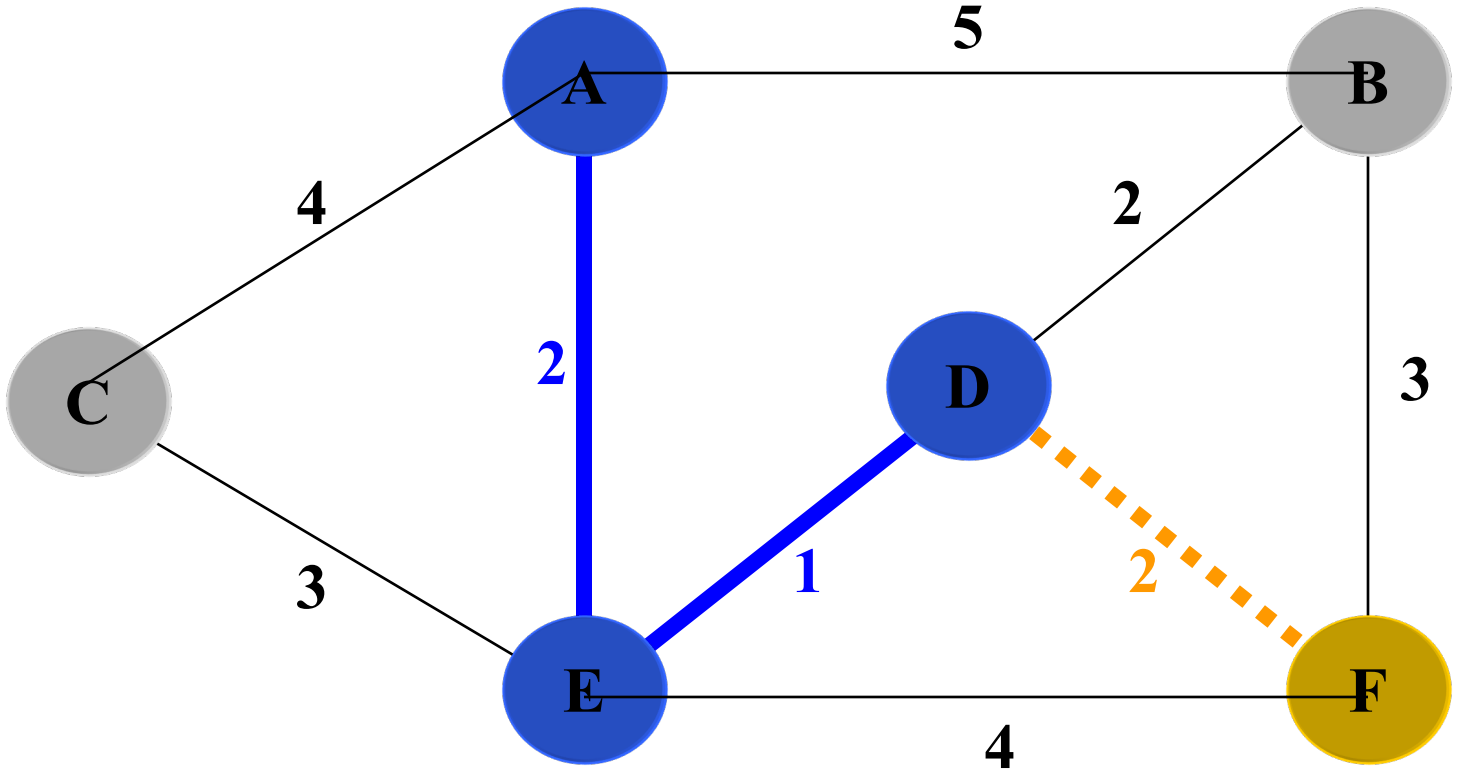

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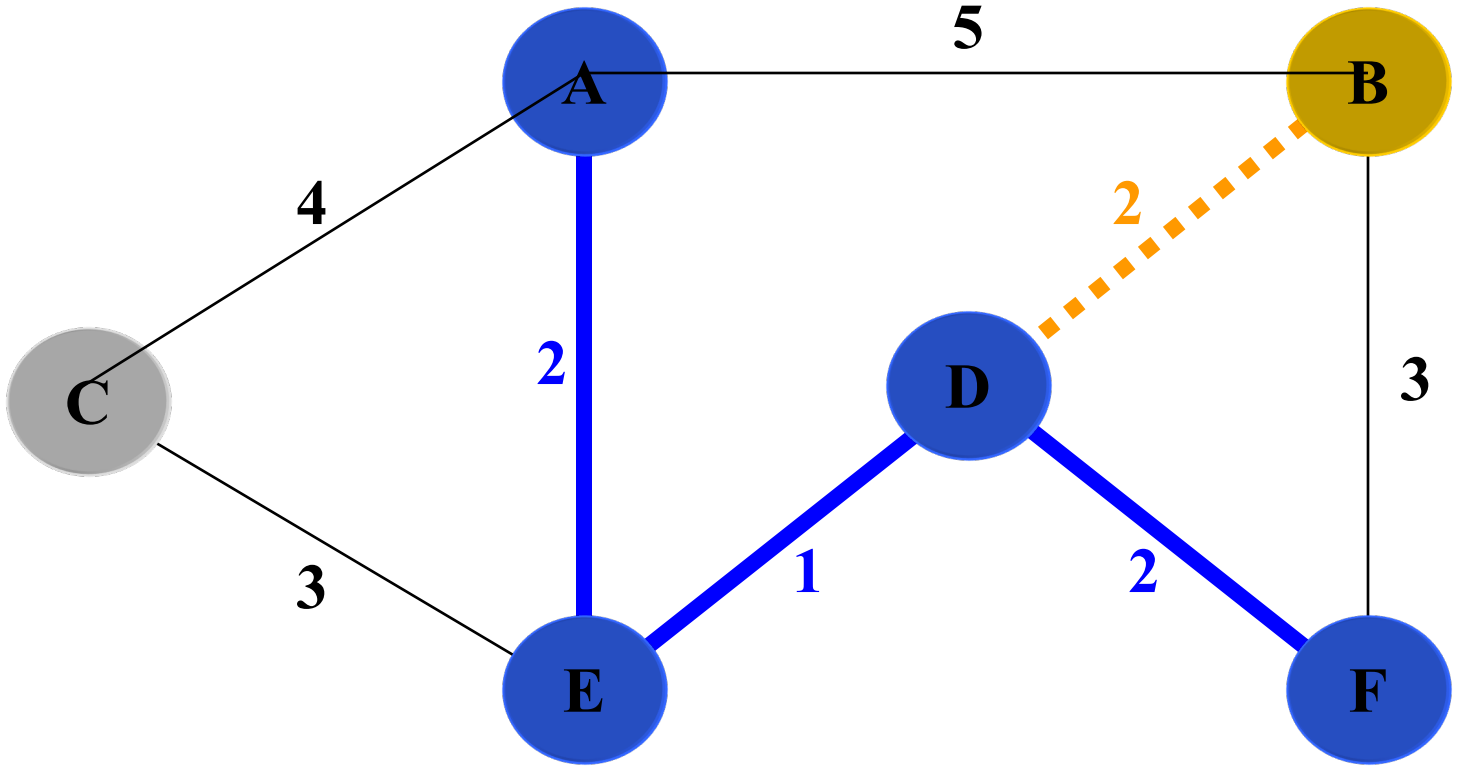

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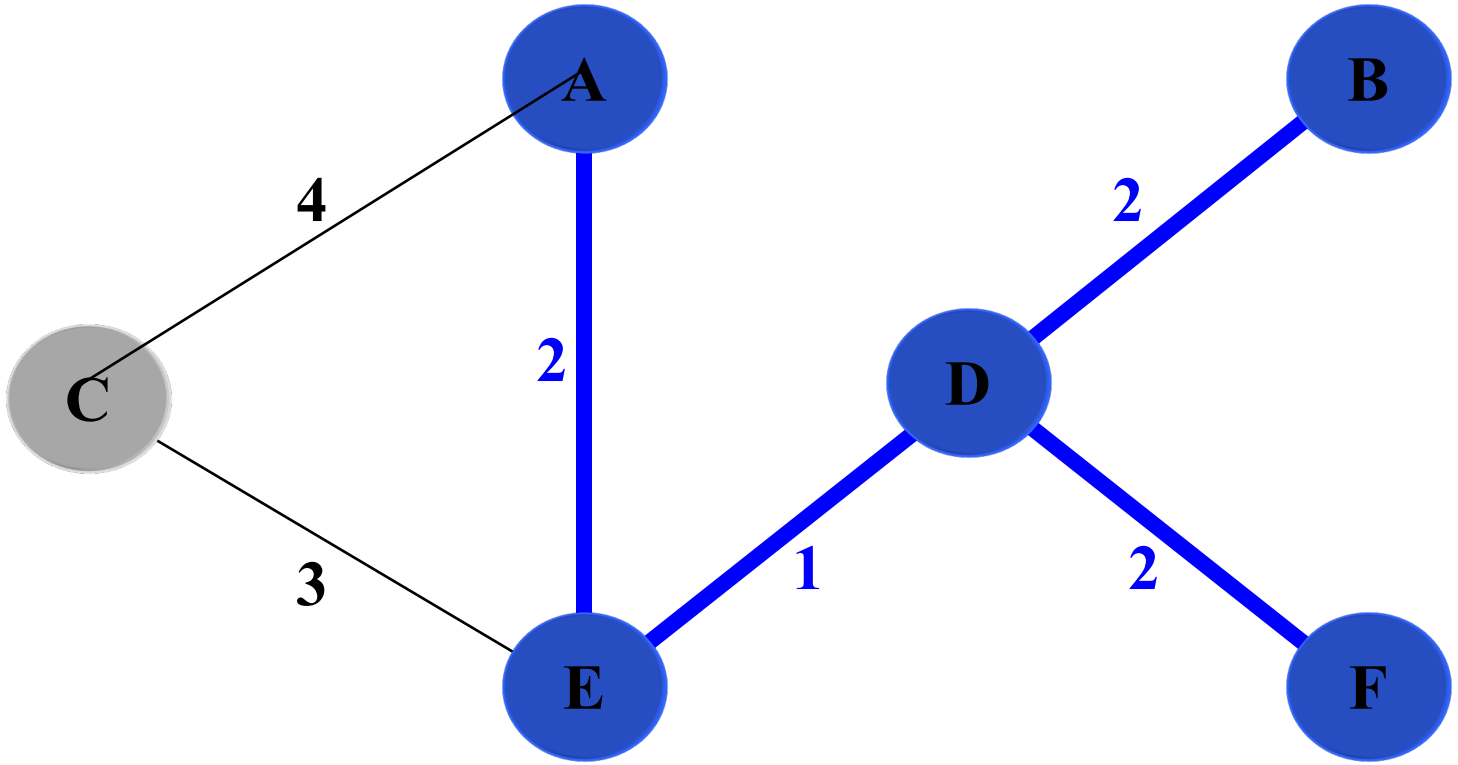

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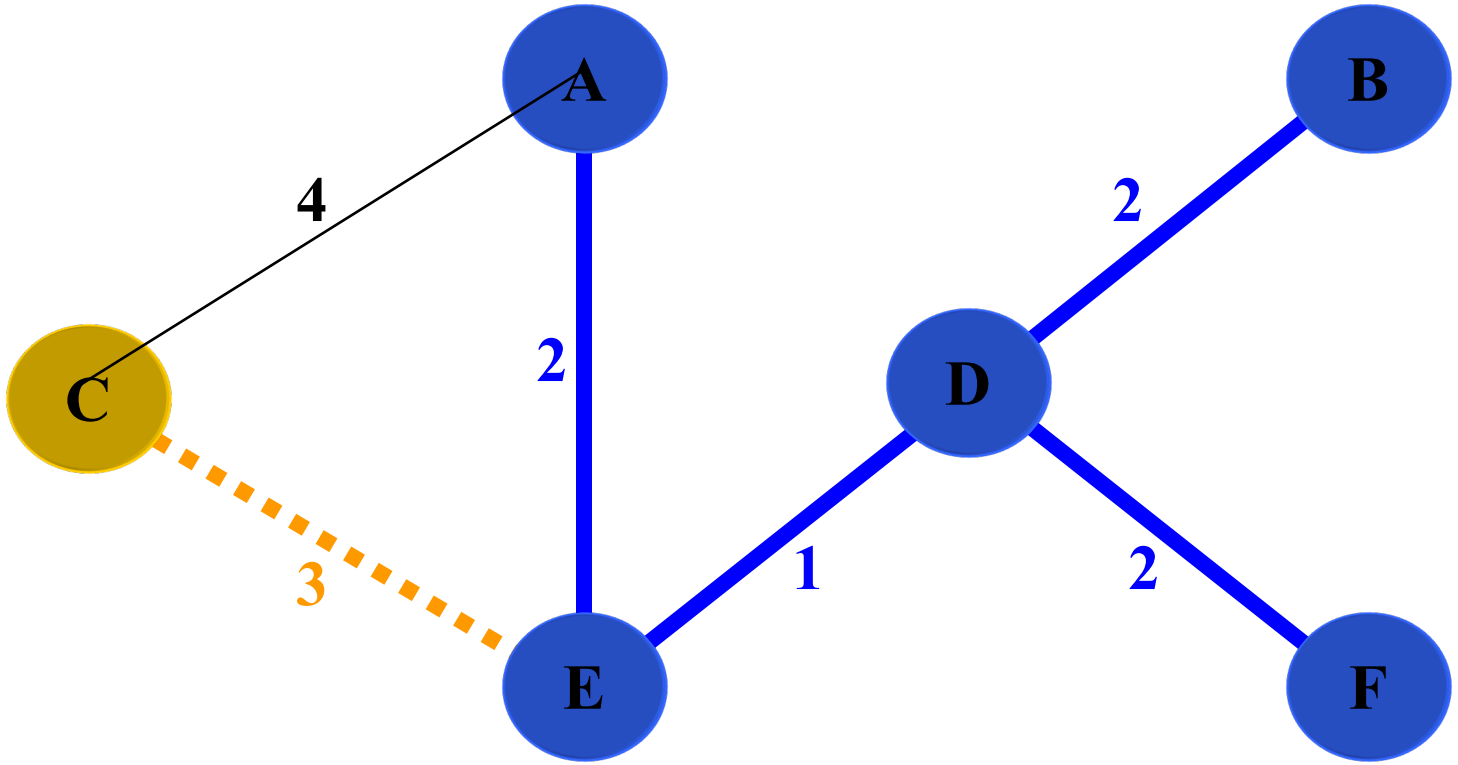
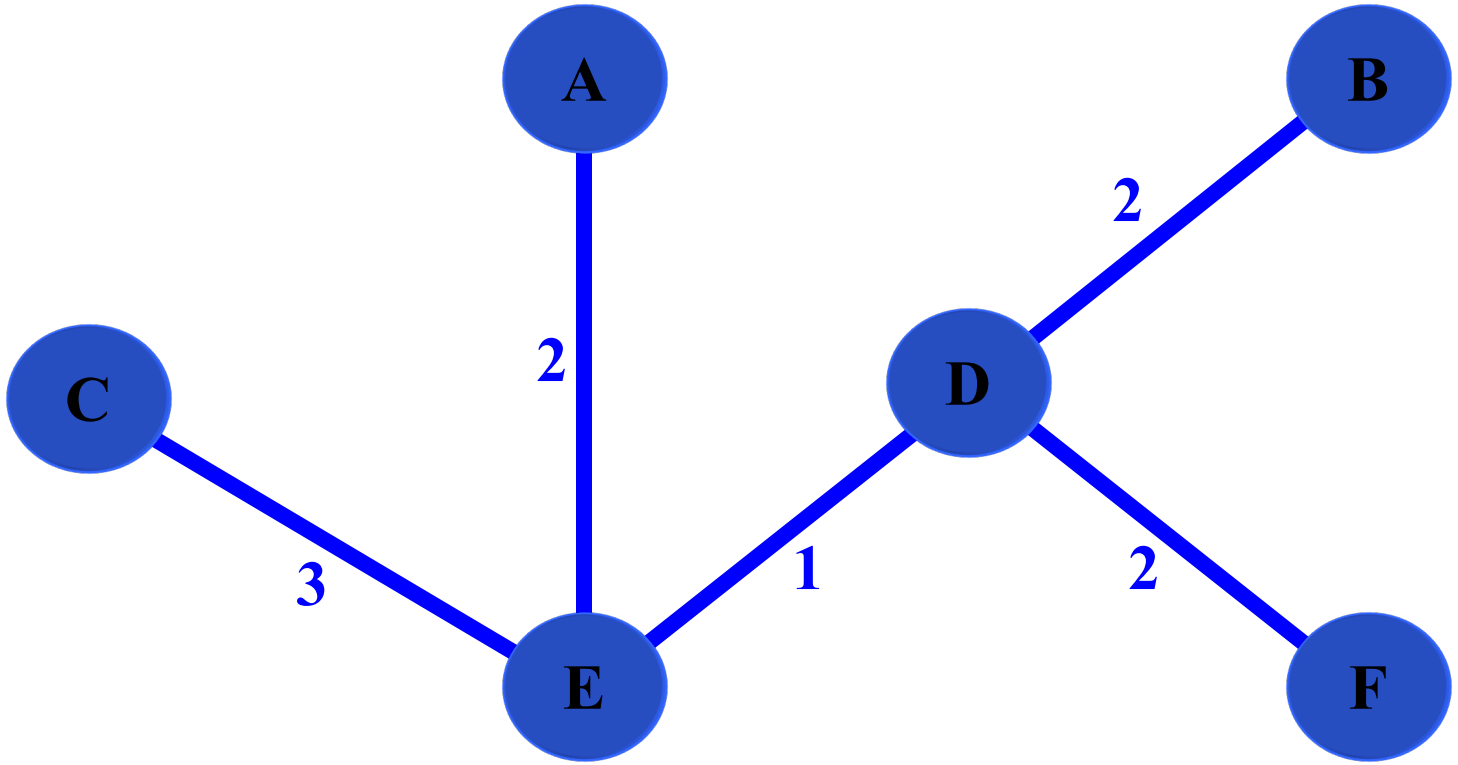


Illustration of Prim's Algorithm

Minimum- Spanning Tree



Prim's Algorithm

ALGORITHM *Prim*(G)

//Prim's algorithm for constructing a minimum spanning tree
//Input: A weighted connected graph $G = \langle V, E \rangle$
//Output: E_T, the set of edges composing a minimum spanning tree of G
$V_T \leftarrow \{v_0\}$ //the set of tree vertices can be initialized with any vertex
$E_T \leftarrow \emptyset$
for $i \leftarrow 1$ **to** $|V| - 1$ **do**
    find a minimum-weight edge $e^* = (v^*, u^*)$ among all the edges (v, u)
    such that v is in V_T and u is in $V - V_T$
    $V_T \leftarrow V_T \cup \{u^*\}$
    $E_T \leftarrow E_T \cup \{e^*\}$
return E_T

Complexity???

