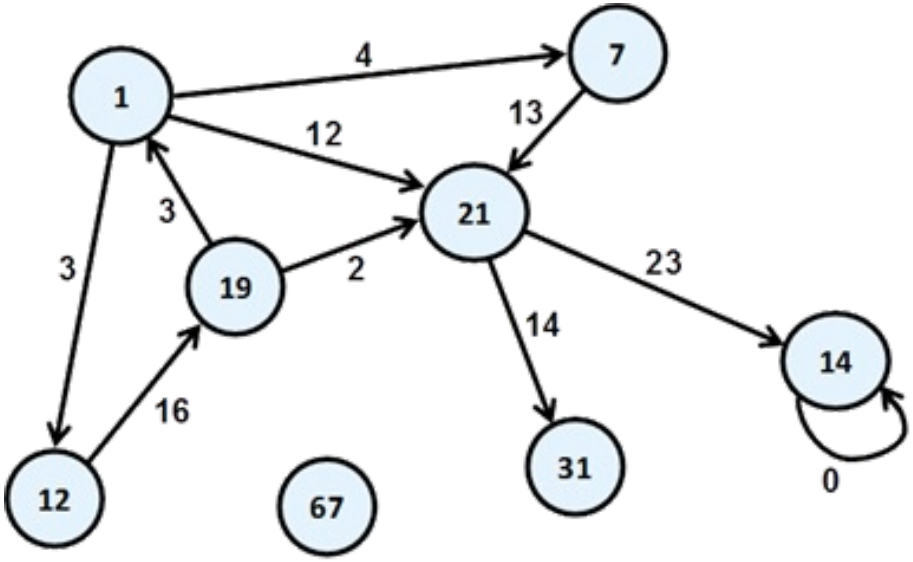
Graph Algorithms 1



NYIT CSCI-651

Graphs and Graph Theory

- Graphs are a really powerful formalism to model the *relationships* between *things*
  - Things can be whatever you want.
  - Relationships can be any relation that makes sense on those things

Graphs and Applications

- Examples:
  - Cities, highways
  - Intersections, streets
  - Airports, flights
  - Webpages, links
  - Facebook ids, friends
  - Computer, network connections
  - Task scheduling in OS
  - etc.

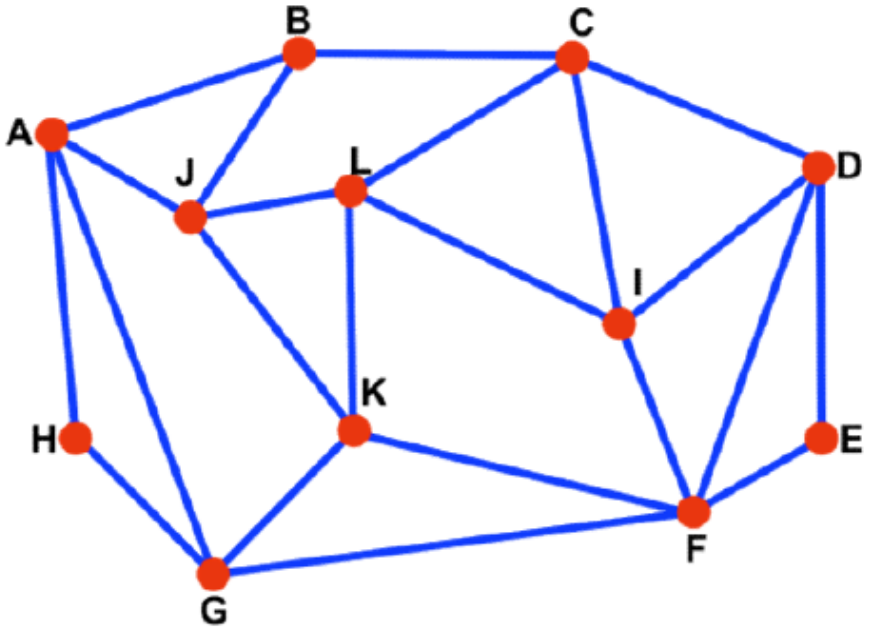
Drawing Graphs

- We draw graphs with circles or dots for the things (called vertices, or nodes) and lines or arrows for the relationships (called edges, or arcs).
  - Vertices usually labeled.
  - Edges can be labeled, too.

Graph Abstract Data Type (ADT)

- Graphs are a formalism, useful for representing relationships between things
  - a graph G is represented as G = (V, E)
    - V is a set of vertices: $\{v_1, v_2, \ldots, v_n\}$
    - E is a set of edges: $\{e_1, e_2, \ldots, e_m\}$ where each e_i connects two vertices (v_{i1}, v_{i2})
      - a set $E \subseteq V \times V$ of ***edges***.

Example



G = <**V**,**E**>

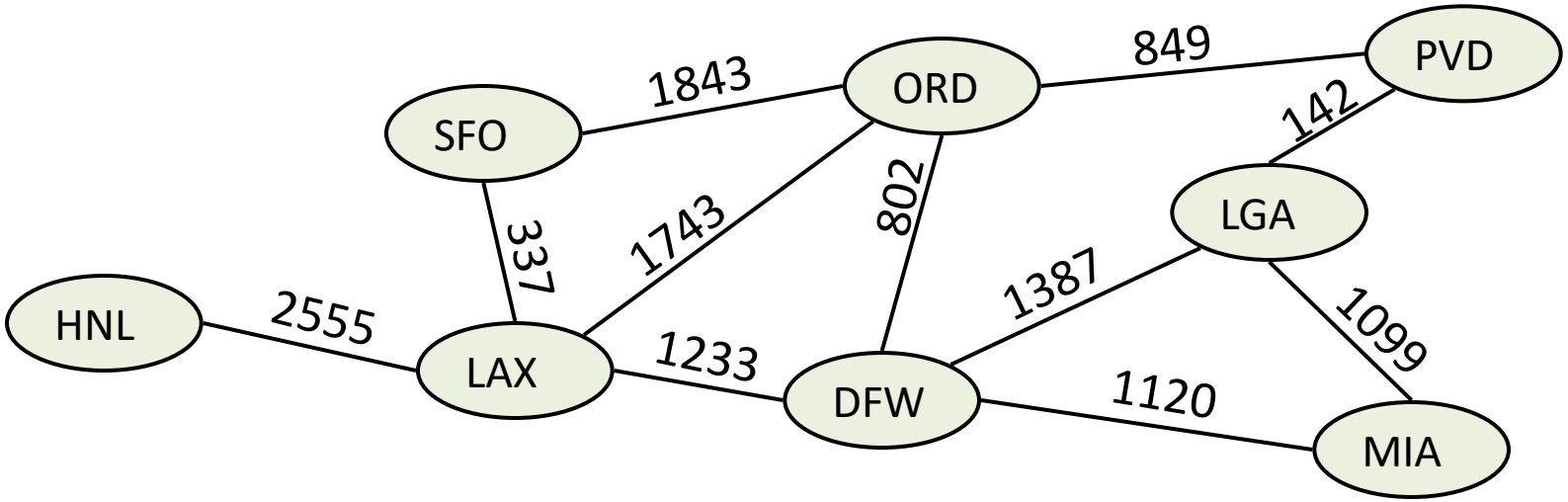
V = vertices =
{A,B,C,D,E,F,G,H,I,J,K,L}

E = edges =
{(A,B),(B,C),(C,D),(D,E),(E,F),
(F,G),(G,H),(H,A),(A,J),(A,G),
(B,J),(K,F),(C,L),(C,I),(D,I),
(D,F),(F,I),(G,K),(J,L),(J,K),
(K,L),(L,I)}

Graphs

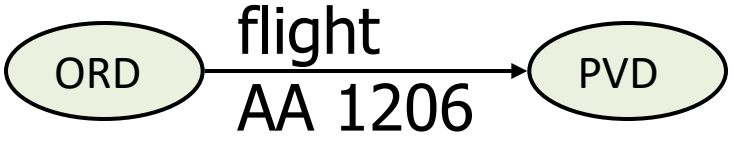
- A graph *G* is a pair (***V, E***), where ***V*** is a set of nodes, called vertices, and ***E*** is a collection of pairs of vertices, called edges. Vertices and edges can store elements

- Example: A vertex represents an airport and stores the three-letter airport code. An edge represents a flight route between two airports and stores the mileage of the route



Edge Types

- Directed edge
  - ordered pair of vertices (u,v)
  - first vertex u is the origin
  - second vertex v is the destination
  - e.g., a flight
- Undirected edge
  - unordered pair of vertices (u,v)
  - e.g., a flight route
- Directed graph
  - all the edges are directed
  - e.g., flight network
- Undirected graph
  - all the edges are undirected
  - e.g., route network



Graph ADT

- Operations might include:
  - creation (with a certain number of vertices)
  - inserting/removing edges
  - iterating over vertices adjacent to a specific vertex
  - asking whether an edge exists connecting two vertices

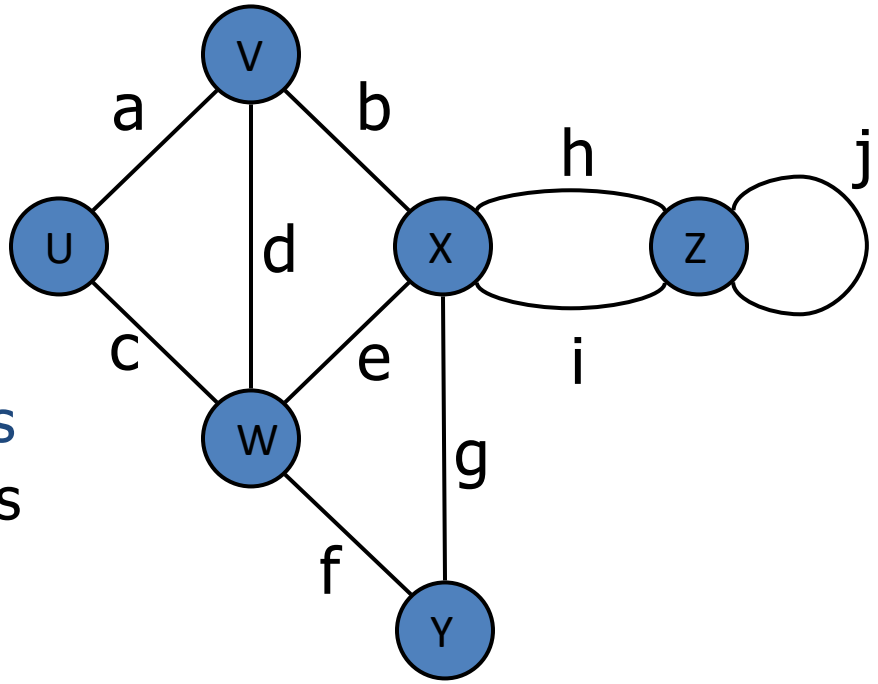
Graph Representation

- 2-D matrix of vertices
  (marking edges in the cells)
  - "adjacency matrix"

