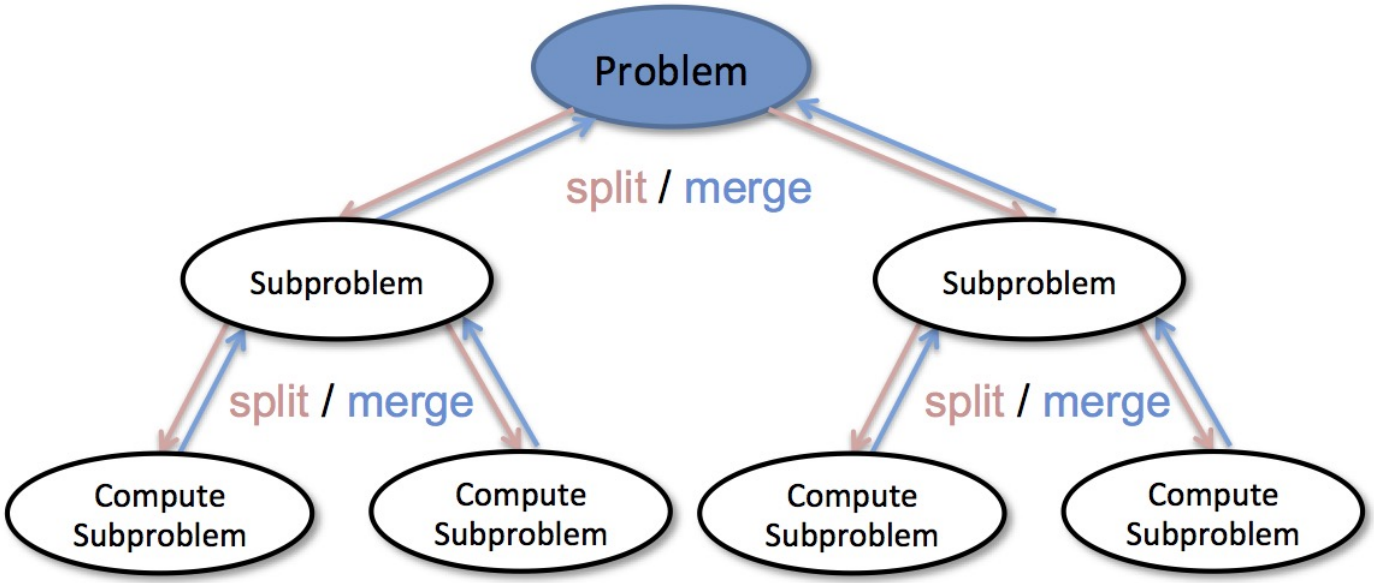
Divide and Conquer Algorithms



NYIT CSCI-651

Review

- Asymptotic analysis is analyzing what happens to the run time (or other performance metric) as the input size n goes to infinity.
  - Rate of growth ≠ fastness
- For large enough inputs, an algorithm whose running time has a lower order of growth , a $O(n^2)$ algorithm, for example, will take less time (run more quickly) in the worst case than a $O(n^3)$ algorithm.

- **Order Notation** – Big-O, Big-Theta and Big-Omega

Review - Asymptotic Analysis Hacks

- These are quick tricks to get big-Θ (or big-O) category.
- Eliminate low order terms (and hence constants)
  - $4n+5 \Rightarrow 4n$
  - $0.5\ n \log n - 2n + 7 \Rightarrow 0.5\ n \log n$
  - $2^n + n^3 + 3n \Rightarrow 2^n$
- Eliminate coefficients
  - $4n \Rightarrow n$
  - $0.5\ n \log n \Rightarrow n \log n$
  - $n \log (n^2) = 2\ n \log n \Rightarrow n \log n$

Solving Recurrences

- Substitution
  - Based on guesswork
  - Requires a lot of experience
- Recursion Tree
  - Can be be good if used done correctly
  - Or used to create an initial guess for the substitution method
- Master Method
  - Can only be used in certain cases
  - Requires memorization of the rules required to identify the three base cases

Solving Recurrences – Substitution Method

The most general method:

1. ***Guess*** the form of the solution.
2. ***Verify*** by induction.
3. ***Solve*** for constants.

Solving Recurrences – Substitution Method

Determine upper bound on recurrence

$$T(n) = 2T(\lfloor n/2 \rfloor) + n\,, \tag{4.19}$$

which is similar to recurrences (4.3) and (4.4). We guess that the solution is $T(n) = O(n \lg n)$. The substitution method requires us to prove that $T(n) \le cn \lg n$ for an appropriate choice of the constant $c > 0$. We start by assuming that this bound holds for all positive $m < n$, in particular for $m = \lfloor n/2 \rfloor$, yielding $T(\lfloor n/2 \rfloor) \le c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &\le 2(c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)) + n \\ &\le cn \lg(n/2) + n \\ &= cn \lg n - cn \lg 2 + n \\ &= cn \lg n - cn + n \\ &\le cn \lg n\,, \end{aligned}$$

Solving Recurrences – Recursion Tree Method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- The recursion-tree method promotes intuition, however.
- The recursion tree method is good for generating guesses for the substitution method.

Solving Recurrences – Master Method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\,,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

Solving Recurrences – Master Method

$$T(n) = a\,T(n/b) + f(n)$$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon})$, constant $\varepsilon > 0$
$\Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a} \lg^k n)$, constant $k \geq 0$
$\Rightarrow T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$, constant $\varepsilon > 0$, and regularity condition
$\Rightarrow T(n) = \Theta(f(n))$.

$$T(n) = a\,T(n/b) + f(n)\ ,$$

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.

2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

   and $f(n)$ satisfies the ***regularity condition*** that $af(n/b) \le cf(n)$ for some constant $c < 1$.

   Solution: $T(n) = \Theta(f(n))$.

Master Method - Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n.$

Case 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2).$

Master Method - Examples

Ex. $T(n) = 4T(n/2) + n^2$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2.$

Case 2: $f(n) = \Theta(n^2 \lg^0 n)$, that is, $k = 0$.

$\therefore\ T(n) = \Theta(n^2 \lg n).$

Master Method - Examples

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

and $4(n/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3).$

Ex. $T(n) = 4T(n/2) + n^2/\lg n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2/\lg n.$

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^\varepsilon = \omega(\lg n)$.

Merge Sort – Master Theorem

$$T(n) = a\,T(n/b) + f(n)$$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon})$, constant $\varepsilon > 0$
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CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$, constant $\varepsilon > 0$,
and regularity condition
$\Rightarrow T(n) = \Theta(f(n))$.

Merge sort: $a = 2$, $b = 2$ $\Rightarrow$ $n^{\log_b a} = n^{\log_2 2} = n$
$\Rightarrow$ CASE 2 ($k = 0$) $\Rightarrow$ $T(n) = \Theta(n \lg n)$.

Divide and Conquer

Divide and Conquer

- In politics and sociology is gaining and maintaining power by breaking up larger concentrations of power into pieces that individually have less power than the one implementing the strategy.

[wikipedia]

Divide and Conquer

In Computer Science:

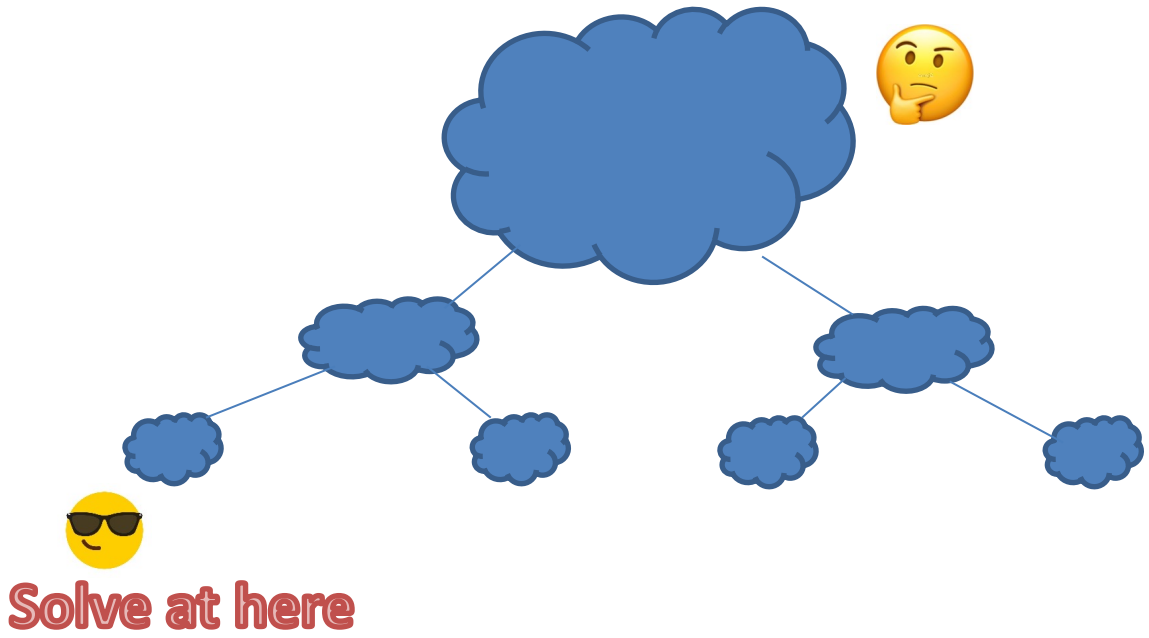
- A divide and conquer algorithm recursively breaks down a problem into two or more sub-problems of the same or related type, until these become simple enough to be solved directly.
- The solutions to the sub-problems are then combined to give a solution to the original problem.

Divide and Conquer

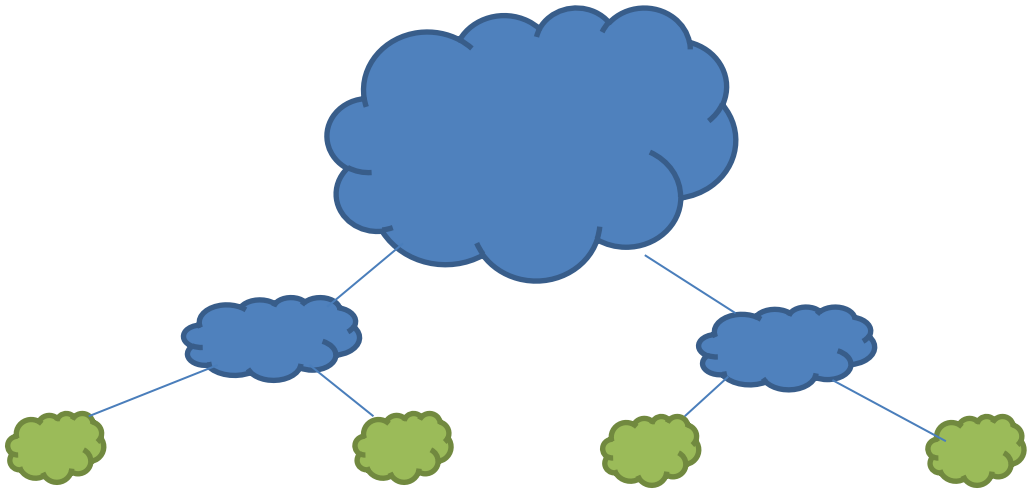
- Involves 3 steps

1. **Divide** the problem into a number of sub-problems that are smaller instances of the same problem
2. **Conquer** the sub-problems by solving them recursively. If the sub-problem is small enough, solve it in a straightforward way
3. **Combine** the solutions to the sub-problems into the solution of the original problem.

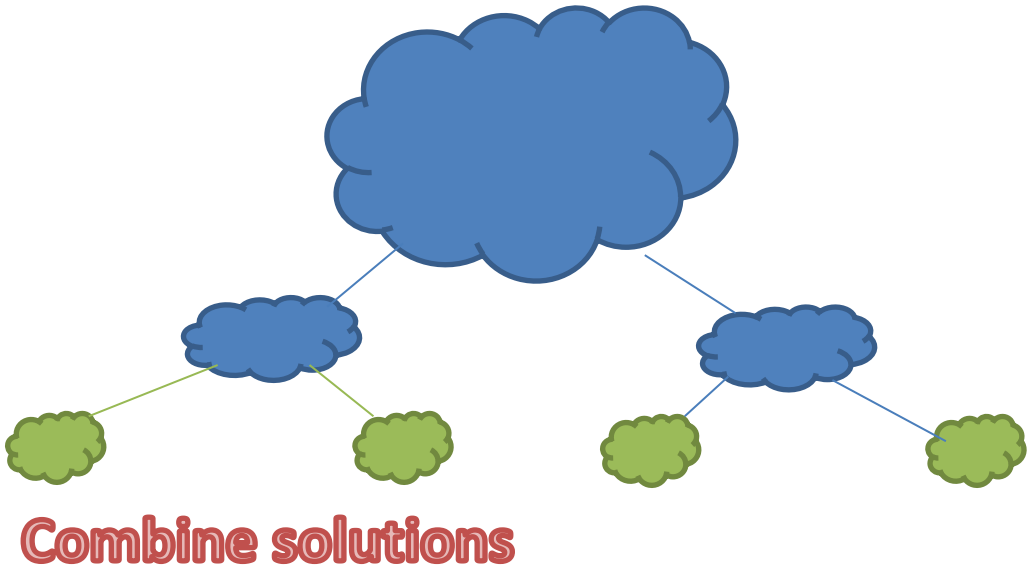
Divide and Conquer



Divide and Conquer



Divide and Conquer



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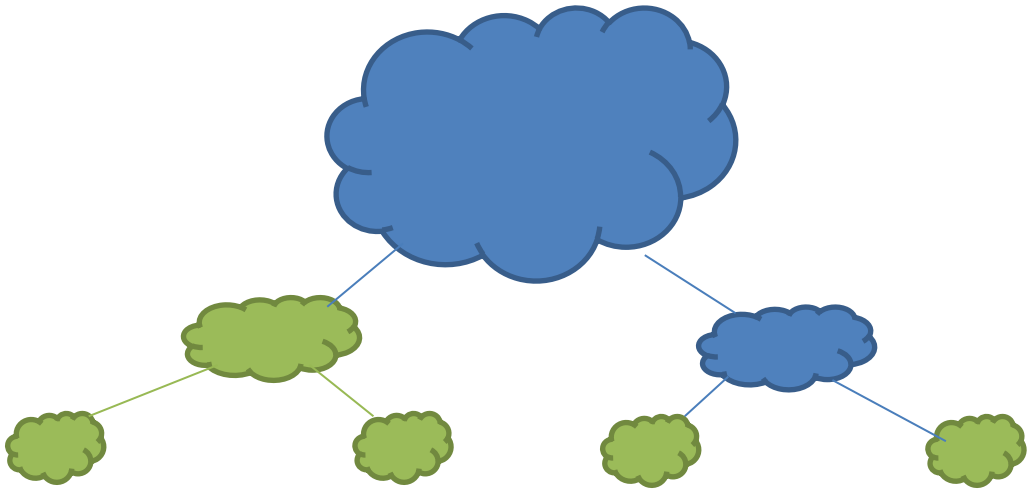