Complexity

- If implemented using adjacency lists and min-heap
  - Time: O(|E| log|V|)

Proof of Correctness of Prim's MST Algorithm

Claim: For $1 \le i \le |V|$, let T_i be the partially built tree in Prim's algorithm on input G = (V, E) after the i^{th} iteration (T_i has i vertices.). T_i is contained in some minimum spanning tree of G.

Proof by induction on i

- **Basis step**: i = 1. T_1 contains one vertex and no edges. Since every minimum spanning tree of G contains every vertex in V, T_1 is clearly contained in some minimum spanning tree of G.

- **Induction hypothesis**: Suppose that T_i is contained in some minimum spanning tree of G, for some fixed $1 \le i \le |V| - 1$.

Proof of Prim's MST Algorithm (cont.)

- **Induction step**: We need to show that T_{i+1} is contained in some minimum spanning tree of G.
  - Let the edge (v, w) be the one that Prim's algorithm chooses and adds to its partially built tree. Note that v is in the known set and w is not in the known set and that T_{i+1} is T_i with the vertex w and the edge (v, w) added.
  - By the induction hypothesis, T_i is contained in some minimum spanning tree. Call this minimum spanning tree T_i^*. If T_i^* contains the edge (v, w), then T_{i+1} is contained in the minimum spanning tree T_i^* and we are done with the induction step.

Proof of Prim's MST Algorithm (cont.)

- So suppose (v, w) is not in T_i*. We now show how to construct a minimum spanning tree that contains T_{i+1}. T_i* contains the edges in T_i. Since T_i * is a tree that spans G, then there is some edge not equal to (v, w), call this edge e, that connects the vertices of T_i to the rest of T_i *. Note that e has one endpoint in the known set after iteration i and one endpoint not in the known set. The following figure illustrates the situation.

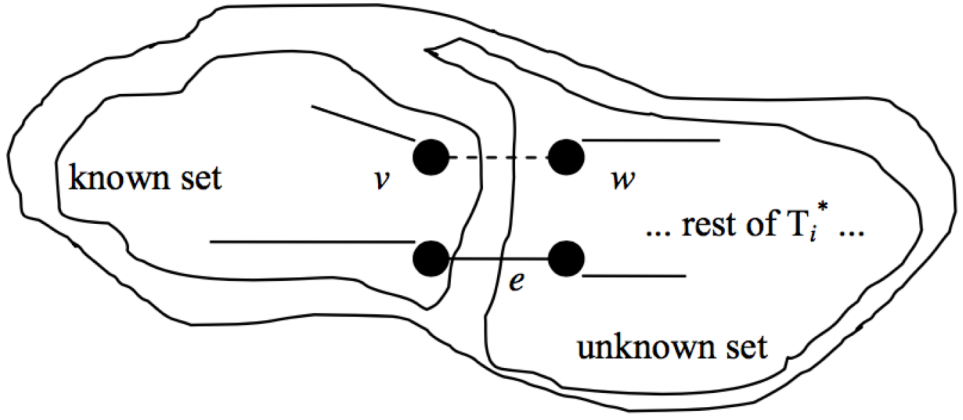
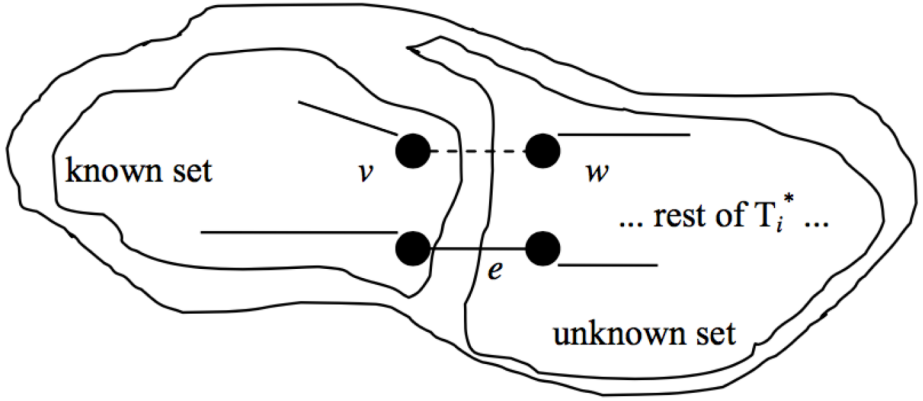


Figure 1. The situation after the *i*th iteration in Prim's algorithm. The known set comprises T_i. The solid black edges are in T_i^*. Prim's algorithm chooses edge (v, w) to add to T_i. There may be other edges from the known set to the unknown set.

Proof of Prim's MST Algorithm (cont.)

- The cost of e cannot be less than that of (v, w) because it would contradict the fact that Prim's algorithm chooses an edge that connects a known vertex to an unknown vertex with least cost. That is, if e had cost less than (v, w), then Prim's algorithm would have chosen e rather than (v, w) to add to its partially built tree.
- The cost of e cannot be greater than that of (v, w) because in T_i^* if we replace e with (v, w), then we have a spanning tree of G with a cost less than that of T_i^*(we have replaced e with an edge with less cost), which contradicts the fact that T_i^* is a minimum spanning tree.



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Proof of Prim's MST Algorithm (cont.)

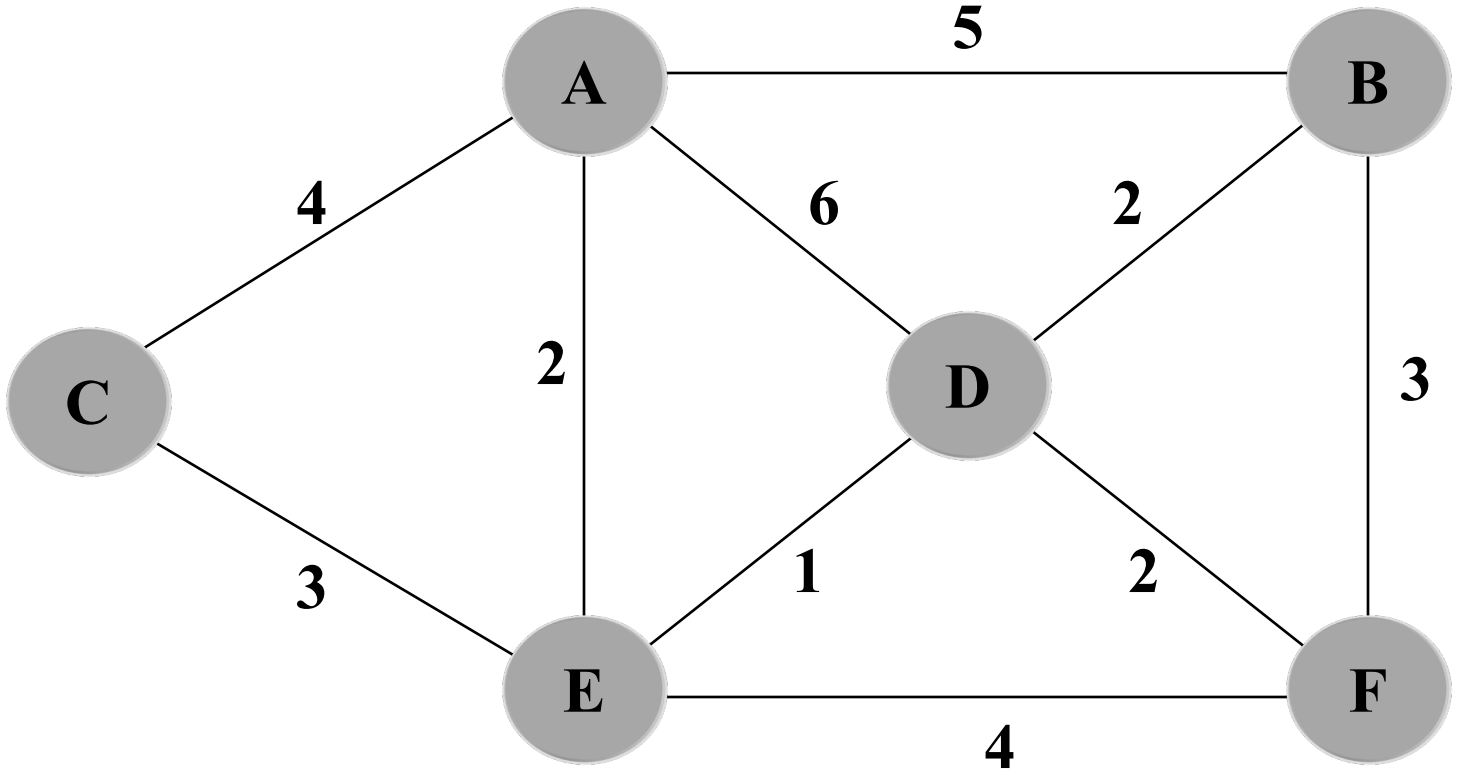
- So, e's cost must be the same as that of (v, w). If we take T_i^* and replace e with (v, w), we have a spanning tree of G with cost the same as that of T_i^*. Hence, this new tree is a minimum spanning tree and it contains T_i and (v, w). That is, this minimum spanning tree contains T_{i+1}.

- This concludes the proof by induction.

Another greedy algorithm for MST: Kruskal's

- Sort the edges in increasing order of lengths
- "Grow" tree one edge at a time to produce MST through a series of expanding forests $F_1, F_2, \ldots, F_{n-1}$
- On each iteration, add the next edge on the sorted list unless this would create a cycle. (If it would, skip the edge.)

Illustration of Kruskal's Algorithm



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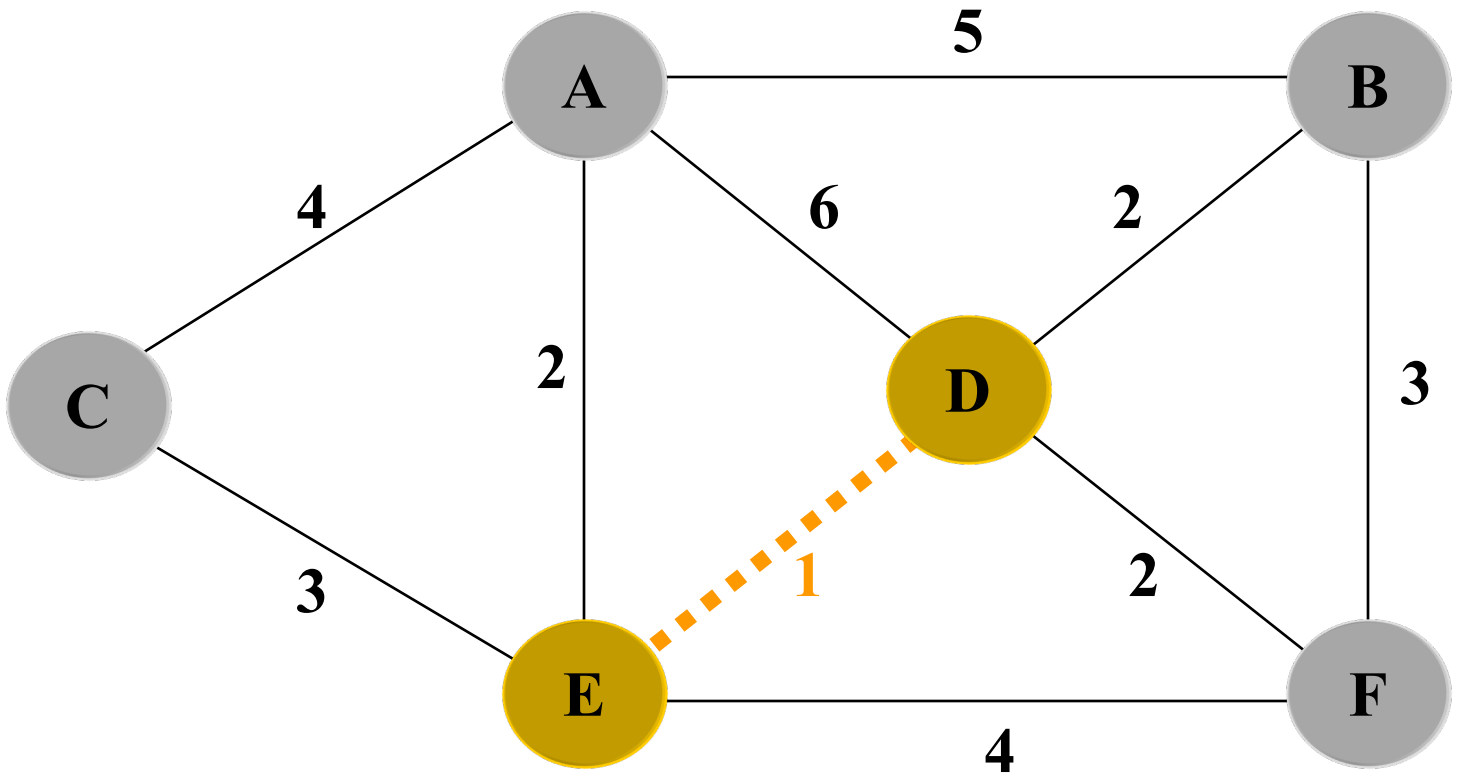

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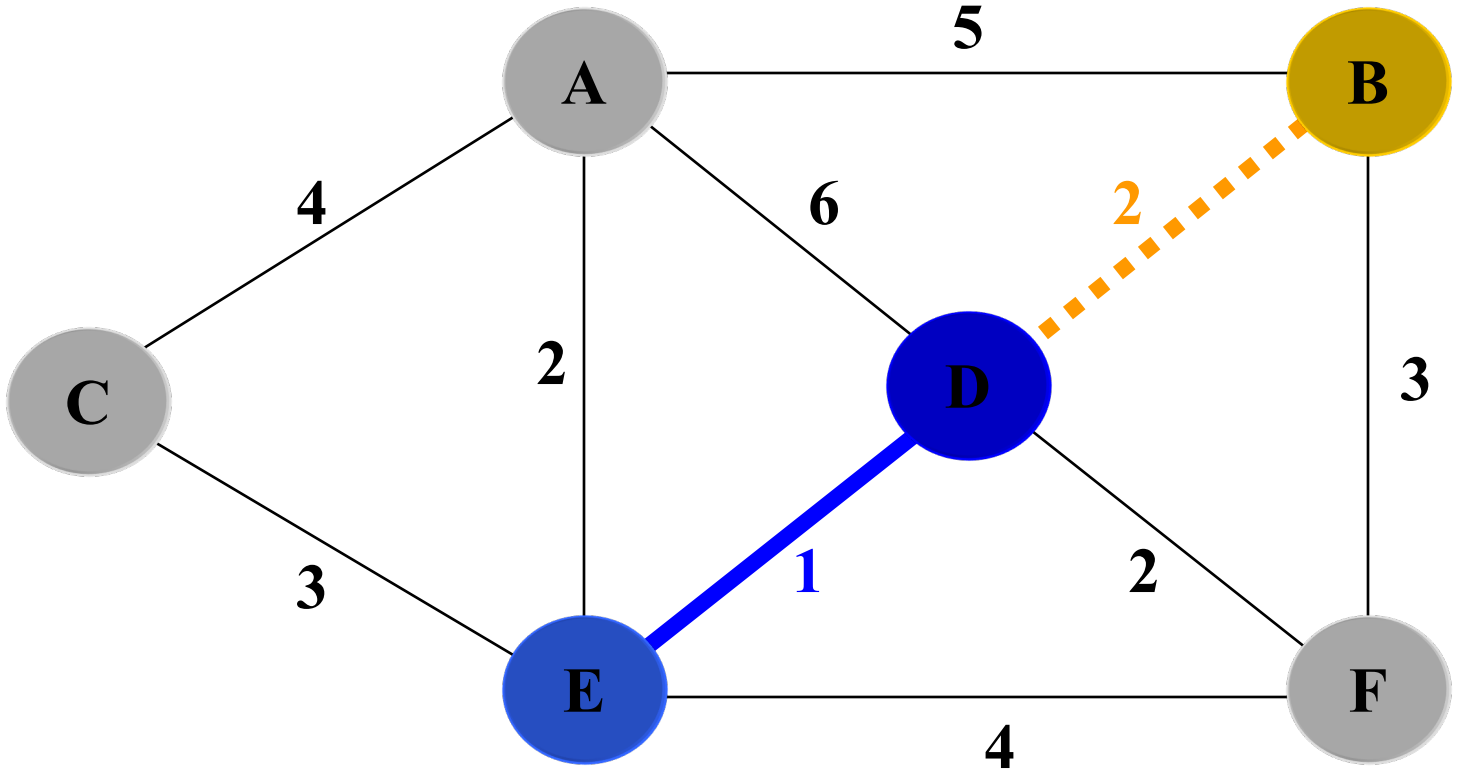