- List of vertices each with a list of adjacent vertices
  - "adjacency list"

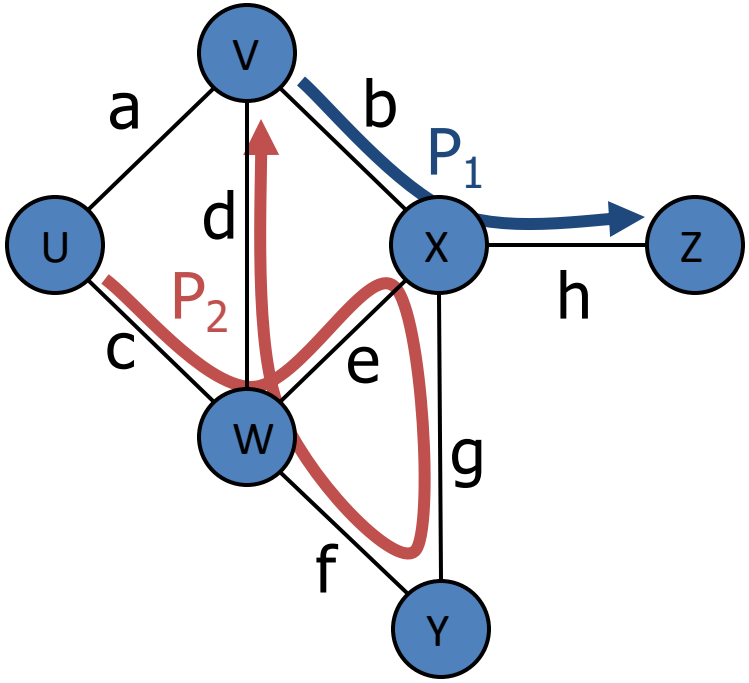
Terminology

- Adjacent vertices
  - U and V are adjacent
- Degree of a vertex
  - X has degree 5
- Parallel edges and self-loops
  - h and i are parallel edges
  - j is a self-loop



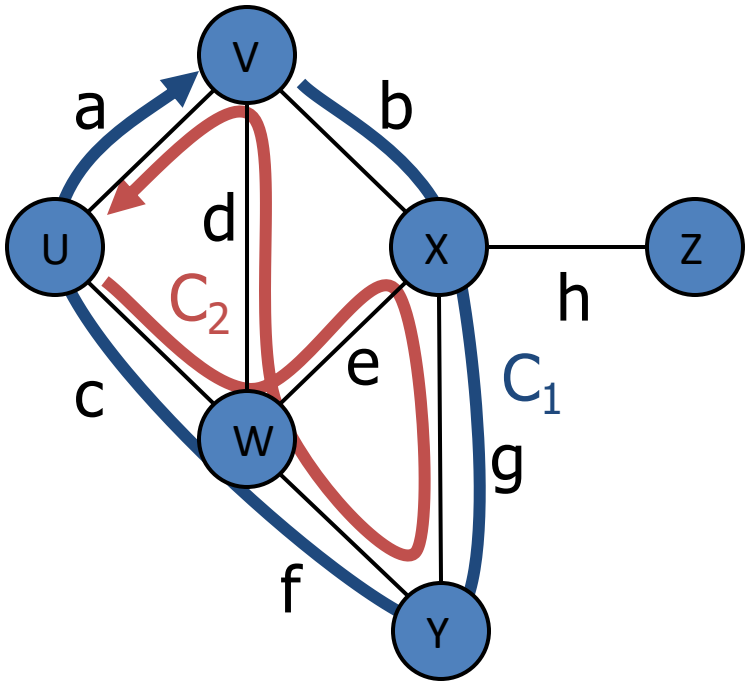
Terminology (cont.)

- Path
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints
- Simple path
  - path such that all its vertices and edges are distinct
  - P_1=(V,b,X,h,Z) is a simple path
  - P_2=(U,c,W,e,X,g,Y,f,W,d,V) is a path that is not simple



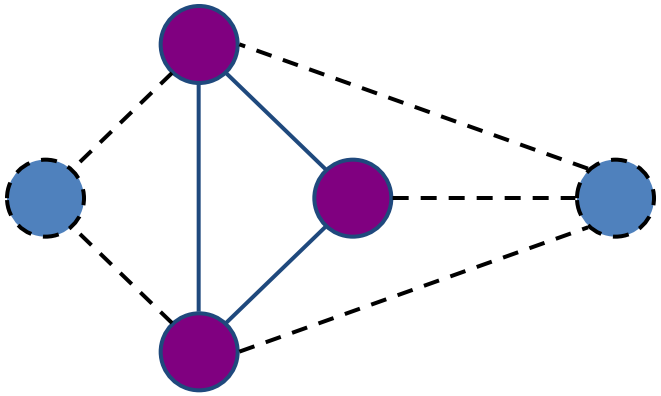
Terminology (cont.)

- Cycle
  - circular sequence of alternating vertices and edges
  - each edge is preceded and followed by its endpoints
- Simple cycle
  - cycle such that all its vertices and edges are distinct
- Examples
  - C_1=(V,b,X,g,Y,f,W,c,U,a,↵) is a simple cycle
  - C_2=(U,c,W,e,X,g,Y,f,W,d,V,a,↵) is a cycle that is not simple

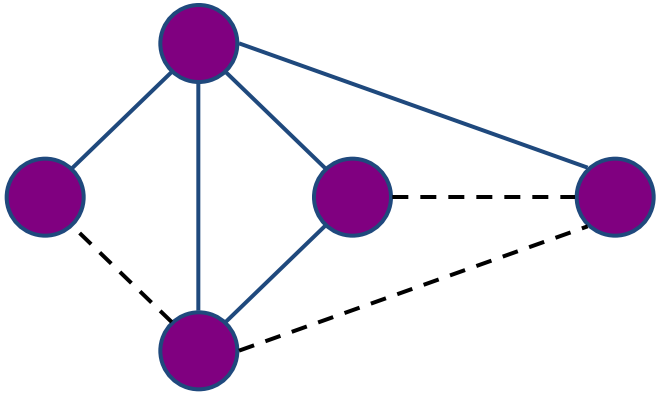


Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



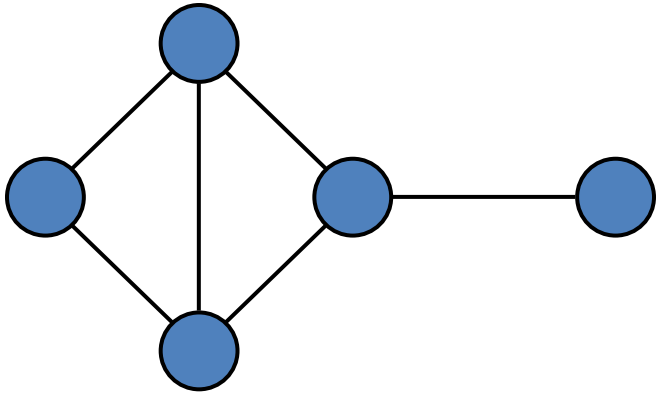
Subgraph



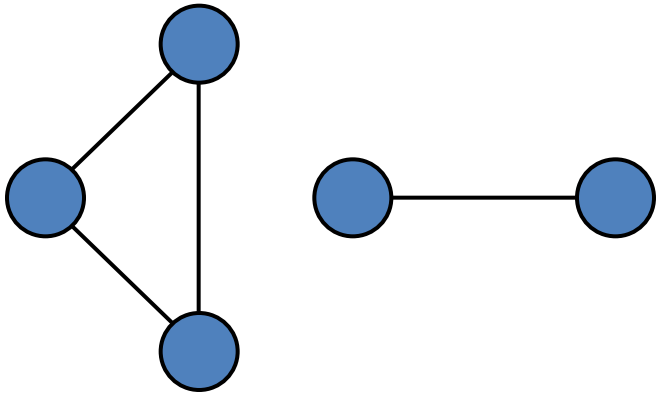
Spanning subgraph

Connectivity

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G



Connected graph

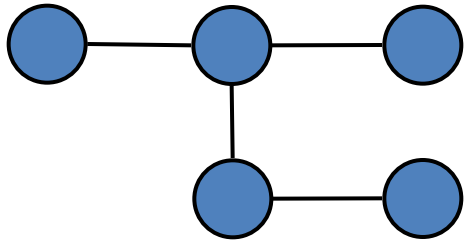


Non connected graph with two connected components

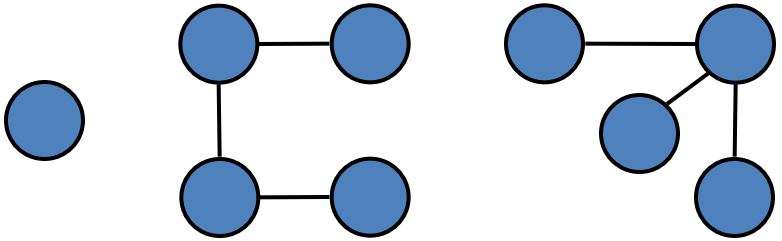
Trees and Forests

- A tree is an undirected graph T such that
  - T is connected
  - T has no cycles

  This definition of tree is different from the one of a rooted tree
- A forest is an undirected graph without cycles
- The connected components of a forest are trees



