作业一

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第一题 重心插值公式 (barycentric interpolation formula)

(a)

(1) 易知

$$
\begin{aligned}
\ell_j(x_k) &= \frac{\ell(x_k)}{\ell'(x_j)(x_k-x_j)} \\
&= \frac{\prod\limits_{i=0}^{n}(x_k-x_i)}{\prod\limits_{i\neq j}(x_j-x_i)\cdot(x_k-x_j)} \\
&= \begin{cases} \frac{0}{\prod\limits_{i\neq j}(x_j-x_i)\cdot(x_k-x_j)} = 0 & k\neq j \\ \frac{\prod\limits_{i\neq j}(x_k-x_i)}{\prod\limits_{i\neq j}(x_j-x_i)} = 1 & k=j \end{cases}
\end{aligned}
\tag{1}
$$

即 $\ell_j(x) = \frac{\ell(x)}{\ell'(x_j)(x-x_j)}$ 可以作为 Lagrange 基函数.

(2)

$$
\begin{aligned}
p(x) &= \sum_{j=0}^{n} \ell_j(x) f_j \\
&= \ell(x)\sum_{j=0}^{n}\frac{1}{\ell'(x_j)}\cdot\frac{f_j}{x-x_j}
\end{aligned}
\tag{2}
$$

令 $\lambda_j = \frac{1}{\ell'(x_j)}$，即得

$$
p(x) = \ell(x)\sum_{j=0}^{n}\frac{\lambda_j}{x-x_j}f_j \tag{3}
$$

(b)

(1) 取 $f(x)\equiv 1$,

$$
f(x) = p(x) + R(x) = \sum_{j=0}^{n} f(j)\ell_j(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{k=0}^{n}(x-x_k) \tag{4}
$$

代入 $f(j)=1$，$f^{(n+1)}(x)=0$，得

$$
\sum_{j=0}^{n}\ell_j(x) = 1 \tag{5}
$$

(2) 由上式 (5)，

$$1=\sum_{j=0}^{n}\ell_j(x)=\sum_{j=0}^{n}\frac{\ell(x)}{\ell'(x_j)(x-x_j)} \tag{6}$$

故 $\ell(x)=1/\sum_{j=0}^{n}\frac{1}{\ell'(x_j)(x-x_j)}=1/\sum_{j=0}^{n}\frac{\lambda_j}{x-x_j}$，代入 (3) 即得

$$\begin{aligned} p(x)&=\ell(x)\sum_{j=0}^{n}\frac{\lambda_j}{x-x_j}f_j \\ &=\frac{\sum\limits_{j=0}^{n}\frac{\lambda_j f_j}{x-x_j}}{\sum\limits_{j=0}^{n}\frac{\lambda_j}{x-x_j}} \end{aligned} \tag{7}$$

(c) 先证明 $\prod\limits_{k=1}^{n-1}\sin\frac{k\pi}{n}=\frac{n}{2^{n-1}}$:

设 $\epsilon=\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}$（$i$ 为虚数单位），则 $1,\epsilon,\epsilon^2,\ldots,\epsilon^{2(n-1)}$ 为 $x^{2n}-1=0$ 的根，且

$$\sin\frac{k\pi}{n}=\frac{\epsilon^k-\epsilon^{-k}}{2i}=\frac{\epsilon^{2k}-1}{2i\epsilon^k}, \tag{8}$$

所以

$$\begin{aligned} &\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} \\ =&\frac{(\epsilon^2-1)(\epsilon^4-1)\ldots[\epsilon^{2(n-1)}-1]}{2^{n-1}i^{n-1}\epsilon^{\frac{1}{2}n(n-1)}} \\ =&\frac{(-1)^{n-1}(\epsilon^2-1)(\epsilon^4-1)\ldots[\epsilon^{2(n-1)}-1]}{2^{n-1}(i^{n-1})^2} \\ =&\frac{(1-\epsilon^2)(1-\epsilon^4)\ldots[1-\epsilon^{2(n-1)}]}{2^{n-1}}, \end{aligned} \tag{9}$$

