The Theory of Isomorphic Physics
Part 4: The Dualistic Algebra

John Henke

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Particle States

In this framework, we define the total state Ω as part of a set of quaternion-valued group elements that form both a group and an algebra. These group elements are represented as matrices, where the real and imaginary parts of Ω interact with quaternionic units i, j, and k. This interaction introduces specific symmetries and a rich algebraic structure. Qualitatively, these elements can be seen as quaternion analogs to the Dirac matrices, as they exhibit cancellation dynamics and symmetries that are useful in describing the physical behavior of both particle and composite-particle states.

The group elements are defined as follows:

$$G = \frac{1}{2}\begin{pmatrix} \Omega & -i\Omega & -j\Omega & -k\Omega \\ i\Omega & \Omega & k\Omega & -j\Omega \\ j\Omega^* & -k\Omega^* & \Omega^* & i\Omega^* \\ k\Omega^* & j\Omega^* & -i\Omega^* & \Omega^* \end{pmatrix}, \quad X = \frac{1}{2}\begin{pmatrix} \Omega & -i\Omega & j\Omega & k\Omega \\ i\Omega & \Omega & -k\Omega & j\Omega \\ -j\Omega^* & k\Omega^* & \Omega^* & i\Omega^* \\ -k\Omega^* & -j\Omega^* & -i\Omega^* & \Omega^* \end{pmatrix}$$

$$Y = \frac{1}{2}\begin{pmatrix} \Omega & i\Omega & -j\Omega & k\Omega \\ -i\Omega & \Omega & -k\Omega & -j\Omega \\ j\Omega^* & k\Omega^* & \Omega^* & -i\Omega^* \\ -k\Omega^* & j\Omega^* & i\Omega^* & \Omega^* \end{pmatrix}, \quad Z = \frac{1}{2}\begin{pmatrix} \Omega & i\Omega & j\Omega & -k\Omega \\ -i\Omega & \Omega & k\Omega & j\Omega \\ -j\Omega^* & -k\Omega^* & \Omega^* & -i\Omega^* \\ k\Omega^* & -j\Omega^* & i\Omega^* & \Omega^* \end{pmatrix}$$

$$I = \begin{pmatrix} \Omega & 0 & 0 & 0 \\ 0 & \Omega & 0 & 0 \\ 0 & 0 & \Omega^* & 0 \\ 0 & 0 & 0 & \Omega^* \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -i\Omega & 0 & 0 \\ i\Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & i\Omega^* \\ 0 & 0 & -i\Omega^* & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & -j\Omega & 0 \\ 0 & 0 & 0 & -j\Omega \\ j\Omega^* & 0 & 0 & 0 \\ 0 & j\Omega^* & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & -k\Omega \\ 0 & 0 & k\Omega & 0 \\ 0 & -k\Omega^* & 0 & 0 \\ k\Omega^* & 0 & 0 & 0 \end{pmatrix}$$

Each element is parameterized by Ω, which characterizes the state of the particle or system. Importantly, when combining group elements (e.g., $G + X$), Ω for G and X need not be identical. By definition, each group element can have its own distinct Ω, representing states with different parameters. This flexibility reflects the physical diversity of particle states and allows for a rich algebraic structure.

A formal proof showing that these elements form a group under matrix multiplication when $\Omega = 1$ is available in the accompanying Mathematica notebook. While this setup forms a finite group for fixed Ω, allowing Ω to vary suggests that the structure generalizes to an infinite group. Further investigation is required to rigorously establish this property.

We define a particle state as an arbitrary group element β:

$$\beta \in \{G, X, Y, Z, I, A, B, C, -G, -X, -Y, -Z, -I, -A, -B, -C\}.$$

This definition encapsulates the group structure and its symmetries, while the freedom to assign different Ω values to each element underscores the flexibility of the formalism. This dynamic approach facilitates the exploration of complex particle and composite-particle behaviors.