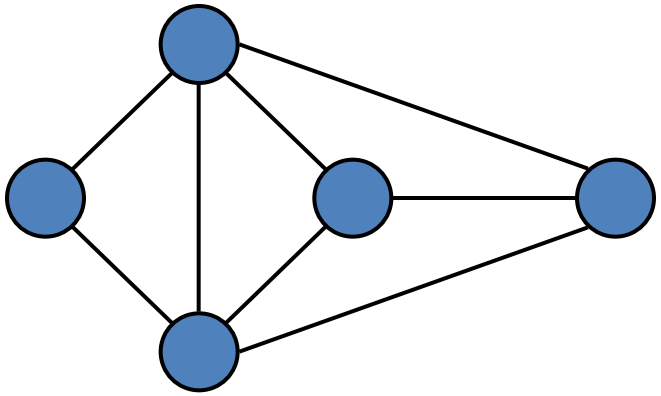
Tree



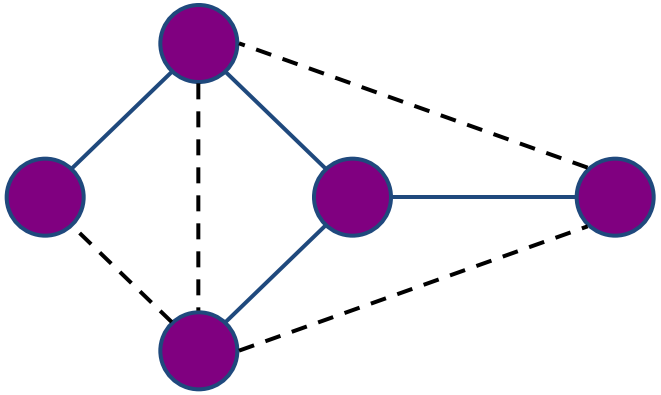
Forest

Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



Graph



Spanning tree

Properties

Property 1

$\sum_v \deg(v) = 2m$

Proof: each edge is counted twice

Property 2

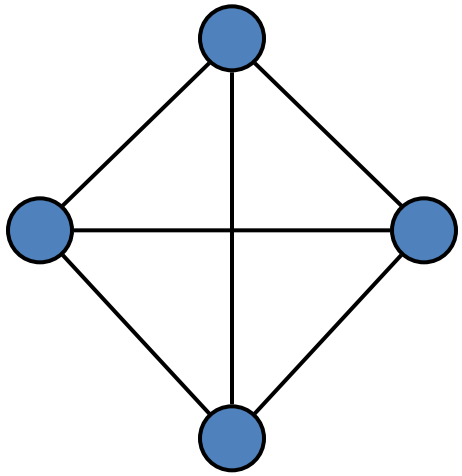
In an undirected graph with no self-loops and no parallel edges

$m \leq n\,(n-1)/2$

Proof: $(n-1) + (n-2) + \ldots + 1$

Notation

n	number of vertices
m	number of edges
$\deg(v)$	degree of vertex v

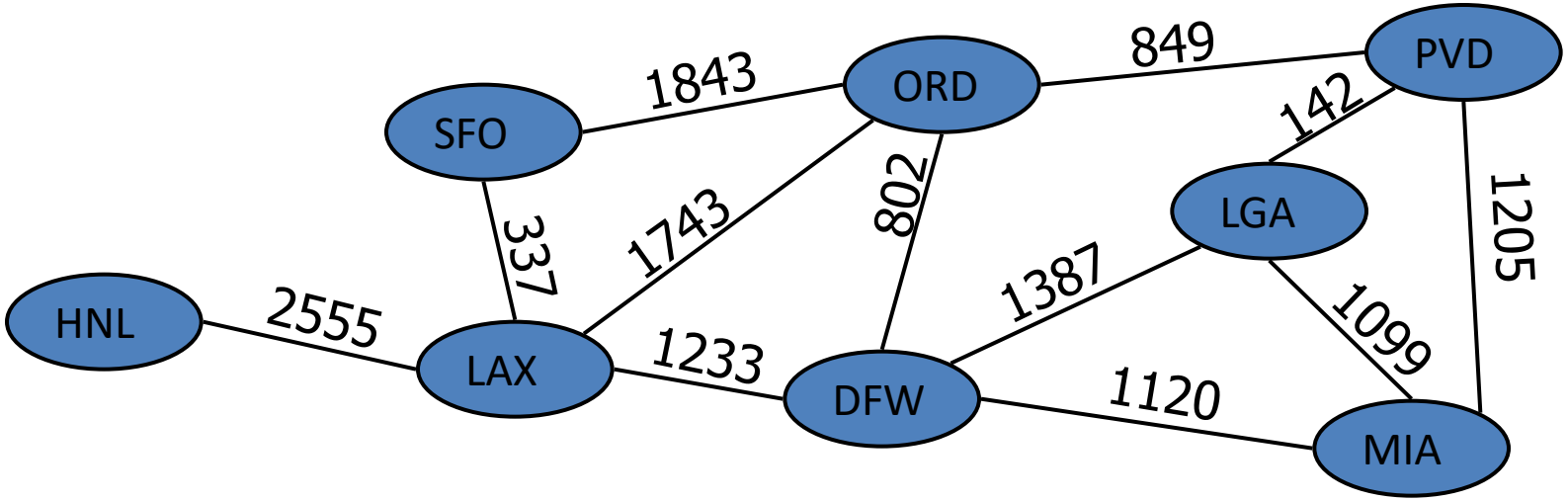


Example

- $n = 4$
- $m = 6$
- $\deg(v) = 3$

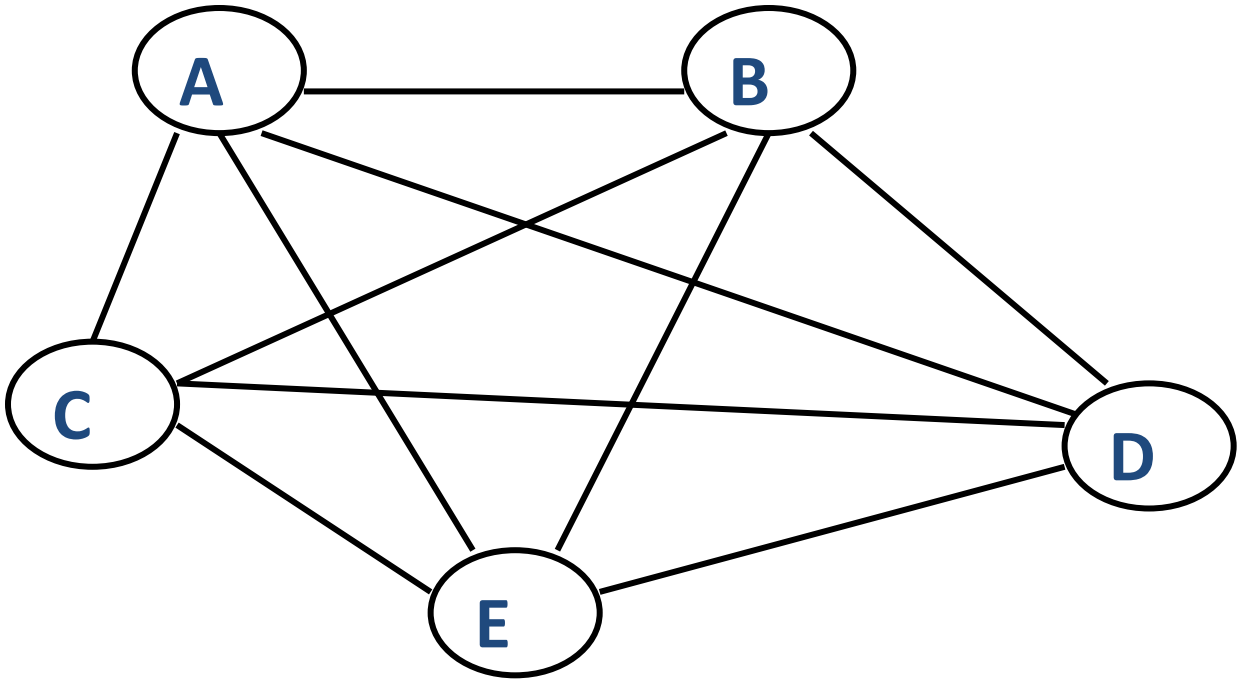
Weighted Graphs

- In a weighted graph, each edge has an associated numerical value, called the weight of the edge
- Edge weights may represent, distances, costs, etc.
- Example:
  - In a flight route graph, the weight of an edge represents the distance in miles between the endpoint airports