又因为

$$(x^2-\epsilon^2)(x^2-\epsilon^4)\ldots[x^2-\epsilon^{2(n-1)}]=x^{2(n-1)}+x^{2(n-2)}+\ldots+x^2+1, \tag{10}$$

所以

$$(1-\epsilon^2)(1-\epsilon^4)\ldots[1-\epsilon^{2(n-1)}]=n, \tag{11}$$

因此，

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}=\frac{n}{2^{n-1}}. \tag{12}$$

下面化简插值权重：

$$\lambda_j=\frac{1}{\ell'(x_j)} \tag{13}$$

而

$$\begin{aligned}
\ell'(x_j) &= \prod_{k\neq j}(x_j - x_k) \\
&= \prod_{k\neq j}(\cos\frac{j\pi}{n} - \cos\frac{k\pi}{n}) \\
&= \prod_{k\neq j} -2\sin\frac{(j+k)\pi}{2n}\sin\frac{(j-k)\pi}{2n} \\
&= (-2)^n \prod_{k\neq j}\sin\frac{(j+k)\pi}{2n}\sin\frac{(j-k)\pi}{2n}
\end{aligned} \tag{14}$$

当 $j=0$ 时，

$$\begin{aligned}
\ell'(x_0) &= (-2)^n \prod_{k=1}^{n}\sin\frac{k\pi}{2n}\sin\frac{-k\pi}{2n} \\
&= 2^n \prod_{k=1}^{n}\sin\frac{k\pi}{2n}\sin\frac{k\pi}{2n} \\
&= 2^n(\sin\frac{\pi}{2})^2 \prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}\sin\frac{k\pi}{2n} \\
&= 2^n \prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}\sin\frac{k\pi}{2n} \\
&= 2^n \prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}\sin\frac{(2n-k)\pi}{2n} \\
&= 2^n \sin\frac{n\pi}{2n}\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}\prod_{k=n+1}^{2n-1}\sin\frac{k\pi}{2n} \\
&= 2^n \prod_{k=1}^{2n-1}\sin\frac{k\pi}{2n}
\end{aligned} \tag{15}$$

由于 $\prod\limits_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}$，所以 $\prod\limits_{k=1}^{2n-1}\sin\frac{k\pi}{2n} = \frac{2n}{2^{2n-1}} = \frac{n}{2^{2n-2}}$，
由推导过程可得，$\prod_{k=1}^{n}(\sin\frac{k\pi}{2n})^2 = \frac{n}{2^{2n-2}}$.

$$\begin{aligned}
\ell'(x_0) &= 2^n \prod_{k=1}^{2n-1}\sin\frac{k\pi}{2n} \\
&= 2^n\frac{n}{2^{2n-2}} \\
&= \frac{n}{2^(n-2)}
\end{aligned} \tag{16}$$

故 $\lambda_0 = \frac{1}{\ell'(x_0)} = \frac{2^{(n-2)}}{n}$

当 $j=n$ 时，

$$
\begin{aligned}
\ell^{'}(x_n) &= (-2)^n \prod_{k=0}^{n-1} \sin\frac{(n+k)\pi}{2n} \sin\frac{n-k\pi}{2n} \\
&= (-2)^n \prod_{k=n}^{2n-1} \sin\frac{k\pi}{2n} \prod_{k=1}^{n} \sin\frac{k\pi}{2n} \\
&= (-2)^n \prod_{k=1}^{2n-1} \sin\frac{k\pi}{2n} \\
&= (-2)^n \frac{n}{2^{2n-2}} \\
&= (-1)^n \frac{n}{2^{n-2}}
\end{aligned}
\tag{17}
$$