Squaring G

The quaternion-valued matrices, such as G, exhibit unique dynamics. For example, if $\Omega = 1$, squaring G yields (leaving the terms in the matrix multiplication seperated by addition):

$$G^2 = \begin{pmatrix} \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} & -\frac{i}{4}-\frac{i}{4}+\frac{i}{4}+\frac{i}{4} & -\frac{j}{4}+\frac{j}{4}-\frac{j}{4}+\frac{j}{4} & -\frac{k}{4}+\frac{k}{4}+\frac{k}{4}-\frac{k}{4} \\ -\frac{i}{4}-\frac{i}{4}+\frac{i}{4}+\frac{i}{4} & \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} & -\frac{k}{4}+\frac{k}{4}+\frac{k}{4}-\frac{k}{4} & -\frac{j}{4}+\frac{j}{4}-\frac{j}{4}+\frac{j}{4} \\ -\frac{j}{4}-\frac{j}{4}+\frac{j}{4}+\frac{j}{4} & -\frac{k}{4}+\frac{k}{4}+\frac{k}{4}-\frac{k}{4} & \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} & -\frac{i}{4}-\frac{i}{4}+\frac{i}{4}+\frac{i}{4} \\ \frac{k}{4}-\frac{k}{4}-\frac{k}{4}+\frac{k}{4} & -\frac{j}{4}+\frac{j}{4}+\frac{j}{4}-\frac{j}{4} & -\frac{i}{4}-\frac{i}{4}+\frac{i}{4}+\frac{i}{4} & \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} \end{pmatrix}$$

From this, we see that all off-diagonal elements cancel, while each diagonal element sums to 1. Thus, $G^2 = I$, showing that G behaves as an involution, i.e., G^2 equals the identity matrix. The same result holds for matrices X, Y, and Z, demonstrating a consistent group behavior across these quaternion-valued elements.

The Dualistic Group

When we set $\Omega = 1$, this effectively quantizes the group, representing a specific set of elements rather than an infinite group of $s.Inthegeneralcase, theseelementsaresubsetsofaninfinitegrou$ $valuedmatrices, butretainthestructurefromwhen\Omega = 1.$

The multiplication table for these elements with $\Omega = 1$ or as subsets of an infinite group is as follows:

	G	X	Y	Z	I	A	B	C
G	I	A	B	C	G	X	Y	Z
X	A	I	$-C$	$-B$	X	G	$-Z$	$-Y$
Y	B	$-C$	I	$-A$	Y	$-Z$	G	$-X$
Z	C	$-B$	$-A$	I	Z	$-Y$	$-X$	G
I	G	X	Y	Z	I	A	B	C
A	X	G	$-Z$	$-Y$	A	I	$-C$	$-B$
B	Y	$-Z$	G	$-X$	B	$-C$	I	$-A$
C	Z	$-Y$	$-X$	G	C	$-B$	$-A$	I

This table maintains the group structure and duality across the elements.

The Dualism of the Group

This multiplication table reveals a dualism in the group, with symmetries between G and I, X and A, Y and B, and Z and C. Each element is an involution, squaring to I. For example, multiplying G by X yields A, and multiplying G by A returns X.

If we associate 0 with G and I, 1 with A and X, 2 with B and Y, and 3 with C and Z, the multiplication table simplifies to a pattern reminiscent of the quaternions with there being four blocks of:

$$0 \quad 1 \quad 2 \quad 31 \quad 0 \quad -3 \quad -22 \quad -3 \quad 0 \quad 13 \quad -2 \quad -1 \quad 0$$

This pattern mirrors the original structure up to a sign of G when $\Omega = 1$ if 0 is associated with 1, 1 with i, 2 with j and 3 with k:

$$G\,(\Omega = 1) = \frac{1}{2}\begin{pmatrix} 1 & -i & -j & -k \\ i & 1 & k & -j \\ j & -k & 1 & i \\ k & j & -i & 1 \end{pmatrix}$$

Although the signs differ, a deep symmetry and duality emerge in the group's structure.

Fermion and Boson Transformation Dynamics

In this group structure, elements from the set $\{G, X, Y, Z\}$ multiply to yield elements from the set $\{I, A, B, C\}$. However, elements within the set $\{I, A, B, C\}$ multiply to form other elements within the same set. Conversely, if we multiply an element from the set $\{G, X, Y, Z\}$ with an element from the set $\{I, A, B, C\}$, the result belongs to the set $\{G, X, Y, Z\}$.