Complete graphs

Graphs with all edges present



A complete graph

Dense graphs: relatively few of the possible edges are missing

Sparse graphs: relatively few of the possible edges are present

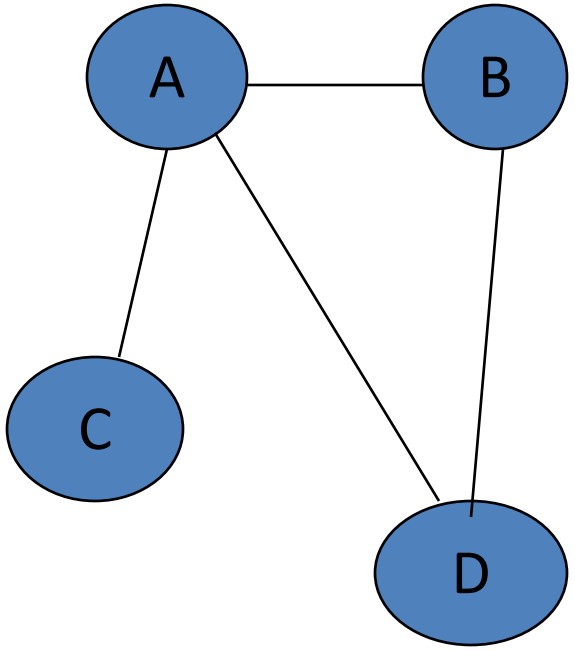
Adjacency matrix – undirected graphs

Vertices: A,B,C,D

Edges: AC, AB, AD, BD

The matrix is symmetrical

	A	B	C	D
A	0	1	1	1
B	1	0	0	1
C	1	0	0	0
D	1	1	0	0

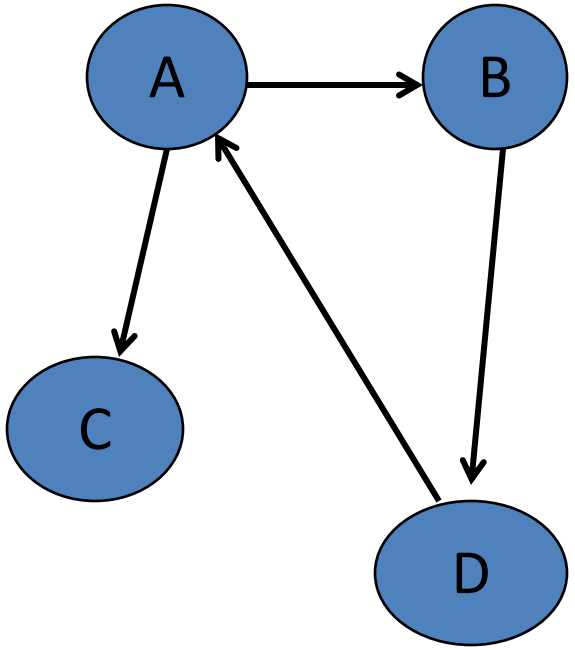


Adjacency matrix – directed graphs

Vertices: A,B,C,D

Edges: AC, AB, BD, DA

	A	B	C	D
A	0	1	1	0
B	0	0	0	1
C	0	0	0	0
D	1	0	0	0



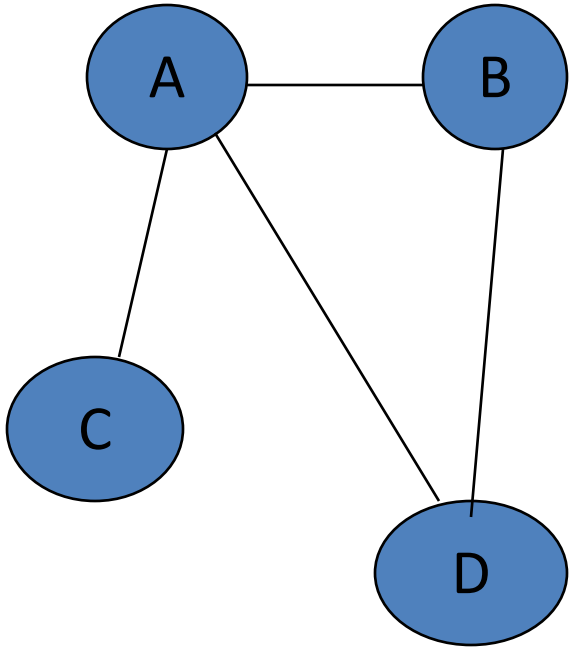
Adjacency lists – undirected graphs

Vertices: A,B,C,D

Edges: AC, AB, AD, BD

Heads lists

A	B C D
B	A D
C	A
D	A B

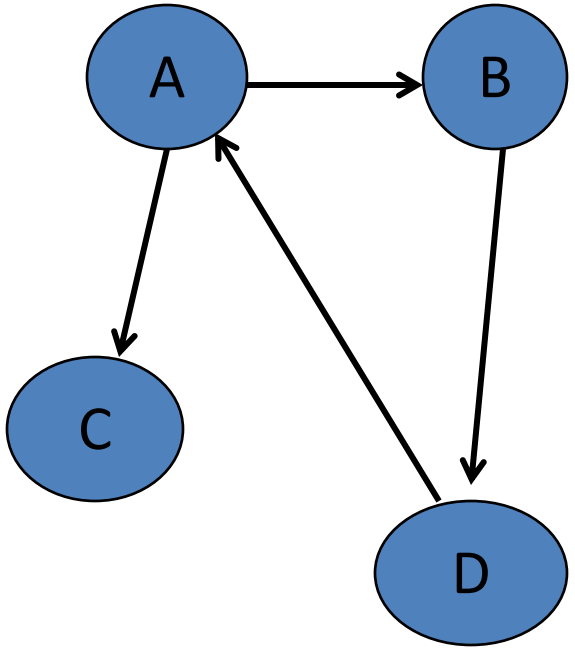


Adjacency lists – directed graphs

Vertices: A,B,C,D

Edges: AC, AB, BD, DA

Heads	lists
A	B C
B	D
C	=
D	A



Graph Representation

- If the graph is sparse, then number of edges is much less than n^2
- If the graph is dense, then the number of edges is close to (the complete) $n(n-1)/2$, or to n^2 if the graph is directed with self-loops.
  - Then there is no advantage of using adjacency list over matrix.
- In terms of space complexity
  Adjacency matrix: $O(n^2)$
  Adjacency list: $O(n+m)$
  where n is the number nodes, m is the number of edges.

Graph Representation

- When the graph is **undirected tree** then
  Adjacency matrix: $O(n$^2)
  Adjacency list: $O(n+n)$ is $O(n)$ (better than n^2)

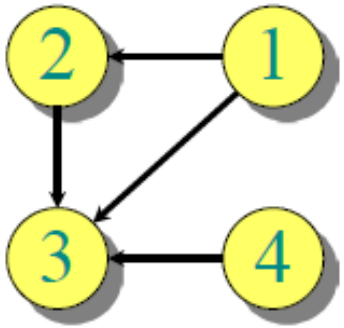
- When the graph is **directed, complete, with self-loops** then
  Adjacency matrix: $O(n$^2)
  Adjacency list: $O(n+n$^2) is $O(n$^2) (no difference)

- And finally, when you implement using matrix, checking if there is an edge between two nodes takes O**(1) times**, while with an adjacency list, it may take linear time in n.

Adjacency-matrix representation

The ***adjacency matrix*** of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1 \ldots n, 1 \ldots n]$ given by

$$A[i,j] = \begin{cases} 1 & \text{if } (i,j) \in \mathrm{E}, \\ 0 & \text{if } (i,j) \notin \mathrm{E}. \end{cases}$$



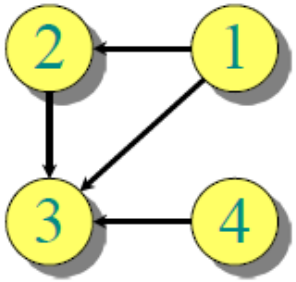
A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

$\Theta(V^2)$ storage

$\Rightarrow$ ***dense*** representation.

Adjacency-list representation

An ***adjacency list*** of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to v.



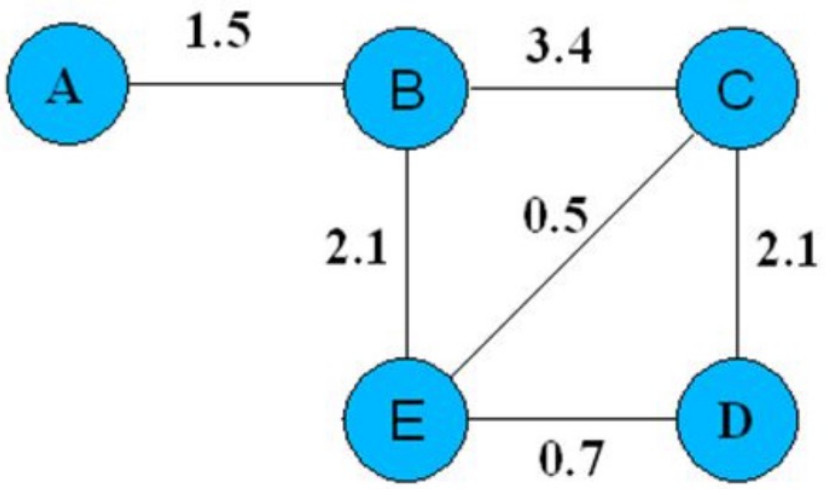
$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3\}$

For undirected graphs, $|Adj[v]| = degree(v)$.
For digraphs, $|Adj[v]| = out\text{-}degree(v)$.

Handshaking Lemma: $\sum_{v \in V} degree(v) = 2|E|$ for undirected graphs $\Rightarrow$ adjacency lists use $\Theta(V + E)$ storage — a ***sparse*** representation.

Weighted Graphs

- How can we store weights in adjacency matrix



	A	B	C	D	E
A	0	1.5	0	0	0
B	1.5	0	3.4	0	0
C	0	3.4	0	2.1	0.5
D	0	0	2.1	0	0.7
E	0	2.1	0.5	0.7	0

Graph Density

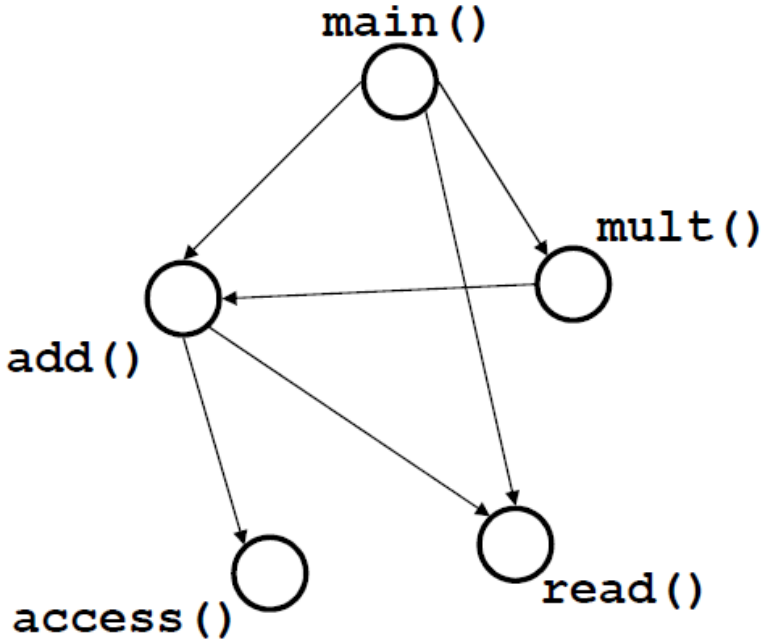
- A *sparse* graph has O(|V|) edges
  - Why is the adjacency list likely to be a better representation than the adjacency matrix for sparse graphs?

- A *dense* graph has $O(|V|^2)$ edges

Anything in between is either on the sparse side or on the dense side, depending critically on context!