故 $\lambda_n = \frac{1}{\ell^{'}(x_n)} = \frac{2^{(n-2)}}{n}(-1)^n$

当 $1 \le j \le n-1$ 时，

$$
\begin{aligned}
\ell'(x_j) &= (-2)^n \prod_{k\neq j} \sin\frac{(j+k)\pi}{2n}\sin\frac{(j-k)\pi}{2n} \\
&= (-2)^n \cdot \prod_{k=j}^{2j-1}\sin\frac{k\pi}{2n}\prod_{k=2j+1}^{n+j}\sin\frac{k\pi}{2n}\prod_{k=j-n}^{-1}\sin\frac{k\pi}{2n}\prod_{k=1}^{j}\sin\frac{k\pi}{2n} \\
&= (-2)^n \cdot (-1)^{n-j}\prod_{k=j}^{2j-1}\sin\frac{k\pi}{2n}\prod_{k=2j+1}^{n+j}\sin\frac{k\pi}{2n}\prod_{k=1}^{n-j}\sin\frac{k\pi}{2n}\prod_{k=1}^{j}\sin\frac{k\pi}{2n} \\
&= (-1)^j\cdot 2^n\sin\frac{j\pi}{2n}\prod_{1}^{n+j}\sin\frac{k\pi}{2n}/(\sin\frac{2j\pi}{2n}\prod_{1}^{n-j}\sin\frac{k\pi}{2n}) \\
&= (-1)^j\cdot 2^n\sin\frac{j\pi}{2n}\prod_{1}^{n}\sin\frac{k\pi}{2n}\prod_{1}^{n}\sin\frac{k\pi}{2n}\prod_{n+1}^{n+j}\sin\frac{k\pi}{2n}/(\sin\frac{2j\pi}{2n}\prod_{n-j+1}^{n}\sin\frac{k\pi}{2n}) \\
&= (-1)^j\cdot 2^n\sin\frac{j\pi}{2n}\prod_{1}^{2n-1}\sin\frac{k\pi}{2n}\prod_{n+1}^{n+j}\sin\frac{(2n-k)\pi}{2n}/(\sin\frac{2j\pi}{2n}\prod_{n-j+1}^{n}\sin\frac{k\pi}{2n}) \\
&= (-1)^j\cdot\frac{n}{2^{n-2}}\sin\frac{j\pi}{2n}\prod_{n-j}^{n-1}\sin\frac{k\pi}{2n}/(\sin\frac{2j\pi}{2n}\prod_{n-j+1}^{n}\sin\frac{k\pi}{2n}) \\
&= (-1)^j\cdot\frac{n}{2^{n-2}}\sin\frac{j\pi}{2n}\sin\frac{(n-j)\pi}{2n}\prod_{n-j+1}^{n-1}\sin\frac{k\pi}{2n}/(\sin\frac{2j\pi}{2n}\prod_{n-j+1}^{n-1}\sin\frac{k\pi}{2n}) \\
&= (-1)^j\cdot\frac{n}{2^{n-2}}\sin\frac{j\pi}{2n}\sin\frac{(n-j)\pi}{2n}/(\sin\frac{2j\pi}{2n}) \\
&= (-1)^j\cdot\frac{n}{2^{n-2}}\sin\frac{(n-j)\pi}{2n}/(2\cos\frac{j\pi}{2n}) \\
&= (-1)^j\cdot\frac{n}{2^{n-2}}\cos\frac{j\pi}{2n}/(2\cos\frac{j\pi}{2n}) \\
&= (-1)^j\frac{n}{2^{n-1}}
\end{aligned}
\tag{18}
$$