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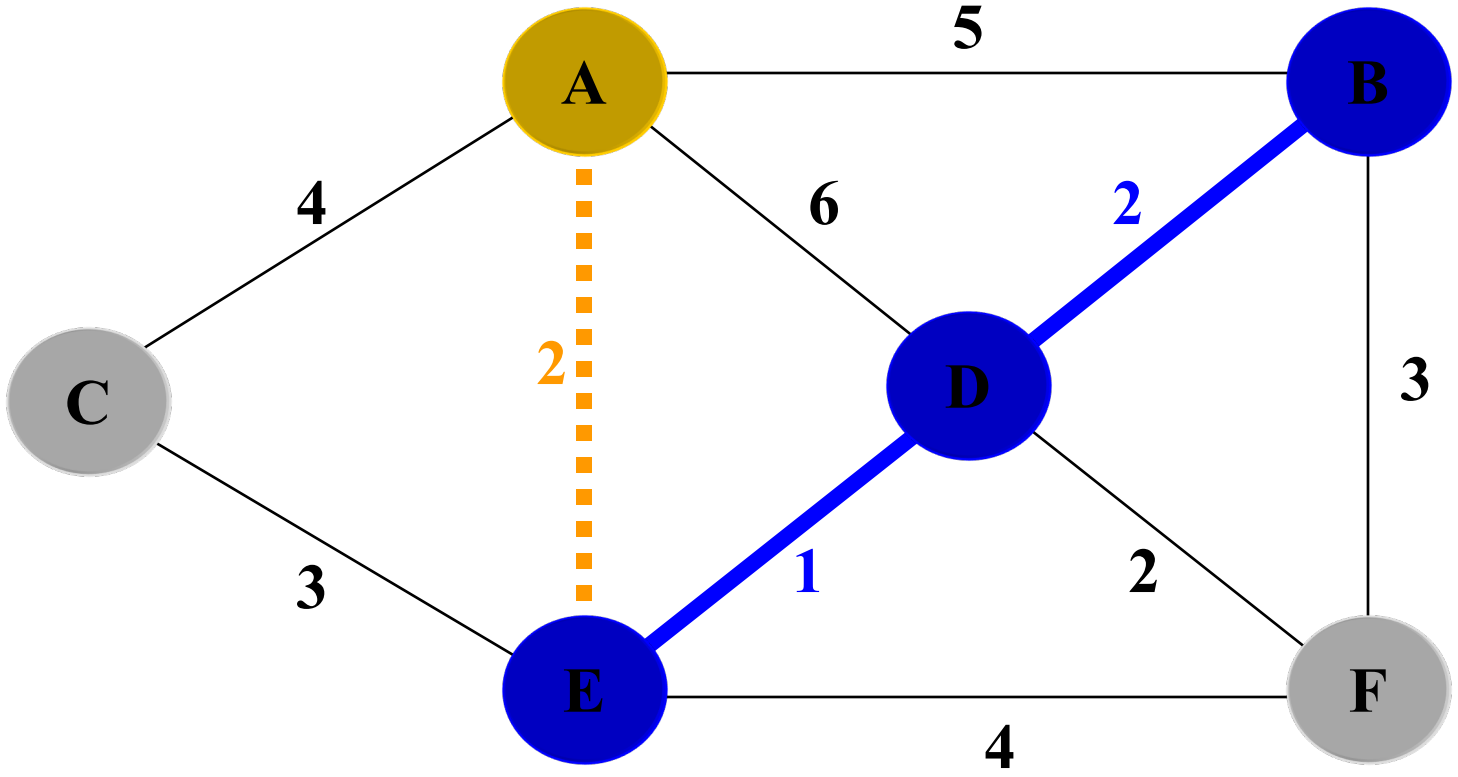

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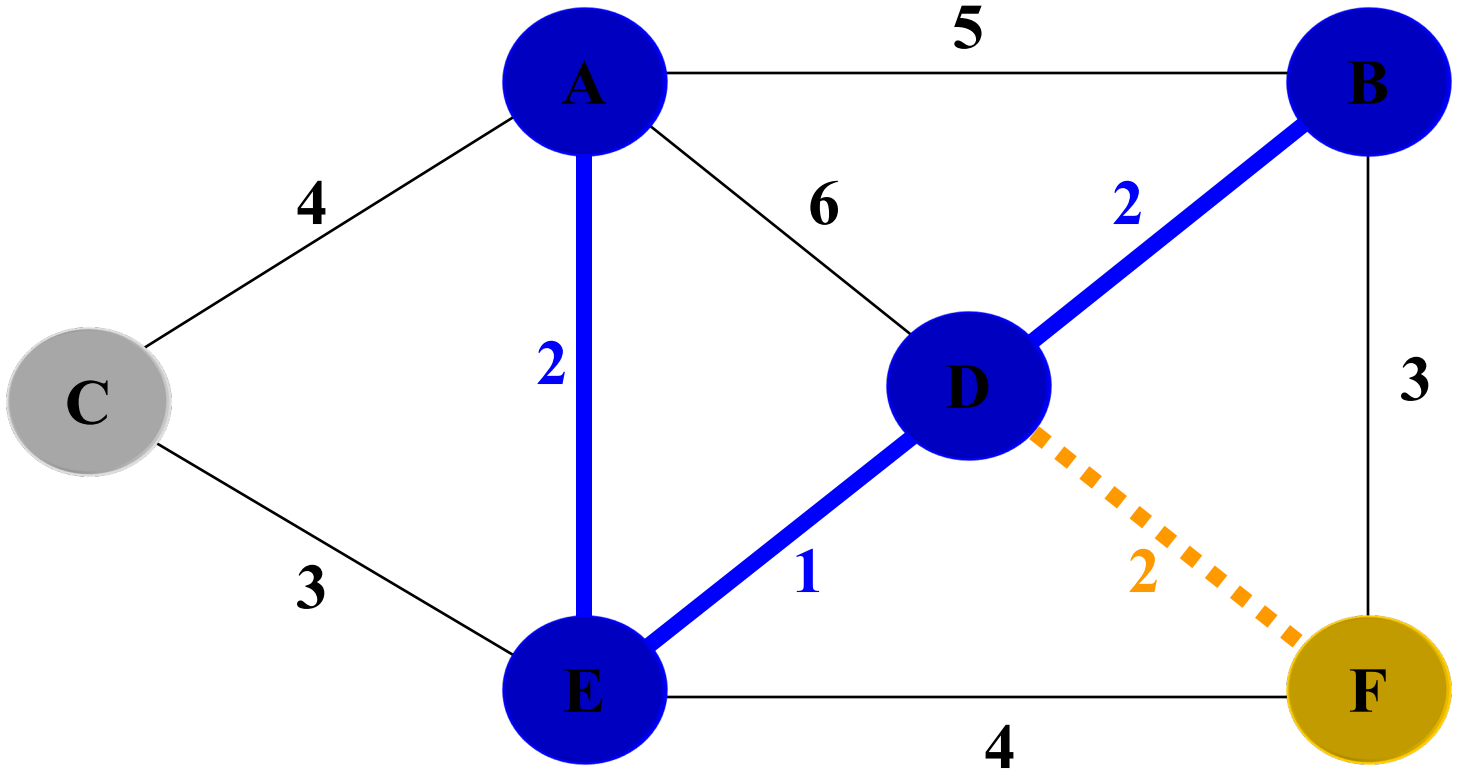

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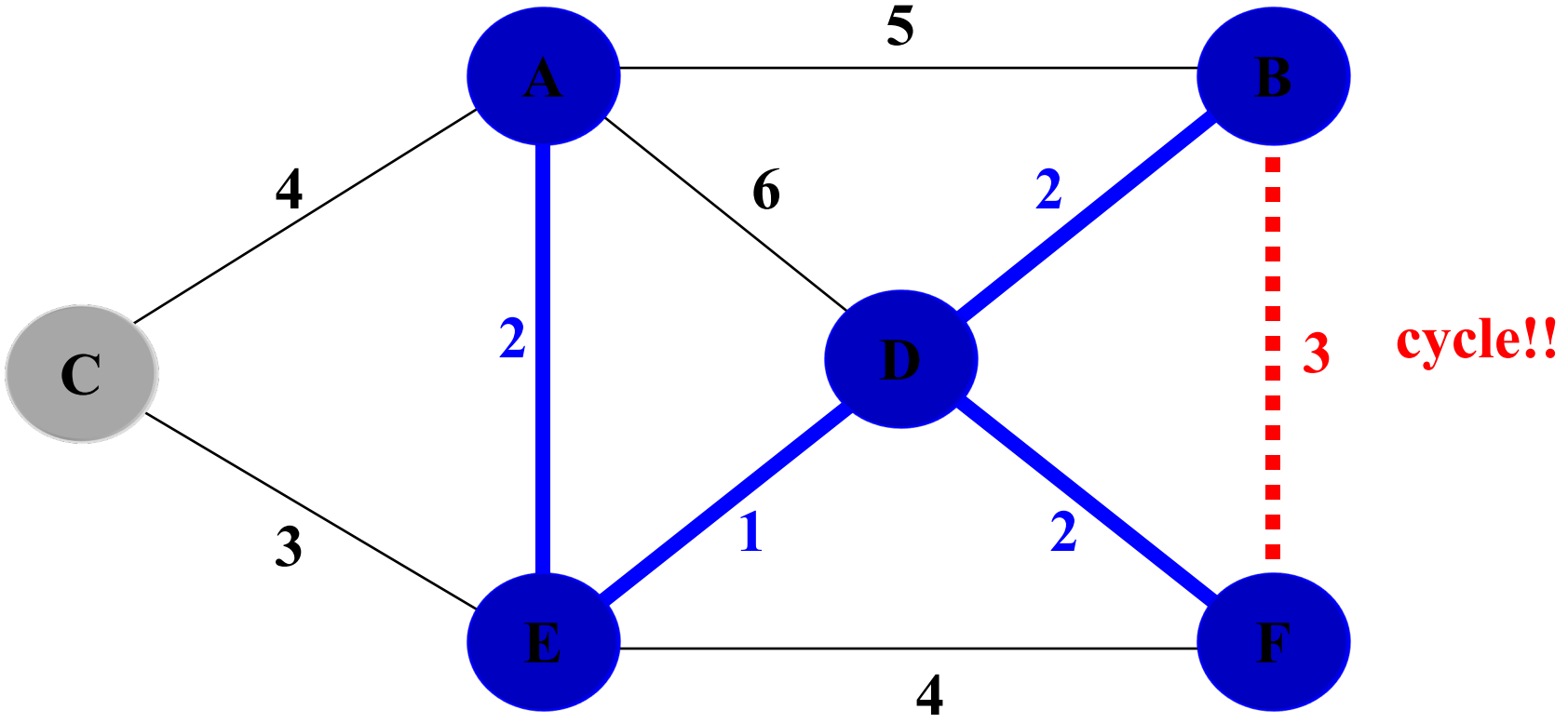

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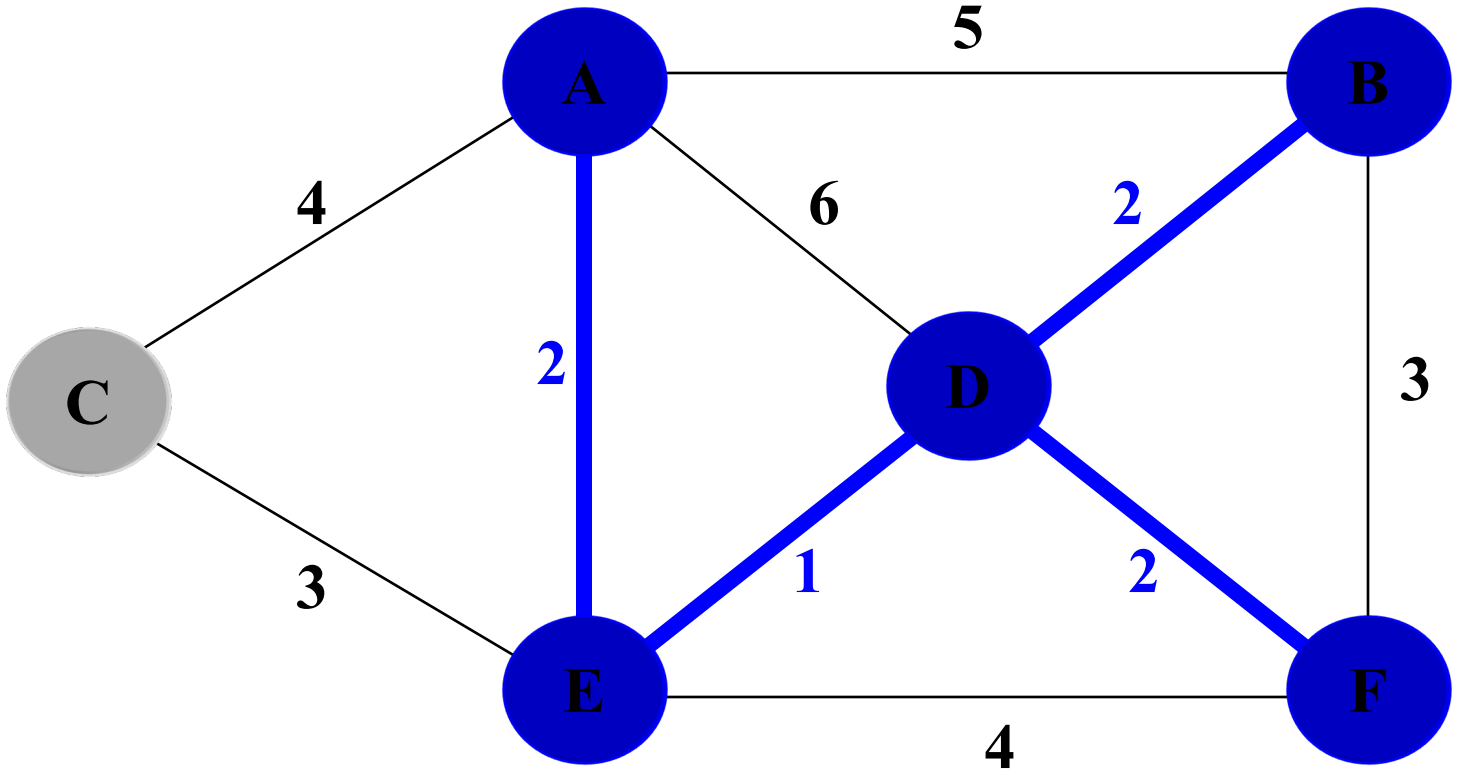

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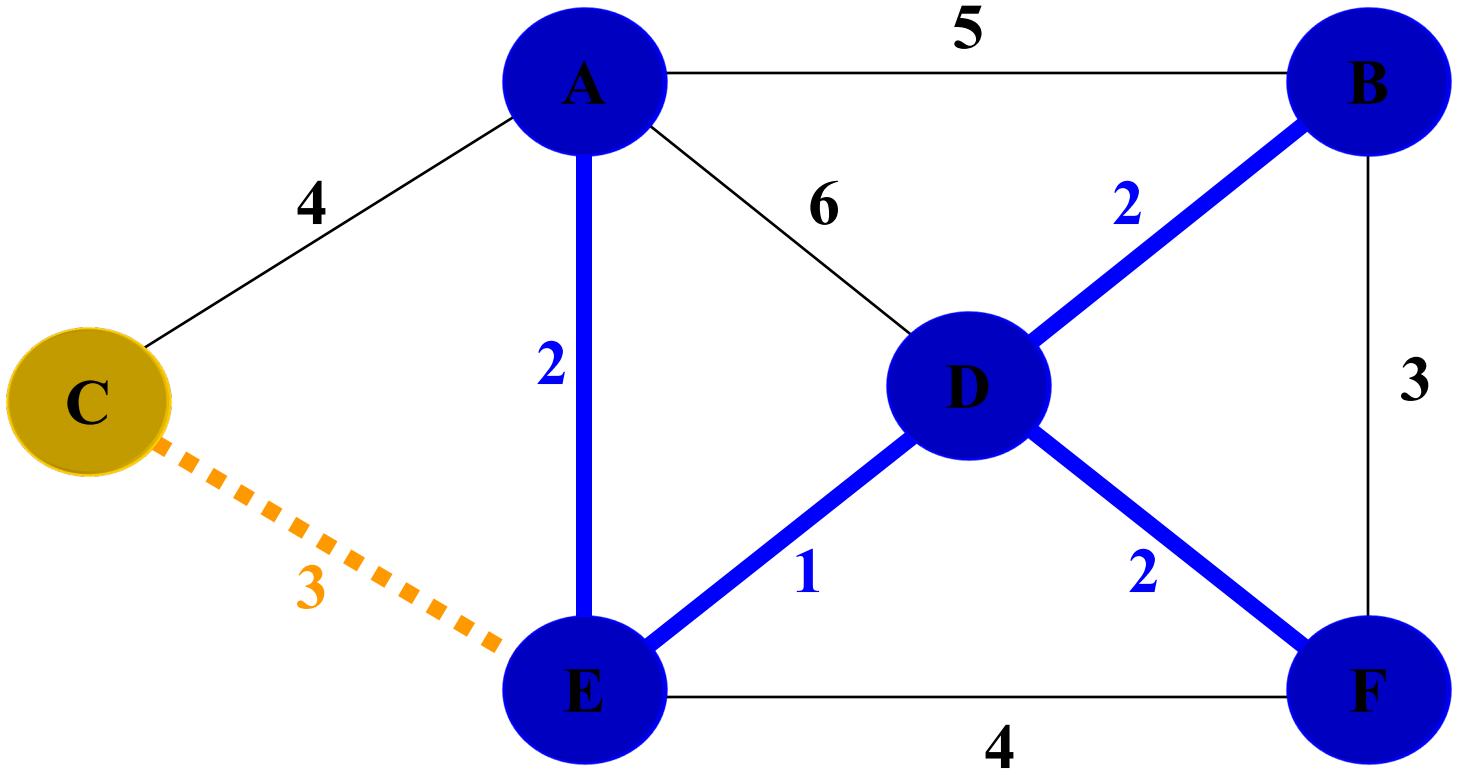
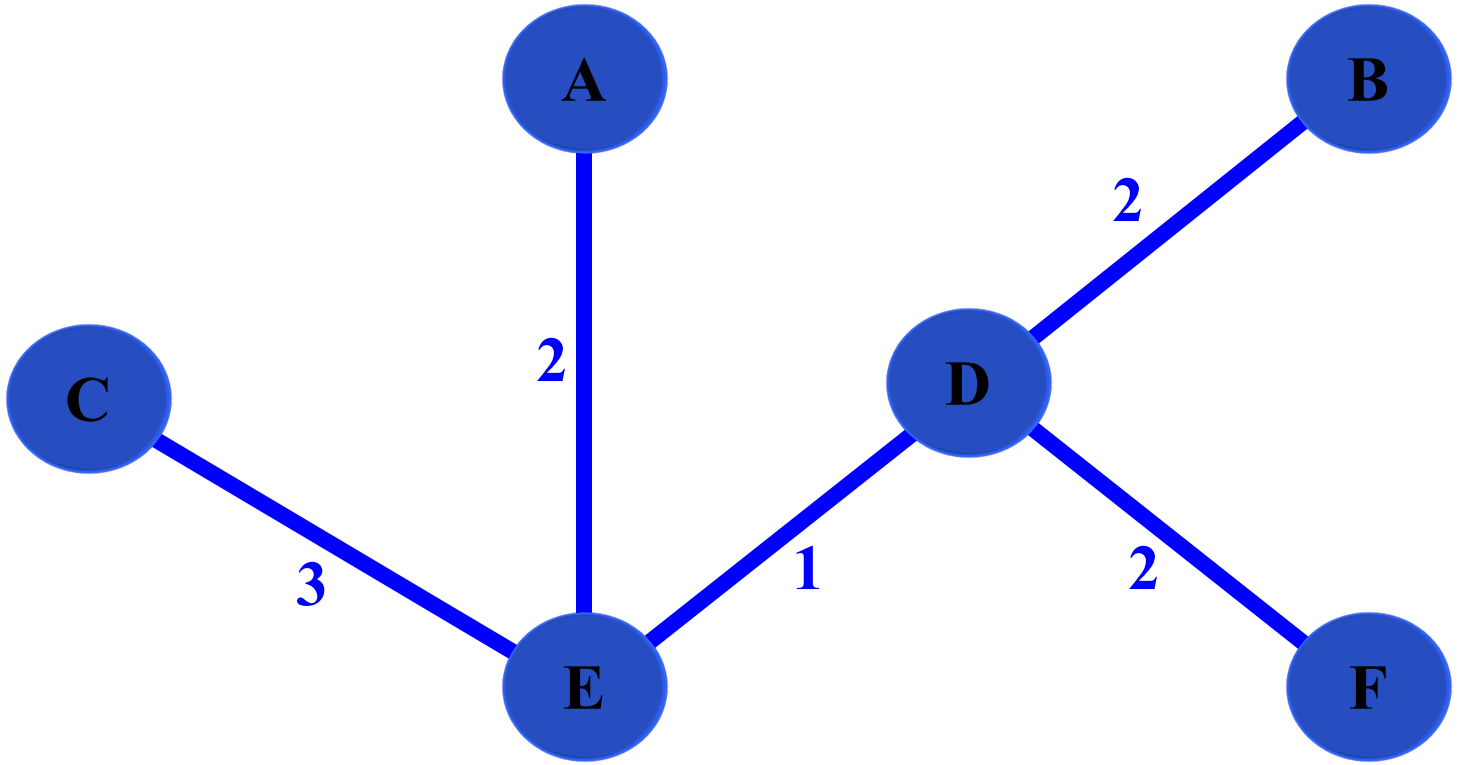


