TMA1301 Computational Methods

Introduction

① Error Analysis

$e = x - x^*$

$e_{abs} = |x - x^*|$

$e_{rel} = \frac{|x - x^*|}{|x|}$

② Error Reduction

1. Nested form
2. Avoid loss of significance in subtraction
   (a) Rationalisation
   (b) Taylor Series: Use first three terms.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$

Locating Roots of Equations

① Bisection Method

$p_n = \frac{b_n + a_n}{2}$ $\quad \frac{b-a}{2^n} \leq$ tolerance

If $f(a_n) \cdot f(p_n) = -ve$, then p in $[a_n, p_n]$.

If $f(a_n) \cdot f(p_n) = +ve$, then p in $[p_n, b_n]$.

stop when $\checkmark < tol$

n	a_n	b_n	p_n	$f(a_n)$	$f(p_n)$	$f(a_n) \cdot f(p_n)$	$\lvert p_n - p_{n-1} \rvert$

② Secant Method

$p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}$

n	p_n	p_{n-1}	$f(p_n)$	$f(p_{n-1})$	p_{n+1}	$f(p_{n+1})$	$\lvert p_{n+1} - p_n \rvert$

③ Newton-Raphson Method

$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$

n	p_n	$f(p_n)$	$f'(p_n)$	p_{n+1}	$f(p_{n+1})$	$\lvert p_{n+1} - p_n \rvert$

Numerical Integration $\int_a^b f(x)\,dx$

① Trapezoidal Rule

width, $h = \frac{b-a}{n}$ $\quad$ Tip: Draw

x	a	...	b
y			

Area $= \frac{h}{2}[f(a) + f(b) + 2f(\text{middles})]$

② Simpson's Rule

$h = \frac{b-a}{n}$

$A = \frac{h}{3}[f(a) + f(b) + 4f(\text{evens}) + 2f(\text{odds})]$

③ Romberg Algorithm

$n \backslash m$	0	1	2	...	
0	$R(0,0)$				
1	$R(1,0)$	$R(1,1)$			
2	$R(2,0)$	$R(2,1)$	$R(2,2)$	$\ddots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
	$R(n,0)$	$R(n,1)$	$R(n,2)$	...	$R(n,m)$

$R(0,0) = \frac{1}{2}(b-a)[f(a) + f(b)]$

$R(n,0) = \frac{1}{2}R(n-1,0) + h\sum_{k=1}^{2^{n-1}} f[a + (2k-1)h]$, $h = \frac{b-a}{2^n}$

$R(n,m) = R(n,m-1) + \frac{1}{4^m - 1}[R(n,m-1) - R(n-1,m-1)]$

Monte Carlo Methods and Simulation

① Random Numbers

1. Linear Congruential Generator (LCG)
   - Mixed Congruential Method $X_n = (aX_{n-1} + c)(\text{mod } m)$
   - Multiplicative Congruential Method $X_n = (aX_{n-1} + 0)(\text{mod } m)$
   - Additive Congruential Method $X_n = (1X_{n-1} + c)(\text{mod } m)$
2. Random Variate Generator for
   - Exponential Distribution $X_n = (-\frac{1}{\lambda})\ln(1 - R_n)$
   - Uniform Distribution on $[a, b]$ $X_n = R_n(b-a) + a$

$f(x) = 2x^3 + 3x^2 - 3x - 5$, $[1, 2]$, with tolerance 1×10^{-3}

Bisection Method $\quad p_n = \dfrac{b_n + a_n}{2}$

n	a_n	b_n	p_n	$f(a_n)$	$f(p_n)$	$f(a_n)f(p_n)$	$\lvert p_n - p_{n-1} \rvert$
1	1.000000	2.000000	1.500000	-3.000000	4.000000	−	-
2	1.000000	1.500000	1.250000	-3.000000	-0.156250	+	0.250000
3	1.250000	1.500000	1.375000	-0.156250	1.746094	−	0.125000
4	1.250000	1.375000	1.312500	-0.156250	0.752441	−	0.062500
5	1.250000	1.312500	1.281250	-0.156250	0.287659	−	0.031250
6	1.250000	1.281250	1.265625	-0.156250	0.063118	−	0.015625
7	1.250000	1.265625	1.257813	-0.156250	-0.047203	+	0.007812
8	1.257813	1.265625	1.261719	-0.047203	0.007796	−	0.003906
9	1.257813	1.261719	1.259766	-0.047203	-0.019743	+	0.001953
10	1.259766	1.261719	1.260743	-0.019743	-0.005977	+	0.000977