故 $\lambda_j = \frac{1}{\ell'(x_j)} = \frac{2^{n-1}}{n}(-1)^j$

综上可得，

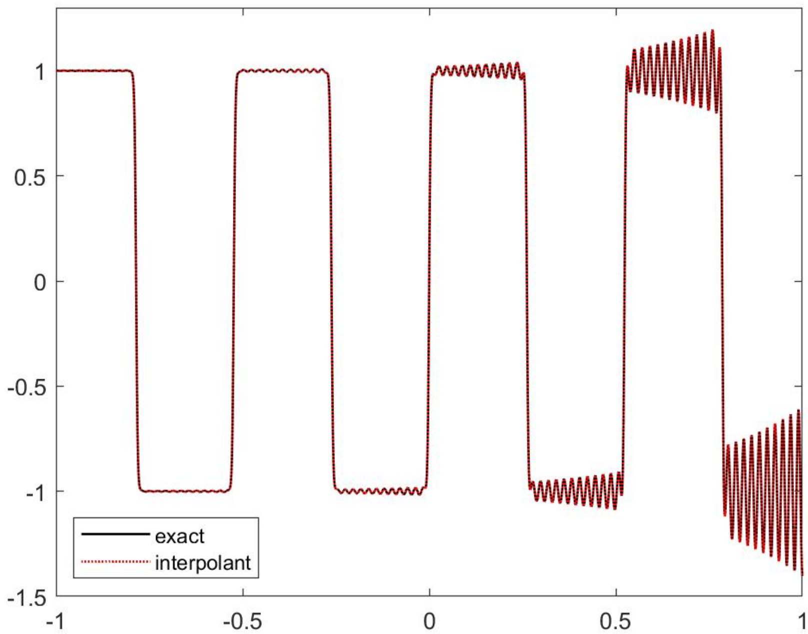
$$
\lambda_j = \begin{cases} \frac{2^{n-2}}{n} & j=0 \\ \frac{2^{n-1}}{n}(-1)^j & 1\le j\le n-1 \\ \frac{2^{n-2}}{n}(-1)^n & j=n \end{cases} \tag{19}
$$

(d) MATLAB 程序如下：

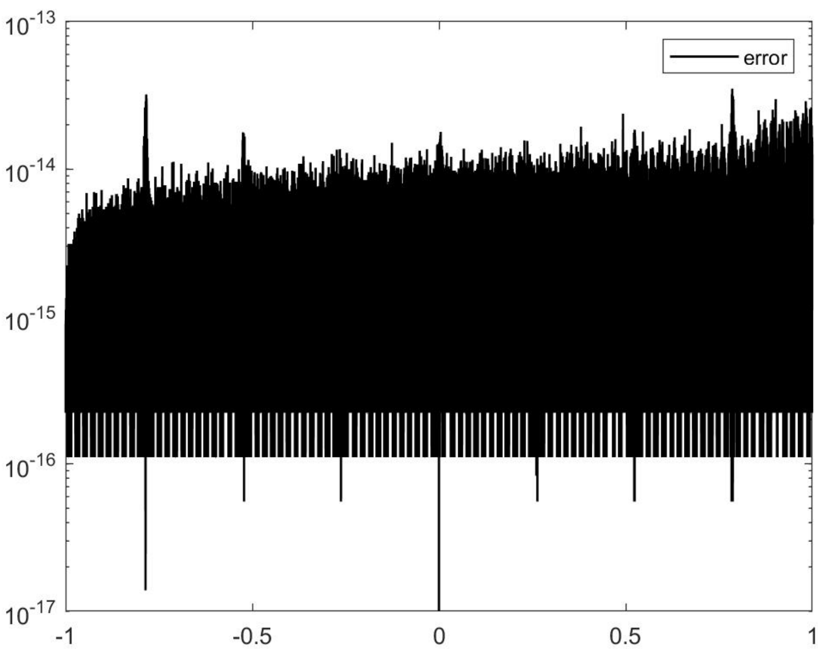
```
clear , clc , clf
LW = 'linewidth'; lw = 1;
```

```
n = 5000;
x = linspace(0, n, n + 1)';
x = cos(x .* pi ./ n);
m = 100000;
xx = linspace(-1, 1, m + 1)';
F = @(x)(tanh(20 * sin(12 * x)) + ...
    0.02 * exp(3 * x) .* sin(300 * x));
f = F(x);

numerator = zeros(m + 1, 1);
denominator = zeros(m + 1, 1);
for j = 1 : n + 1
    l = (-1)^(j-1) ./ (xx - x(j));
    if j == 1 || j == n + 1
        l = l ./ 2;
    end
    numerator = numerator + l .* f(j);
    denominator = denominator + l;
end
p = numerator ./ denominator;

figure(1)
plot(xx, F(xx), 'k', LW, lw), hold on
plot(xx, p, 'r:', LW, lw)
legend('exact', 'interpolant', 'location', 'sw')
axis([-1, 1, -1.5, 1.3])

figure(2)
plot(2)
semilogy(xx, abs(F(xx) - p), 'k', LW, lw)
legend('error', 'location', 'ne')
axis([-1, 1, 10^(-17), 10^(-13)])
```