Directed Acyclic Graphs (DAGs)

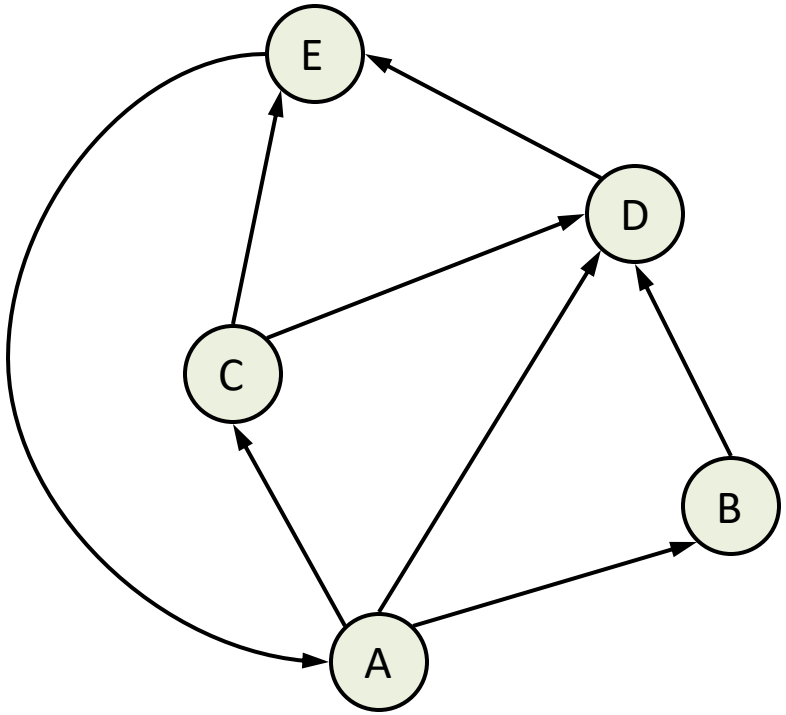
- DAGs are directed graphs with no (directed) cycles.



Trees ⊂ DAGs ⊂ Graphs

Directed Graphs (Digraphs)

- A digraph is a graph whose edges are all directed
  - Short for "directed graph"
- Applications
  - one-way streets
  - flights
  - task scheduling



DAG and Topological Sort

- Directed acyclic graph is a partially ordered set
- A topological sort is the process of sorting items over which a partial order is defined.
- **Partial order** is a relation on a set A that is reflexive, transitive and antisymmetric.
- **Total order** is a partial order that has one additional property - any two elements in the set should be related.

DAGs and Topological Ordering

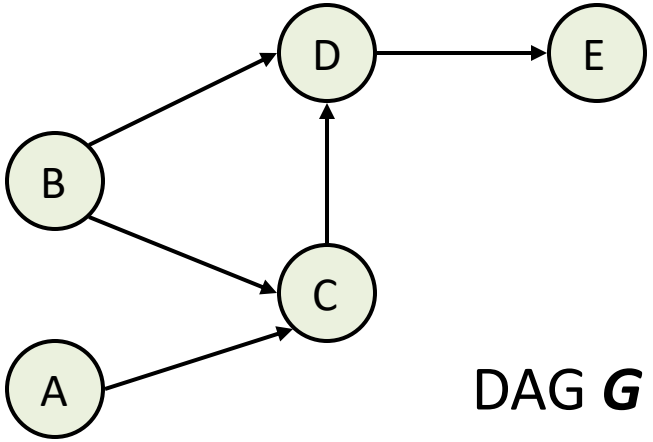
- A directed acyclic graph (DAG) is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering

  $v_1, \ldots, v_n$

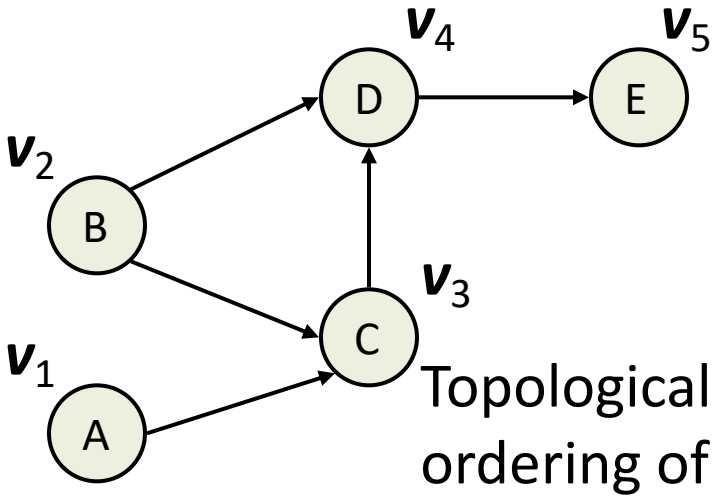
  of the vertices such that for every edge (v_i, v_j), we have $i < j$
- Example: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints

Theorem

A digraph admits a topological ordering if and only if it is a DAG



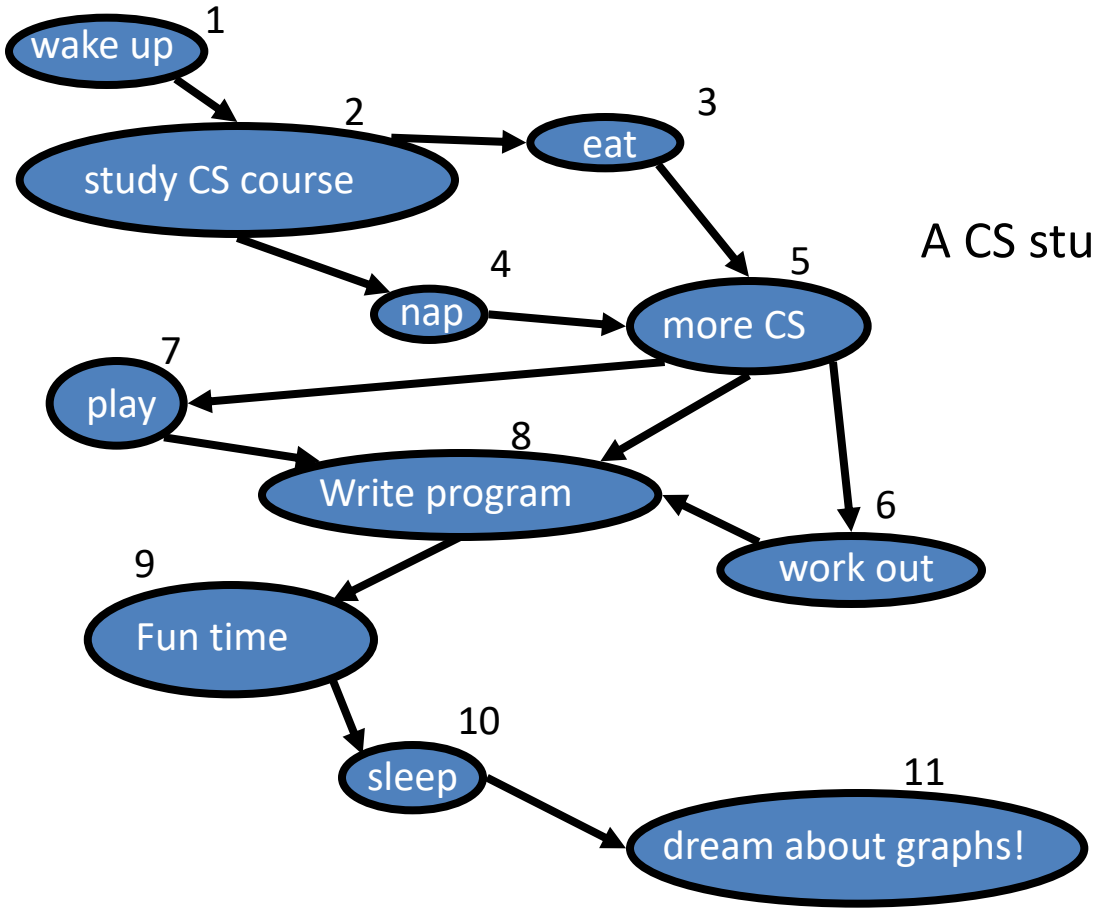
DAG ***G***



Topological ordering of ***G***

Topological Sorting

- Number vertices, so that (u,v) in E implies u < v



A CS student day

Topological Sort

RULE: If there is a path from ***u*** to ***v***, then ***v*** appears after ***u*** in the ordering.

Graphs: directed, acyclic (DAG)

OutDegree of a vertex **U**: the number of **outgoing** edges
Indegree of a vertex **U**: the number of **incoming** edges

The algorithm for topological sort uses "indegrees" of vertices.
Implementation: with a queue

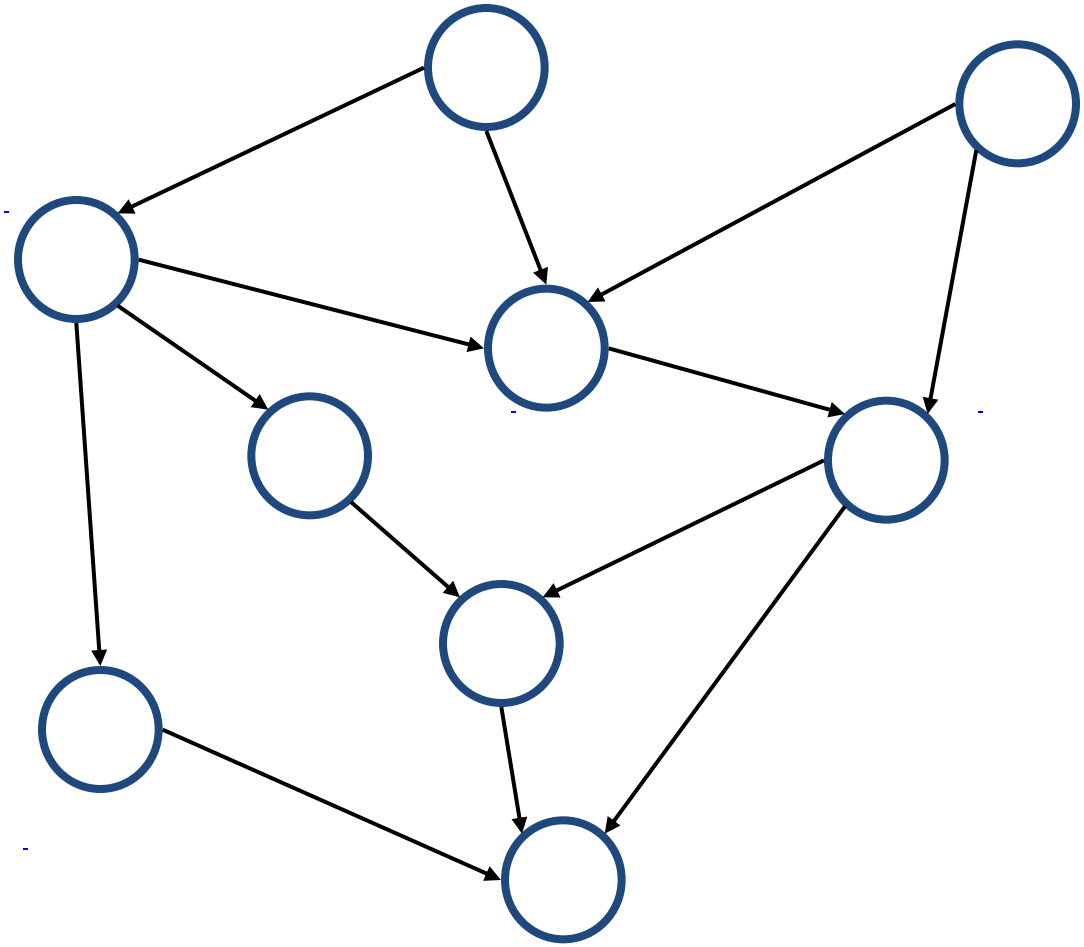
Complexity: **Adjacency lists: O(|E| + |V|),**
Matrix representation: O(|V|²)