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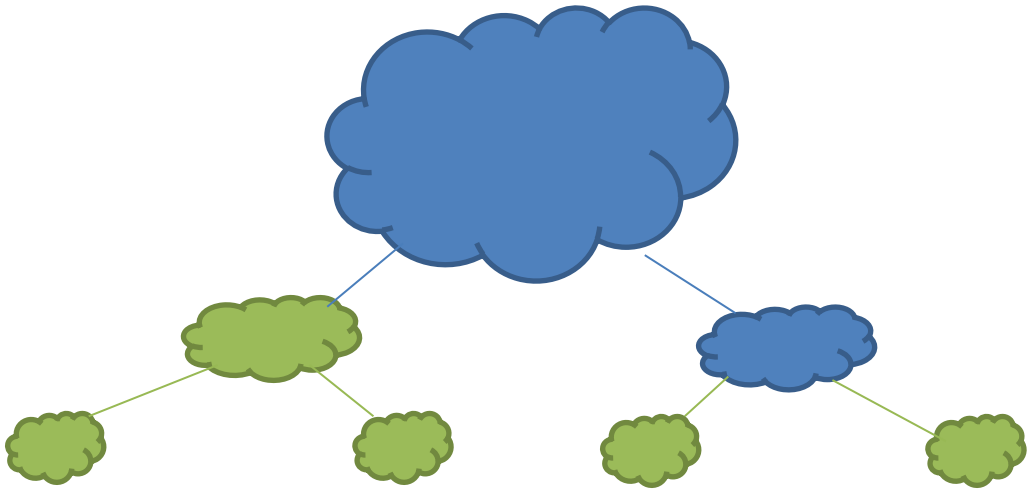
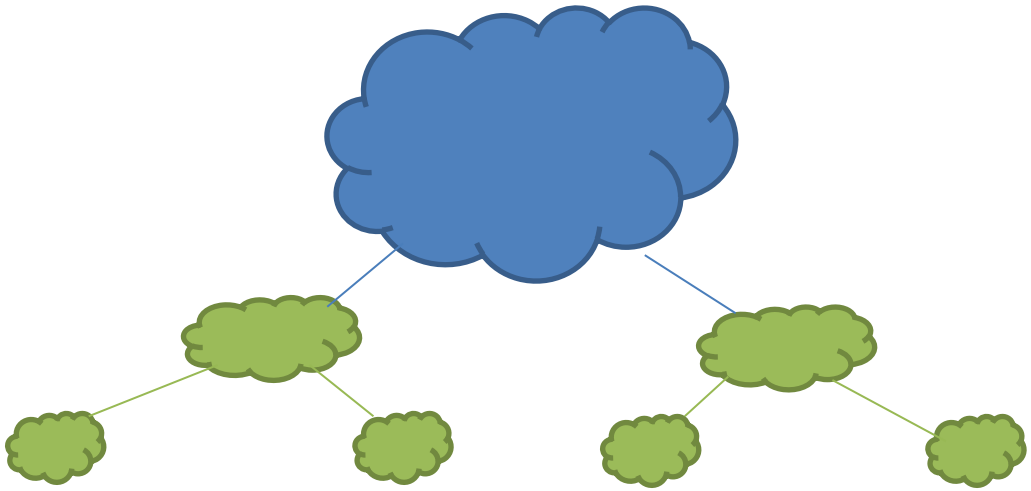


Divide and Conquer



Binary Search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

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Example: Find 9

3	5	7	8	9	12	15

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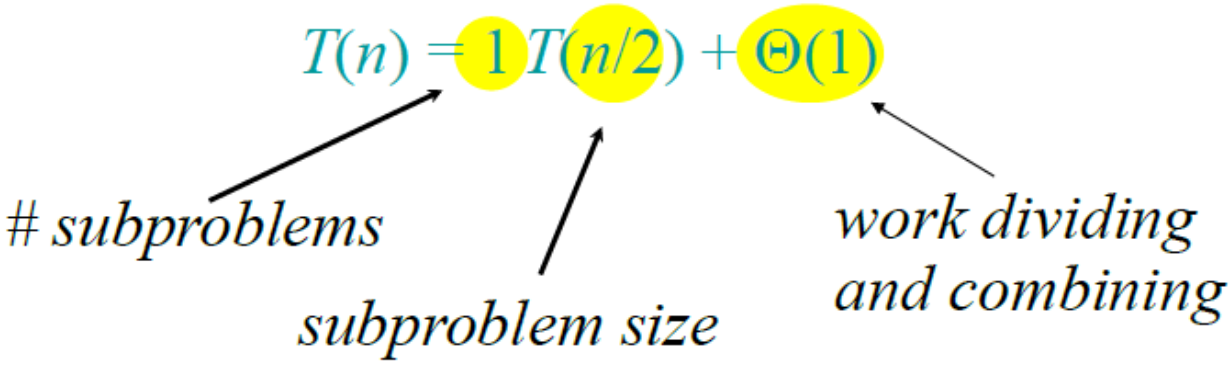
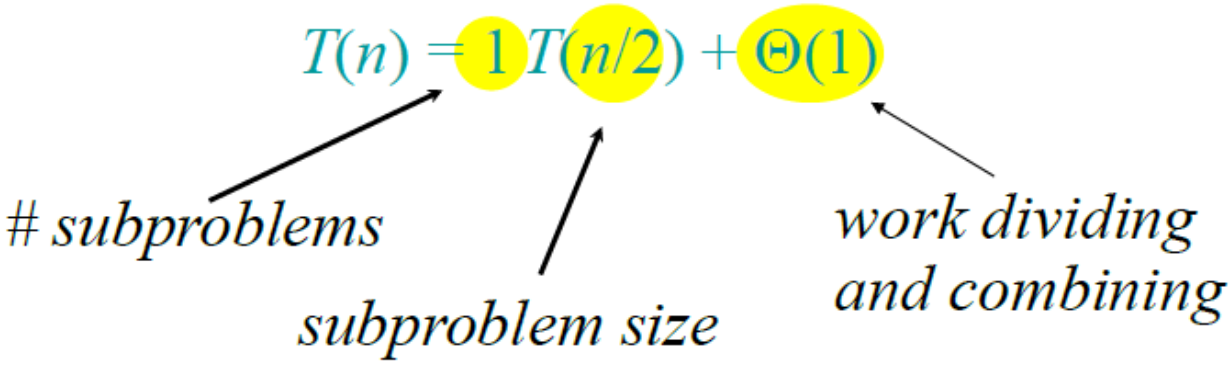
Binary Search - Recurrence



Binary Search - Recurrence



$n^{\log_b a} = n^{\log_2 1} = n^0 = 1 \Rightarrow$ CASE 2 ($k = 0$)

$\Rightarrow T(n) = \Theta(\lg n)$.

Algorithm – Sorting Problem

- **Computational Problem:** *Sort a sequence of numbers into nondecreasing order*
- **Input and Output:**

Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

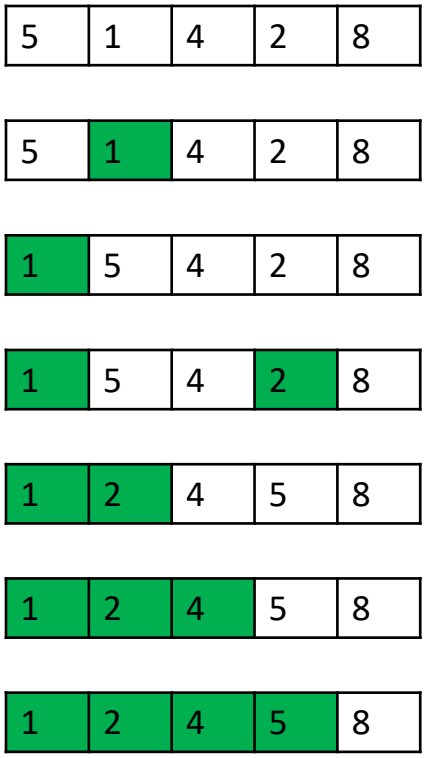
Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Example:

Input: 8 2 4 9 3 6

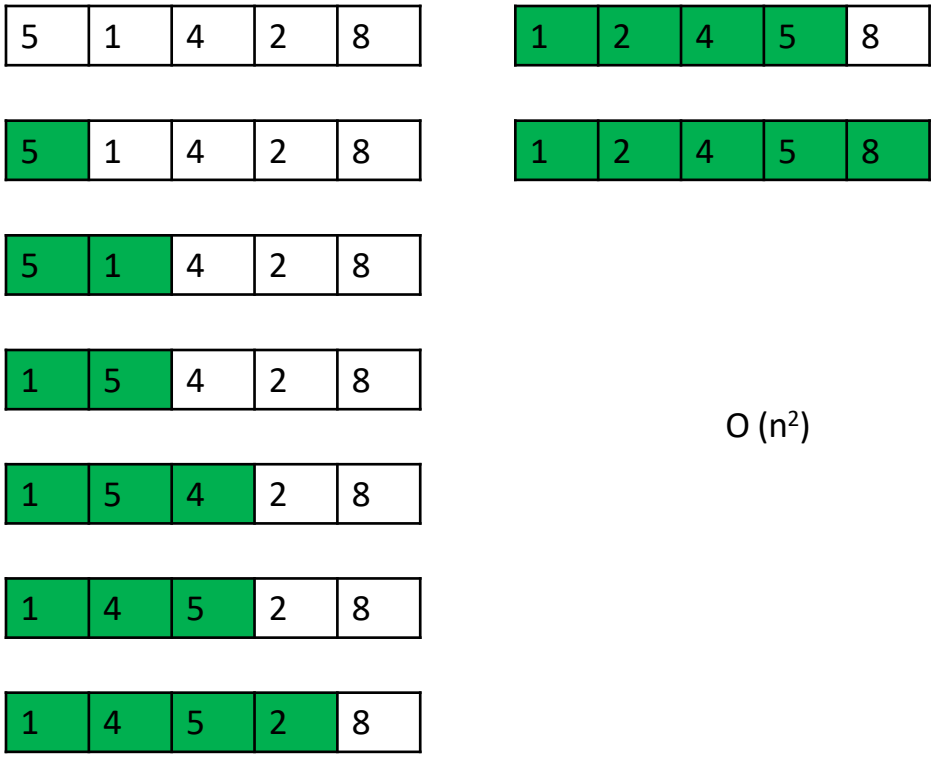
Output: 2 3 4 6 8 9

Selection Sort



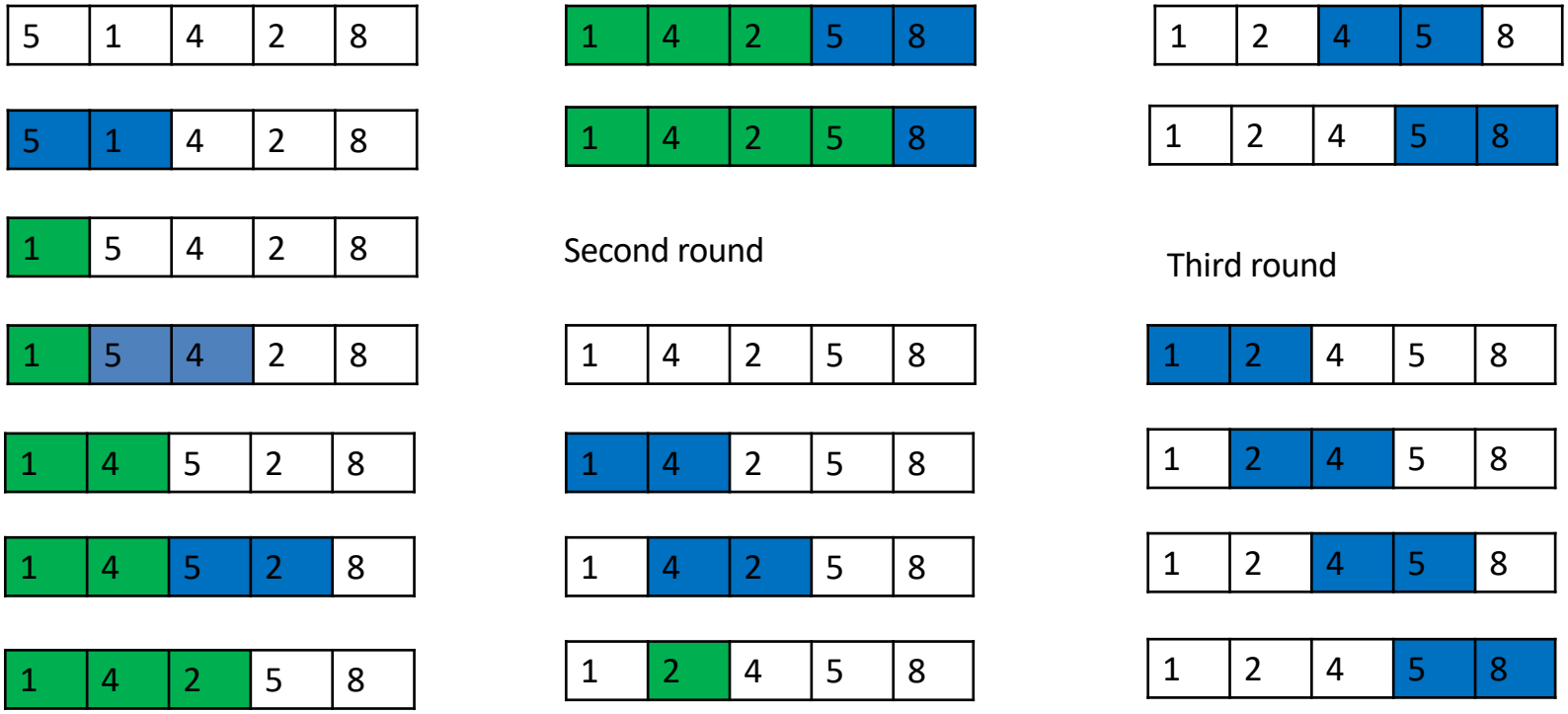
$O(n^2)$

Insertion Sort



$O(n^2)$

Bubble Sort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5 1 4 2 8
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1 4 5 2 8
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1 4 2 5 8
Second round
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1 4 2 5 8
1 4 2 5 8
1 2 4 5 8
1 2 4 5 8
1 2 4 5 8
Third round
1 2 4 5 8
1 2 4 5 8
1 2 4 5 8
1 2 4 5 8

Merge Sort

- Merge sort algorithm follows the divide and conquer paradigm
  - The problem space is continually split in half, recursively applying the algorithm to each half until the base case is reached.

- Merge sort improves over Selection, Insertion and Bubble sorts

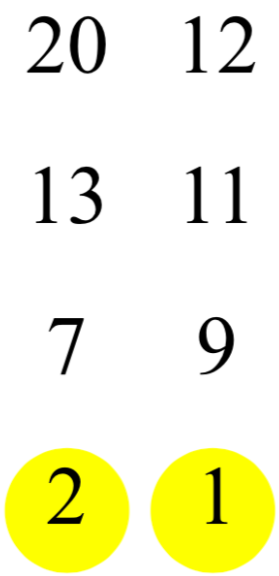
Merge Sort

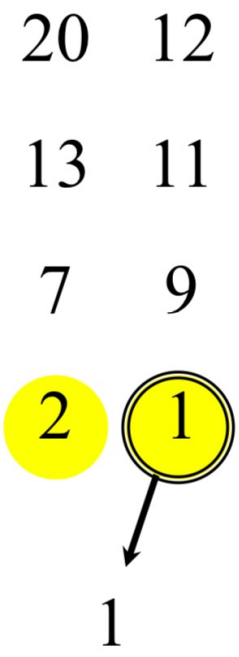
MERGE-SORT $A[1 \ldots n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1 \ldots \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \ldots n\,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: **MERGE**

