CSC 222: AUTOMATA THEORY

Lecture III: Recursive Definitions

Set Definition

- A precise definition aids in carrying out proof
- A concise definition assists in understanding
- Recursive definitions are precise and concise

Recursive Definitions

- A recursive definition is a three-step process:
  - Define basic set members
  - Give rules for constructing more set members from the ones already known
  - Exclude all set members except those created by the first two steps

Example

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EVEN is the set of all positive whole numbers divisible by 2

Or

EVEN is the set of all $2n$ where $n = 1, 2, 3, 4$…..

Example of a Recursive Definition

- The set EVEN (of positive even integers) is defined recursively as follows:

Rule 1: 2 is in EVEN

Rule 2: if x is in EVEN then so is x + 2

Rule 3: Only elements defined by rules 1 and 2 above are in EVEN

- The last rule is called an EXCLUSION RULE (it prevents members from sneaking into a set by unknown means, and it is tacitly presumed in recursive definitions).

Proving membership using definitions

- Is 14 in set EVEN?
- Which definition is "better"?
- Note, there are some sets that have no better definition than a recursive one

Definition of Set EVEN

- How else can the set EVEN be defined recursively?
- If EVEN_INTS is the set of positive and negative integers, how would you define EVEN_INTS recursively?

Definition aides Proof

- Supposing you want to prove that the sum of any two numbers in EVEN is also in EVEN, which definition would best serve your purpose?

Recap

- A recursive definition is better than another if it produces shorter proofs for set membership.
- Set EVEN has fine definitions that are non-recursive – later in the course, we shall be interested in certain sets that have no better definition than the recursive one.
- The choice of whether or not to use a recursive definition depends on:
  - How easy is it to understand the other possible definitions?
  - What types of theorems do we wish to prove about the set?

Example – Set of Integers

- Following is a recursive definition of the set of positive integers:

Rule I: 1 is in INTEGERS.
Rule II: If x is in INTEGERS, then so is $x + 1$.

- How would you define INTEGERS to include both positive and negative integers?

Example – Set POLYNOMIAL

- *Polynomial is a finite sum of terms each of which is of the form of a real number times a power of x (that may be $x^0 = 1$)*
- Definition of the set POLYNOMIAL :

Rule 1: Any natural number is in POLYNOMIAL

Rule 2: The variable x is in POLYNOMIAL

Rule 3: If p and q are in POLYNOMIAL then so are $p + q$, $p – q$, (p) and (pq)

Rule 4: Exclusion rule

Set POLYNOMIAL

- Use the definition of POLYNOMIAL to prove that 3x + 2 is in POLYNOMIAL
- Use the definition of POLYNOMIAL to prove that $4x^2 + 2x – 6$ is in POLYNOMIAL

The Power of Recursive Definitions

- Suppose in a calculus class we prove that:
  - the derivative of the sum of two functions is the sum of the derivatives
  - And that the derivative of the product fg is $f'g + fg'$
  - And that the derivative of a number is always 0
  - And that the derivative of x is 1
- Using the recursive definition of POLYNOMIAL and these proofs it is now a trivial exercise to show that all POLYNOMIALS can be differentiated. how?
- We don't yet know the most efficient algorithm for doing this but we know that it can be done.

The Power of Recursive Definitions

- Perhaps even more importantly recursive definitions can be used to show that certain tasks are theoretically impossible for any computer to perform…
- But that is beyond the scope of this course…

Recursion in the Real World

- Rule 1: The children of ABRAHAM are all members of the set DESCENDANTS_OF_ABRAHAM
- Rule 2: if x is an element of DESCENDANTS_OF_ABRAHAM then so are x' s children.

Recursion in Mathematics

- Definition of set FACTORIAL:

Rule 1: 0! = 1

Rule 2: n! = n (n – 1)!

Recursive Definitions of earlier Languages

$L_1 = x^+ = \{x\ xx\ xxx\ xxxx\ ...\}$

$L_{odd} = x^{2n-1}$ *for* $n > 0 = \{x\ xxx\ xxxxx\ ...\}$

$L_4 = x^* = \{\Lambda\ x\ xx\ xxx\ xxxx\ ...\}$

Kleene Closure e.g. S*

An Important Language – Arithmetic Expressions

- What constitutes a valid arithmetic expression (AE) that can be typed on one line, in a computer-digestible form?
- What is the alphabet?
- Are the following strings in language AE?
  -)6 +)5 + 3
  -)9 + 8/(2 –
  - (4 + (5 -) 8)
- What rules govern formation of correct AE strings?

Example – Arithmetic Expressions

- Recursive definition of AE most natural way rather than a list of forbidden substrings, parentheses requirements and rules!

Rule 1: Any number (positive, negative or zero) is in AE

Rule 2: If x is in AE then so are (x) and $-(x)$

Rule 3: If x and y are in AE then so are
- $x + y$, (if the first symbol in y is not -)
- $x - y$, (if the first symbol in y is not -)
- $x * y$,
- x / y,
- $x ** y$

Arithmetic Expressions

- Is the arithmetic expression (4 + 25 / (10 ** 2) / 5 in AE?
- Is the arithmetic expression (2 + 5) * (6 * (9 – 2)/5)/4 * (2 + 9) – 1 in AE?
- How does the recursive definition allow us to simply carry out such proofs?

Proving Theorems

- THEOREM 2: No arithmetic expression can contain the character $
- THEOREM 3: No arithmetic expression can begin or end with the symbol /

Example from LOGIC

- Definition of the set WFF (well-formed formulae):

$\Sigma = \{\rightarrow\ \neg\ (\ )\ a\ b\ c\ d\ ....\}$

Rule 1: Any single Latin letter is a WFF e.g. *a b c d*

Rule 2: If p is a WFF, then so are (p) and $\neg p$

Rule 3: If p and q are WFFs then so is p → q

- Are the following in WFF?
  1. $p \rightarrow ((p \rightarrow p) \rightarrow q)$
  2. $p \rightarrow$
  3. $\rightarrow p$
  4. $p \rightarrow \neg p$
  5. $p) \rightarrow p($

Exercises:

1. Using the fact that $3x^2 + 7x - 9 = (((((3)x) + 7)x) - 9)$, show how to produce this polynomial from the rules for POLYNOMIAL using multiplication only twice. What is the smallest number of steps needed for producing $x^8 + x^4$?
2. Show that if n is less than 31, then xn can be shown to be in POLYNOMIAL in fewer than eight steps.
3. What is the full list of substrings of length 2 that cannot occur in arithmetic expressions?
4. The rules given for the set AE allow for the peculiar expressions (((((9))))) and -(-(-(-(9)))). These are not harmful, but is there some modified definition of AE that eliminates this problem?

Exercises:

5. Give a recursive definition for the language EVENPALINDROME of all palindromes of even length.
6. Give a recursive definition for the set POWERS-OF-TWO = {1 2 4 8 16 ....} and use your definition to prove that the product of two POWERS-OF-TWO is also a POWER-OF-TWO.