This pattern suggests a deeper symmetry that mirrors fermionic and bosonic dynamics: elements from $\{G, X, Y, Z\}$ correspond to fermions, while elements from $\{I, A, B, C\}$ correspond to bosons. In this context, a fermion-fermion interaction produces a boson, a boson-boson interaction results in another boson, and a fermion-boson interaction returns a fermion, as predicted by this mapping.

A Unique Group Property

This group exhibits a remarkable and seemingly unique symmetry property. Specifically, if three group elements satisfy the relation:

$$\beta_1\beta_2 = \beta_3$$

then the following symmetry relations also hold:

$$\beta_3\beta_2 = \beta_1 \quad \text{and} \quad \beta_3\beta_1 = \beta_2$$

For example:

$$GX = A, \quad AX = G, \quad AG = X$$

or:

$$XY = -C, \quad -CY = X, \quad -CX = Y$$

This symmetry is intrinsic to the group and reflects its deep structure, arising from the dualistic dynamics of the sets $\{G, X, Y, Z\}$ and $\{I, A, B, C\}$. This property will later be used to explain the weak force symmetries.

Additive Symmetries

When we set $\Omega = 1$, the group's additive symmetries become evident. The following relations hold for the group elements:

$$2I = G + X + Y + Z$$
$$2A = G + X - Y - Z$$
$$2B = G - X + Y - Z$$
$$2C = G - X - Y + Z$$
$$2G = I + A + B + C$$
$$2X = I + A - B - C$$
$$2Y = I - A + B - C$$
$$2Z = I - A - B + C$$

These expressions reveal a strong duality in the additive symmetries. The symmetry pairs $G \leftrightarrow I$, $X \leftrightarrow A$, $Y \leftrightarrow B$, and $Z \leftrightarrow C$ consistently appear across both types of operations, suggesting a deeper underlying structure within the group.

Example: The Relation $2G = I + A + B + C$

Consider the elements I, A, B, C, each containing four non-zero components. When we sum them, zero-valued elements always add to zero, and non-zero elements add to other non-zero elements of the same scale. Since the elements in I, A, B, C all have a scale of 1, and the elements in G have a scale of $1/2$, the result of the summation is $2G$, maintaining the appropriate scaling.

Example: The Relation $2I = G + X + Y + Z$

To visualize the addition process more clearly, consider the following sum table, where the terms are aligned as in basic arithmetic:

$$
\begin{array}{ccccccccc}
\frac{I}{2} & + & \frac{A}{2} & + & \frac{B}{2} & + & \frac{C}{2} & = (G) \\
\frac{I}{2} & + & \frac{A}{2} & - & \frac{B}{2} & - & \frac{C}{2} & = (X) \\
\frac{I}{2} & - & \frac{A}{2} & + & \frac{B}{2} & - & \frac{C}{2} & = (Y) \\
\frac{I}{2} & - & \frac{A}{2} & - & \frac{B}{2} & + & \frac{C}{2} & = (Z) \\
\hline
2I & & 0 & & 0 & & 0 &
\end{array}
$$

In this table, the positive and negative contributions of A, B, and C effectively cancel out, leaving us with the result:

$$G + X + Y + Z = 2I$$

This demonstrates how the off-diagonal terms cancel, while the diagonal terms sum to produce the final result of $2I$. These additive symmetries will interact with the multiplicative symmetries in an algebraic way in the parentheses notation, allowing us to have superpositions of particles as observed in protons and atoms.

Additive Symmetries with General Ω

When Ω is not set to 1, the additive symmetries still follow a similar structure, but the resulting sums depend on the specific values of Ω for each group element. The notation in parentheses is used to indicate the influence of the beta particle state Ω on the symmetry relations:

$$(G + X + Y + Z)(\Omega) = 2I(\Omega)$$

This highlights that the sum of elements is scaled by Ω, resulting in different outcomes depending on the beta particle state value.