Merge Sort – Merging Two Sorted Arrays

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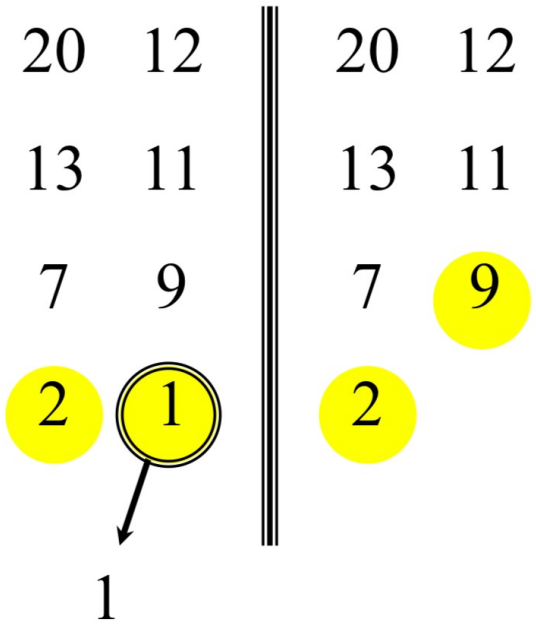
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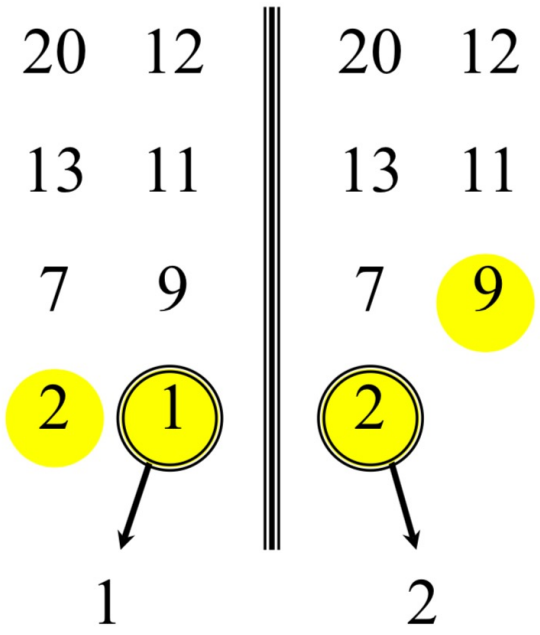
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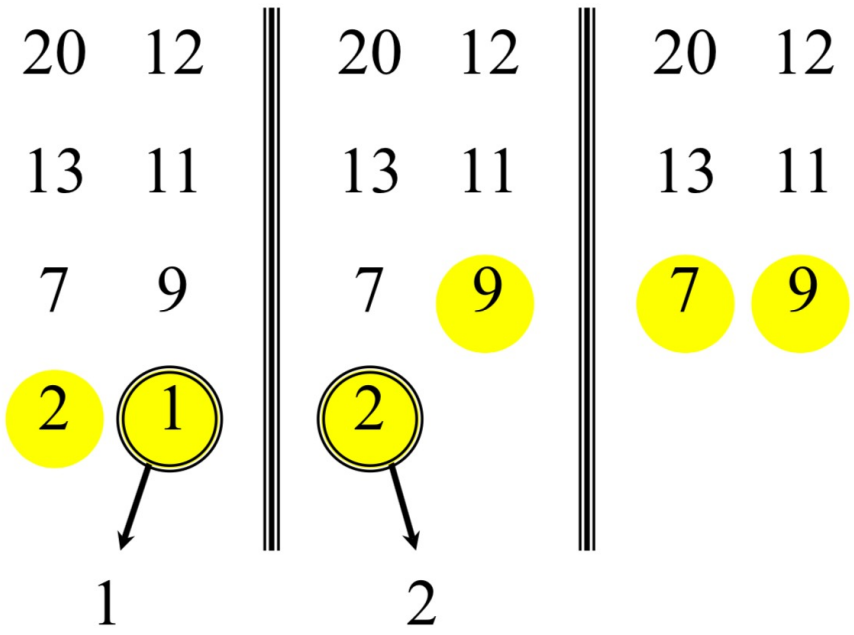
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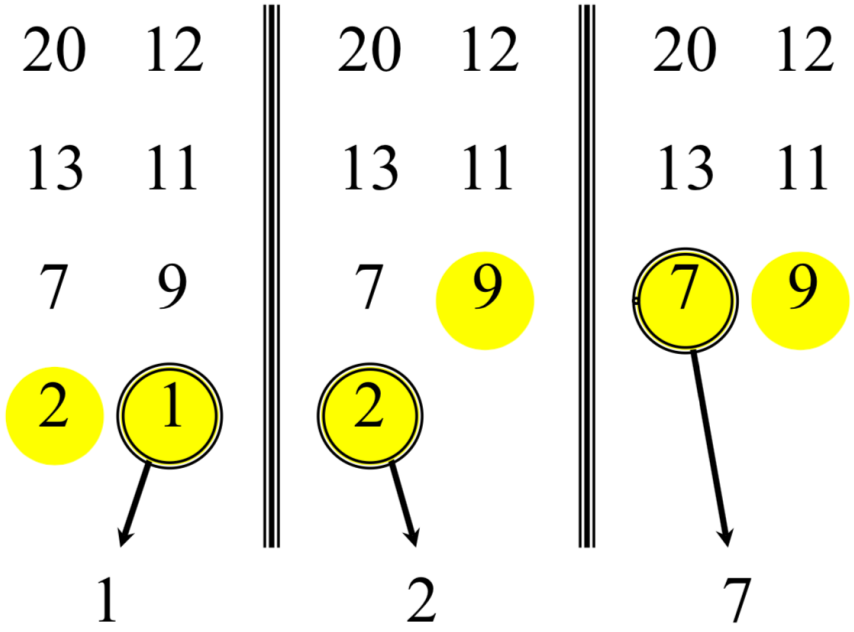
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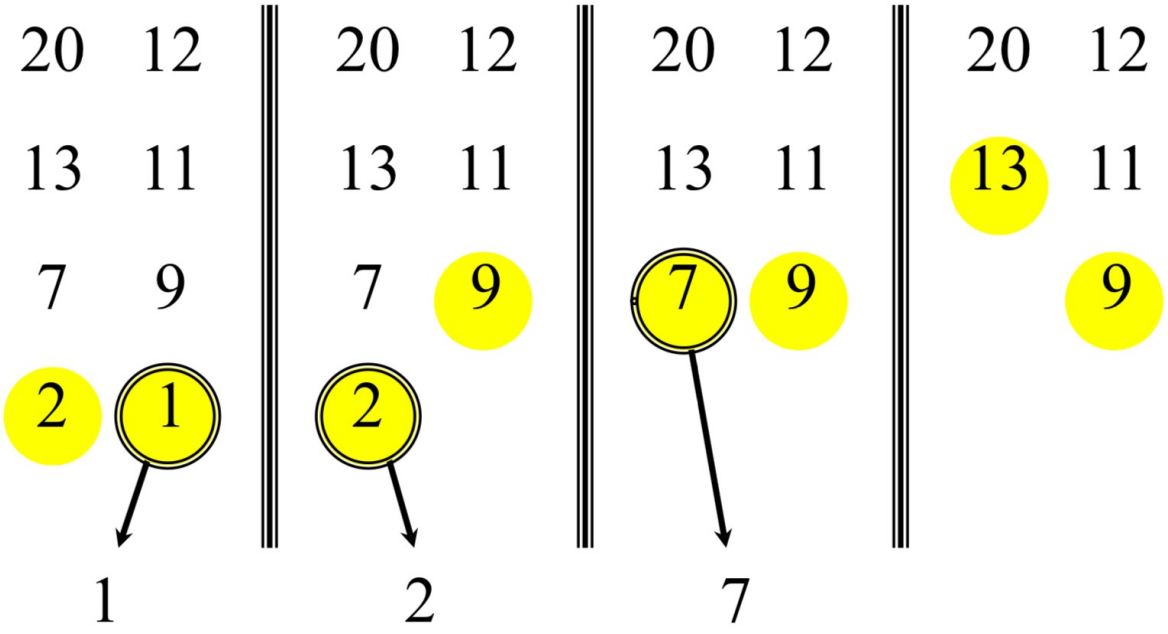
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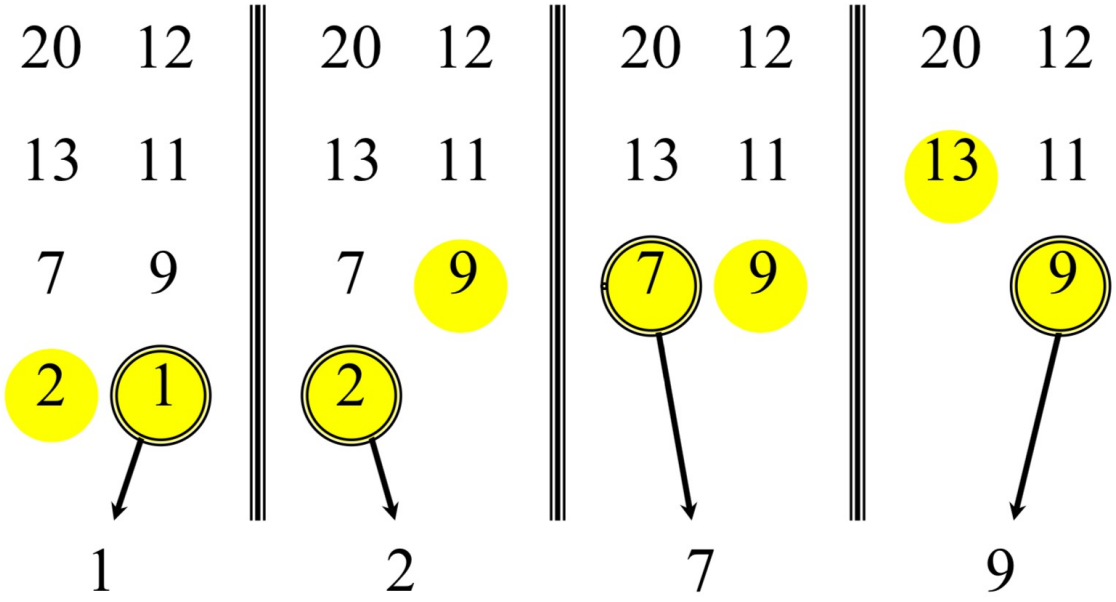
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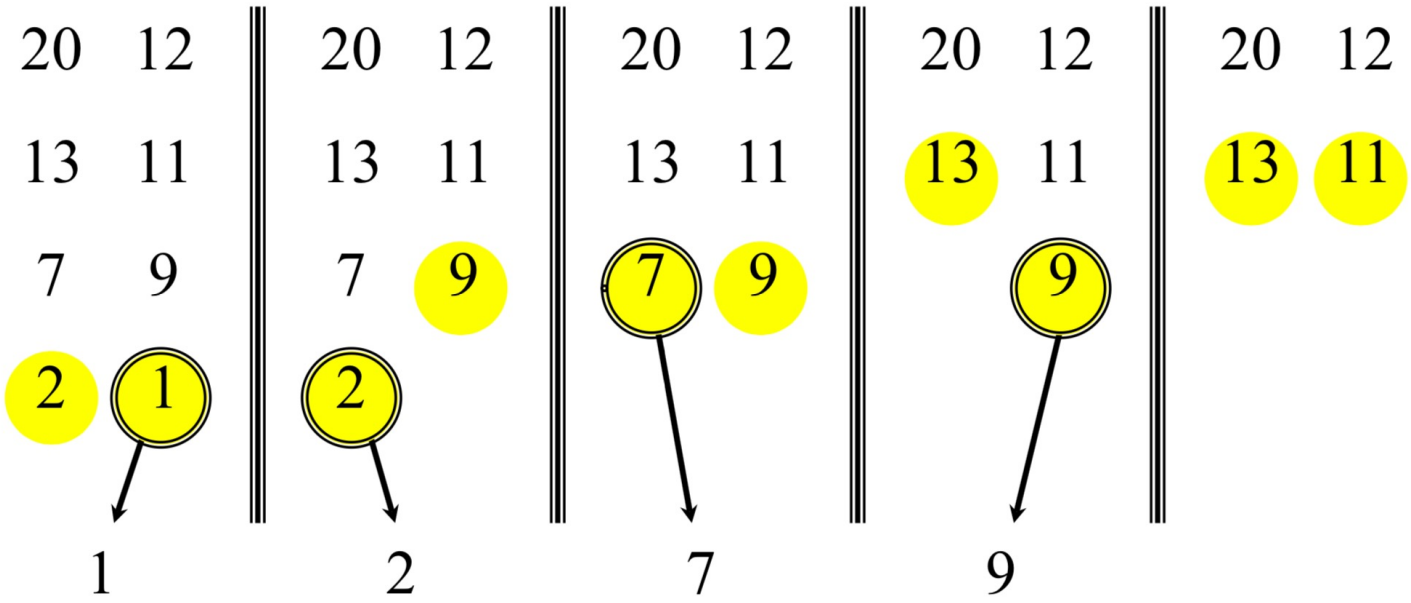
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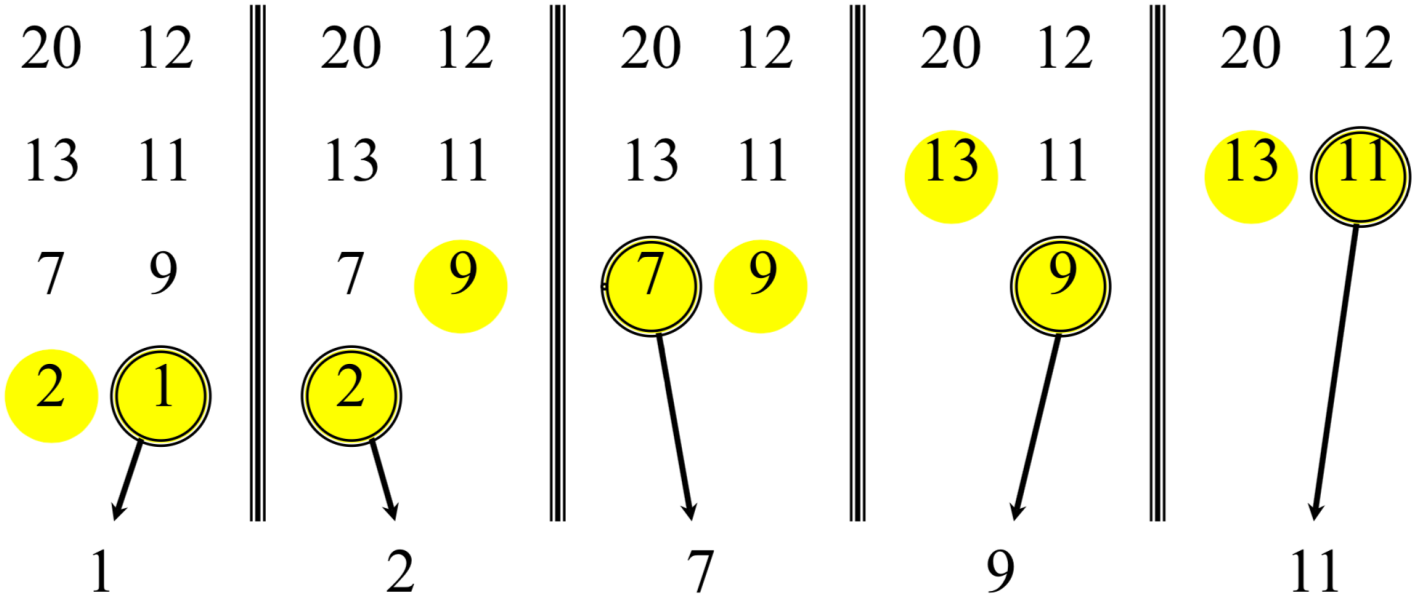
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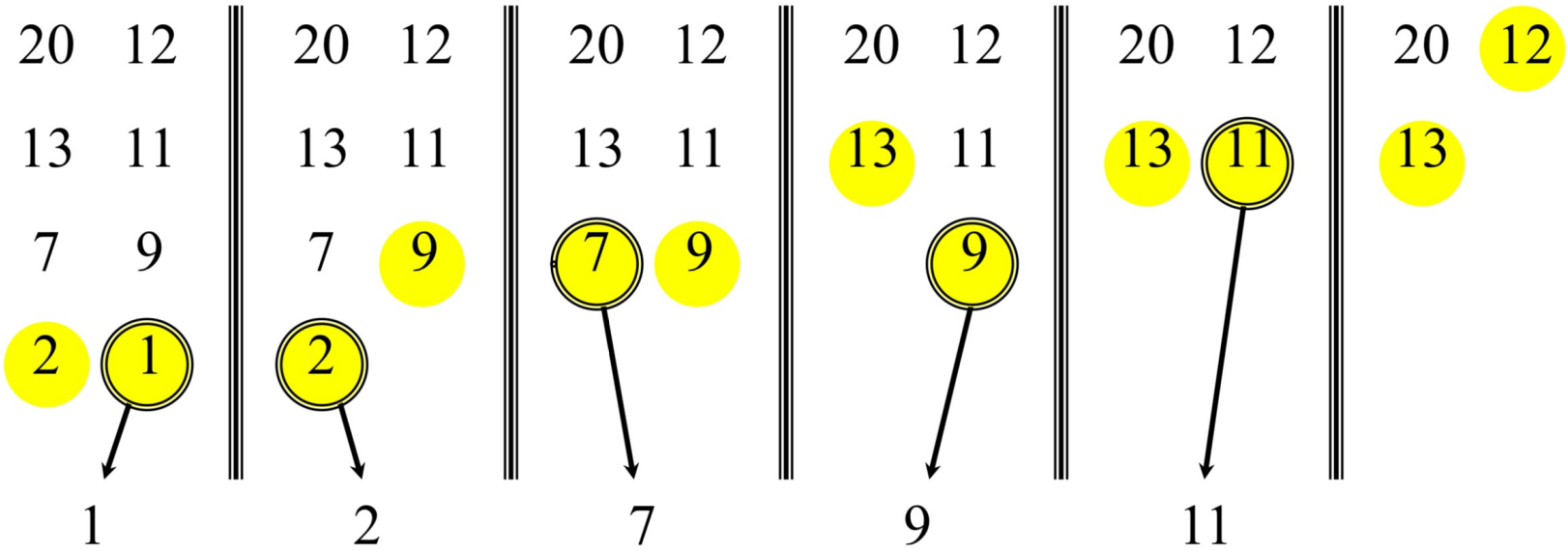
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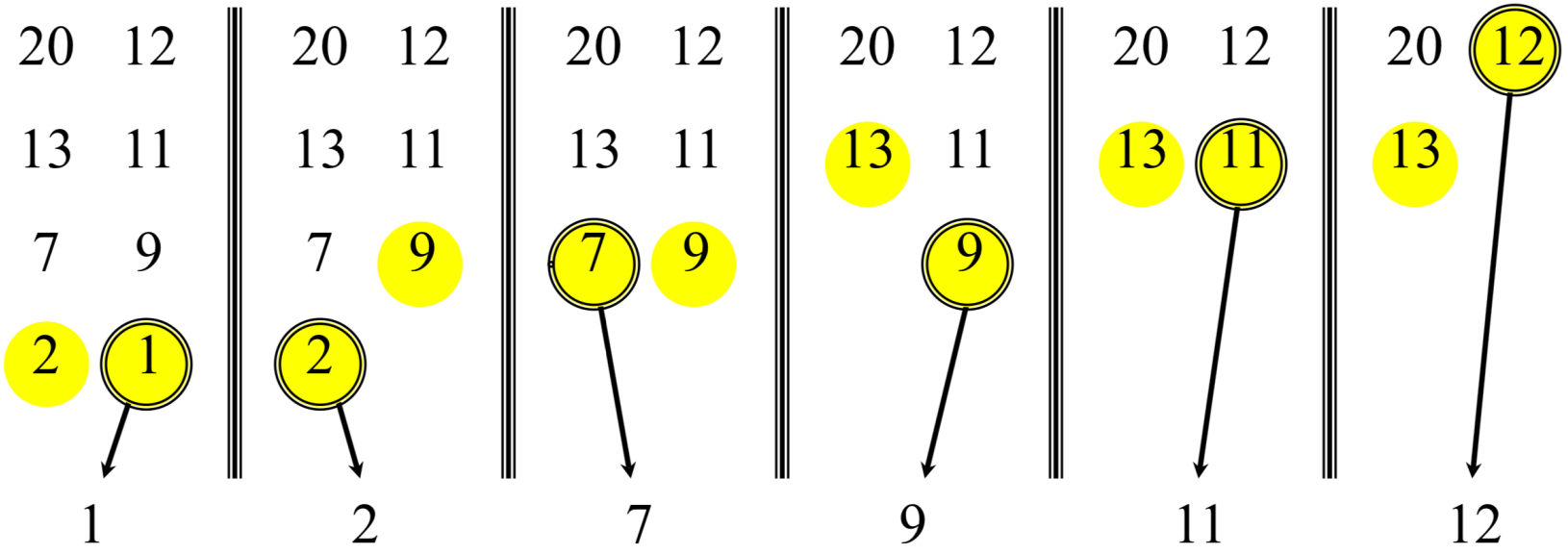
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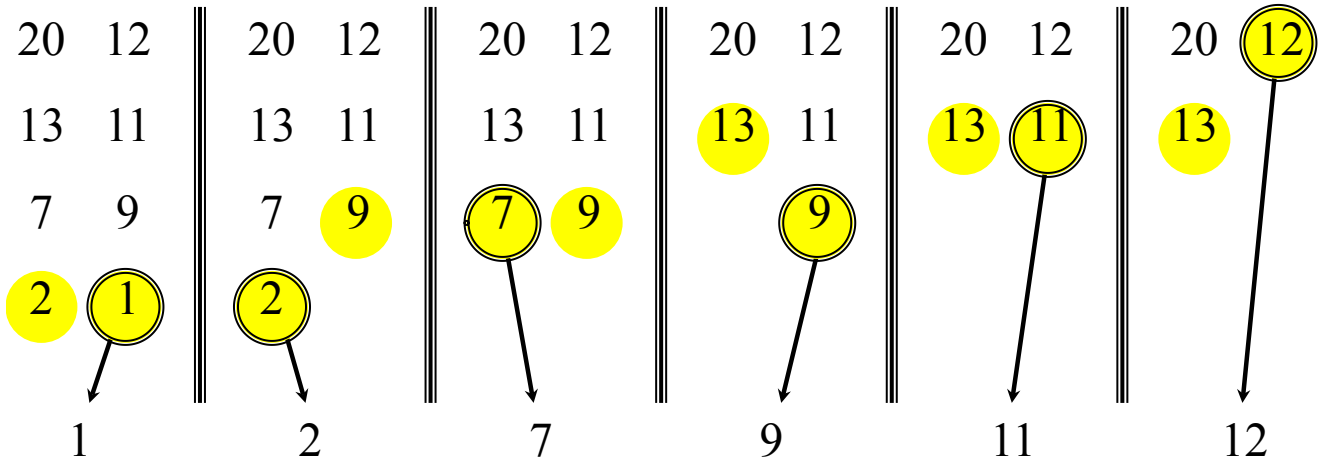
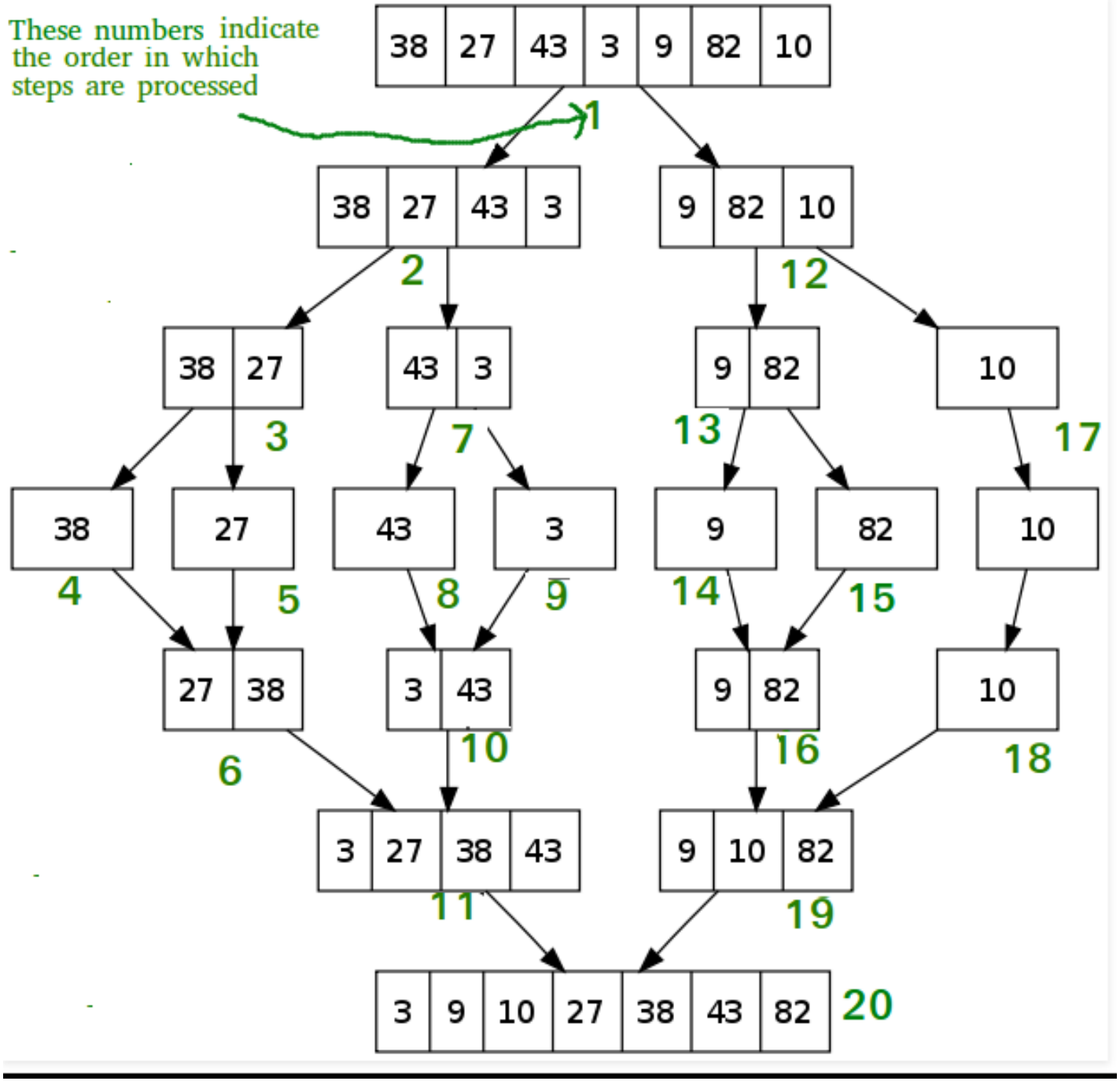