Algorithm for Topological Sorting

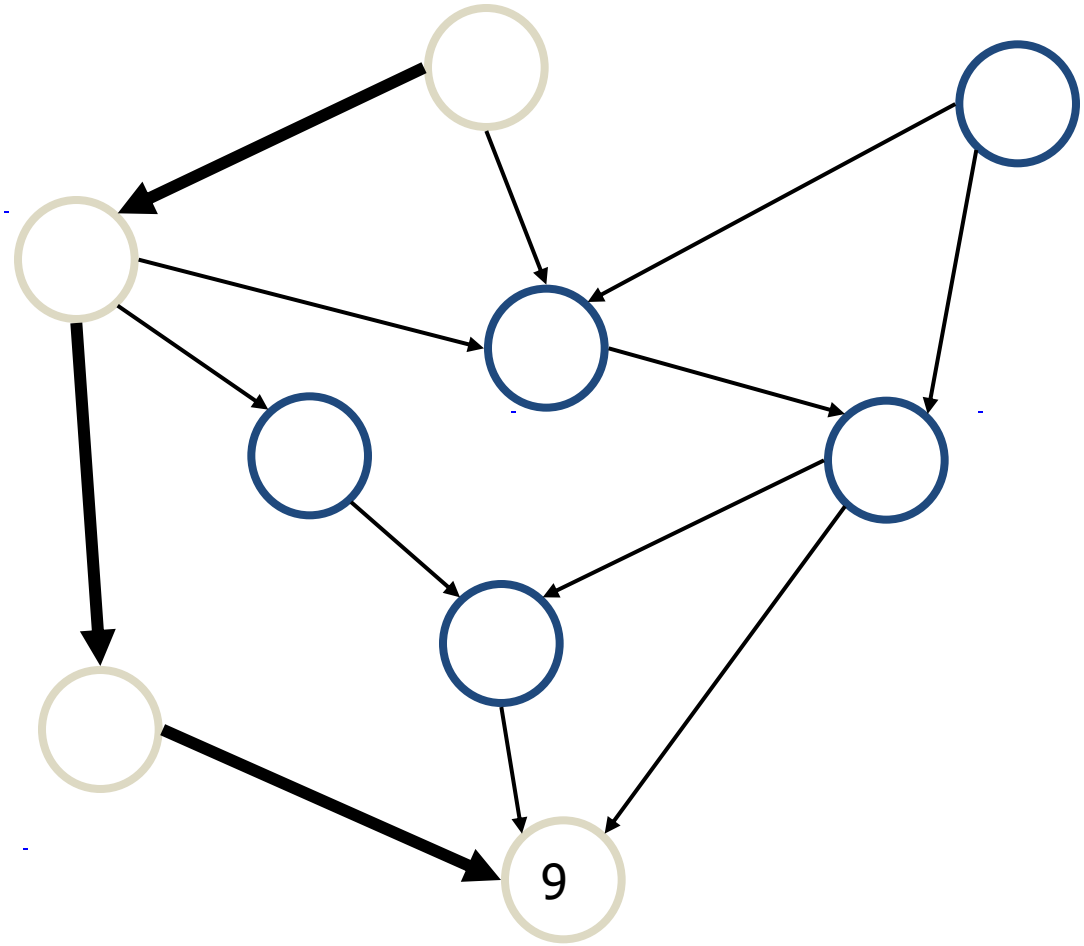
```
Algorithm TopologicalSort(G)
    H ← G    // Temporary copy of G
    n ← G.numVertices()
    while H is not empty do
     Let v be a vertex with no outgoing edges
     Label v ← n
     n ← n – 1
     Remove v from H
```

- Running time: O(n + m)