$\therefore p_{10} = 1.260743$

Secant Method $\quad p_{n+1} = p_n - \dfrac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}$

n	p_n	p_{n-1}	$f(p_n)$	$f(p_{n-1})$	p_{n+1}	$f(p_{n+1})$	$\lvert p_{n+1} - p_n \rvert$
1	2.000000	1.000000	17.000000	-3.000000	1.150000	-1.440750	0.850000
2	1.150000	2.000000	-1.440750	17.000000	1.216409	-0.610553	0.066409
3	1.216409	1.150000	-0.610553	-1.440750	1.265248	0.057764	0.048839
4	1.265248	1.216409	0.057764	-0.610553	1.261027	-0.001971	0.004221
5	1.261027	1.265248	-0.001971	0.057764	1.261161	-0.000080	0.001334
6	1.261161	1.261027	-0.000080	-0.001971	1.261167	0.000004	0.000006

$\therefore p_7 = 1.261167$

Newton Method $\quad p_{n+1} = p_n - \dfrac{f(p_n)}{f'(p_n)}$

$f'(x) = 6x^2 + 6x - 3$

n	p_n	$f(p_n)$	$f'(p_n)$	p_{n+1}	$f(p_{n+1})$	$\lvert p_{n+1} - p_n \rvert$
1	2.000000	17.000000	33.000000	1.484848	3.707286	0.515152
2	1.484848	3.707286	19.137732	1.291132	0.432460	0.193716
3	1.291132	0.432460	14.748924	1.261811	0.010986	0.029321
4	1.261811	0.010986	14.123868	1.261033	-0.001887	0.000778

$\therefore p_5 = 1.261033$

5 points / 4 equal parts

$f(x)=\int_3^4 \frac{1}{x^2+x-6}\,dx$ $\quad n=5-1=4 \quad h=\frac{b-a}{n}=\frac{4-3}{4}=0.25$

Composite Trapezoidal Rule

$$A=\frac{0.25}{2}\left[f(3)+f(4)+2\left[f(3.25)+f(3.5)+f(3.75)\right]\right]$$
$$=0.125\left[\frac{1}{9+3-6}+\frac{1}{16+4-6}+2\left(\frac{1}{10.5625+3.25-6}+\frac{1}{12.25+3.5-6}+\frac{1}{14.0625+3.75-6}\right)\right]$$
$$=0.125\left(\frac{1}{6}+\frac{1}{14}+2\left(\frac{1}{7.8125}+\frac{1}{9.75}+\frac{1}{11.8125}\right)\right)$$
$$=0.125(0.166667+0.071429+2(0.128+0.102564+0.084656))$$
$$=0.125(0.238096+2(0.31522))$$
$$=0.125(0.868536)$$
$$=0.108567$$

Simpson's Rule

$$A=\frac{0.25}{3}\left[f(3)+f(4)+4\left[f(3.25)+f(3.75)\right]+2f(3.5)\right]$$
$$\vdots$$
$$=0.107820$$

$\int_0^2 \frac{4}{1+x^2}\,dx$ by evaluating $R(2,2)$

Romberg Algorithm

n	0	1	2
0	4.800000		
1	4.400000	4.266667	
2	4.415385	4.420513	4.430769

$$R(0,0)=\frac{1}{2}(b-a)\left[f(a)+f(b)\right]$$
$$=\frac{1}{2}(2-0)\left[f(0)+f(2)\right]$$
$$\vdots$$
$$=4.8$$

$$R(n,0)=\frac{1}{2}R(n-1,0)+h\sum_{k=1}^{2^{n-1}} f\left[a+(2k-1)h\right],\quad h=\frac{b-a}{2^n}$$

$$R(1,0)=\frac{1}{2}(\underset{R(0,0)}{4.8})+\sum_{k=1}^{2^0} f(2k-1),\quad h=\frac{2-0}{2^1}=1$$
$$=2.4+f(1)$$
$$\vdots$$
$$=4.4$$

$$R(n,m)=R(n,m-1)+\frac{1}{4^m-1}\left[R(n,m-1)-R(n-1,m-1)\right]$$

$$R(1,1)=\underset{R(1,0)}{4.4}+\frac{1}{4^1-1}\left[\underset{R(1,0)}{4.4}-\underset{R(0,0)}{4.8}\right]$$
$$\vdots$$
$$=4.266667$$