T(n) = Θ(n) to merge a total of n elements (linear time)

Merge Sort – Pseudocode

```
MERGE-SORT(A, p, r)
1   if p < r
2       q = ⌊(p + r)/2⌋
3       MERGE-SORT(A, p, q)
4       MERGE-SORT(A, q + 1, r)
5       MERGE(A, p, q, r)
```


These numbers indicate the order in which steps are processed
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5
8
9
14
15
27 38
3 43
9 82
10
6
10
16
18
3 27 38 43
9 10 82
11
19
3 9 10 27 38 43 82
20

Merge Sort – Pseudocode

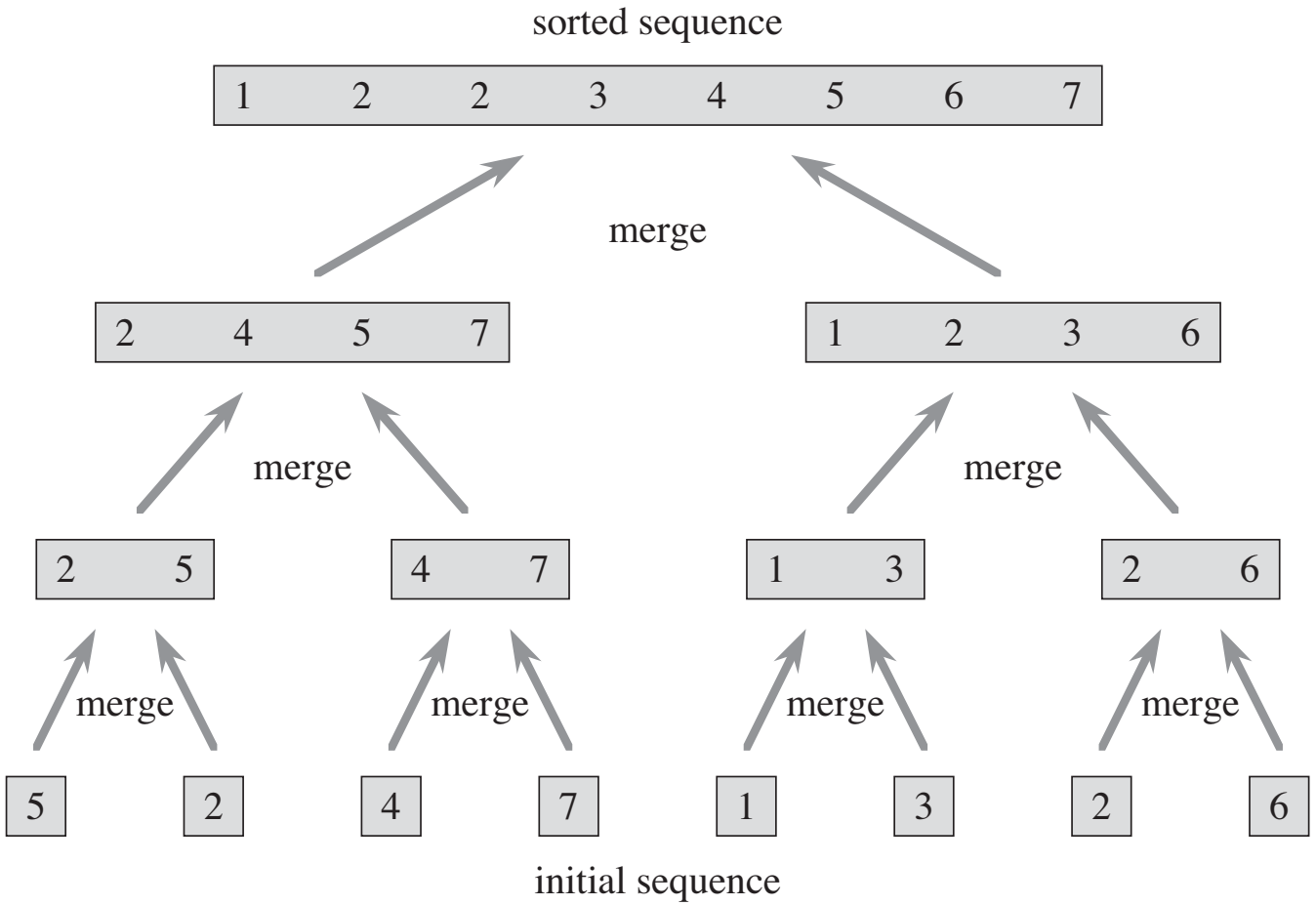
```
MERGE(A, p, q, r)
 1  n1 = q − p + 1
 2  n2 = r − q
 3  let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
 4  for i = 1 to n1
 5      L[i] = A[p + i − 1]
 6  for j = 1 to n2
 7      R[j] = A[q + j]
 8  L[n1 + 1] = ∞
 9  R[n2 + 1] = ∞
10  i = 1
11  j = 1
12  for k = p to r
13      if L[i] ≤ R[j]
14          A[k] = L[i]
15          i = i + 1
16      else A[k] = R[j]
17          j = j + 1
```

```
merge ( array a, array b )
    array c

    while ( a and b have elements )
        if ( a[0] > b[0] )
            add b[0] to the end of c
            remove b[0] from b
        else
            add a[0] to the end of c
            remove a[0] from a

    // At this point either a or b is empty

    while ( a has elements )
        add a[0] to the end of c
        remove a[0] from a

    while ( b has elements )
        add b[0] to the end of c
        remove b[0] from b

    return c
```

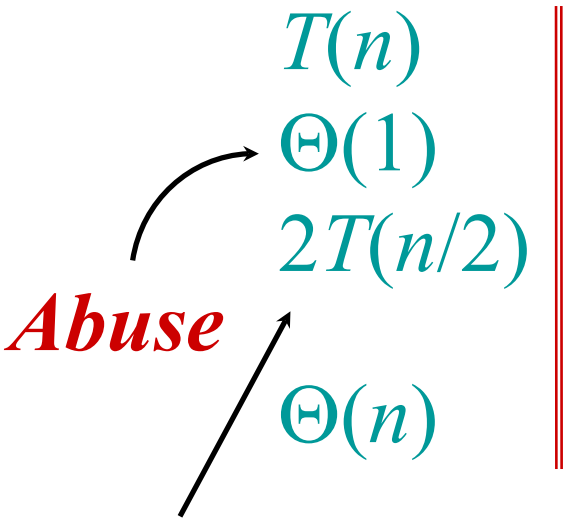
Example

- Illustrate the operation of merge sort on the array:

A = {5 , 2, 4, 7, 1, 3, 2, 6}



Merge Sort - Analysis



MERGE-SORT $A[1\,..\,n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1\,..\lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1\,..\,n\,]$.
3. ***"Merge"*** the 2 sorted lists

Sloppiness: Should be $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$, but it turns out not to matter asymptotically.

Merge Sort - Analysis (Recurrence)

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.