Illustration of Kruskal's Algorithm

Minimum- Spanning Tree



Kruskal's Algorithm

ALGORITHM *Kruskal*(G)

//Kruskal's algorithm for constructing a minimum spanning tree
//Input: A weighted connected graph $G = \langle V, E \rangle$
//Output: E_T, the set of edges composing a minimum spanning tree of G
sort E in nondecreasing order of the edge weights $w(e_{i_1}) \leq \ldots \leq w(e_{i_{|E|}})$
$E_T \leftarrow \emptyset$; $ecounter \leftarrow 0$ //initialize the set of tree edges and its size
$k \leftarrow 0$ //initialize the number of processed edges
while $ecounter < |V| - 1$ **do**
    $k \leftarrow k + 1$
    if $E_T \cup \{e_{i_k}\}$ is acyclic
        $E_T \leftarrow E_T \cup \{e_{i_k}\}$; $ecounter \leftarrow ecounter + 1$
return E_T

Notes about Kruskal's algorithm

- Algorithm looks easier than Prim's but is harder to implement (checking for cycles!)

- Cycle checking: a cycle is created iff added edge connects vertices in the same connected component

Disjoint-set Data Structure

- More efficient approach:
  - Employ *Disjoint Subsets and Union-Find* Algorithms

- First studied by Bernerd Galler and Michael J. Fischer in 1964

Bernie Galler, 2004

Professor, University of Michigan

Michael J. Fischer
Professor, Yale Univ.

Disjoint-set Data Structure

- Two sets are said to be **disjoint** if they have no elements in common
- A set of elements partitioned into a number of disjoint subsets
- Initially, each element *e* is a set in itself:
  - e.g., { $\{e_1\}$, $\{e_2\}$, $\{e_3\}$, $\{e_4\}$, $\{e_5\}$, $\{e_6\}$, $\{e_7\}$}

Operations: Union

- Union(*x*, *y*) – Combine or merge two sets *x* and *y* into a single set
  - Before:

    {{e_3, e_5, e_7} , {e_4, e_2, e_8}, {e_9}, {e_1, e_6}}

  - After Union(e_5, e_1):

    {{e_3, e_5, e_7, e_1, e_6} , {e_4, e_2, e_8}, {e_9}}

Operations: Find

- Determine which set a particular element is in
  - Useful for determining if two elements are in the same set

- Each set has a unique name
  - name is arbitrary; what matters is that find(*a*) == find(*b*) is true only if *a* and *b* in the same set
  - one of the members of the set is the "representative" (i.e. name) of the set: {{e_3, e_5, e_7, e_1, e_6} , {e_4, e_2, e_8}, {e_9}}

Operations: Find (cont.)

- Find(*x*) – return the name of the set containing *x*.
- Examples
  - {{e_3, e_5, e_7, e_1, e_6} , {e_4, e_2, e_8}, {e_9}}
  - Find(e_1) = e_5
  - Find(e_4) = e_8

Kruskal's Algorithm Implementation (Revisited)

Kruskals():

sort edges in increasing order of length (e_1, e_2, e_3, ..., e_m).

initialize disjoint sets.

T := {}.

for i = 1 to m

let e_i = (u, v).

if find(u) != find(v)

union(find(u), find(v)).

add e_i to T.

return T.

What does the disjoint set initialize to?
How many times do we do a union?
How many time do we do a find?
What is the total running time if we have *n* nodes and *m* edges?

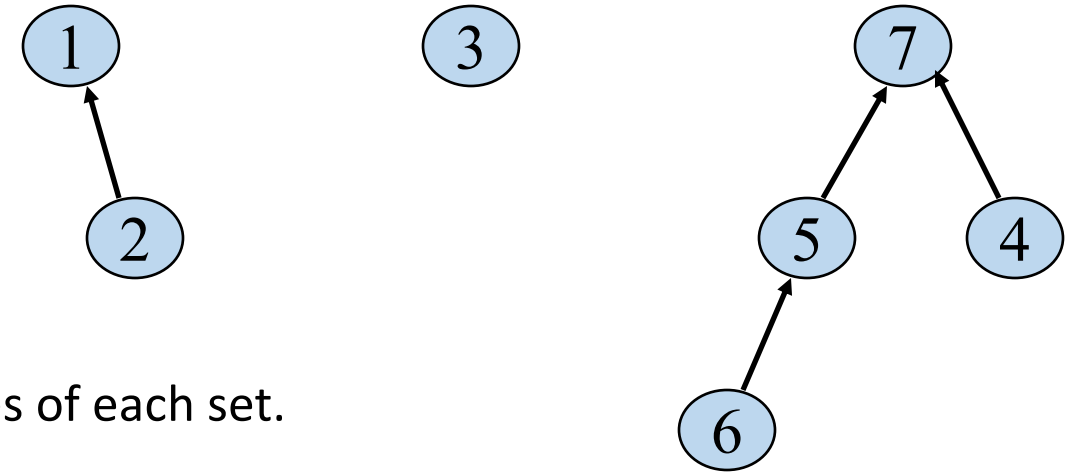
Disjoint Sets with Trees

- Observation: *trees* let us find many elements given one root (i.e. representative)…
- Idea: if we *reverse* the pointers (make them point up from child to parent), we can find a single root from many elements…
- Idea: Use one tree for each subset. The name of the class is the tree root.

Up-Tree for Disjoint Sets

Initial state

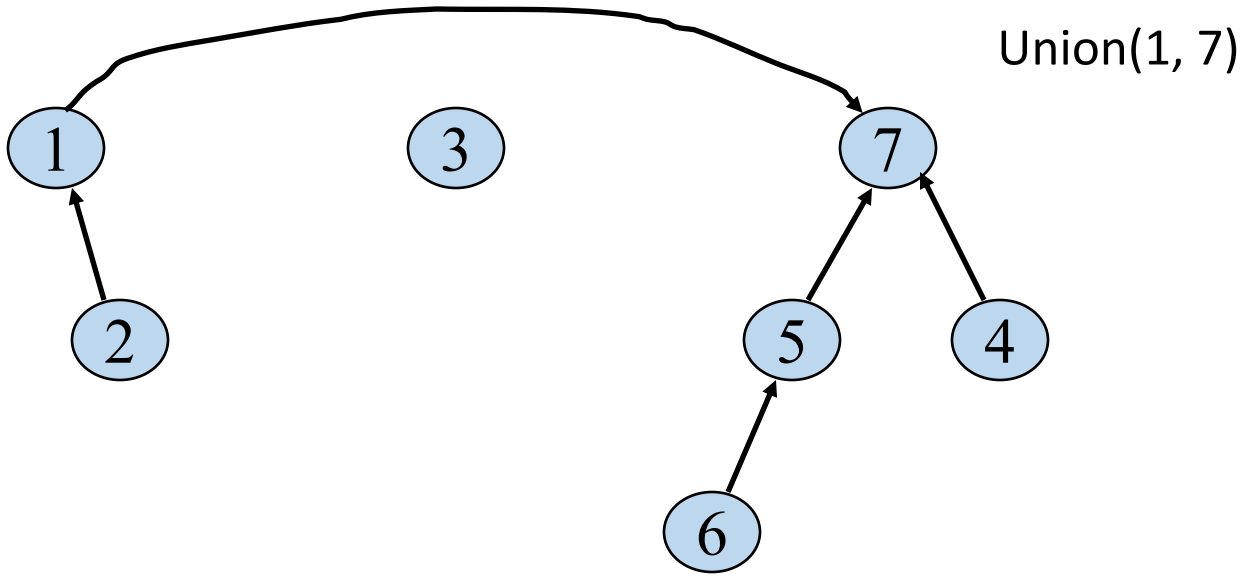
Intermediate state



Roots are the names of each set.

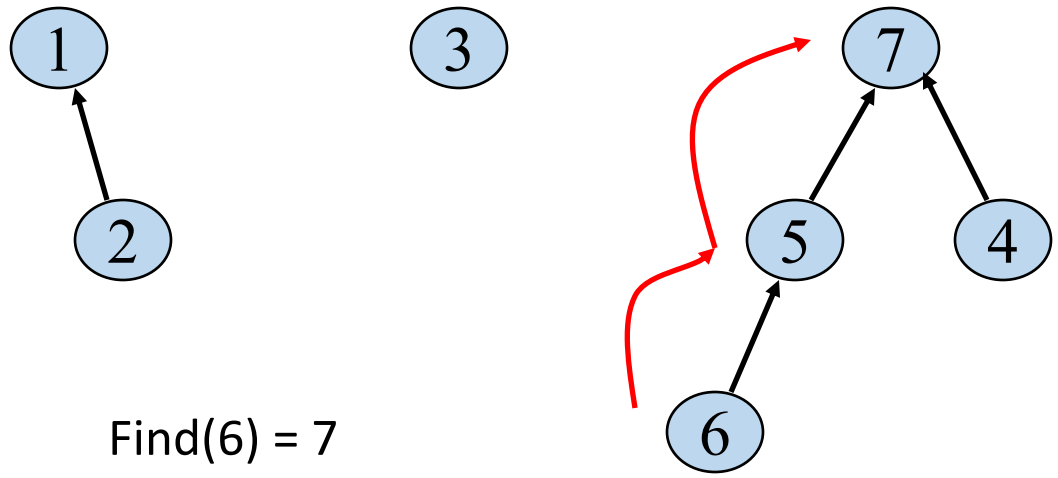
Union Operation

- Union(x, y) – assuming x and y roots, point x to y.



Find Operation

- Find(*x*): follow *x* to root and return root



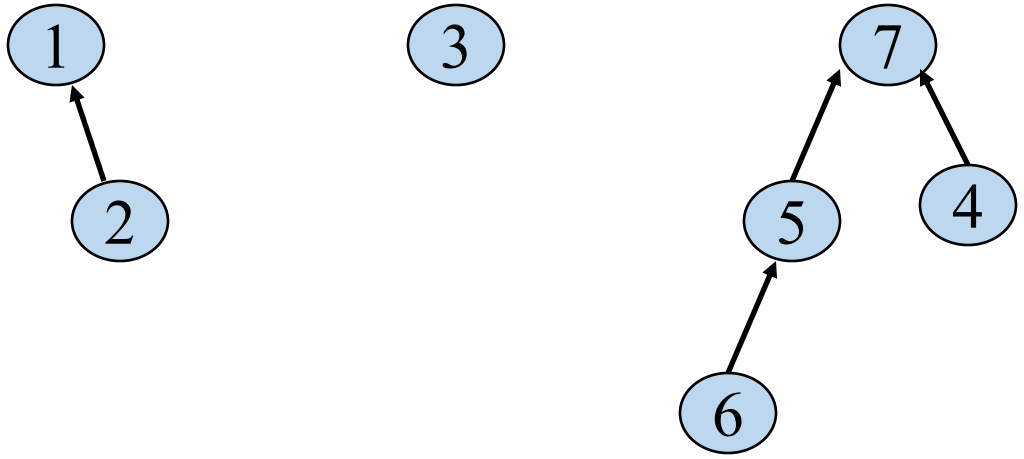
Find(6) = 7

Simple Implementation

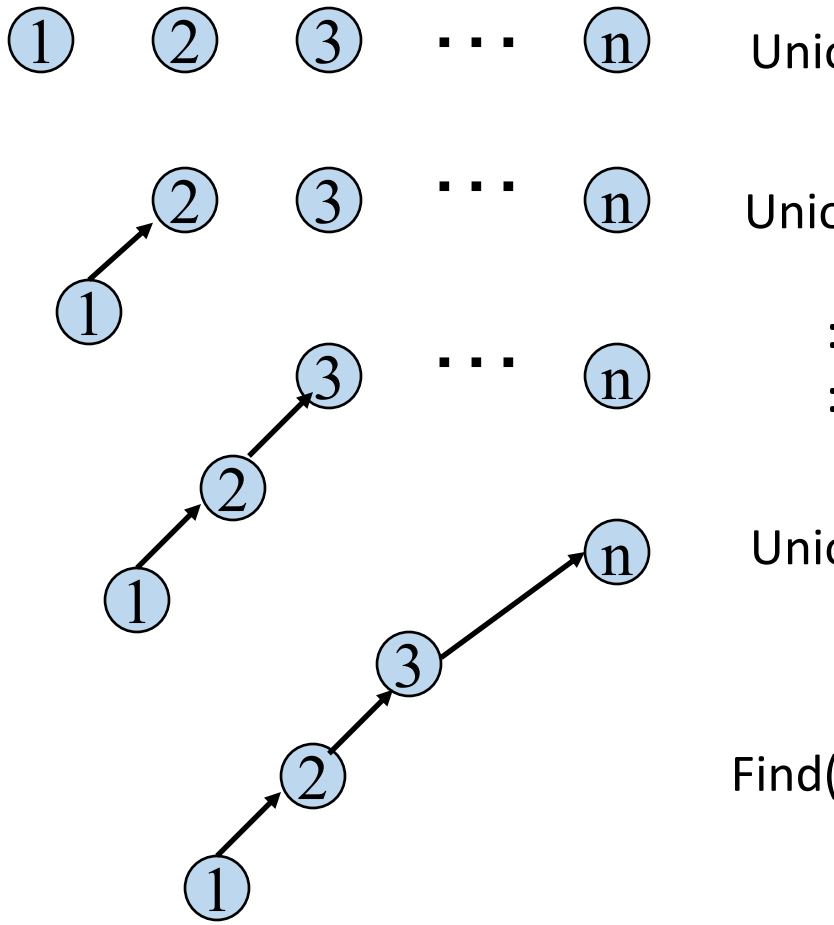
- Array of indices

	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0

Up[x] = 0 means
x is a root.