$$R(2,0)=\frac{1}{2}R(1,0)+\frac{1}{2}\sum_{k=1}^{2^1} f\left[1k-0.5\right],\quad h=\frac{2-0}{2^2}=\frac{1}{2}$$
$$=\frac{1}{2}(4.4)+0.5\left[f(0.5)+f(1.5)\right]$$
$$\vdots$$
$$=4.415385$$

$$R(2,1)=R(2,0)+\frac{1}{4^1-1}\left[R(2,0)-R(1,0)\right]$$
$$\vdots$$
$$=4.420513$$

$$R(2,2)=R(2,1)+\frac{1}{4^2-1}\left[R(2,1)-R(1,1)\right]$$
$$\vdots$$
$$=4.430769$$

Generate a sequence of 5 two-digit random numbers

$$X_n = (13X_{n-1} + 53) \pmod{100}$$
$$X_0 = 46$$

Mixed congruential method

$X_n = (13X_{n-1} + 53) \pmod{100}$

$X_1 = (13(46) + 53) \pmod{100}$
$= 651 \bmod 100$
$= 51$

$X_2 = (13(51) + 53) \pmod{100}$
$= 716 \bmod 100$
$= 16$

$X_3 = (13(16) + 53) \pmod{100}$
$= 261 \bmod 100$
$= 61$

$X_4 = (13(61) + 53) \pmod{100}$
$= 846 \bmod 100$
$= 46$

$X_5 = (13(46) + 53) \pmod{100}$
$= 651 \bmod 100$
$= 51$

Multiplicative congruential method

$X_n = (13X_{n-1} + 0) \pmod{100}$

$X_1 = (13(46) + 0) \pmod{100}$
$= 598 \bmod 100$
$= 98$

$X_2 = (13(98) + 0) \pmod{100}$
$= 1274 \bmod 100$
$= 74$

$X_3 = (13(74) + 0) \pmod{100}$
$= 962 \bmod 100$
$= 62$

$X_4 = (13(62) + 0) \pmod{100}$
$= 806 \bmod 100$
$= 06 = 6.0$

$X_5 = (13(6.0) + 0) \pmod{100}$
$= 78 \bmod 100$
$= 78$

Additive congruential method

$X_n = (1X_{n-1} + 53) \pmod{100}$

$X_1 = (46 + 53) \pmod{100}$
$= 99 \bmod 100$
$= 99$

$X_2 = (99 + 53) \pmod{100}$
$= 152 \bmod 100$
$= 52$

$X_3 = (52 + 53) \pmod{100}$
$= 05 = 5.0$

$X_4 = (5.0 + 53) \pmod{100}$
$= 58 \bmod 100$
$= 58$

$X_5 = (58 + 53) \pmod{100}$
$= 111 \bmod 100$
$= 11$

0.1306, 0.0422, 0.6597, 0.7965, 0.7696

Generate 5 exponential random variates using the generator with $\lambda = 2$

$\lambda = 2$ $R_1 = 0.1306$ $R_2 = 0.0422$ $R_3 = 0.6597$ $R_4 = 0.7965$ $R_5 = 0.7696$

$x_n = -\frac{1}{\lambda}[\ln(1-R_n)]$

$x_1 = -\frac{1}{2}\ln(1-0.1306)$

$= 0.0700$

$x_2 = -\frac{1}{2}\ln(1-0.0422)$

$= 0.0216$

$x_3 = -\frac{1}{2}\ln(1-0.6597)$

$= 0.5390$

$x_4 = -\frac{1}{2}\ln(1-0.7965)$

$= 0.7960$

$x_5 = -\frac{1}{2}\ln(1-0.7696)$

$= 0.7340$

0.4887, 0.0369, 0.8805, 0.6032, 0.1861, 0.7964, 0.7694, 0.9796, 0.3726, 0.2891

Generate a sequence of 5 uniformly distributed random variates in the interval [-5, 10]