Recurrence Equation or Recurrence

When an algorithm contains a recursive call to itself, we can often describe its running time by a recurrence equation or recurrence, which describes the **overall running time on a problem of size n in terms of the running time on smaller inputs**

Merge Sort – Recursion Tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

Merge Sort – Recursion Tree

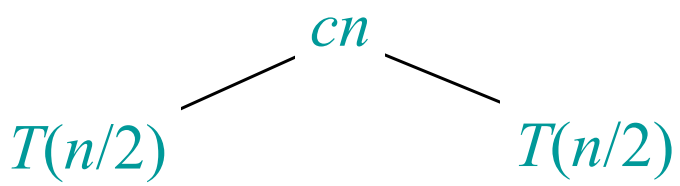
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$T(n)$

The total running time progressively expands to form a recursion tree

Merge Sort – Recursion Tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



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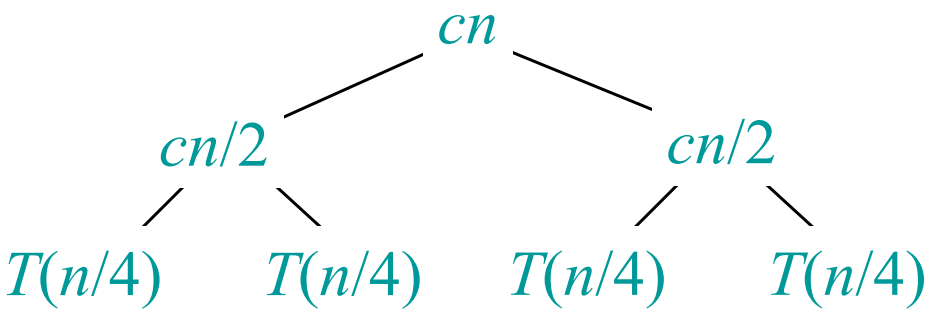

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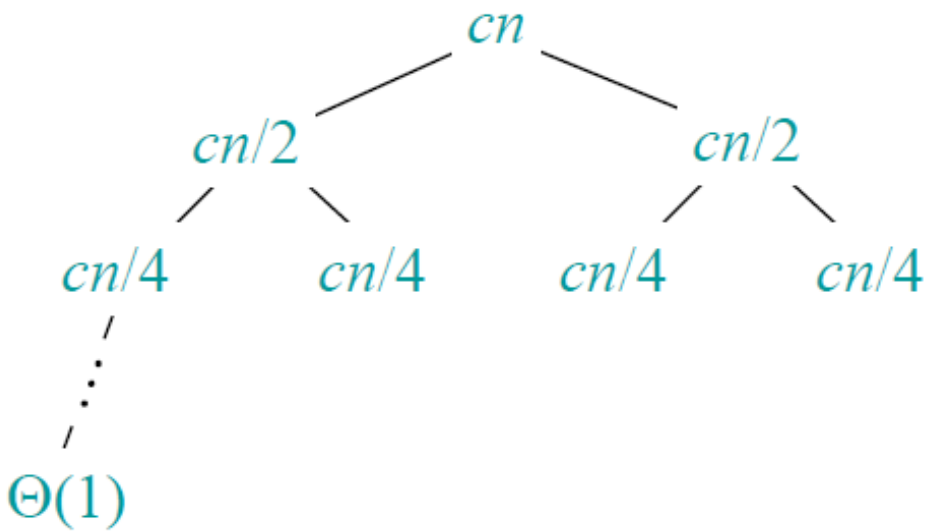

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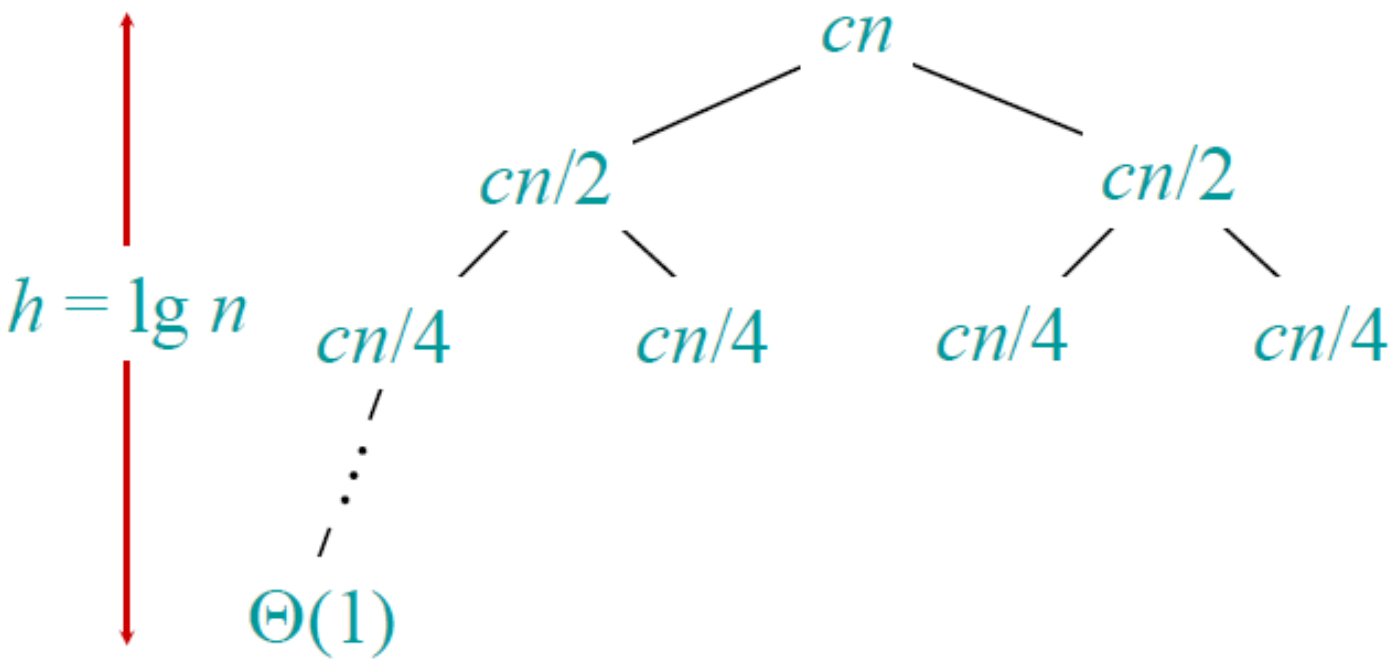

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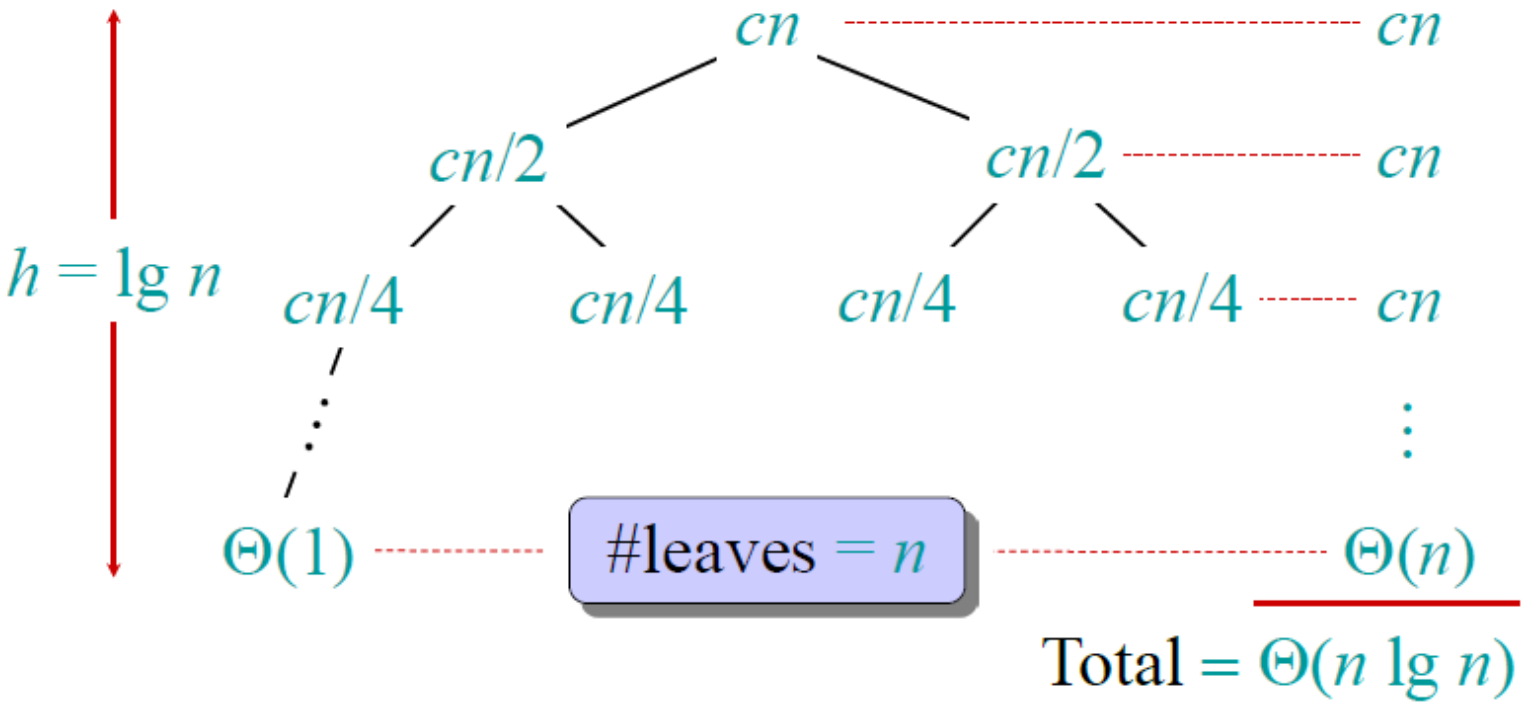
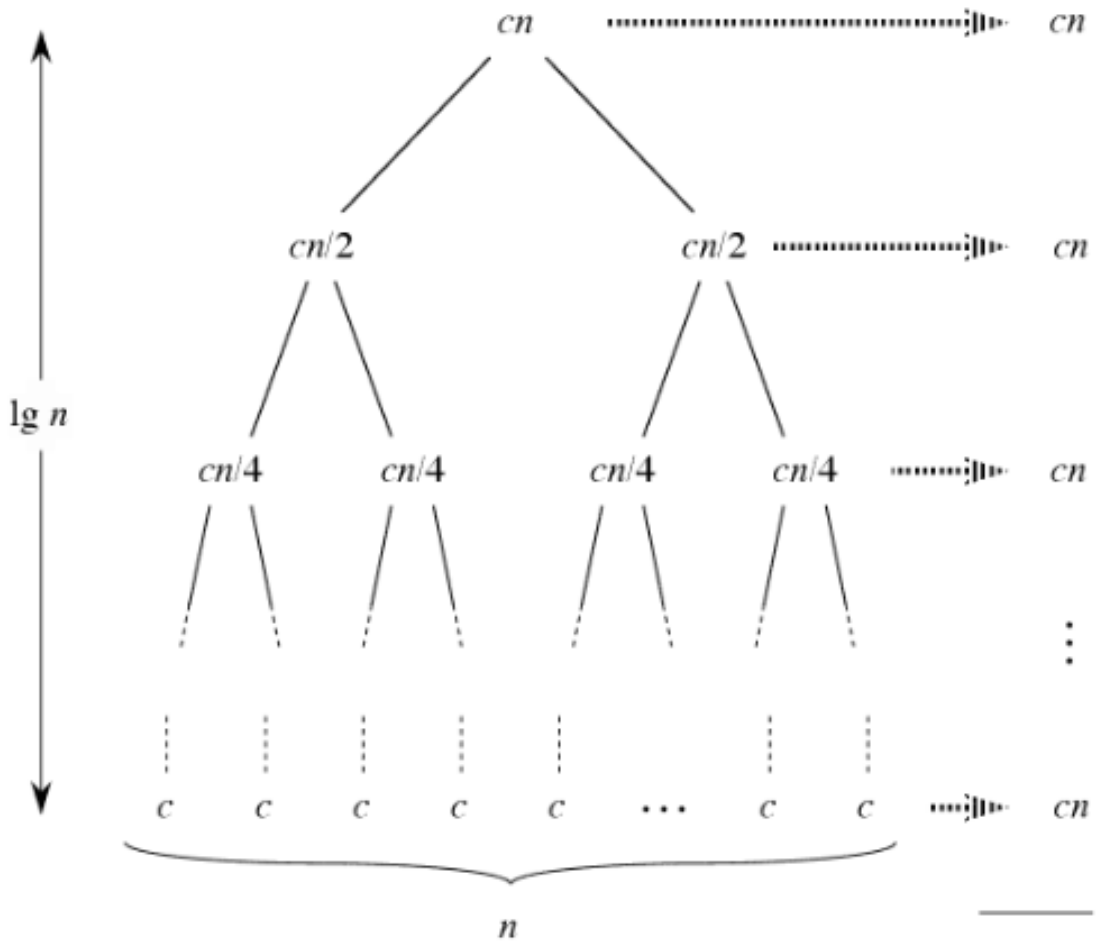


Recursion tree



Analyzing Run Time

- What is the level of the leaves?
- At a given level, j, exactly how **many distinct sub-problems are there**, as a function of the level j?
- For each of those distinct sub-problems at level j, what is the input size? (So, what is the size of the the array, which is passed to a sub-problem residing at level j of this recursion tree.)

Merge Sort

- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so.

Divide and Conquer – Other Problems

- Quicksort
- Powering a number
- Fibonacci Numbers
- Matrix Multiplication (Strassen's Algorithm)

Quick Sort

- One of the most prevalent algorithms based on divide and conquer paradigm
- It is fast (as fast as Merge sort)
- It has the advantage of sorting in place (It does not require additional memory)

Quick Sort

- Partition an array around a pivot element
- For now let's select the first element as the pivot element
- Re-arrange the array such that:
  - All the elements at the left of the pivot are less than the pivot
  - All the elements at the right of the pivot are greater than the pivot

Quick Sort

Input:

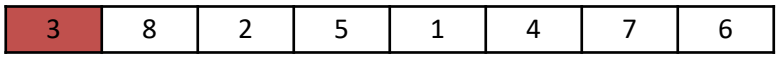
3	8	2	5	1	4	7	6

Output:

1	2	3	4	5	6	7	8

Quick Sort

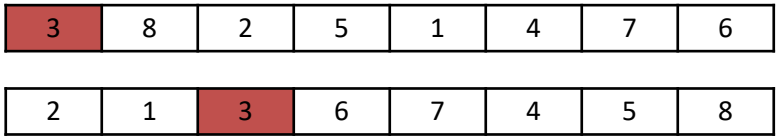
Input:



- Key idea: partition the array around the pivot element
- Re-arrange array so that
  - Left of pivot => less than pivot
  - Right of pivot => greater than pivot
- Puts pivot in its rightful position

Quick Sort

Input:



- Key idea: partition the array around the pivot element
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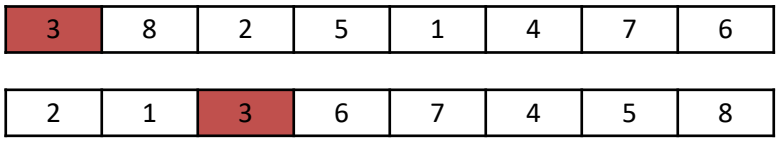
Quick Sort

Input:



- Partitioning:
  - It is linear (O (n))
  - It reduces the problem size
    - Enables divide and conquer

Quick Sort

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.