插值函数图像：



误差:



第二题 MATLAB 程序如下:

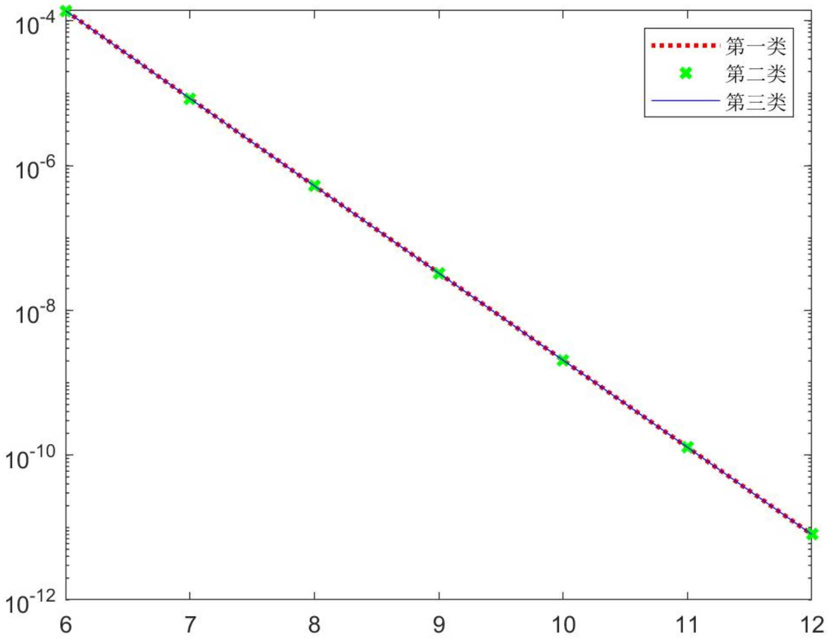
```
clear, clc, clf
LW = 'linewidth'; lw = 2;
MaxError1 = zeros(7, 1);
```

```matlab
MaxError2 = zeros(7, 1);
MaxError3 = zeros(7, 1);
for k = 6:12
    %%
    n = 2 ^ k;
    x = linspace(-n, n, n+1)' ./ n;
    F = @(x)exp(3*cos(pi*x));
    DF = @(x)-3*pi*exp(3*cos(pi*x)).*sin(pi*x);
    D2F = @(x)9*pi^2*exp(3*cos(pi*x)).*sin(pi*x).^2 - ...
        3*pi^2*exp(3*cos(pi*x)).*cos(pi*x);
    f = F(x);
    Df = DF(x);
    D2f = D2F(x);
    %%
    h = diff(x);
    df = diff(f);
    d1 = df ./ h;
    lambda = h(2:n) ./ (h(2:n) + h(1:n-1));
    d = 6 * (df(2:n)./h(2:n) - df(1:n-1)./h(1:n-1)) ./ ...
        (h(2:n) + h(1:n-1));
    mu = 1-lambda;
    %%
    % 第一类边界条件
    M0 = D2f(1);
    Mn = D2f(n + 1);
    A1 = diag(2*ones(n-1,1)) + diag(lambda(1:n-2), 1) +...
        diag(mu(2:n-1), -1);
    D1 = [d(1)-mu(1)*M0; d(2:n-2); d(n-1)-lambda(n-1)*Mn];
    M1 = A1 \ D1;
    M1 = [M0; M1; Mn];
    %%
    % 第二类边界条件
    m0 = Df(1);
    mn = Df(n + 1);
    lambda2 = [1; lambda];
```

```
    mu2 = [mu; 1];
    d0 = 6 * ( df(1) / h(1) - m0 ) / h(1);
    dn = 6 * ( mn - df(n) / h(n) ) / h(n);
    D2 = [d0; d; dn];
    A2 = diag(2*ones(n+1,1))+diag(lambda2,1)+diag(mu2,-1);
    M2 = A2 \ D2;
    %%
    % 第三类边界条件
    lambda0 = h(1) / (h(1) + h(n));
    lambda3 = [lambda0; lambda(1:n-2)];
    mu0 = 1 - lambda0;
    d0 = 6*(df(1)./h(1) - df(n)./h(n)) / (h(1)+h(n));
    D3 = [d0; d];
    A3 = diag(2*ones(n,1)) + diag(lambda3,1) +diag(mu,-1);
    A3(1, n) = mu0;
    A3(n, 1) = lambda(n-1);
    M3 = A3 \ D3;
    M3 = [M3; M3(1)];
    %%
    MaxError1(k - 5) = CubicSpline(x, F, h, M1);
    MaxError2(k - 5) = CubicSpline(x, F, h, M2);
    MaxError3(k - 5) = CubicSpline(x, F, h, M3);
end
%%
xx = linspace(6, 12, 7);
semilogy(xx, MaxError1, 'r:', LW, lw), hold on
semilogy(xx, MaxError2, 'gx', LW, lw), hold on
semilogy(xx, MaxError3, 'b')
legend('第一类', '第二类', '第三类')
%%
function MaxError = CubicSpline(x, F, h, M)
n = size(x) - 1;
f = F(x);
MaxError = 0;
for k = 1:n
```

```
    m = 6;
    xx = linspace(x(k), x(k+1), m)';
    S = ((x(k+1)-xx).^3*M(k) + (xx-x(k)).^3*M(k+1)) / ...
        (6*h(k)) + ...
        ((x(k+1)-xx)*f(k) + (xx-x(k))*f(k+1)) / h(k) - ...
        h(k) * ((x(k+1)-xx)*M(k) + (xx-x(k))*M(k+1)) / 6;
    MaxError = max([MaxError, abs(F(xx) - S)']);
end
end
```