$x_i = R_i(b-a)+a$

$= R_i(10-(-5))+(-5)$

$= 15R_i - 5$

$x_1 = 15(0.4887)-5$

$= 2.3305$

$x_2 = 15(0.0369)-5$

$= -4.4465$

$x_3 = 15(0.8805)-5$

$= 8.2075$

$x_4 = 15(0.6032)-5$

$= 4.0480$

$x_5 = 15(0.1861)-5$

$= -2.2085$

$x_6 = 15(0.7964)-5$

$= 6.9460$

$x_7 = 15(0.7694)-5$

$= 6.5410$

$x_8 = 15(0.9796)-5$

$= 9.6940$

$x_9 = 15(0.3726)-5$

$= 0.5890$

$x_{10} = 15(0.2891)-5$

$= -0.6635$

$$x_1 - 3x_3 = 8$$
$$2x_1 + 2x_2 + 9x_3 = 7$$
$$x_2 + 5x_3 = -2$$

Gaussian Elimination with Back-Substitution

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & -\frac{5}{2} & \frac{5}{2} \end{array}\right]$$

$$-\frac{5}{2}x_3 = \frac{5}{2}$$
$$\therefore x_3 = -1$$

$$2x_2 + 15x_3 = -9$$
$$2x_2 + 15(-1) = -9$$
$$\therefore x_2 = 3$$

$$x_1 - 3x_3 = 8$$
$$x_1 - 3(-1) = 8$$
$$\therefore x_1 = 5$$

Gauss-Jordan Elimination

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{array}\right] \xrightarrow{\frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[-5R_3+R_2]{3R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\therefore x_1 = 5 \quad \therefore x_2 = 3 \quad \therefore x_3 = -1$$

Matrix Inversion

Let $A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 9 \\ 0 & 1 & 5 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 2 & 2 & 9 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 2 & 15 & -2 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \\ 0 & 2 & 15 & -2 & 1 & 0 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \\ 0 & 0 & 5 & -2 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{array}\right] \xrightarrow[-5R_3+R_2]{3R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & \frac{3}{5} & -\frac{6}{5} \\ 0 & 1 & 0 & 2 & -1 & 3 \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} & -\frac{6}{5} \\ 2 & -1 & 3 \\ -\frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

$$x = A^{-1}b$$
$$= \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} & -\frac{6}{5} \\ 2 & -1 & 3 \\ -\frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 8 \\ 7 \\ -2 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$

$$\therefore x_1 = 5 \quad \therefore x_2 = 3 \quad \therefore x_3 = -1$$

$$A=\begin{bmatrix}2 & -1 & 2\\ -6 & 0 & -2\\ 8 & -1 & 5\end{bmatrix} \qquad b=\begin{bmatrix}1\\0\\4\end{bmatrix}$$

LU Factorisation

$$\begin{bmatrix}2 & -1 & 2\\ -6 & 0 & -2\\ 8 & -1 & 5\end{bmatrix} \xrightarrow[-4R_1+R_3]{3R_1+R_2} \begin{bmatrix}2 & -1 & 2\\ 0 & -3 & 4\\ 0 & 3 & -3\end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix}2 & -1 & 2\\ 0 & -3 & 4\\ 0 & 0 & 1\end{bmatrix}$$

$$U=\begin{bmatrix}2 & -1 & 2\\ 0 & -3 & 4\\ 0 & 0 & 1\end{bmatrix} \qquad L=\begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 4 & -1 & 1\end{bmatrix}$$

$Ly=b$

$$\begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 4 & -1 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}1\\0\\4\end{bmatrix}$$

$y_1=1$

$-3y_1+y_2=0$
$-3(1)+y_2=0$
$y_2=3$

$4y_1-y_2+y_3=4$
$4(1)-3+y_3=4$
$y_3=3$

$Ux=y$

$$\begin{bmatrix}2 & -1 & 2\\ 0 & -3 & 4\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\3\\3\end{bmatrix}$$

$2x_1-x_2+2x_3=1$
$2x_1-3+2(3)=1$
$\therefore x_1=-1$

$-3x_2-4x_3=3$
$-3x_2-4(3)=3$
$\therefore x_2=3$

$\therefore x_3=3$

Compute the first two steps of the following with starting vector [0,...,0].