A Bad Case



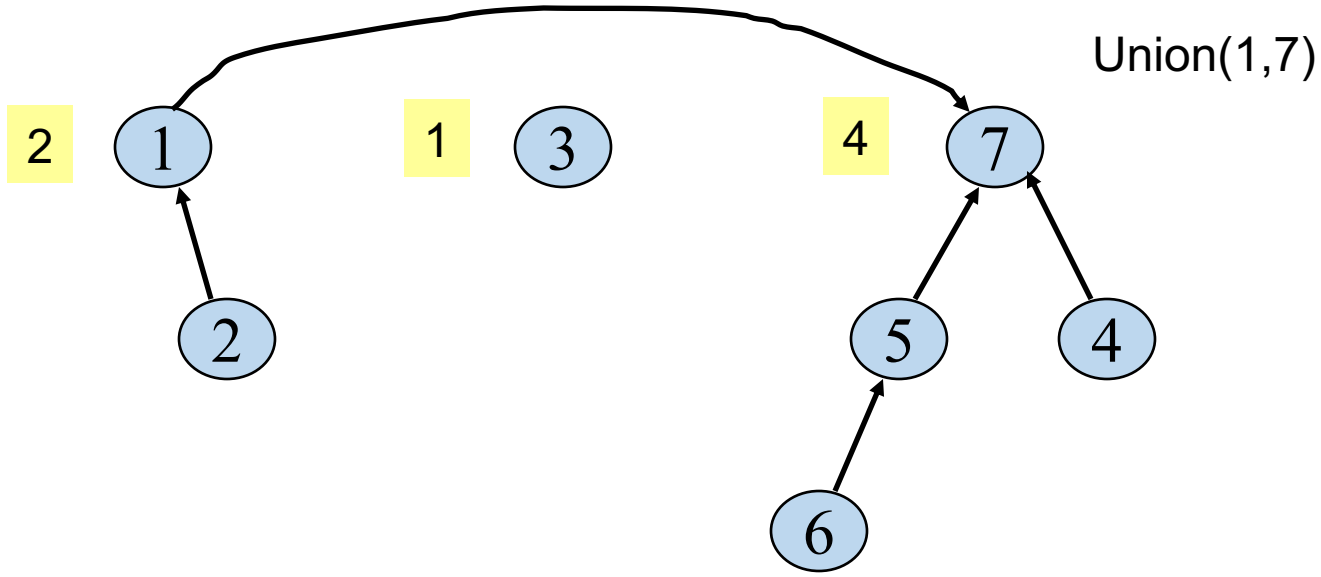
Improving Find

1. Improve union so that *find* only takes $\Theta(\log n)$
   - Union-by-size
   - Reduces complexity to $\Theta(m \log n + n)$

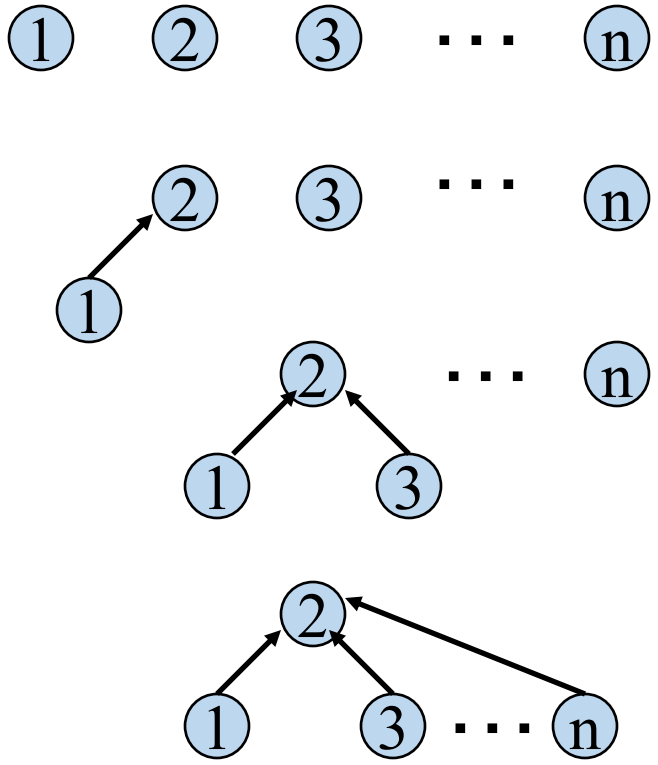
2. Improve find so that it becomes even better!
   - Path compression
   - Reduces complexity to <u>almost</u> $\Theta(m + n)$

Union by Rank

- Union by Rank (also called Union by Size)
  - Always point the smaller tree to the root of the larger tree



Example Again



Union(1,2)

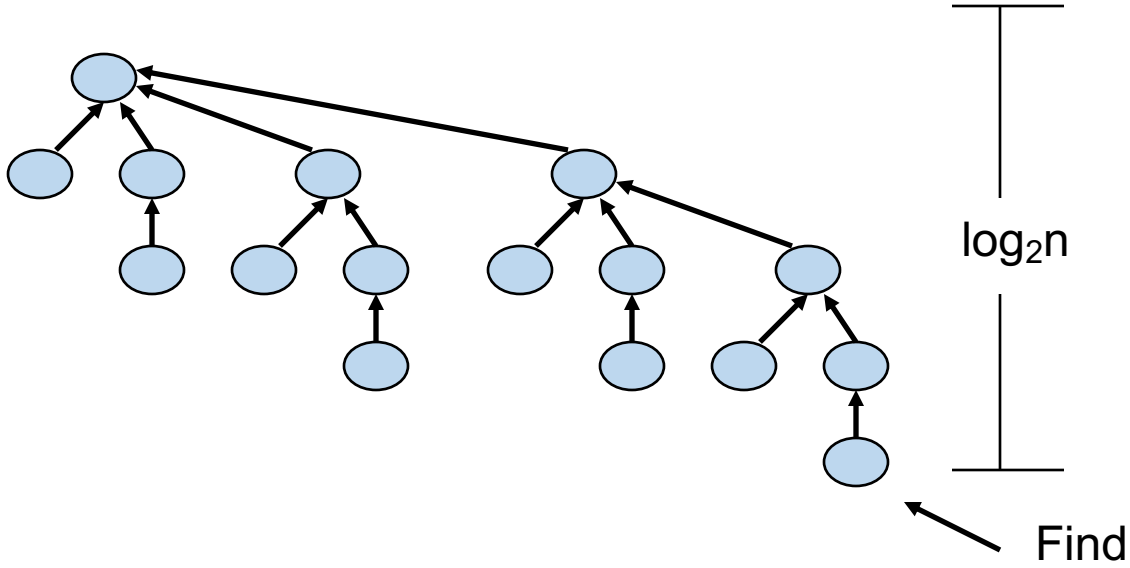
Union(2,3)

Union(n-1,n)

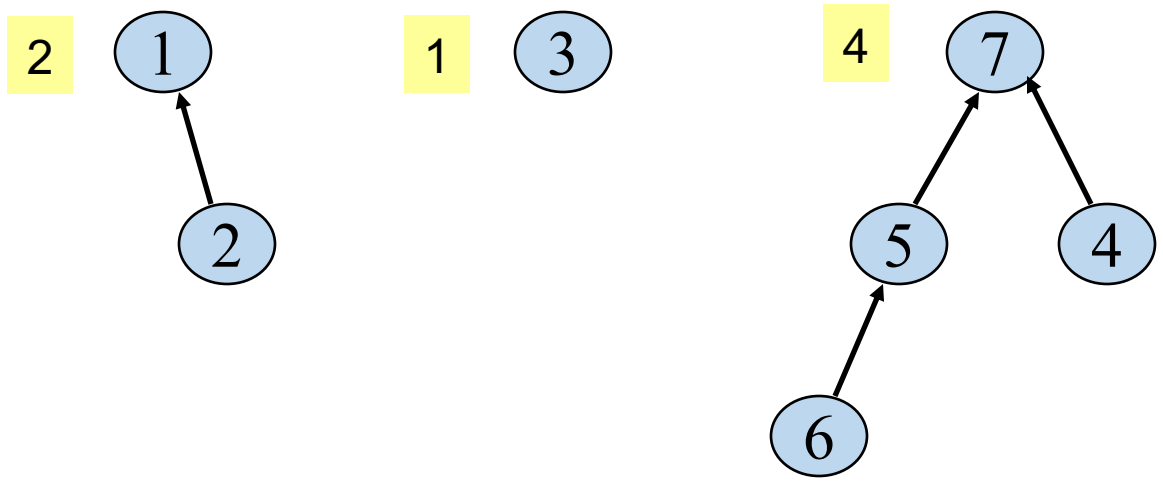
Find(1) constant time

Improved Runtime for Find via Union by Rank

- Depth of tree affects running time of Find
- Union by rank only increases tree depth if depth were equal
- Results in O(log *n*) for Find



Elegant Array Implementation



	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0
weight	2		1				4

Union by Rank

```
Algorithm Union(index i,j)
  //i and j are roots
  w_i <- weight[i]
  w_j <- weight[j]
  if w_i < w_j then
    up[i] <- j
    weight[j] <- w_i + w_j
  else
    up[j] <- i
    weight[i] <- w_i + w_j
```

Kruskal's Algorithm Implementation (Revisited)

Kruskals():
  sort edges in increasing order of length (e_1, e_2, e_3, ..., e_m).

  initialize disjoint sets.

  T := {}.

  for i = 1 to m
    let e_i = (u, v).
    if find(u) != find(v)
      union(find(u), find(v)).
      add e_i to T.

  return T.

How many times do we do a union?
How many time do we do a find?
What is the total running time if we have *n* nodes and *m* edges?

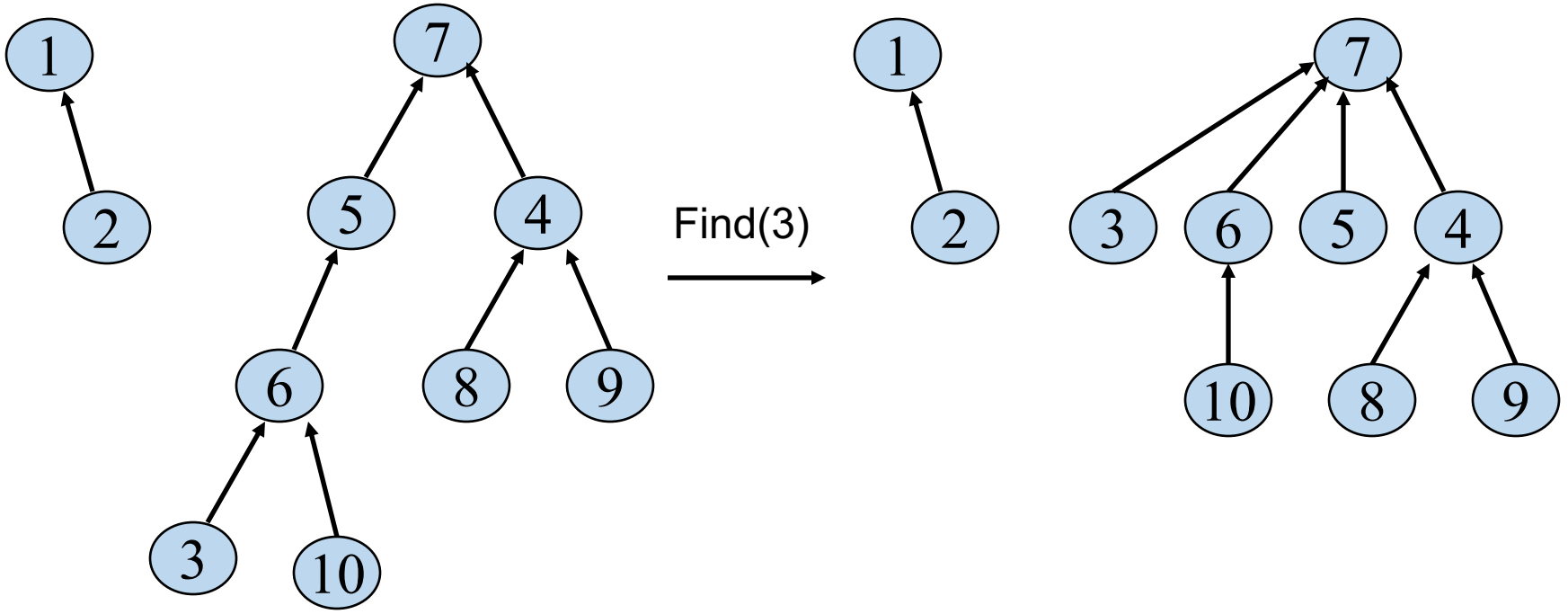
Kruskal's Algorithm Running Time

- Assuming |E| = m edges and |V| = n nodes
- Sort edges: O(m log m)
- Initialization: O(n)
- Finds: O(2 * m * log n) = O(m log n)
- Unions: O(m)

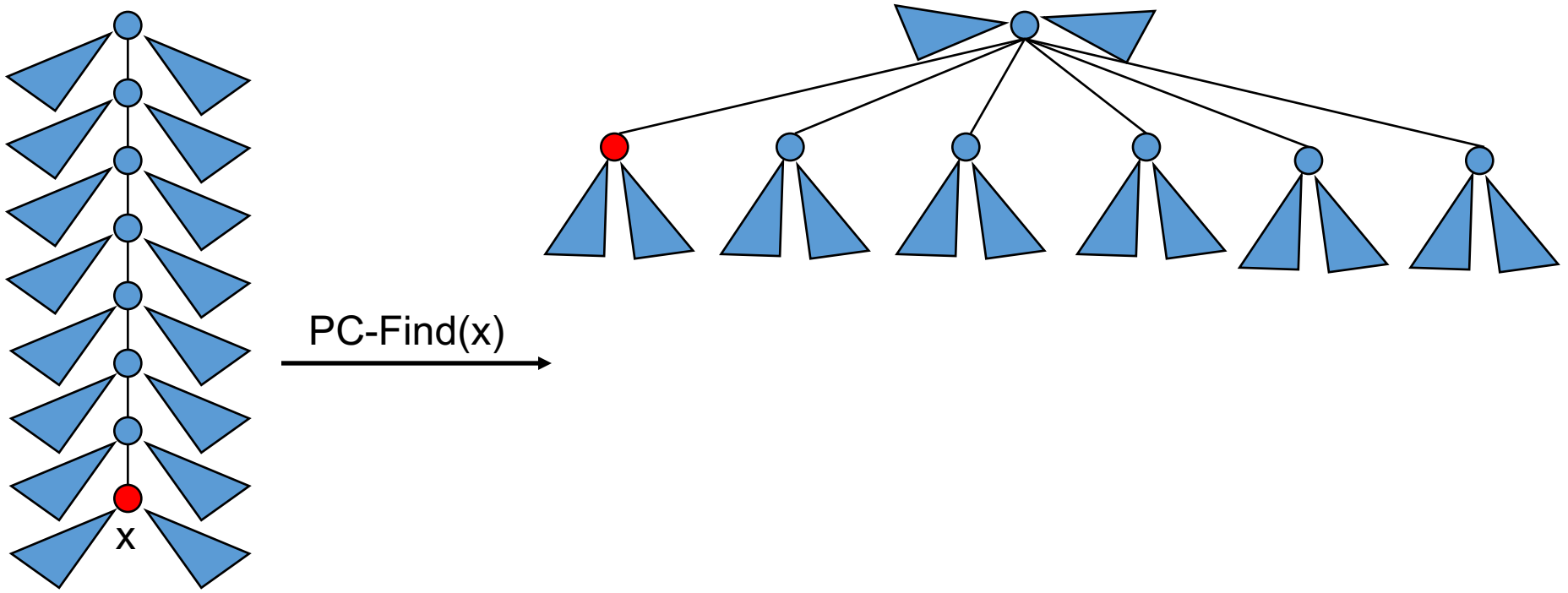
- Total running time: O (m log n + n + m log n + m) = O(m log n)
  - note: log n and log m are within a constant factor of one another

Path Compression

- On a Find operation point all the nodes on the search path directly to the root.



Self-Adjustment Works



Disjoint Union / Find with Union By Rank and Path Compression

- Worst case time complexity
  - Union (using Union by Rank): $\Theta(1)$
  - Find (using Path Compression): $\Theta(\log n)$.

- Time complexity for $m \geq n$ operations on n elements is $\Theta(m \log^* n)$
  - log * is the number of times you need to apply the log function before you get to a number <= 1
  - log * n < 5 for all reasonable n. Essentially constant time per operation!

n	$\lg^* n$
1	0
2	1
$(3, 4]$	2
$[5, 16]$	3
$[17, 65536]$	4
$[65537, 2^{65536}]$	5

Amortized Complexity

- For disjoint union / find with union by rank and path compression
  - average time per operation is essentially a constant
  - worst case time for a Find is $\Theta(\log n)$

- An individual operation can be costly, but over time the average cost per operation is not

- This means the bottleneck of Kruskal's actually becomes the sorting of the edges

Outline

- Dijkstra's algorithm
- Huffman's algorithm

Shortest paths – Dijkstra's algorithm

Single Source Shortest Paths Problem: Given a weighted connected graph G, find shortest paths from source vertex *s* to each of the other vertices

Dijkstra's algorithm was published in 1959.

Implementation of Dijkstra's algorithm:

- $O(m + n^2)$ Dijkstra'59

Edsger Wybe Dijkstra (1930 – 2002), PhD, Professor Emeritus and Chair of Computer Sciences at University of Texas at Austin, Turing Award recipient (1972)

http://acm.math.spbu.ru/~sk1/download/papers/Thorup-1.pdf
https://en.wikipedia.org/wiki/Edsger_W._Dijkstra

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Implementation of Dijkstra's algorithm:

- $O(m + n^2)$ Dijkstra'59
- O(m log n) William'64
- O(m + n log n) Fredman and Tarjan'87
- ……
- linear time and linear space algorithm (undirected graph), Mikkel Thorup'99

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Shortest paths – Dijkstra's algorithm

Dijkstra's algorithm: Similar to Prim's MST algorithm, with a different way of computing numerical labels: Among vertices not already in the tree, it finds vertex u with the smallest sum

$$d_v + w(v,u)$$

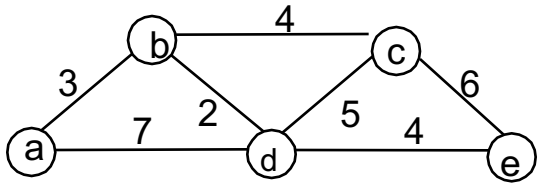
where

v is a vertex for which shortest path has already been found on preceding iterations (such vertices form a tree)

d_v is the length of the shortest path from source to v

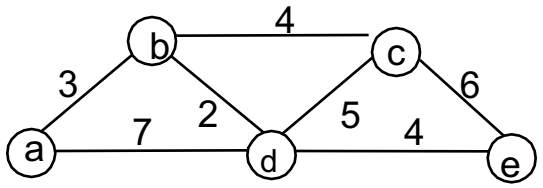
$w(v,u)$ is the length (weight) of edge from v to u

Example



Find shortest path from *s* to nearest *v*, then to 2nd nearest, etc.