最大误差：

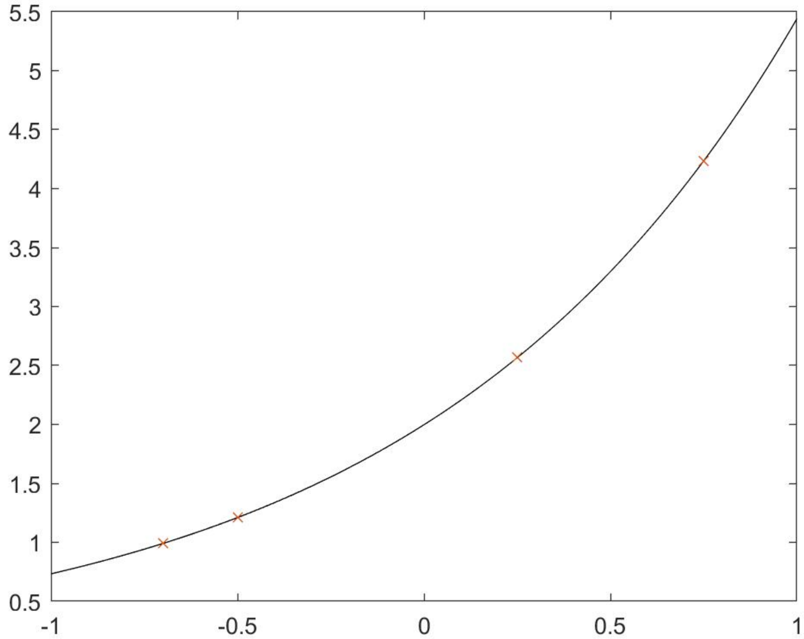


第三题 MATLAB 程序如下：

```
x = [-0.7, -0.5, 0.25, 0.75];
y = [0.99, 1.21, 2.57, 4.23];
% y=a*exp(bx)
% lny=bx+lna
lny = log(y);
R = [x' ones(4, 1)];
% solve A
A = R \ lny';
```

```
a = exp(A(2));
b = A(1);
F = @(x)a*exp(b*x);
f = F(x);
% error
M2 = sqrt((y - f) * (y - f)')

xx = linspace(-1, 1, 1000);
plot(xx, F(xx), 'k', x, y, 'x')
```

拟合函数图像：



误差的 2-范数为 0.0062.