Algebraic Structure of the Group

We have shown that the set of elements $\{G, X, Y, Z, I, A, B, C\}$ and their negatives form a group under multiplication when $\Omega = 1$, as proven in the accompanying Mathematica notebook. The generators of this group are the elements I, A, B, C, from which the elements G, X, Y, Z can be constructed through their linear combinations:

$$G = \frac{1}{2}(I+A+B+C), \quad X = \frac{1}{2}(I+A-B-C), \quad Y = \frac{1}{2}(I-A+B-C), \quad Z = \frac{1}{2}(I-A-B+C)$$

Each of the eight elements is unitary, and all are involutions when $\Omega = 1$, indicating that they may form an algebraic structure.

Testing Algebraic Properties

We conjecture that this group forms an algebra. Based on computational tests using Mathematica, we observe:

- **Closure**: The product of any two elements in the set yields another element within the set. Similarly, the sum of any two elements also remains within the set.
- **Distributivity**: Computational verification confirms that multiplication distributes over addition.
- **Associativity**: Both multiplication and addition appear to be associative in all cases tested.

These properties suggest that the group exhibits algebraic behavior. However, given the complexity of fully formalizing this algebra, we present these findings as a probable structure supported by computational evidence.

Generators and Normalization

The elements I, A, B, C serve as generators of the group, while the complete set of elements, including their negatives, represents the normalized points of the group. These elements behave as unitary transformations, as demonstrated by their involution property when $\Omega = 1$.

A Composite-Particle State

An element α of the algebra, referred to as a systematic state, is defined as a sum of up to four distinct particle states β_k, each chosen exclusively from one of the following sets: $\{G, X, Y, Z\}$ or $\{I, A, B, C\}$. These particle states are combined through addition, with the following conditions:

1. **Distinctness**: Each β_k within α must be unique, so no repeated elements are allowed (e.g., combinations like $G+G$ or $A+A$ are not permitted). 2. **Set Exclusivity**: All particle states within α must come from the same set, meaning combinations across sets, such as $X + I$ or $Z + A$, are not allowed.

Formally, we define α as:

$$\alpha = \sum_k \beta_k, \quad \text{where } \beta_k \in \{\{G, X, Y, Z\} \text{ or } \{I, A, B, C\}\}$$

with $\beta_k \neq \beta_l$ for $k \neq l$, and α can include up to four distinct particle states from the same set.

The particle state β_k represents an individual element within the chosen set (either $\{G, X, Y, Z\}$ or $\{I, A, B, C\}$) and must satisfy the condition of distinctness in the sum.

Additionally, while β exclusively denotes particle states, α may refer to either a particle state or a composite-particle state. When α contains only a single element (i.e., $\alpha = \beta$ for some β), it represents an individual particle state. Conversely, when α contains multiple distinct elements, it represents a composite-particle state.

Parentheses Notation

The parentheses notation defines objects through scalar products, with each object representing a measurable quantity—such as probability or energy—relevant to a particle or composite system. If α is a general element of the algebra representing a complete particle or composite particle, then the object $(\alpha|\alpha)$ is given by:

$$(\alpha|\alpha) = \frac{1}{4}\int_{\mu=0}^{4} d\xi_\mu \sum_{i,j} \alpha_{ij}^* \alpha_{ij}$$

where:

- α_{ij}^* represents the conjugate components of the state α.
- Integration across μ from 0 to 4 accounts for spatial, temporal, and scalar dimensions.
- The summation $\sum_{i,j}$ over matrix elements captures the composite nature of α, ensuring all matrix components are incorporated.

Objects in this notation are defined as the complete expression within parentheses, resulting in a scalar quantity. Examples include:

1. **Single Particle Object**: If $\alpha = G$, then $(G|G)$ measures the probability or presence of a single particle within its category.
2. **Composite Particle Object**: If $\alpha = G + X + Y + Z$, representing an atom, then $(G+X+Y+Z|G+X+Y+Z)$ captures the collective properties of the atom, including interference effects between constituent particles. By definition, this represents the synchronized collapse of four separate states into a unified outcome.

An illustration showing why $G^\dagger G$ forms a scalar when summed over is included in Appendix A while the same is done for $G + X$ in Appendix B. The surprising thing is that the first column of G contains two Ω terms and two $\Omega*$ terms while the first row of $G^\dagger$ only contains $\Omega*$. However, it is because of the way that quaternions in G distribute across the real and complex components of Ω that leads to a surpisingly scalar result. In overly simplistic terms, this happens because 1 and i affect Ω differently than j and k because 1 and i do not anticommute with the complex component of Ω but j and k do. In short, though this may spark incredulity, the math works.