$$\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}0\\2\\0\end{bmatrix}$$

Jacobi Method

$2u-v=0$
$u=\frac{v}{2}$
$u_{k+1}=\frac{v_k}{2}$

$-u+2v-w=2$
$v=\frac{2+u+w}{2}$
$v_{k+1}=\frac{2+u_k+w_k}{2}$

$-v+2w=0$
$w=\frac{v}{2}$
$w_{k+1}=\frac{v_k}{2}$

$$k=1 \quad \begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}=\begin{bmatrix}\frac{0}{2}\\ \frac{2+0+0}{2}\\ \frac{0}{2}\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix} \qquad k=2 \quad \begin{bmatrix}u_2\\v_2\\w_2\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{2+0+0}{2}\\ \frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\1\\ \frac{1}{2}\end{bmatrix}$$

k	u_k	v_k	w_k
0	0	0	0
1	0	1	0
2	$\frac{1}{2}$	1	$\frac{1}{2}$

Gauss-Siedel Method

$2u-v=0$
$u=\frac{v}{2}$
$u_{k+1}=\frac{v_k}{2}$

$-u+2v-w=2$
$v=\frac{2+u+w}{2}$
$v_{k+1}=\frac{2+u_{k+1}+w_k}{2}$

$-v+2w=0$
$w=\frac{v}{2}$
$w_{k+1}=\frac{v_{k+1}}{2}$

$$k=1 \quad \begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}=\begin{bmatrix}\frac{0}{2}\\ \frac{2+0+0}{2}\\ \frac{1}{2}\end{bmatrix}=\begin{bmatrix}0\\1\\ \frac{1}{2}\end{bmatrix} \qquad k=2 \quad \begin{bmatrix}u_2\\v_2\\w_2\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{2+\frac{1}{2}+\frac{1}{2}}{2}\\ \frac{3}{4}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{3}{2}\\ \frac{3}{4}\end{bmatrix}$$

k	u_k	v_k	w_k
0	0	0	0
1	0	1	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{4}$

- Find eigenvalues and its corresponding eigenvectors, and bases for matrix A.

$A=\begin{bmatrix}3&-2\\1&0\end{bmatrix}$

$A-\lambda I=\begin{bmatrix}3&-2\\1&0\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}3-\lambda&-2\\1&-\lambda\end{bmatrix}$

$\det(A-\lambda I)=(3-\lambda)(-\lambda)-(-2)(1)=\lambda^2-3\lambda+2$

characteristic equation: $(\lambda-2)(\lambda-1)=0$

eigenvalues: $\lambda_1=2 \quad \lambda_2=1$

$A-2I=\begin{bmatrix}3&-2\\1&0\end{bmatrix}-2\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}1&-2\\1&-2\end{bmatrix}$

$\left[\begin{array}{cc|c}1&-2&0\\1&-2&0\end{array}\right]\xrightarrow{-R_1+R_2}\begin{bmatrix}1&-2\\0&0\end{bmatrix}$

$x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}-2\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$x_1-2x_2=0$

$x_1=2x_2$

eigenvector for $\lambda_1=2$: $\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_2\begin{bmatrix}2\\1\end{bmatrix}$

basis for $\lambda_1=2$: $v_1=\begin{bmatrix}2\\1\end{bmatrix}$ or $\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$

$A-I=\begin{bmatrix}3&-2\\1&0\end{bmatrix}-\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}2&-2\\1&-1\end{bmatrix}$

$\left[\begin{array}{cc|c}2&-2&0\\1&-1&0\end{array}\right]\xrightarrow{-\frac{1}{2}R_1+R_2}\left[\begin{array}{cc|c}2&-2&0\\0&0&0\end{array}\right]$

$x_1\begin{bmatrix}2\\0\end{bmatrix}+x_2\begin{bmatrix}-2\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$2x_1-2x_2=0$

$x_1=x_2$

eigenvector for $\lambda_2=1$: $\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_2\begin{bmatrix}1\\1\end{bmatrix}$

basis for $\lambda_2=1$: $v_2=\begin{bmatrix}1\\1\end{bmatrix}$ or $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

x	1	2	3	4	5
y	0.6	1.9	4.3	7.6	12.6

Find least square exponential of form $y=ax^b$.

Least Square Method

$a=\dfrac{\Sigma x^2\Sigma y-\Sigma xy\Sigma x}{n\Sigma x^2-(\Sigma x)^2} \qquad b=\dfrac{n\Sigma xy-\Sigma x\Sigma y}{n\Sigma x^2-(\Sigma x)^2}$

$y=ax^b$

$\ln y=\ln(ax^b)$

$\quad=\ln a+b\ln x$

Let $Y=\ln y$, $A=\ln a$, $X=\ln x$

$Y=A+bX$

x	y	X	Y	X^2	XY
1	0.6	⋮	⋮	⋮	⋮
2	1.9				
3	4.3				
4	7.6				
5	12.6				
$\Sigma x=15$	$\Sigma y=27$	$\Sigma X=4.787491$	$\Sigma Y=6.151488$	$\Sigma X^2=6.199503$	$\Sigma XY=8.936788$