2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

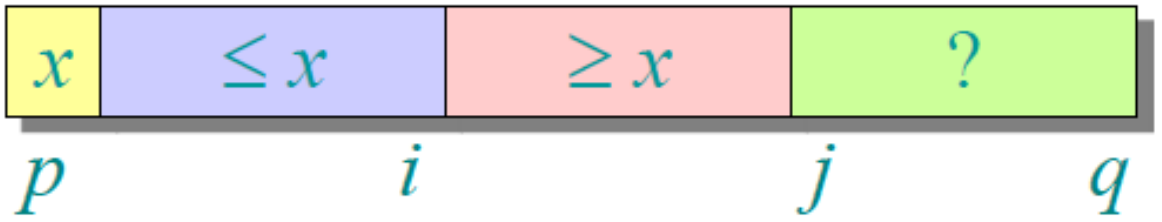
Key: *Linear-time partitioning subroutine.*

Partitioning Subroutine

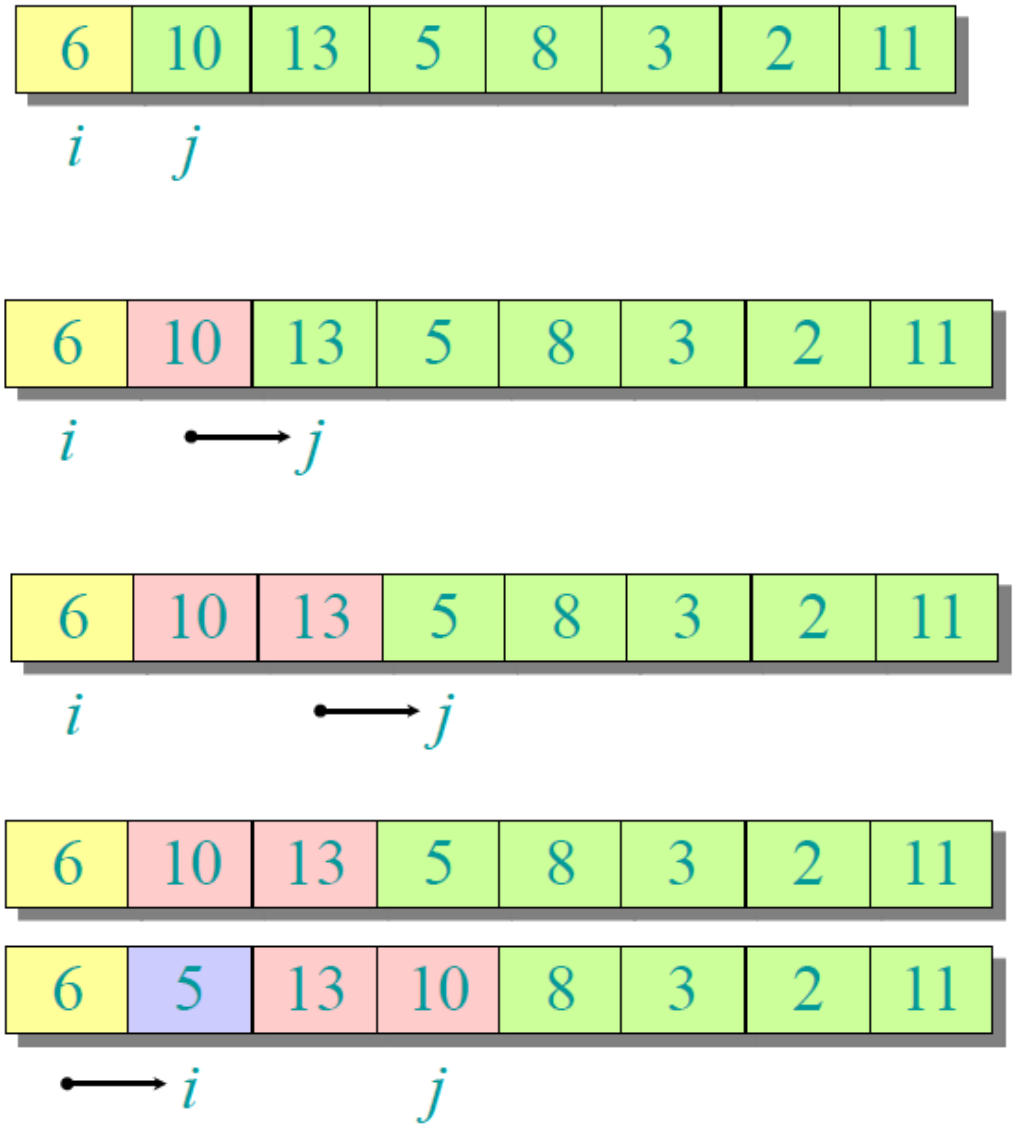
```
PARTITION(A, p, q)   ▷ A[p . . q]
    x ← A[p]         ▷ pivot = A[p]
    i ← p
    for j ← p + 1 to q
        do if A[j] ≤ x
                then  i ← i + 1
                      exchange A[i] ↔ A[j]
    exchange A[p] ↔ A[i]
    return i
```

Running time $= O(n)$ for n elements.

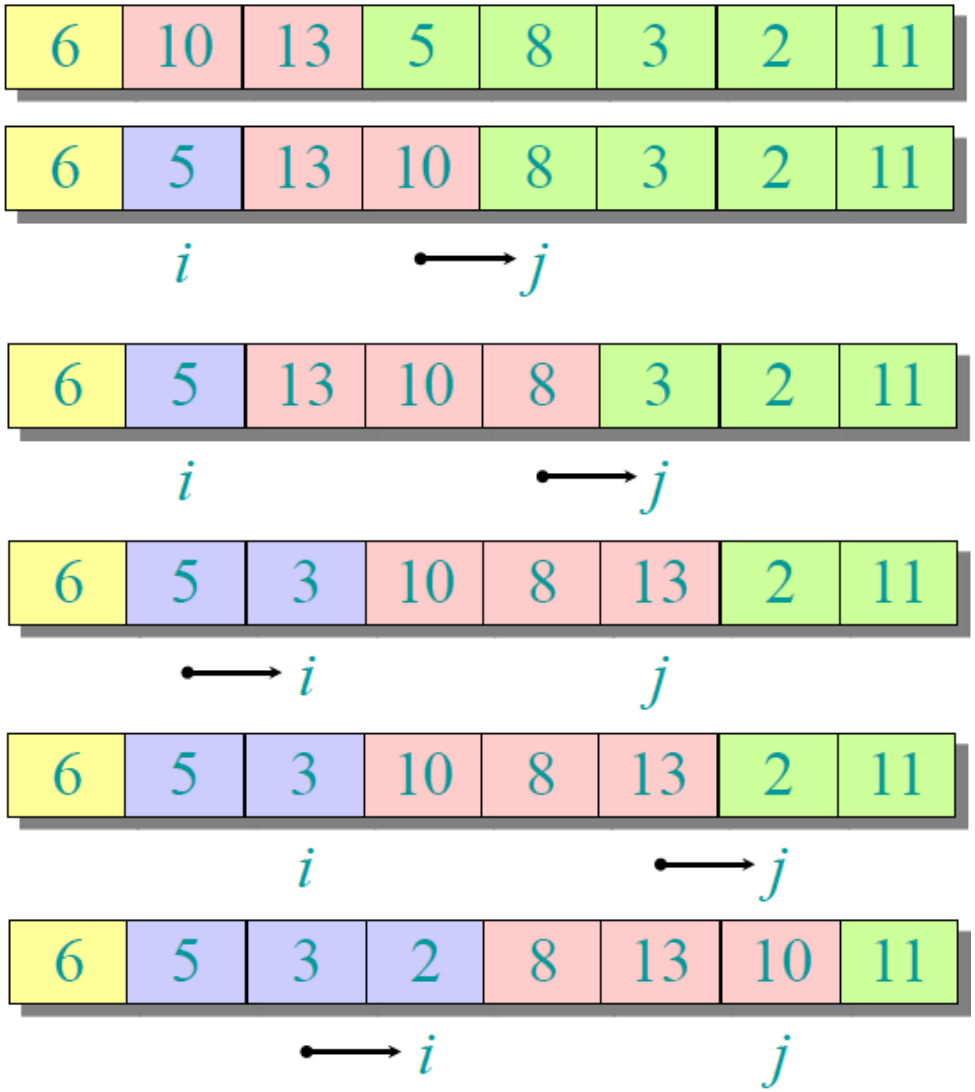
Invariant:



Example of Partitioning



Example of Partitioning



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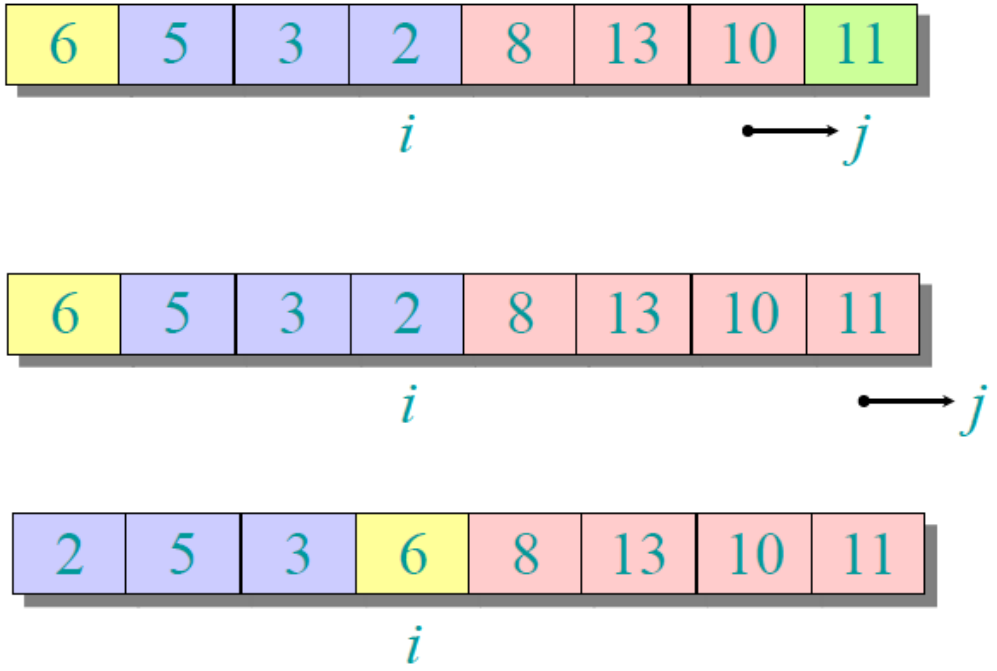


Pseudo Code for Quick Sort

```
QUICKSORT(A, p, r)
    if p < r
        then q ← PARTITION(A, p, r)
             QUICKSORT(A, p, q–1)
             QUICKSORT(A, q+1, r)
```

Initial call: QUICKSORT(*A*, 1, *n*)

Quick Sort

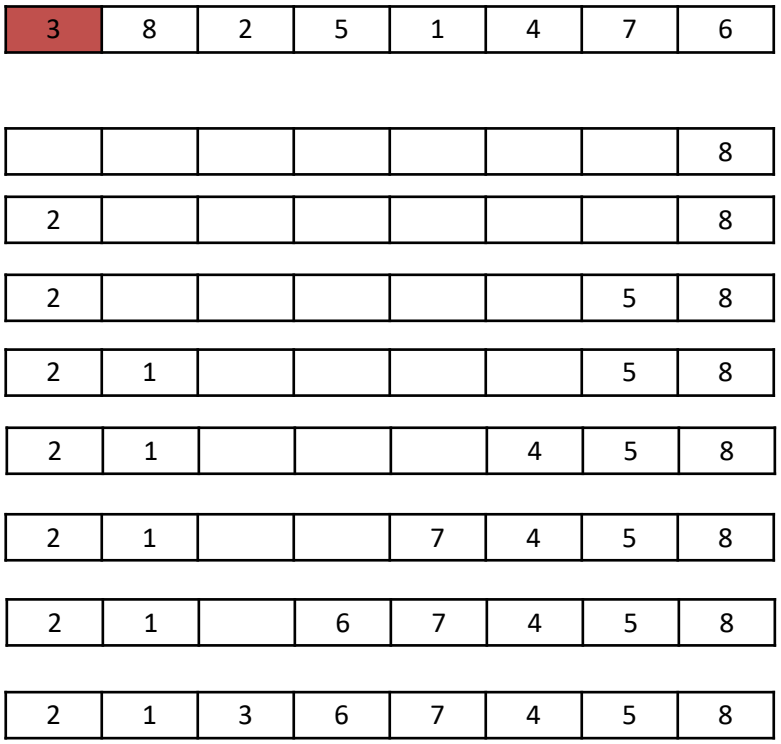
High level description:

Quicksort(array, length n)

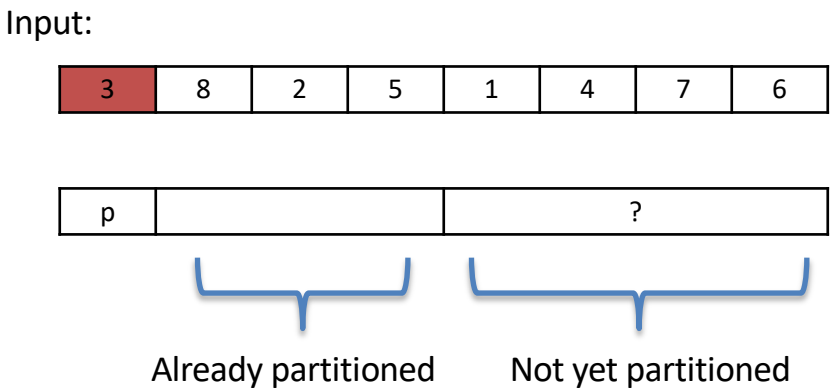
- If n ==1 return
- p = choose pivot (array, n)
- Partition array around p
- Recursively sort 1s part
- Recursively sort 2nd part

Quick sort – partitioning around a pivot – additional array

Input:

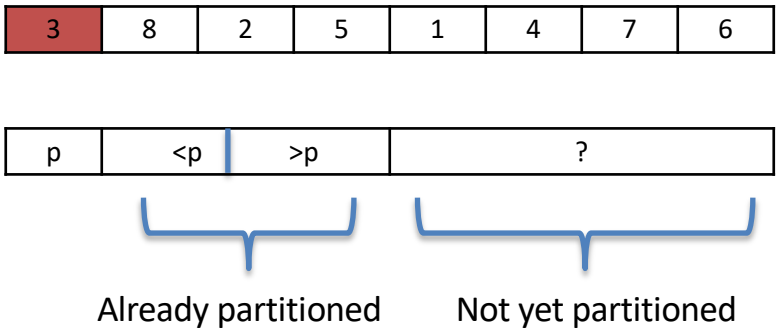


Quick sort – partitioning around a pivot – in-place



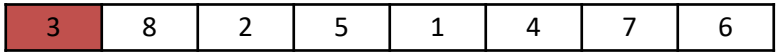
Quick sort – partitioning around a pivot – in-place

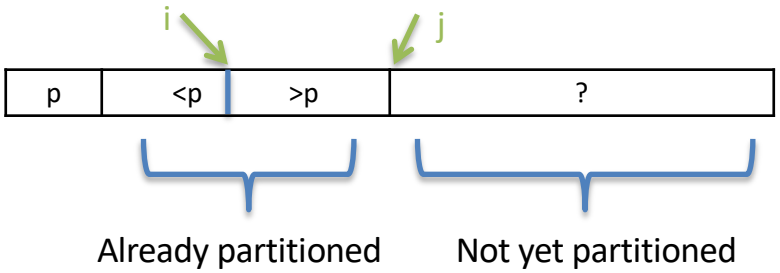
Input:



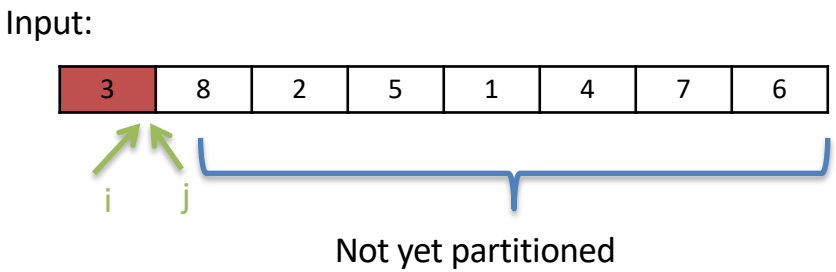
Quick sort – partitioning around a pivot – in-place

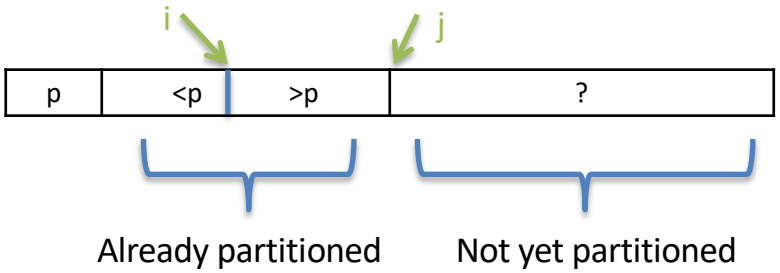
Input:





Quick sort – partitioning around a pivot – in-place





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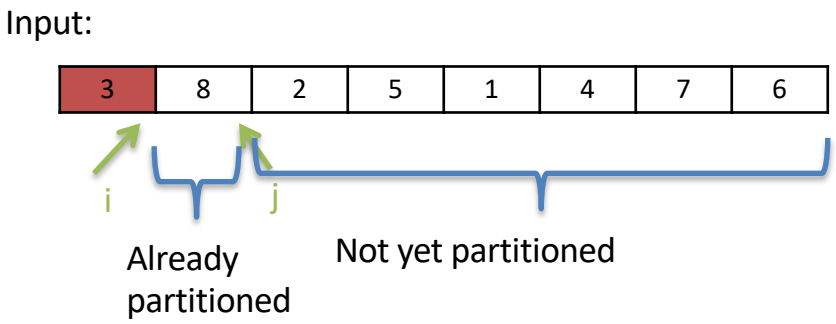
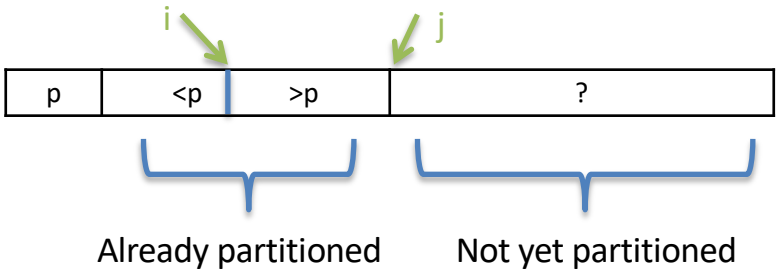
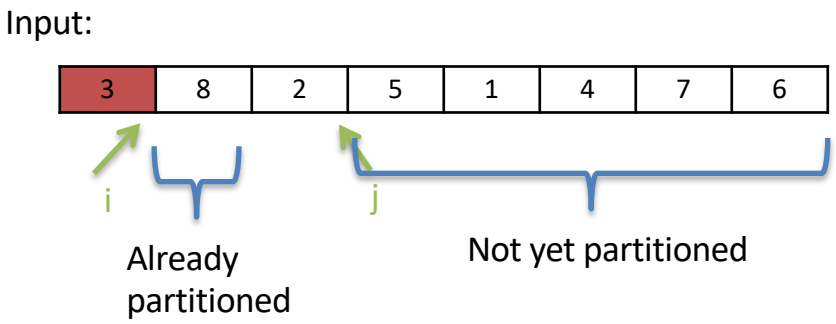
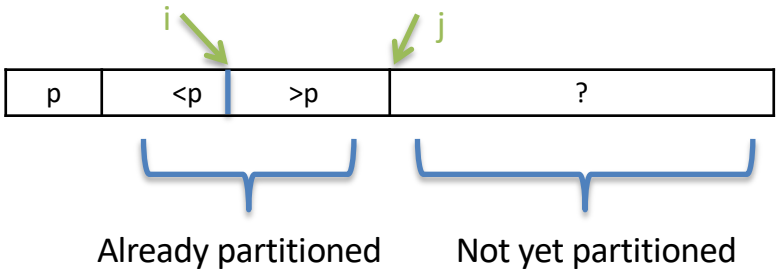


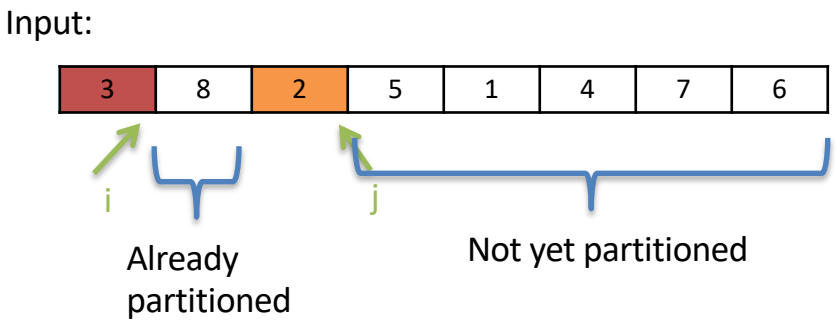


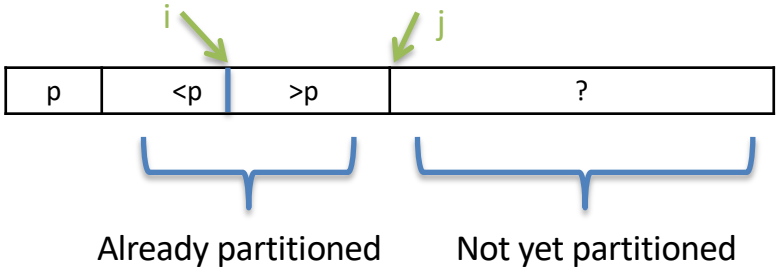
Quick sort – partitioning around a pivot – in-place



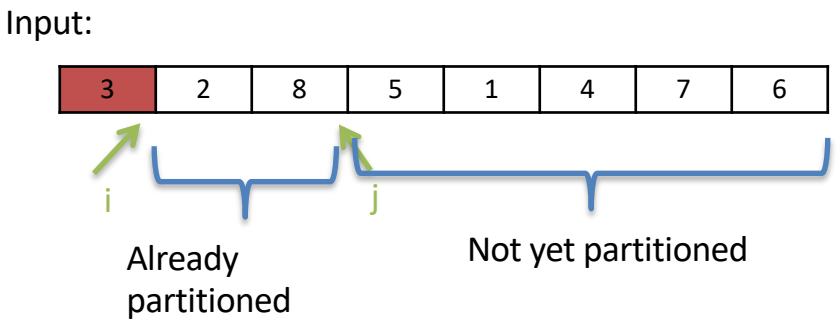


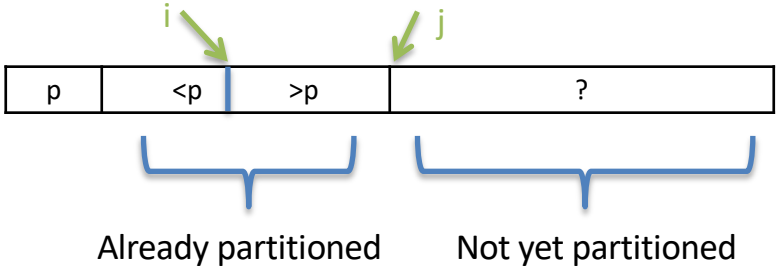
Quick sort – partitioning around a pivot – in-place





Quick sort – partitioning around a pivot – in-place





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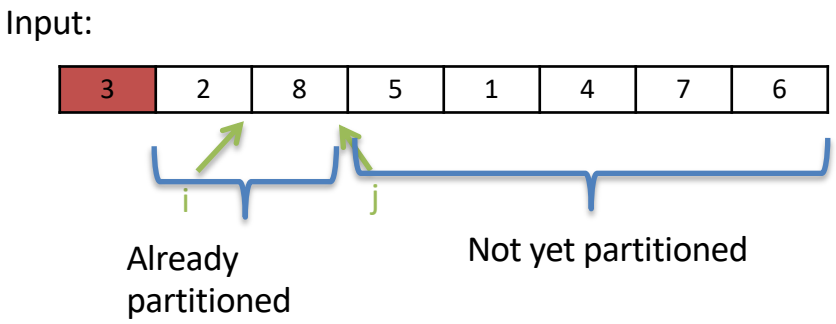
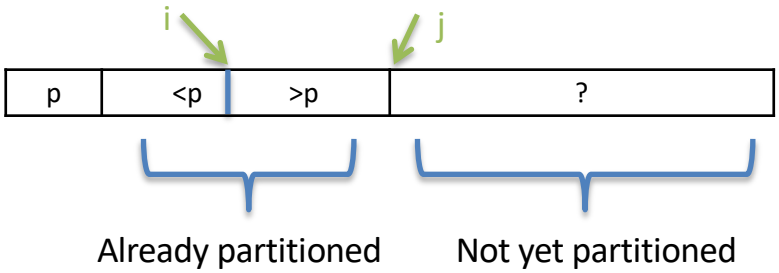
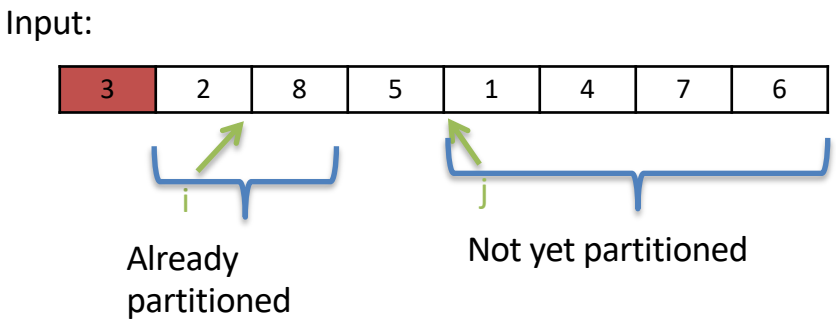
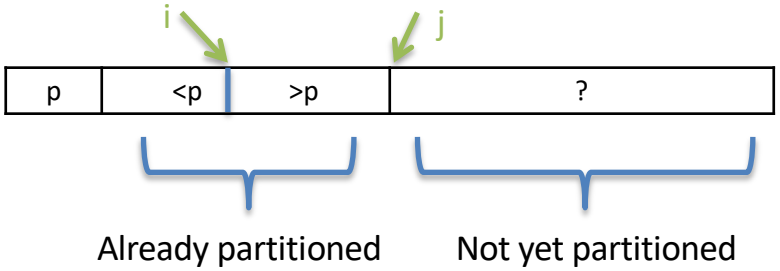




Quick sort – partitioning around a pivot – in-place





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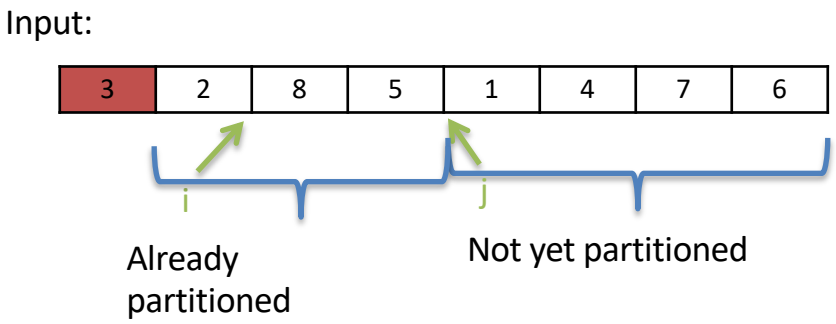

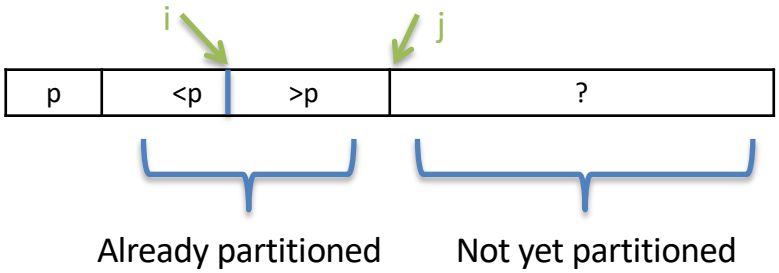

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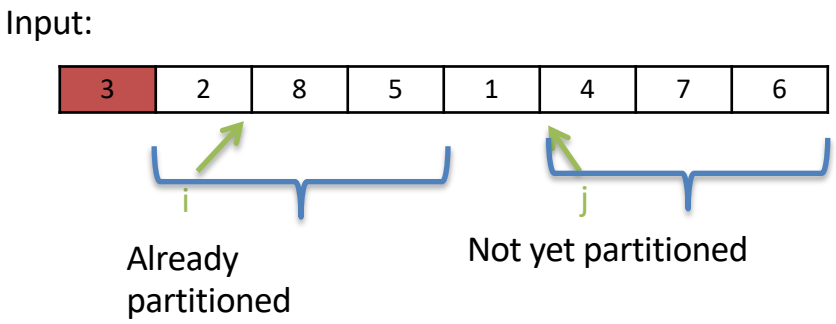
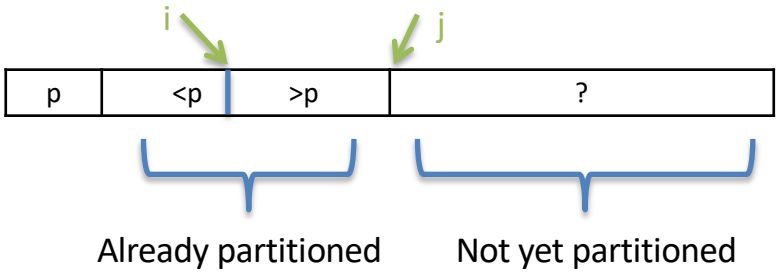
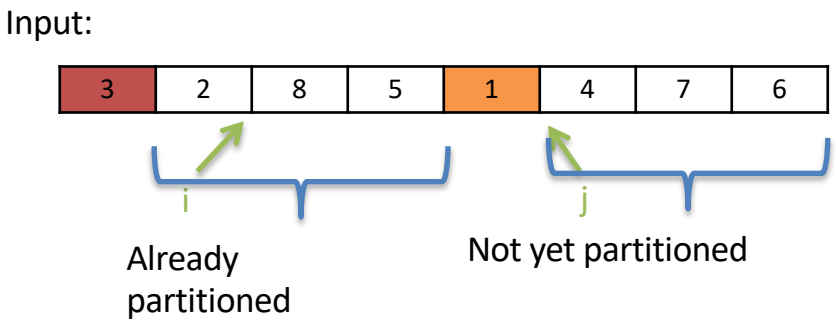
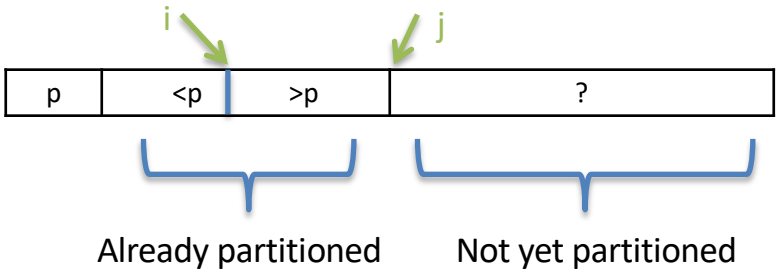




Quick sort – partitioning around a pivot – in-place





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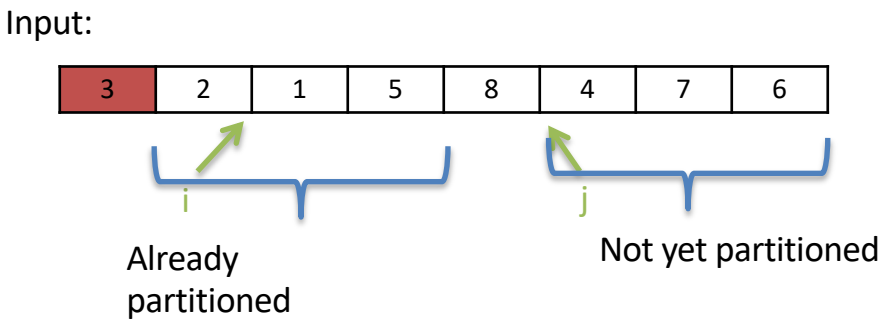

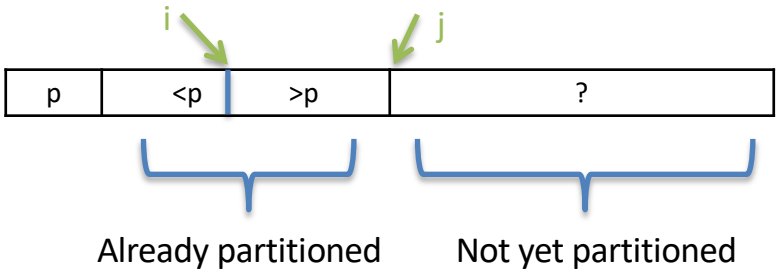

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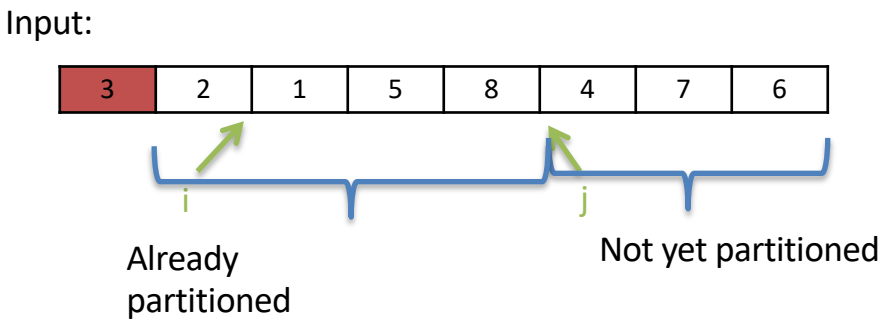