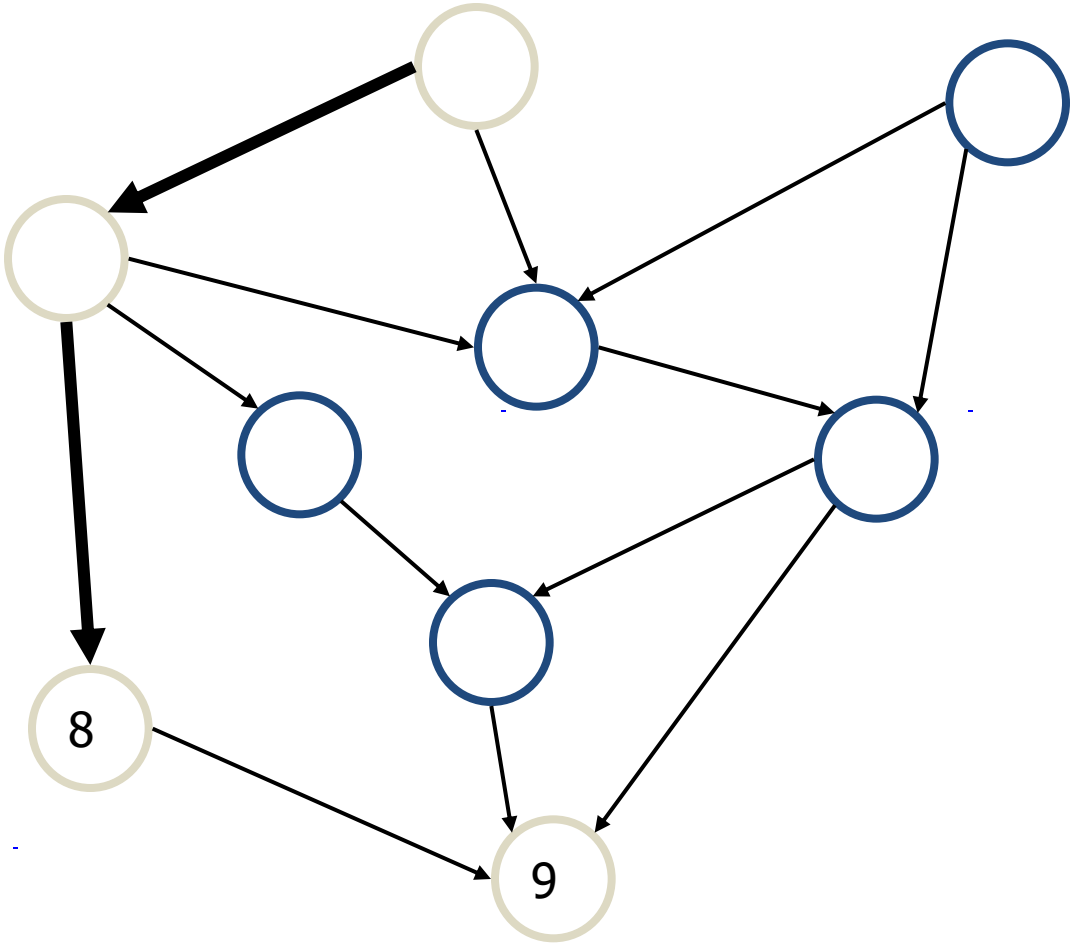
Topological Sorting Example



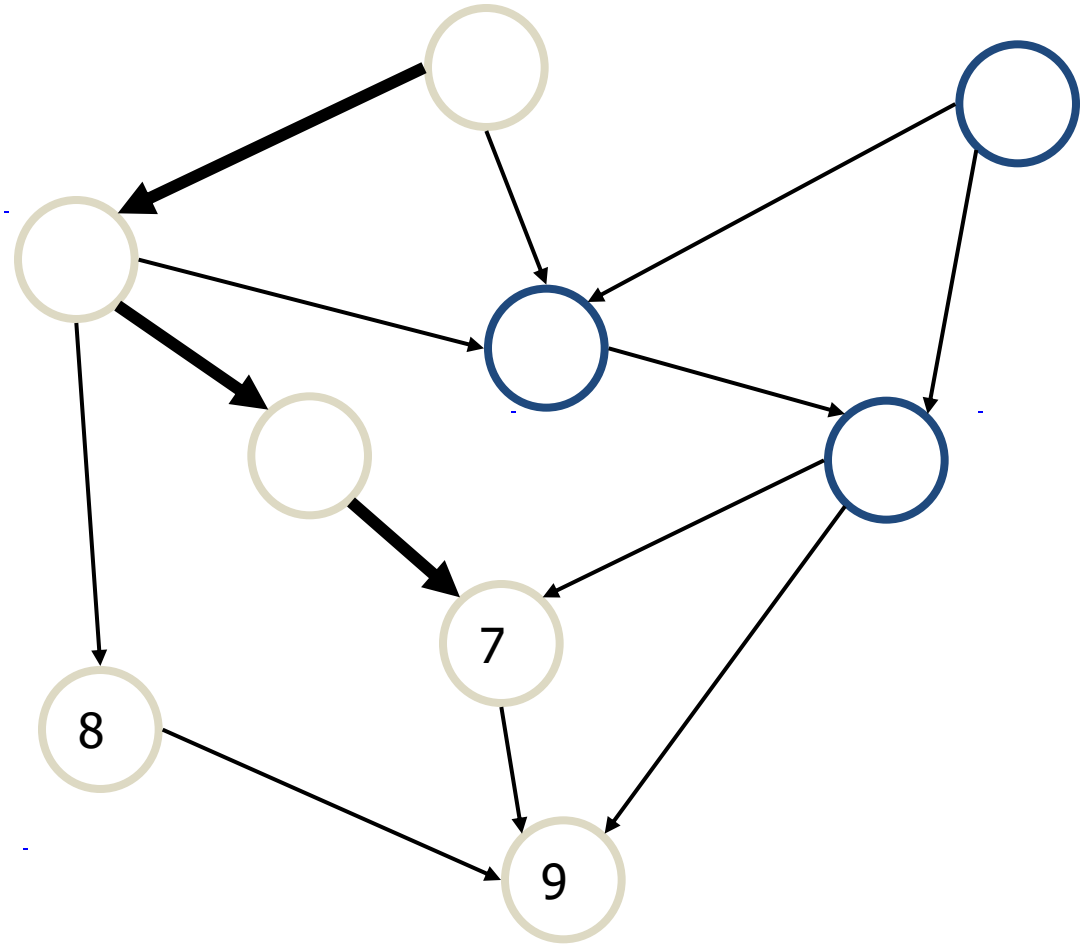
Topological Sorting Example



Topological Sorting Example



Topological Sorting Example



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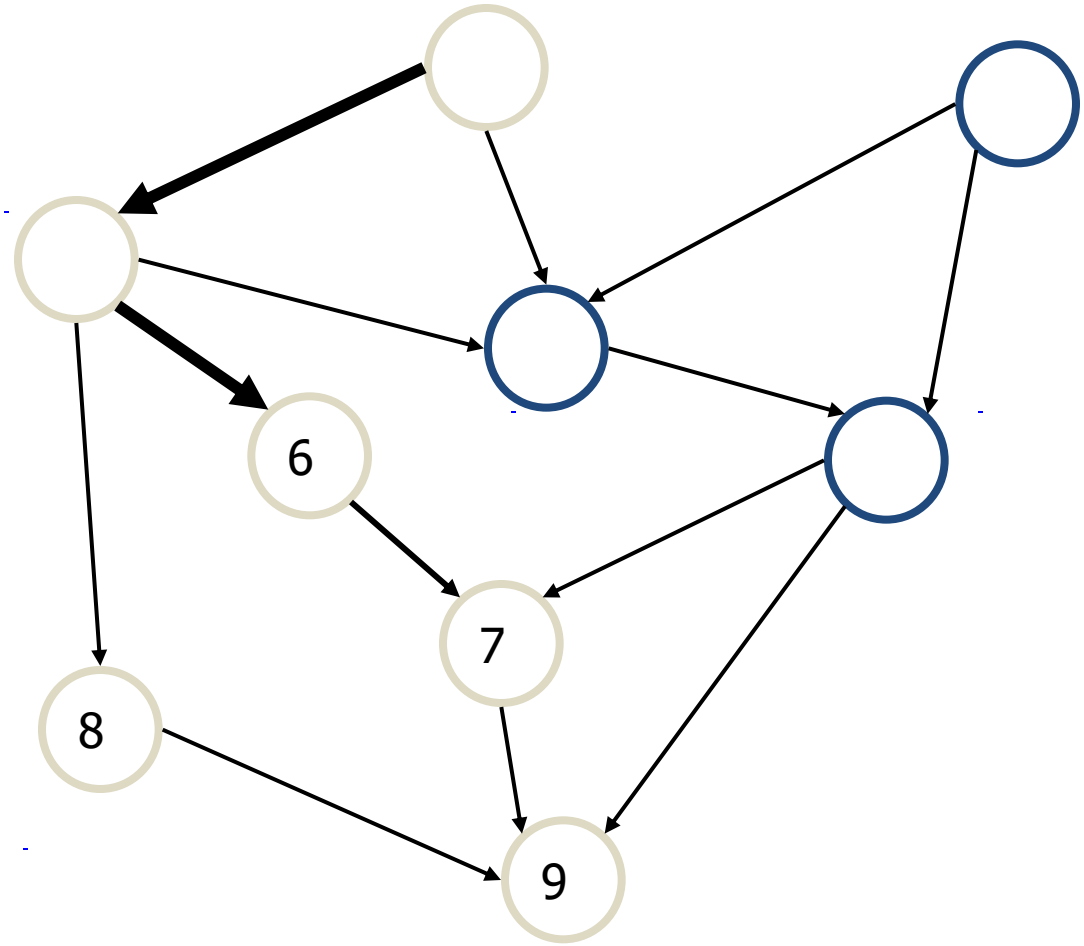

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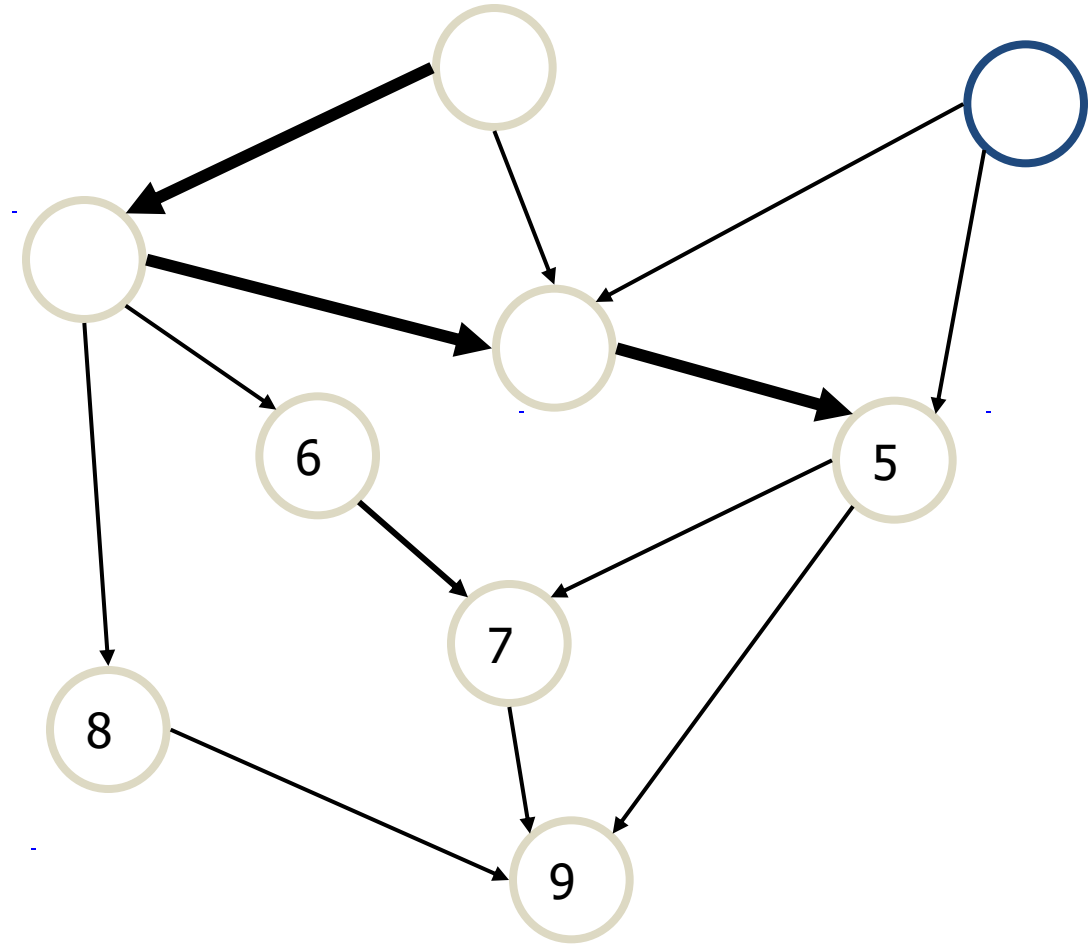
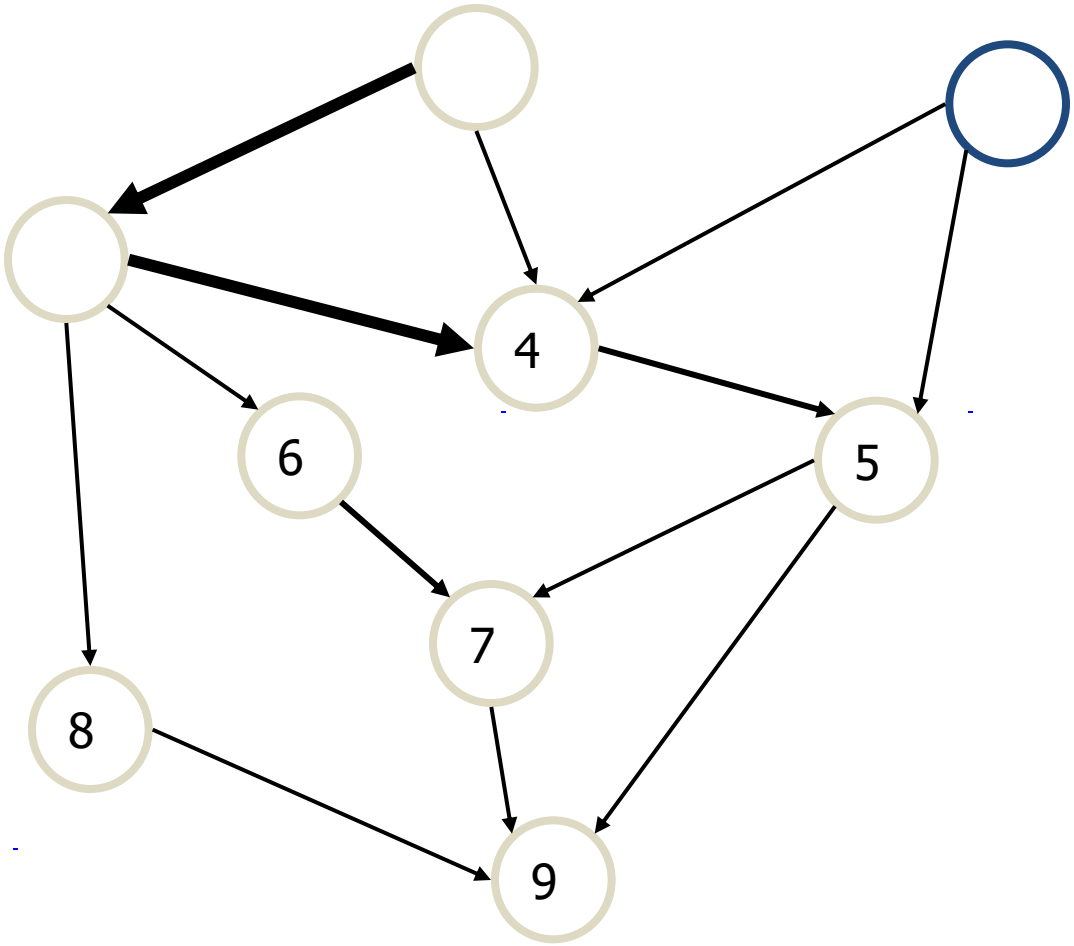