Example



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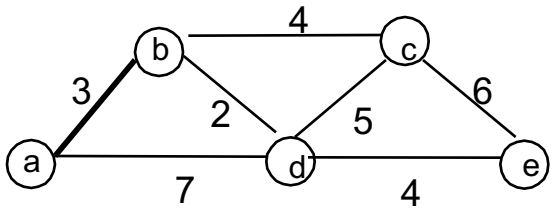
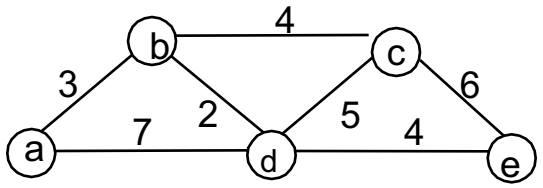
Tree vertices	Remaining vertices
a(-,0)	**b(a,3)** c(-,∞) d(a,7) e(-,∞)



Example



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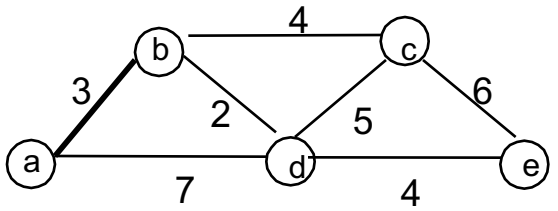
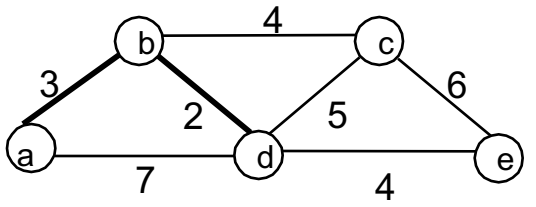
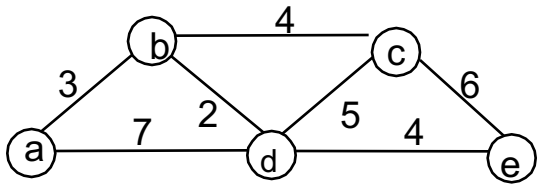




Example



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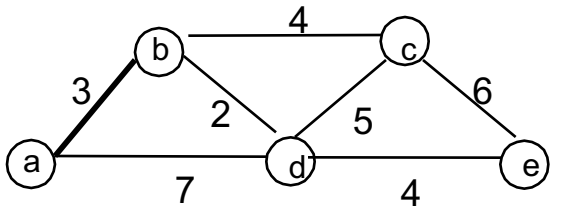
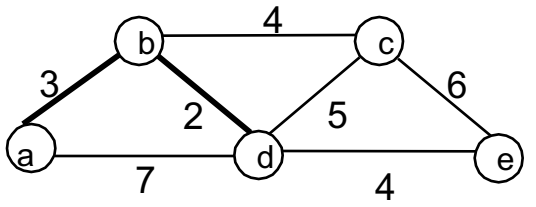
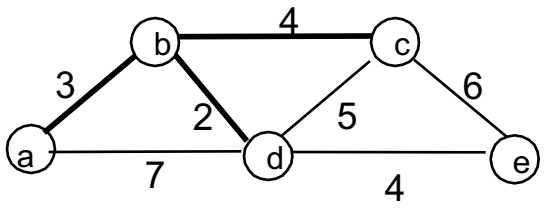
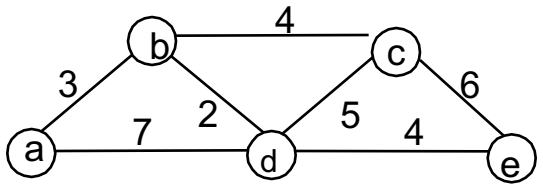
d(b,5)	**c(b,7)** e(d,5+4)







Example



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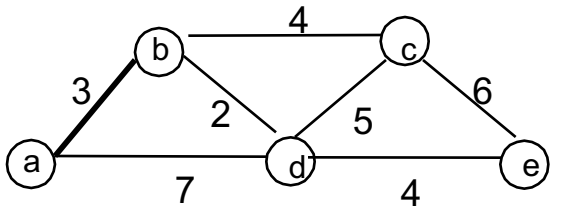
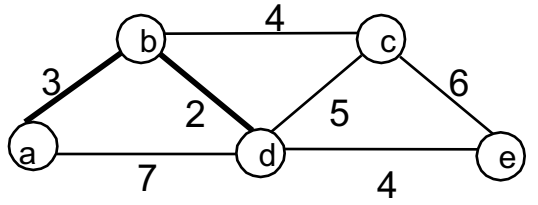
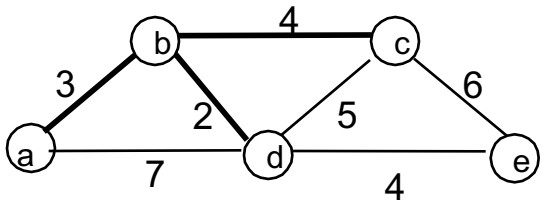
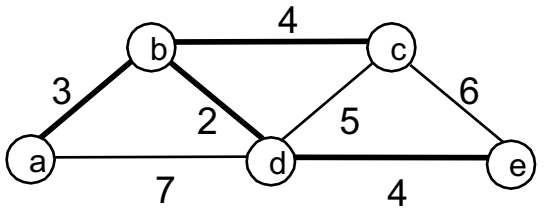
c(b,7)	**e(d,9)**

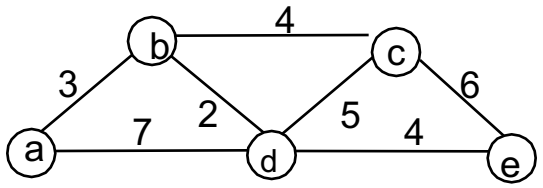








Example



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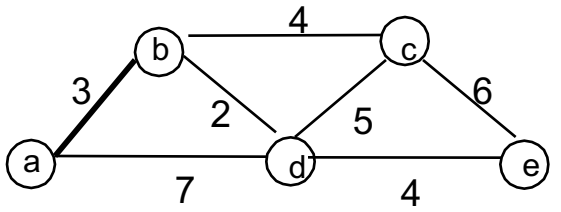
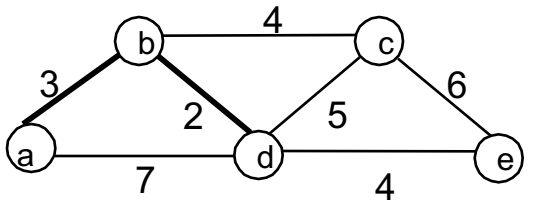
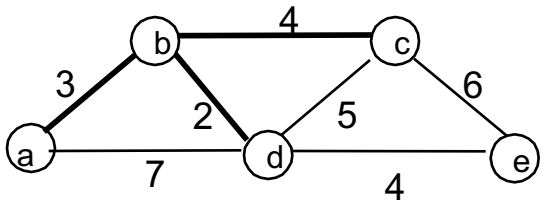
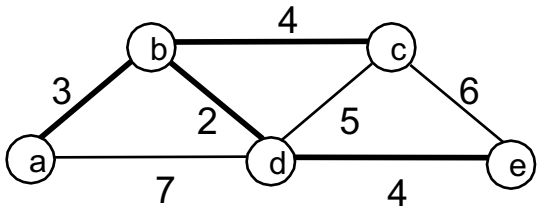
c(b,7)	**e(d,9)**
e(d,9)	









ALGORITHM *Dijkstra*(G, s)

```
//Dijkstra's algorithm for single-source shortest paths
//Input: A weighted connected graph G = ⟨V, E⟩ with nonnegative weights
//       and its vertex s
//Output: The length d_v of a shortest path from s to v
//        and its penultimate vertex p_v for every vertex v in V
Initialize(Q)  //initialize vertex priority queue to empty
for every vertex v in V do
    d_v ← ∞;  p_v ← null
    Insert(Q, v, d_v)  //initialize vertex priority in the priority queue
d_s ← 0;  Decrease(Q, s, d_s)  //update priority of s with d_s
V_T ← ∅
for i ← 0 to |V| − 1 do
    u* ← DeleteMin(Q)  //delete the minimum priority element
    V_T ← V_T ∪ {u*}
    for every vertex u in V − V_T that is adjacent to u* do
        if d_u* + w(u*, u) < d_u
            d_u ← d_u* + w(u*, u);  p_u ← u*
            Decrease(Q, u, d_u)
```

Notes on Dijkstra's algorithm

- Doesn't work for graphs with negative weights
- Applicable to both undirected and directed graphs
- Efficiency
  - $O(|V|^2)$ for graphs represented by weight matrix and array implementation of priority queue
  - $O(|E|\log|V|)$ for graphs represented by adj. lists and min-heap implementation of priority queue
- Don't mix up Dijkstra's algorithm with Prim's algorithm!

Dijkstra's Algorithm: Proof of Correctness

Invariant. For each node $u \in S$ (here we let $S=V_T$), d(u) is the length of the shortest s-u path.

Dijkstra's Algorithm: Proof of Correctness

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Proof by induction on |S|
Base case: |S| = 1 is trivial.

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Base case: |S| = 1 is trivial.
Inductive hypothesis: Assume true for $|S| = k \geq 1$.

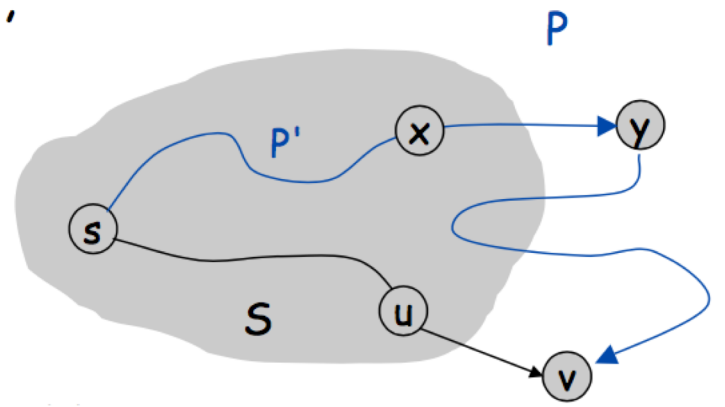
- Let v be next node added to S, and let u-v be the chosen edge.
- The shortest s-u path plus (u, v) is an s-v path of length d(v).

Dijkstra's Algorithm: Proof of Correctness

Invariant. For each node $u \in S$ (here we let $S=V_T$), d(u) is the length of the shortest s-u path.

Proof by induction on |S|
Base case: |S| = 1 is trivial.
Inductive hypothesis: Assume true for $|S| = k \geq 1$.

- Let v be next node added to S, and let u-v be the chosen edge.
- The shortest s-u path plus (u, v) is an s-v path of length d(v).
- Consider any s-v path P. We'll see that it's no shorter than d(v).
- Let x-y be the first edge in P that leaves S, and let P' be the subpath to x.
- P is already too long as soon as it leaves S.



$$d(P) \geq d(P') + w(x,y) \geq d(u) + w(u,y) \geq d(v)$$

Coding Problem

Coding: assignment of bit strings to alphabet characters

Codewords: bit strings assigned for characters of alphabet

Two types of codes:

- *fixed-length encoding* (e.g., ASCII)
- *variable-length encoding* (e,g., Morse code)

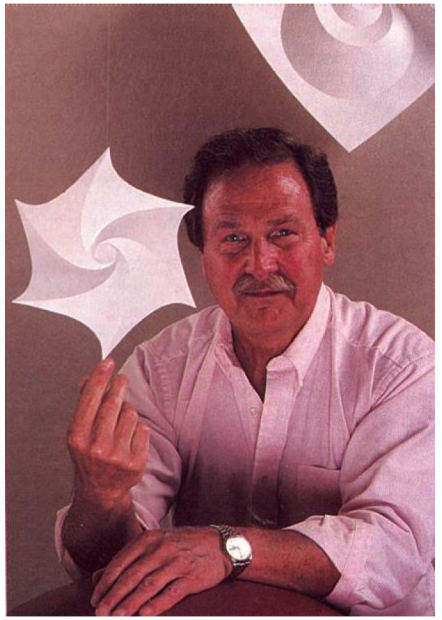
Prefix-free codes: no codeword is a prefix of another codeword

Problem: If frequencies of the character occurrences are known, what is the best binary prefix-free code?

Huffman Codes

- Proposed by Dr. David A. Huffman in 1952.

- Used in nearly every application involving compression and transmission of digital data, e.g., fax machines, modems, computer networks, and high-definition television.



David A. Huffman, PhD
(1925 - 1999)

Video Compression and Huffman Codes

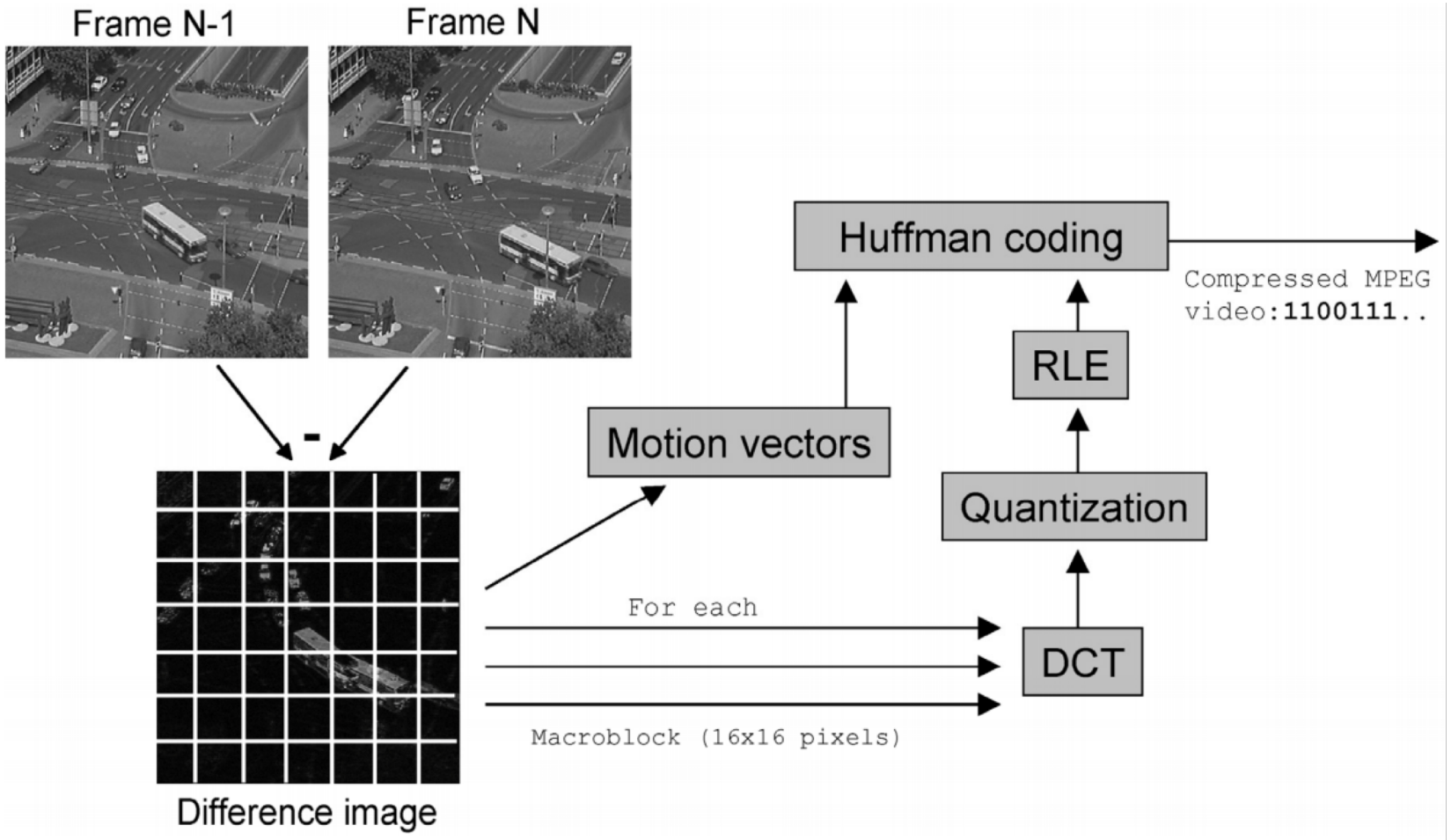


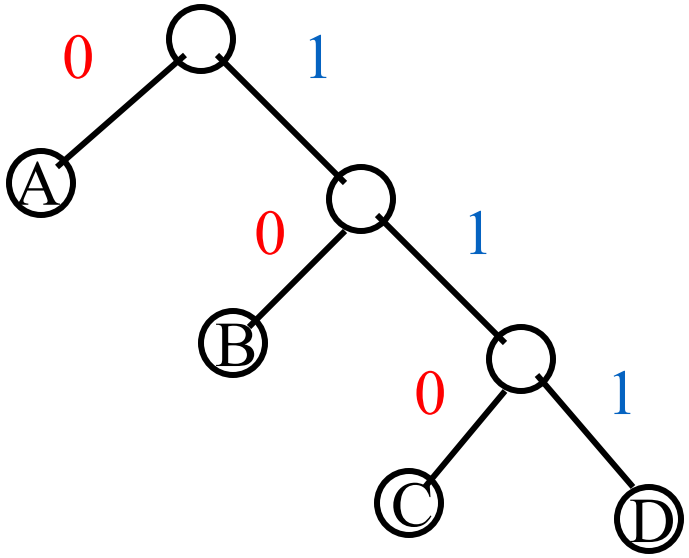
Figure 9: Illustration of the discussed 5 steps for a standard MPEG encoding.

Idea

Consider a binary tree, with:

0 meaning a left turn

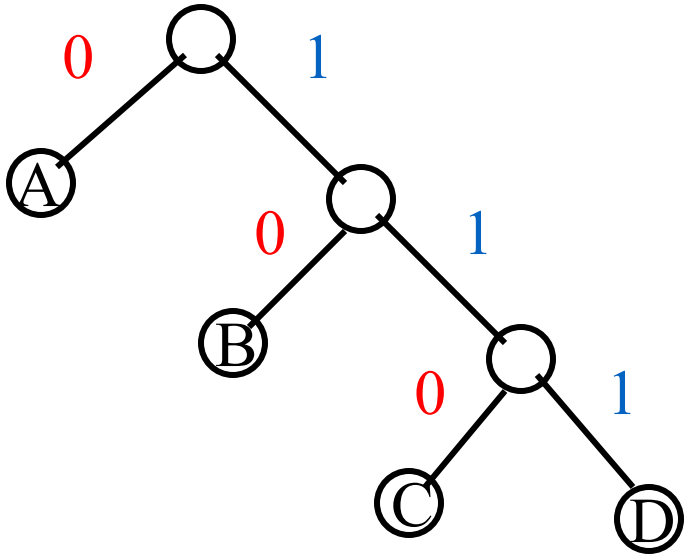
1 meaning a right turn.



Idea (cont.)

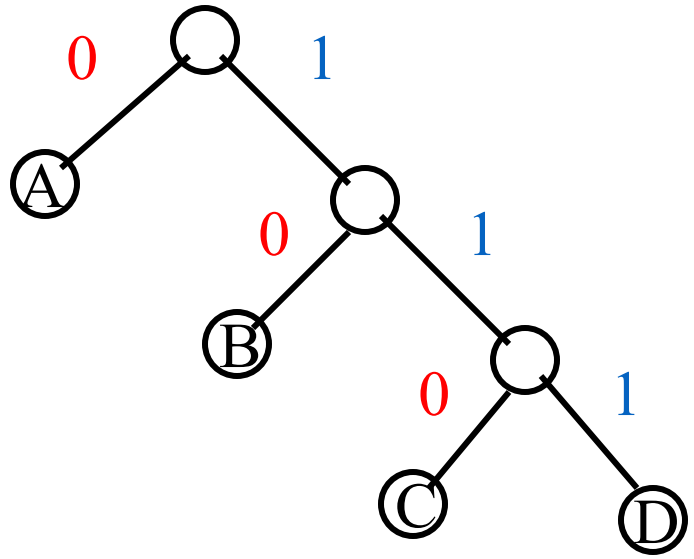
Consider the paths from the root to each of the leaves A, B, C, D:

A : 0
B : 10
C : 110
D : 111



Observe:

1. This is a prefix code, since each of the leaves has a path ending in it, without continuation.
2. How to make sure that the more frequent symbols are closer to the root so that they have a smaller code?