Normalization Condition

For any element α, the normalization condition is given by:

$$\left(\alpha^\flat|\alpha^\flat\right) = 1$$

This normalization ensures that each object, when evaluated in its flat state, represents a coherent and consistent quantity. For composite structures, normalization scales with the number of constituent objects:

$$\left(G^\flat + X^\flat + Y^\flat + Z^\flat \middle| G^\flat + X^\flat + Y^\flat + Z^\flat\right) = 4$$

indicating the collective presence of four distinct particles within a unified notation.

Additivity in Parentheses Notation

Due to the additive symmetries of the algebra, summing multiple normalized elements yields a result proportional to the number of elements added, without needing renormalization:

$$\left(\sum_n \alpha_n^\flat \middle| \sum_n \alpha_n^\flat\right) = n$$

Conditions for this additivity include: 1. **No Repetition**: Elements like G_1 and G_2 cannot be combined within the same set. 2. **Set Consistency**: Elements from different sets (e.g., $\{G, X, Y, Z\}$ vs. $\{I, A, B, C\}$) cannot be mixed.

As an example, combining elements from the same set:

$$(G^\flat|G^\flat) = 1, \quad (X^\flat|X^\flat) = 1, \quad (Y^\flat|Y^\flat) = 1, \quad (Z^\flat|Z^\flat) = 1$$

and

$$(G^\flat + X^\flat + Y^\flat + Z^\flat|G^\flat + X^\flat + Y^\flat + Z^\flat) = 4$$

Scalar Products in Parentheses Notation

In parentheses notation, scalar products evaluate a measurable result or probability distribution. For example, the probability of a system collapsing to a specific location (such as a position) at a given input ξ is expressed as:

$$P(\tau_\mu) \propto (\beta \mid \beta)_{\xi_\mu}$$

where τ_μ is the localized point for each element of the input ξ, establishing a spatial zero-point for position magnitudes. This expression ensures that $(\beta|\beta)$, analogous to $\langle\Omega|\Omega\rangle$ in quantum mechanics, reflects the probability amplitude for system collapse to τ_μ under a position measurement.

Applying an energy-momentum-mass (Epm) operator, for instance, provides the probability of observing specific E, p, or m magnitudes, with normalization constants reflecting the probability of measuring each magnitude under the Epm operator.

Illustrating the Dynamics

Consider elements G, X, Y, Z with wavefunctions $e^{i\theta_n}$ and phase shifts $\theta_G = 0.25$, $\theta_X = 1.4$, $\theta_Y = 3.9$, and $\theta_Z = 0.34$. Representing each as a quaternion matrix, we can express their sum as:

$$\begin{bmatrix} 0.68+0.79i & -0.88-1.23i & -0.43j+0.14k & -0.44j+0.46k \\ 0.88+1.23i & 0.68+0.79i & 0.44j-0.46k & -0.43j+0.14k \\ 0.43j+0.14k & 0.44j+0.46k & 0.68-0.79i & -0.88+1.23i \\ -0.44j-0.46k & 0.43j+0.14k & 0.88-1.23i & 0.68-0.79i \end{bmatrix}$$

and their conjugates as:

$$\begin{bmatrix} 0.68-0.79i & 0.88-1.23i & -0.43j-0.14k & 0.44j+0.46k \\ -0.88+1.23i & 0.68-0.79i & -0.44j-0.46k & -0.43j-0.14k \\ 0.43j-0.14k & -0.44j+0.46k & 0.68+0.79i & 0.88+1.23i \\ 0.44j-0.46k & 0.43j-0.14k & -0.88-1.23i & 0.68+0.79i \end{bmatrix}$$

Their product yields:

$$\begin{bmatrix} 4 & -0.55i & -0.73j & 2.64k \\ 0.55i & 4 & -2.64k & -0.73j \\ 0.73j & 2.64k & 4 & 0.55i \\ -2.64k & 0.73j & -0.55i & 4 \end{bmatrix}$$

where off-diagonal terms exhibit symmetry, canceling when summed, while each diagonal element totals 4. Dividing by 4 yields the initial count of summed elements.