$A=\dfrac{\Sigma X^2\Sigma Y-\Sigma XY\Sigma X}{n\Sigma X^2-(\Sigma X)^2}=\ldots=-0.575507$

$b=\dfrac{n\Sigma XY-\Sigma X\Sigma Y}{n\Sigma X^2-(\Sigma X)^2}=\ldots=1.885961$

$A=\ln a \Rightarrow a=0.562420$

$\therefore\ y=0.562420x^{1.885961}$

Find taylor polynomial of degree 3 and approximate $f(x)$ when $x=0.1$.

$f(x)=\sqrt{x}$ at $x_0=1$

Taylor Polynomial $P_n(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)(x-a)^k$

Maclaurin Polynomial $P_n(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(0)(x-0)^k$

$f(x)=\sqrt{x} \qquad f'(x)=\frac{1}{2\sqrt{x}} \qquad f''(x)=-\frac{1}{4\sqrt{x^3}} \qquad f^{(3)}(x)=\frac{3}{8\sqrt{x^5}}$

$f(1)=1 \qquad f'(1)=\frac{1}{2} \qquad f''(1)=-\frac{1}{4} \qquad f^{(3)}(1)=\frac{3}{8}$

$P_3(x)=\sum_{n=0}^{3}\frac{1}{k!}f^{(k)}(1)(x-1)^k$

$=\frac{1}{0!}f(1)(x-1)^0+\frac{1}{1!}f'(1)(x-1)^1+\frac{1}{2!}f''(1)(x-1)^2+\frac{1}{3!}f^{(3)}(1)(x-1)^3$

$\vdots$

$=\frac{1}{16}x^3-\frac{5}{16}x^2+\frac{15}{16}x+\frac{5}{16}$ #

$P_3(0.1)=\ldots=0.403188$ #

Find second interpolating polynomial $P_2(x)$.

$f(x)=\sqrt{1+x}$ $\quad x_0=0 \quad x_1=0.3 \quad x_2=0.6$

$\boxed{\text{Lagrange Polynomial } P_n(x)=\sum_{i=0}^{n} f(x_i)L_i(x)}$

$$L_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{(x-0.3)(x-0.6)}{(0-0.3)(0-0.6)} \vdots = \frac{1}{0.18}(x^2-0.9x+0.18)$$

$$L_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\frac{(x-0)(x-0.6)}{(0.3-0)(0.3-0.6)} \vdots = -\frac{1}{0.09}(x^2-0.6x)$$

$$L_2(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{(x-0)(x-0.3)}{(0.6-0)(0.6-0.3)} \vdots = \frac{1}{0.18}(x^2-0.3x)$$

$$P_2(x)=\sum_{i=0}^{2} f(x_i)L_i(x)$$
$$= f(x_0)L_0(x)+f(x_1)L_1(x)+f(x_2)L_2(x)$$
$$\vdots$$
$$= -0.085772x^2+0.492982x+1 \;\#$$

Find newton polynomial $P_3(x)$.

x_k	$y_k=f(x_k)$	1st divided difference	2nd divided difference	3rd divided difference
0	$a_0=0$			
		$a_1=1.5$		
1	1.5		$a_2=-0.75$	
		0		$a_3=0.1875$
2	1.5		-0.1875	
		0.375		
3	1.125			

$$f(x_0,x_1)=\frac{f(x_1)-f(x_0)}{x_1-x_0}$$
$$1.5=\frac{f(x_1)-0}{1-0}$$
$$f(x_1)=1.5$$

$$f(x_1,x_2)=\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{1.5-1.5}{2-1}=0$$

$$f(x_0,x_1,x_2)=\frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}=\frac{0-1.5}{2-0}=-0.75$$

$$f(x_0,x_1,x_2,x_3)=\frac{f(x_1,x_2,x_3)-f(x_0,x_1,x_2)}{x_3-x_0}=\frac{-0.1875-(-0.75)}{3-0}=0.1875$$

$$P_3(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+a_3(x-x_0)(x-x_1)(x-x_2)$$
$$=0+1.5(x-0)-0.75(x-0)(x-1)+0.1875(x-0)(x-1)(x-2)$$
$$\vdots$$
$$=0.1875x^3-1.3125x^2+2.625x \;\#$$