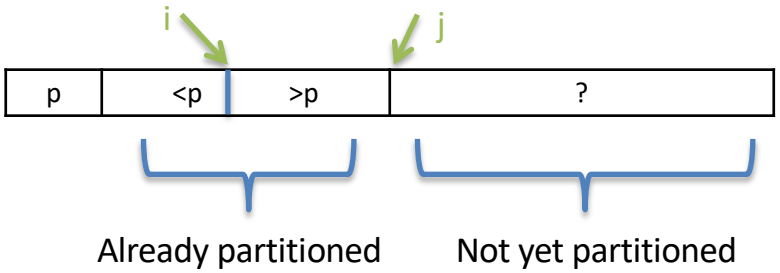

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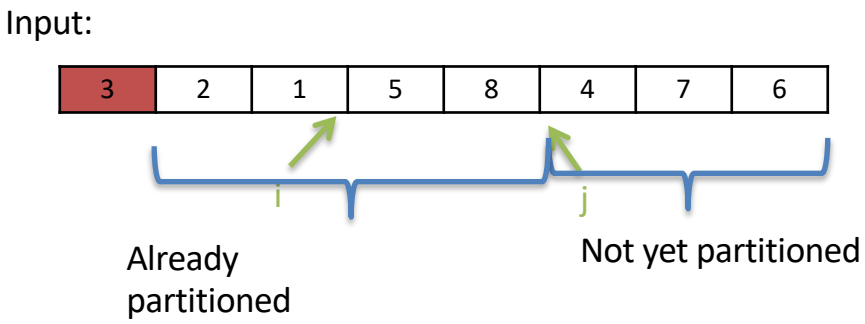

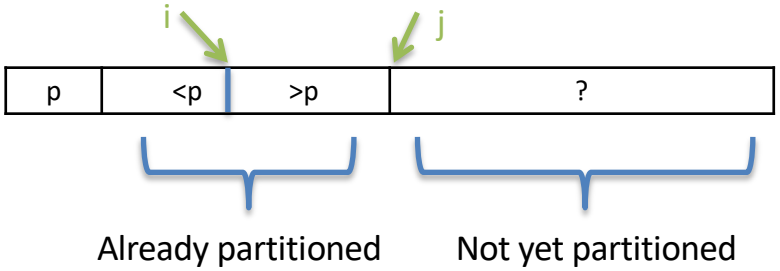

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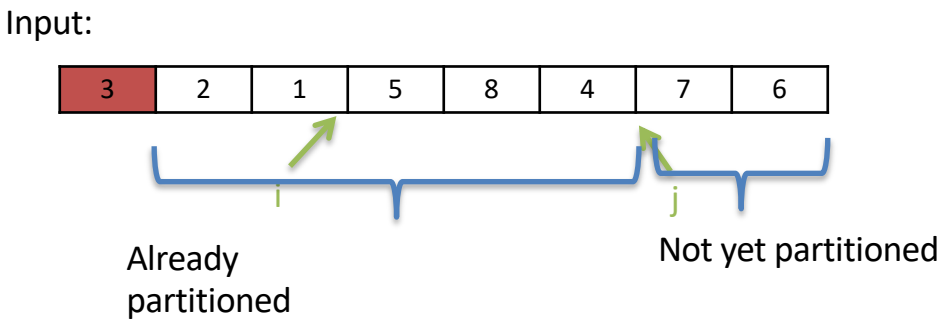

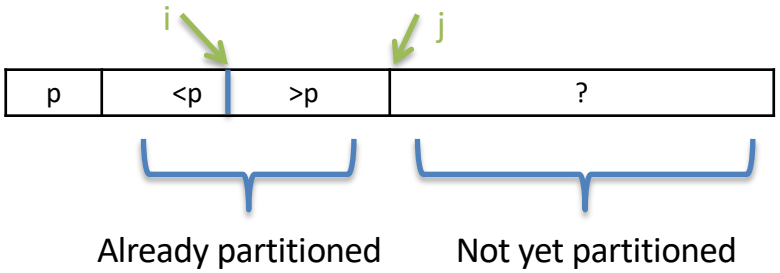

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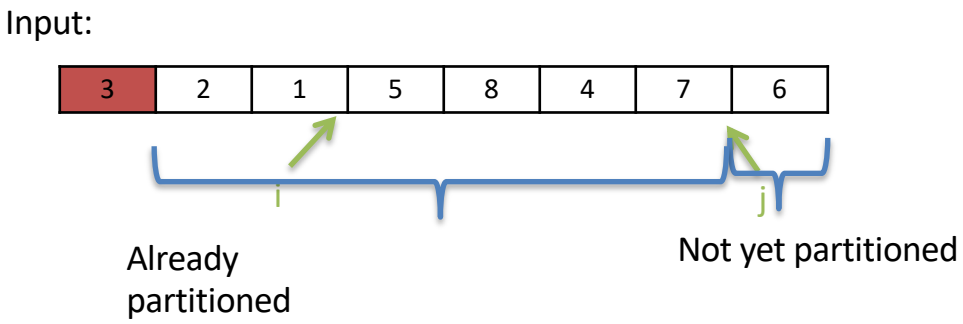

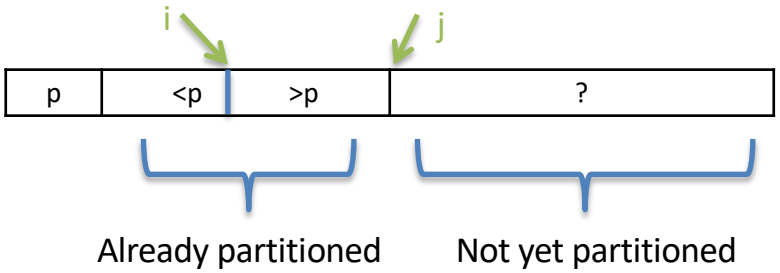

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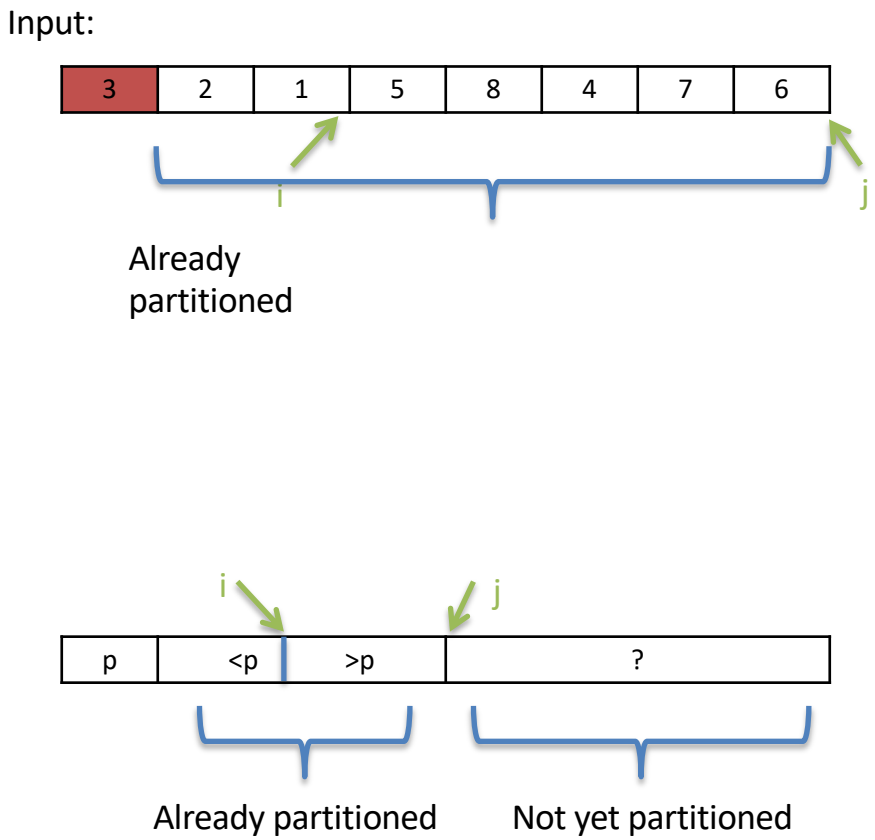
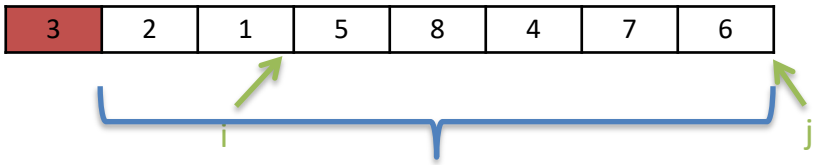


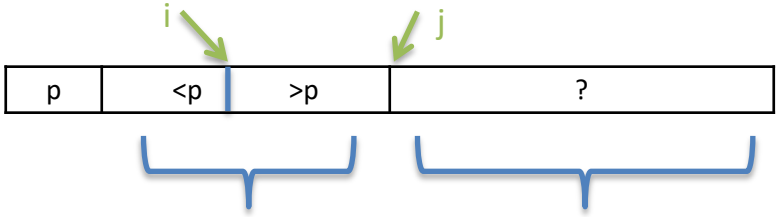
Quick sort – partitioning around a pivot – in-place

Input:



Already partitioned

now take care of the pivot position

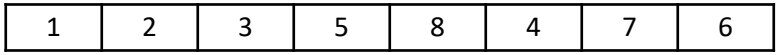


Already partitioned

Not yet partitioned

Quick sort – partitioning around a pivot – in-place

Input:



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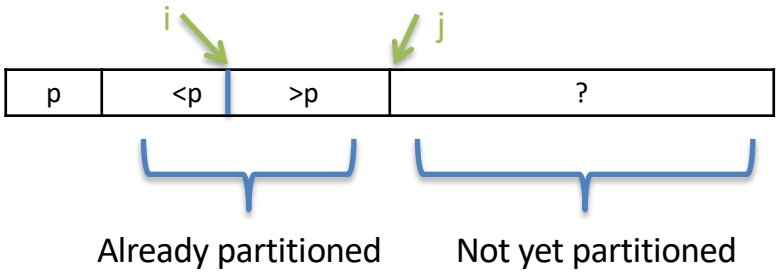


Analysis of Quick Sort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.

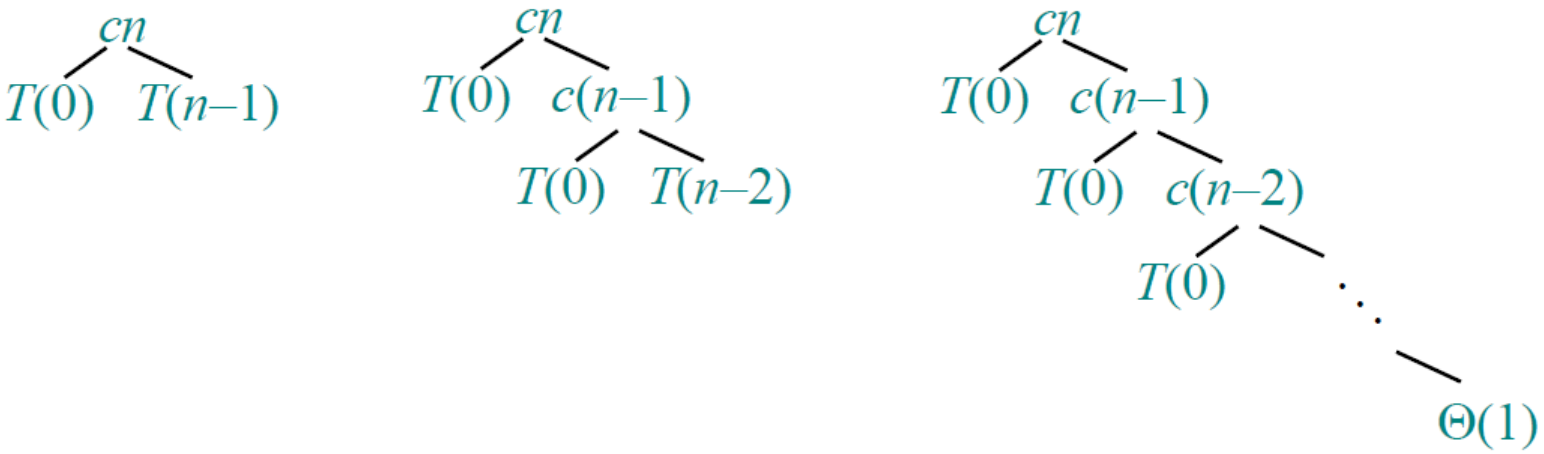
Worst Case of Quick Sort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

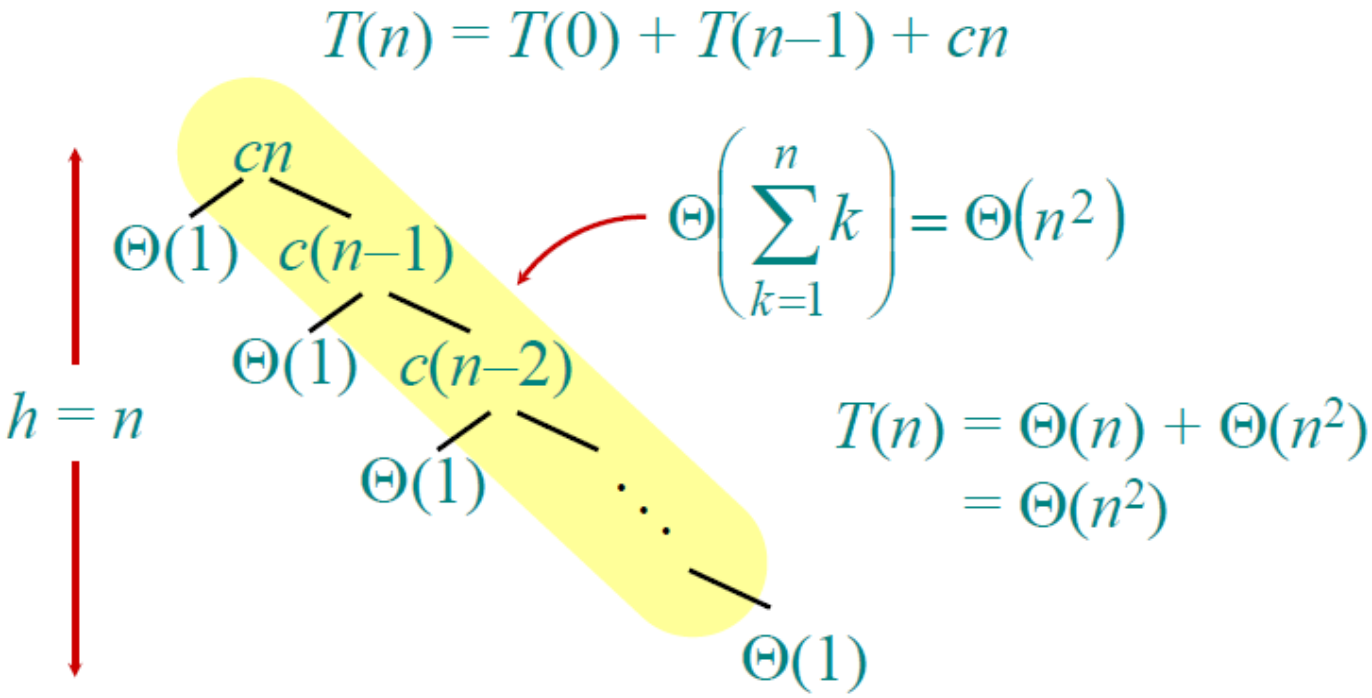
$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \Theta(n^2) \qquad \textit{(arithmetic series)}
\end{aligned}
$$

Worst Case Recursion Tree

$$T(n) = T(0) + T(n-1) + cn$$



Worst Case Recursion Tree



Best-case analysis

If we're lucky, PARTITION splits the array evenly:

$$T(n) = 2T(n/2) + \Theta(n)$$
$$= \Theta(n \lg n) \qquad \text{(same as merge sort)}$$

In the Average Case, PARTITION produces a mix of "good" and "bad" splits. The running time of quicksort, when levels alternate between good and bad splits is like the running time for good splits alone: still O(n lg n), but with a slightly larger constant hidden by the O-notation

Exercise

- Illustrate the operation of merge sort on the array:

A = {3 , 41, 52, 26, 38, 57, 9, 49}

Exercise

- Illustrate the operation *Partition* on the array :

A = { 13, 19, 9, 5, 12, 8, 7, 4, 11, 2, 6, 21}

Powering a number

Problem: Compute a^n, where $n \in \mathbb{N}$.

Naive algorithm: $\Theta(n)$.

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Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

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$$T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = \Theta(\lg n) .$$