Greedy Algorithm:

1. Consider all pairs: <frequency, symbol>.

2. Choose the two lowest frequencies, and make them brothers, with the root having the combined frequency.

3. Iterate.

Huffman algorithm

Initialize trees of a single node each.

Keep the roots of all subtrees in a priority queue.

Iterate until only one tree left:
Merge the two smallest frequency subtrees into a single subtree with two children, and insert into priority queue.

Greedy Algorithm Example:

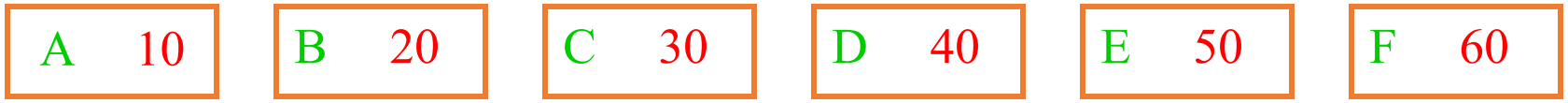
Alphabet: A, B, C, D, E, F

Frequency table:

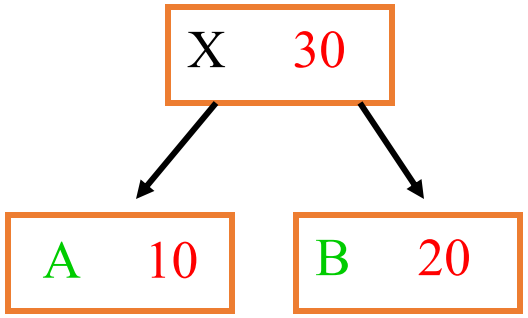
A	B	C	D	E	F
10	20	30	40	50	60

Total File Length: 210

Algorithm Run:

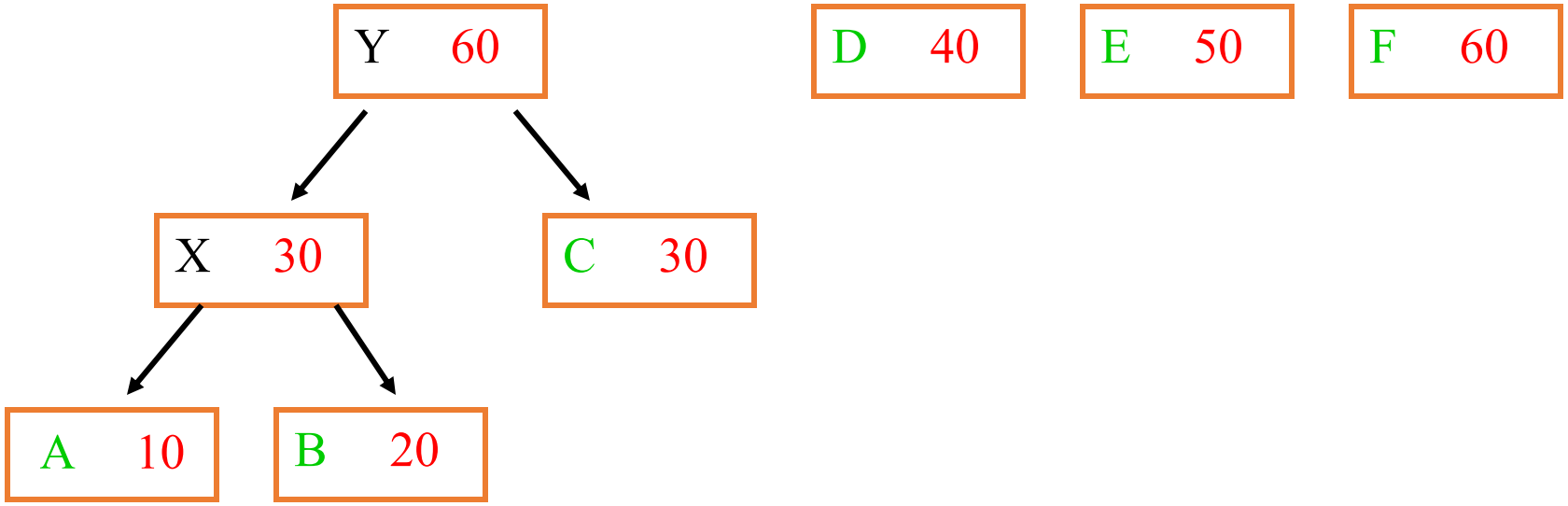


Algorithm Run:

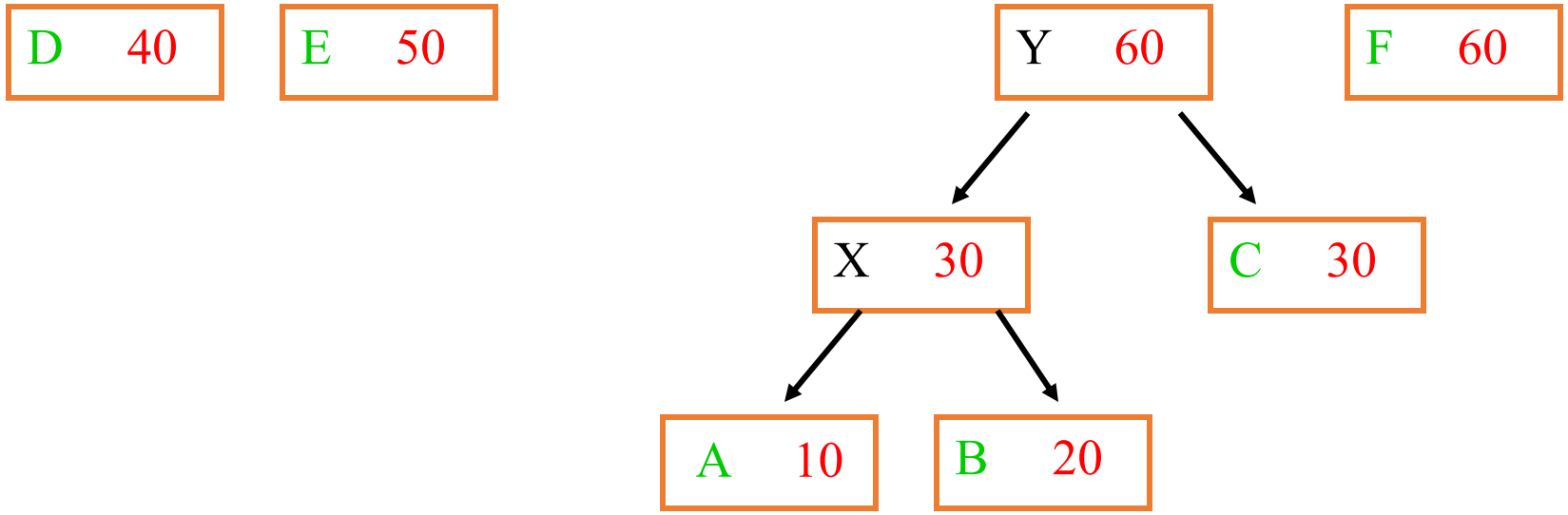


C 30 D 40 E 50 F 60

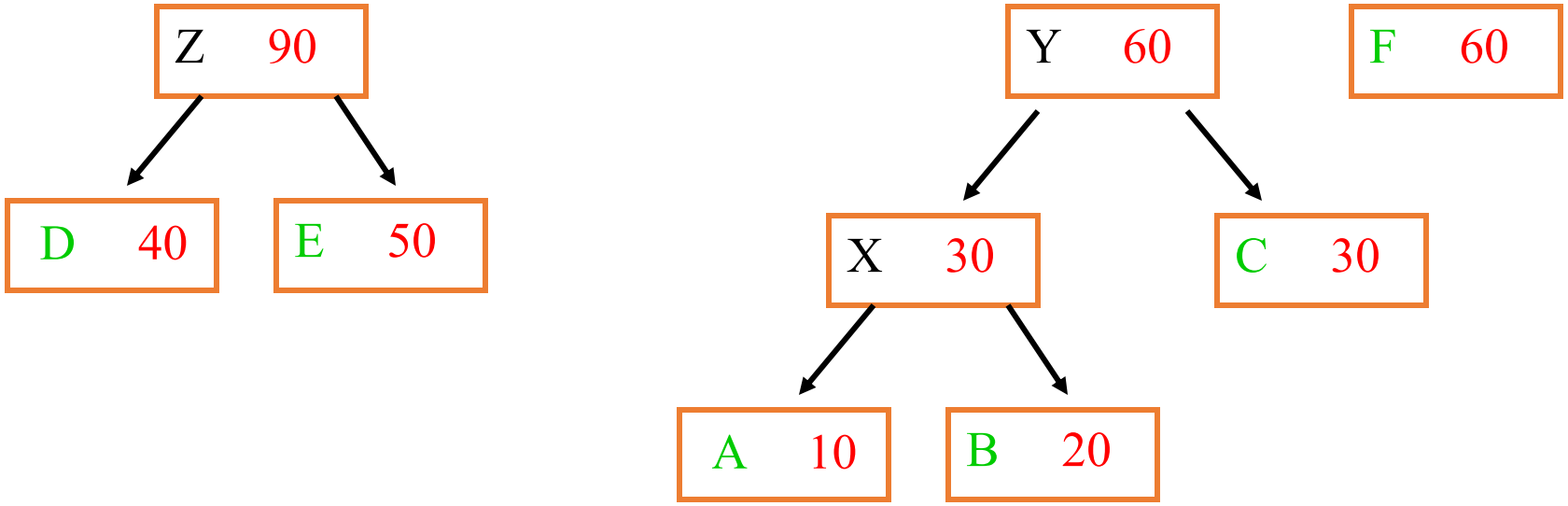
Algorithm Run:



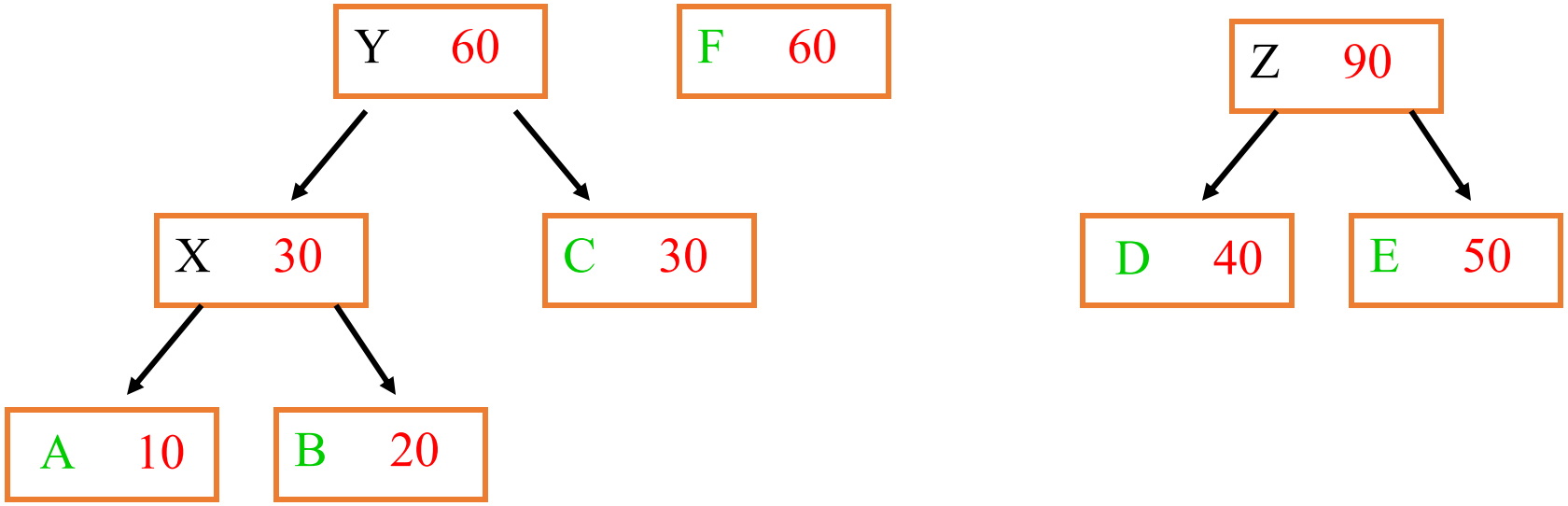
Algorithm Run:



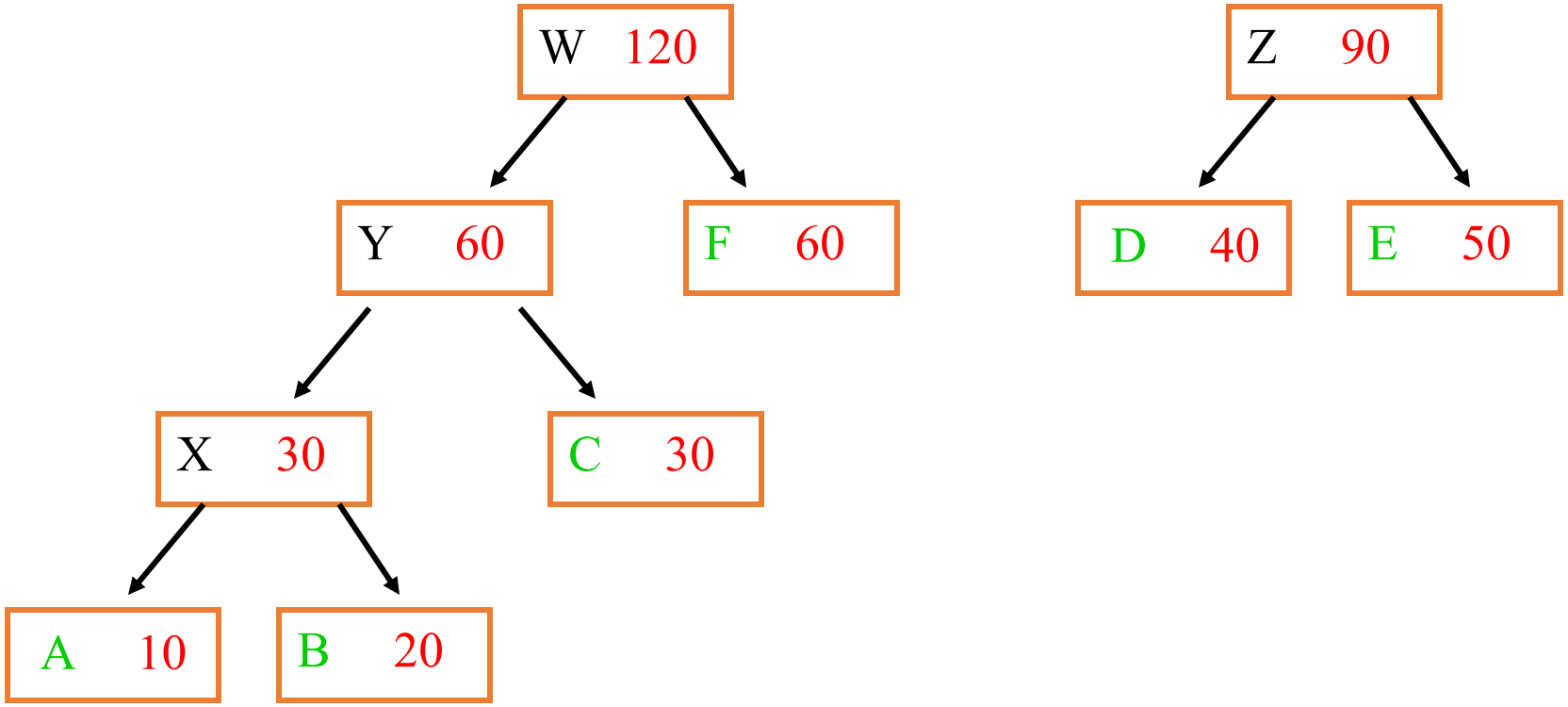
Algorithm Run:



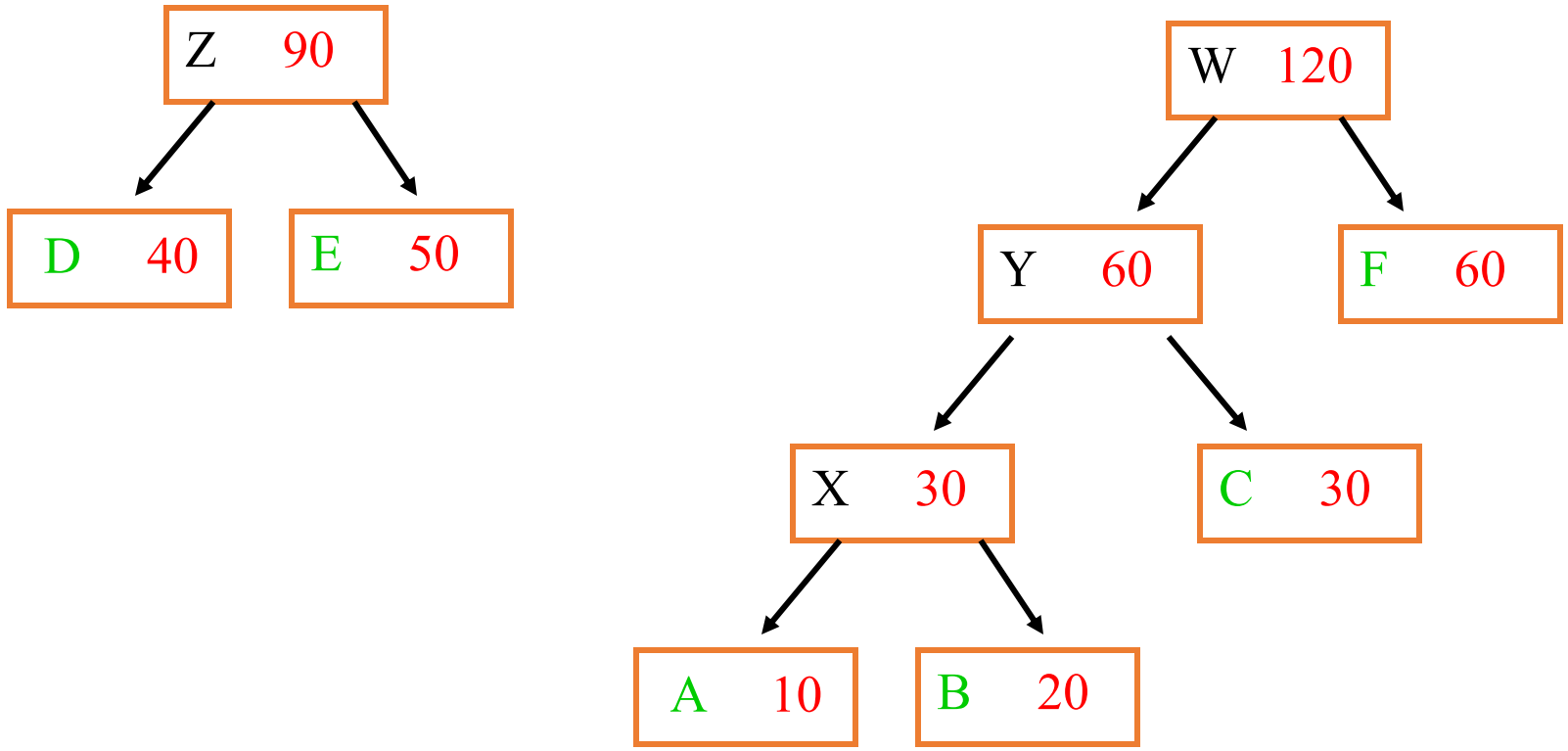
Algorithm Run:



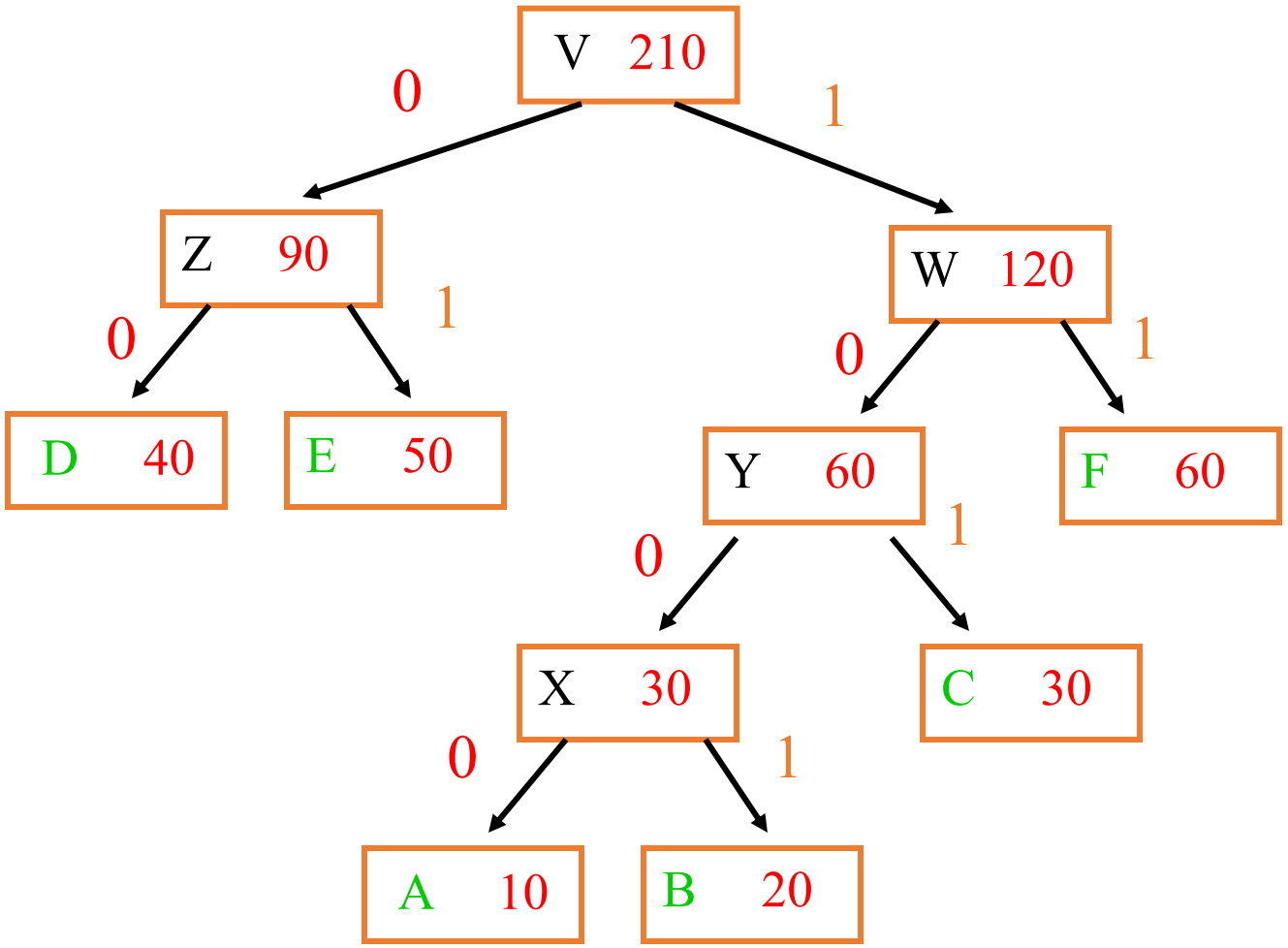
Algorithm Run:



Algorithm Run:

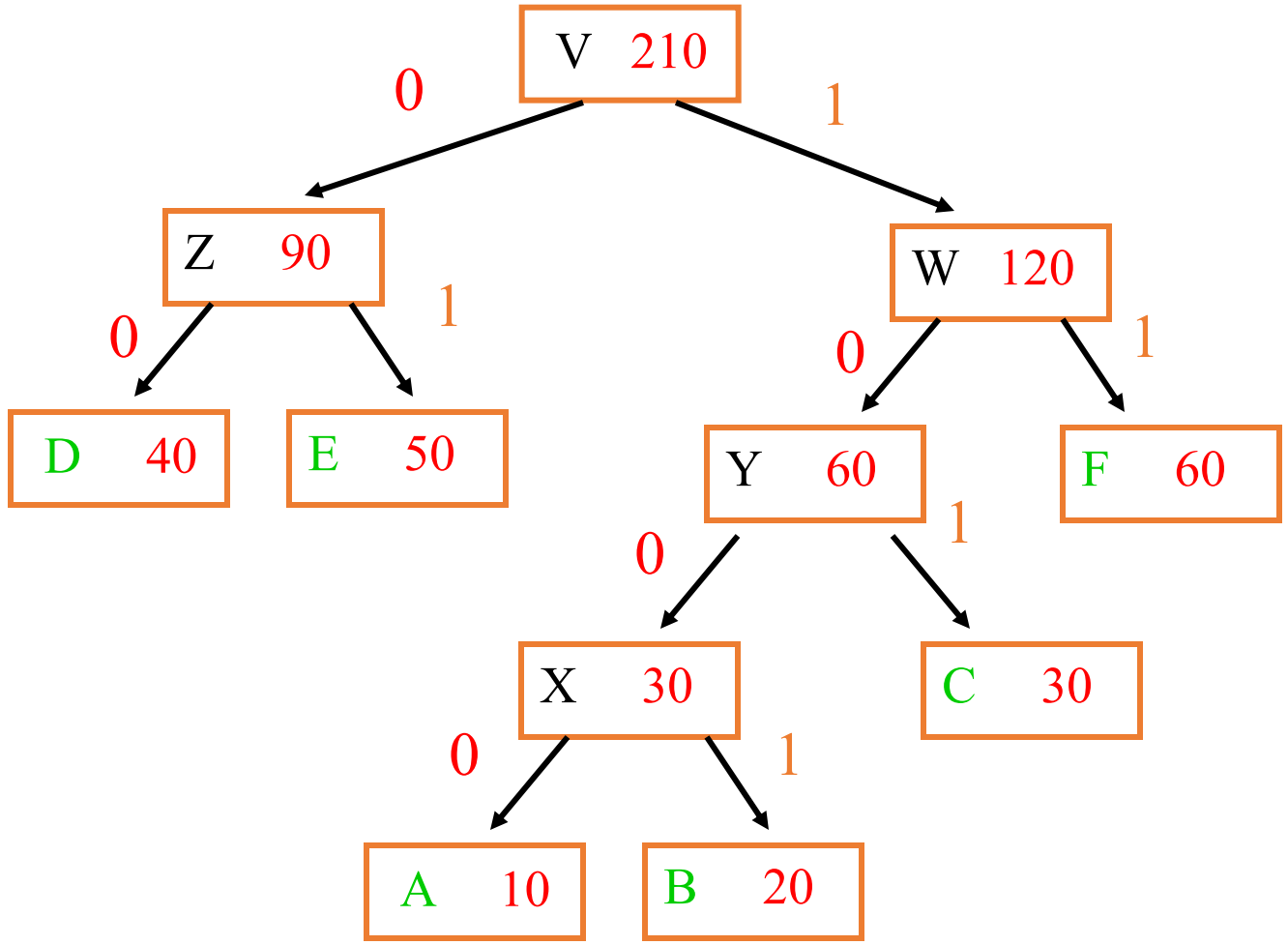


Algorithm Run:



The Huffman encoding:

A: 1000
B: 1001
C: 101
D: 00
E: 01
F: 11



File Size ?

The Huffman encoding:

A: 1000
B: 1001
C: 101
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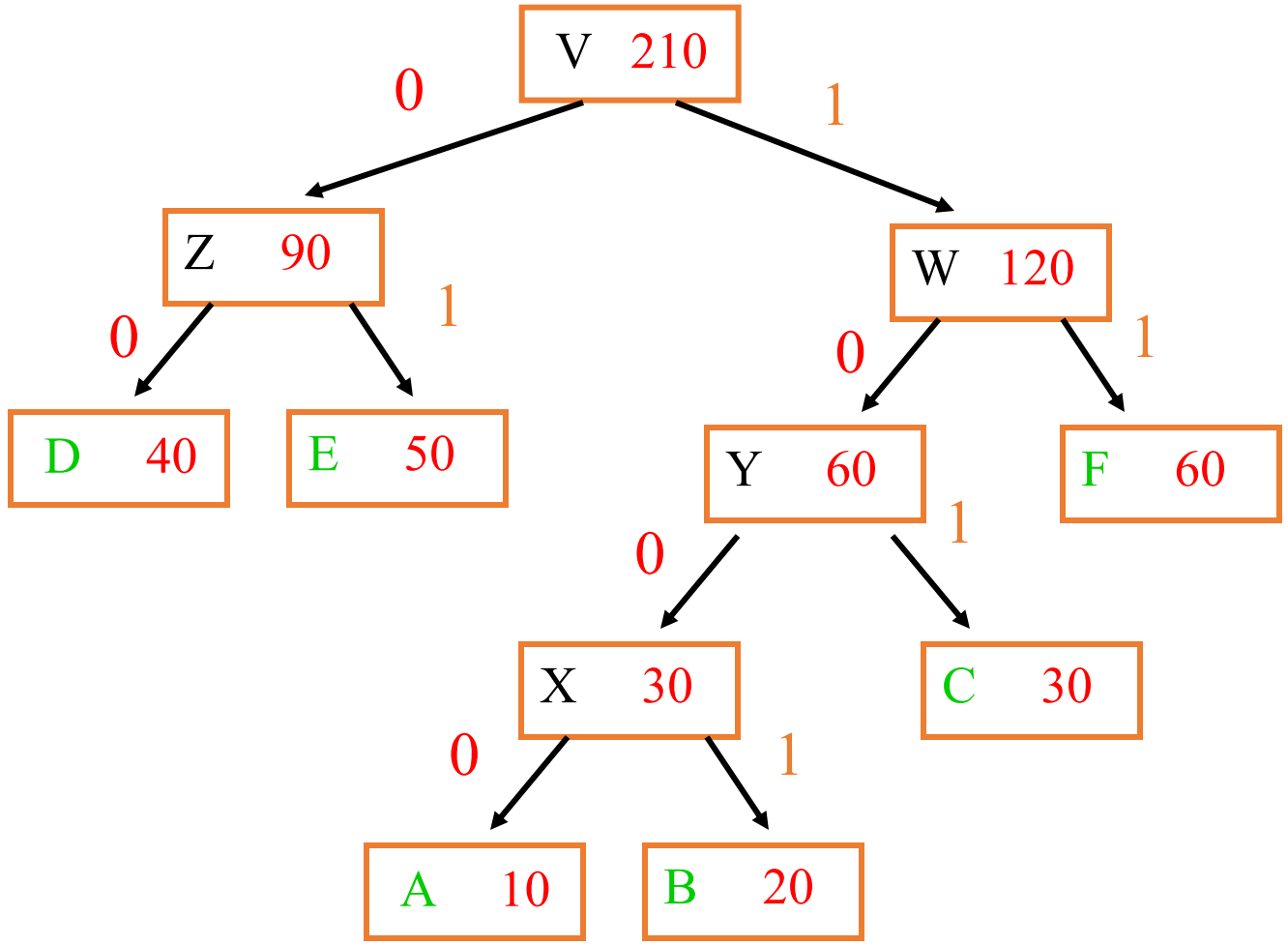


File Size:

10x4 + 20x4 + 30x3 + 40x2 + 50x2 + 60x2 =40 + 80 + 90 + 80 + 100 +120 = 510 bits

Note the savings:

The Huffman code:
Required 510 bits for the file.

Fixed length code:
Need 3 bits for 6 characters.
File has 210 characters.
Total: 630 bits for the file.

Compression ratio = (630-510)/630 = 20%

Algorithm time:

Each priority queue operation (e.g. heap):
$O(\log n)$

In each iteration: one less subtree.

Initially: n subtrees.

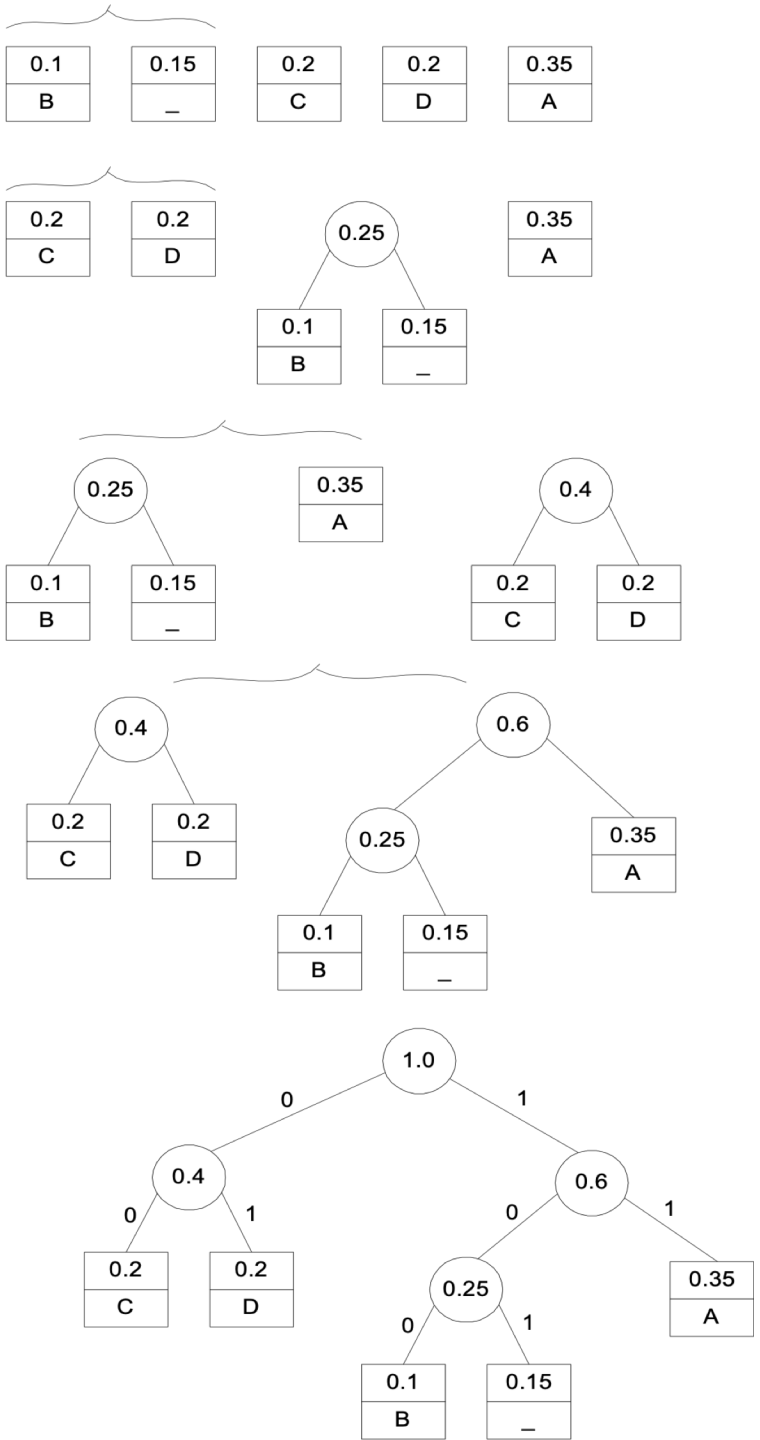
Total: $O(n \log n)$ time.

Example

character	A	B	C	D	_
frequency	0.35	0.1	0.2	0.2	0.15

Example

character	A	B	C	D	_
frequency	0.35	0.1	0.2	0.2	0.15
codeword	11	100	00	01	101



Summary

- Huffman tree is **optimal**

- Drawback: It requires the coding table into the encoded text to make its decoding possible

- The drawback can be overcome by ***dynamic Huffman encoding,*** which updates coding tree each time a new symbol is read from the source text