Caveats in Simplification

This example simplifies by using "metaphorical states" $e^{i\theta_n}$ rather than the full total states, and by evaluating at specific inputs rather than integrating. However, these simplifications do not alter the dynamics this example illustrates.

General Rule for Parentheses Notation

In parentheses notation, instead of squaring the sum of terms, each term is squared individually, with cross-terms canceling out, reminiscent of the Dirac matrices. For example:

$$(G^\flat + X^\flat + Y^\flat + Z^\flat | G^\flat + X^\flat + Y^\flat + Z^\flat) = 4$$

Though one might expect $(1+1+1+1)^2 = 16$, parentheses notation instead yields $1^2 + 1^2 + 1^2 + 1^2 = 4$. This result applies to any valid combination of distinct terms. If repeated terms are present, they contribute quadratically:

$$(G_1^\flat + G_2^\flat + X^\flat + Y^\flat + Z^\flat | G_1^\flat + G_2^\flat + X^\flat + Y^\flat + Z^\flat) = 7$$

Thus, repeated terms are squared, while distinct terms sum directly. Additionally, combining elements from separate sets (e.g., G and A) does not yield a straightforward particle count, reinforcing the constraints within parentheses notation that ensure normalized and non-mixed elements.

Connection to Probability Distributions

When the elements inside parentheses represent particle or composite-particle states, the notation can be interpreted as producing a probability distribution, analogous to the bra-ket notation in quantum mechanics. For a normalized state α, the expression $(\alpha|\alpha)$ reflects the probability of observing the system in that state.

Superposition and Particle Interpretation

In this framework, we can represent multi-particle systems in superposition, such as protons or atoms, while preserving individual normalization for each component. When several states (e.g., $G^\flat, X^\flat, Y^\flat, Z^\flat$) combine in superposition, their joint measurement results in constructive or destructive interference.

Unlike traditional quantum mechanics, which typically normalizes a superposition to 1, this framework allows each state to retain its normalization, avoiding the need to rescale. The resulting superposition represents the total number of particles without collapsing, supporting multi-particle interference patterns.

Retaining Individual Normalization in Superposition

Each state element is individually normalized to 1:

$$(G^\flat|G^\flat) = 1, \quad (X^\flat|X^\flat) = 1, \quad (Y^\flat|Y^\flat) = 1, \quad (Z^\flat|Z^\flat) = 1$$

When these elements form a superposition, their combined normalization reflects the sum of their individual contributions, such as:

$$(G^\flat + X^\flat + Y^\flat + Z^\flat | G^\flat + X^\flat + Y^\flat + Z^\flat) = 4$$

This approach ensures that the superposition retains the total particle count, reflecting the number of components directly.

Preservation of Particle Count

In this theory, normalization reflects both the probability distribution and particle count. For example, the normalization condition for a single state $G^\flat$ is:

$$(G^\flat|G^\flat) = 1$$

For a superposition of multiple states:

$$(G^\flat + X^\flat + Y^\flat + Z^\flat|G^\flat + X^\flat + Y^\flat + Z^\flat) = 4$$

Each state's individual normalization remains intact within the superposition, preserving the total count of particles.

Multiplication with Conjugates

When individually normalized states multiply by their conjugates, parentheses notation retains the total particle count. Whether for single states or superpositions, the notation ensures consistent normalization across different configurations. For instance:

$$(G^\flat|G^\flat) = 1$$
$$(G^\flat + X^\flat + Y^\flat + Z^\flat|G^\flat + X^\flat + Y^\flat + Z^\flat) = 4$$

Both cases demonstrate that particle count and normalization remain consistent without collapsing to a single value.

Implications of Unique Normalization

This unique approach to normalization allows superpositions to represent multiparticle systems while preserving individual normalization, diverging from traditional quantum mechanics' requirement to normalize probability to 1. In this framework, normalization reflects both probability and particle count, providing new avenues for interpreting superpositions and particle interactions within complex systems.