Topological Sorting Example



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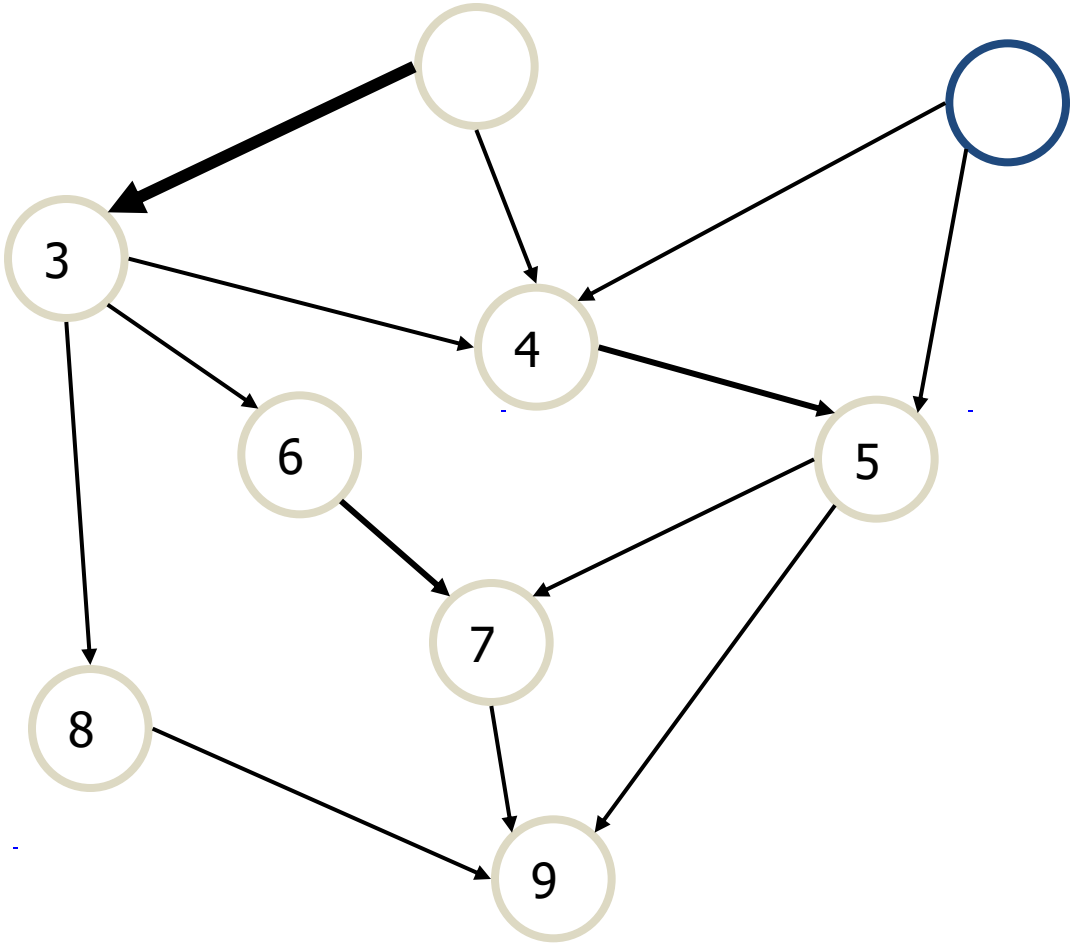
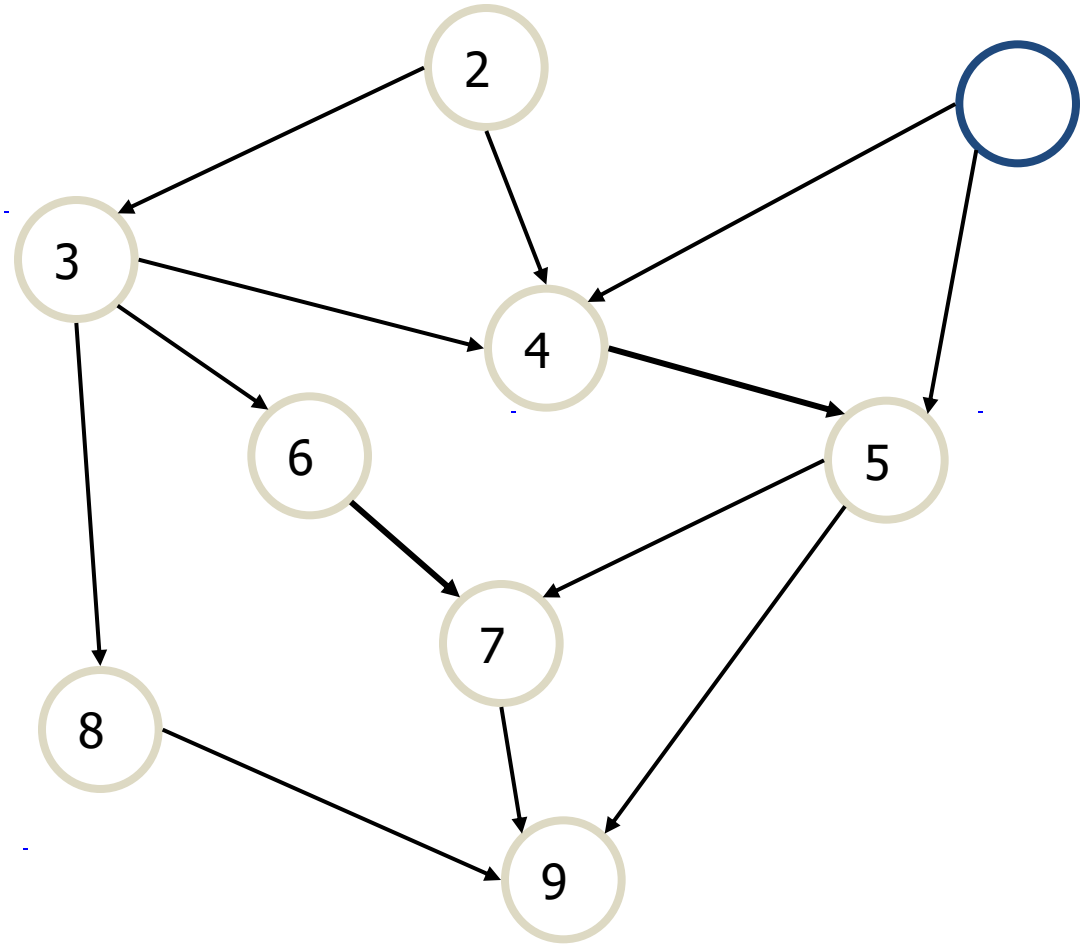
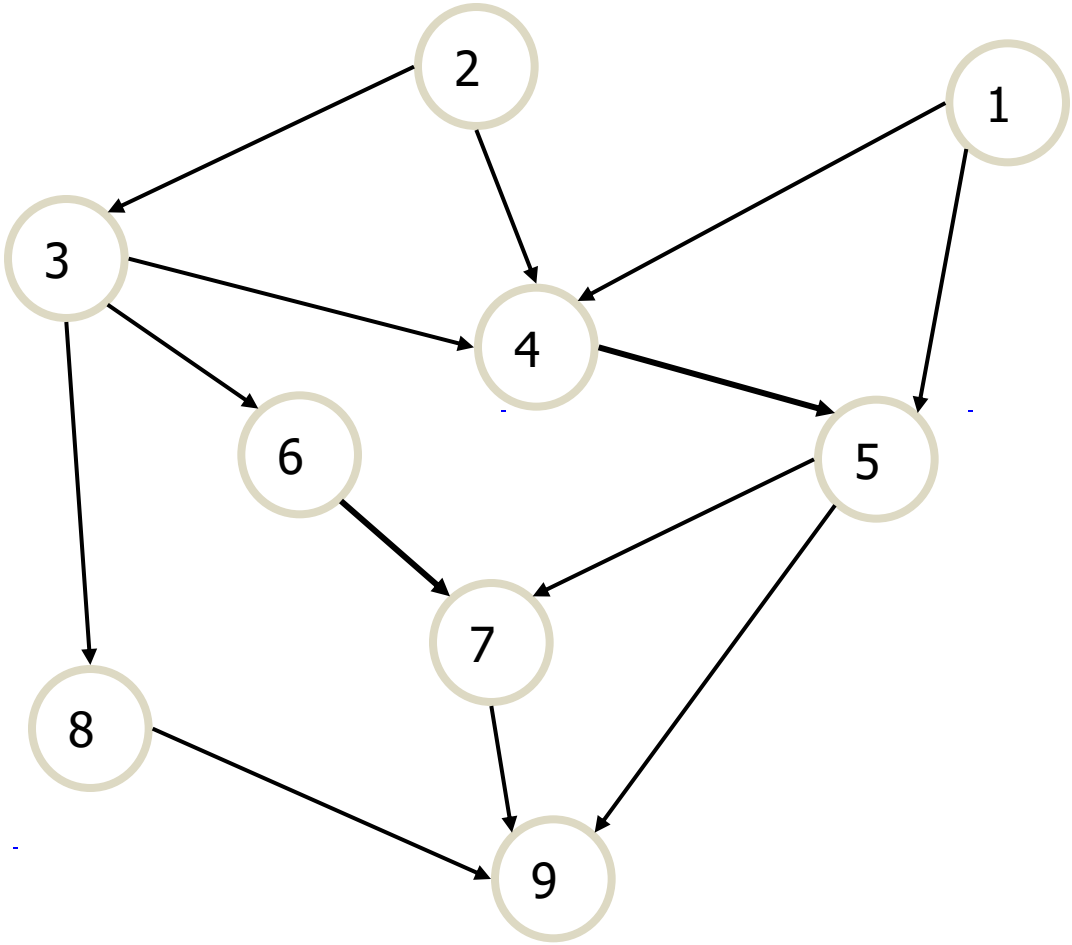


Topological Sorting Example



Topological Sorting Example



Minimum Spanning tree

Spanning Tree

- A spanning tree of a connected weighted undirected graph G is a tree that has all the vertices of the graph connected by some edges.
  - A graph can have one or more number of spanning trees.
  - If the graph has N vertices, then the spanning tree will have N-1 edges.

Minimum Spanning Trees

Spanning subgraph

- Subgraph of a graph G containing all the vertices of G

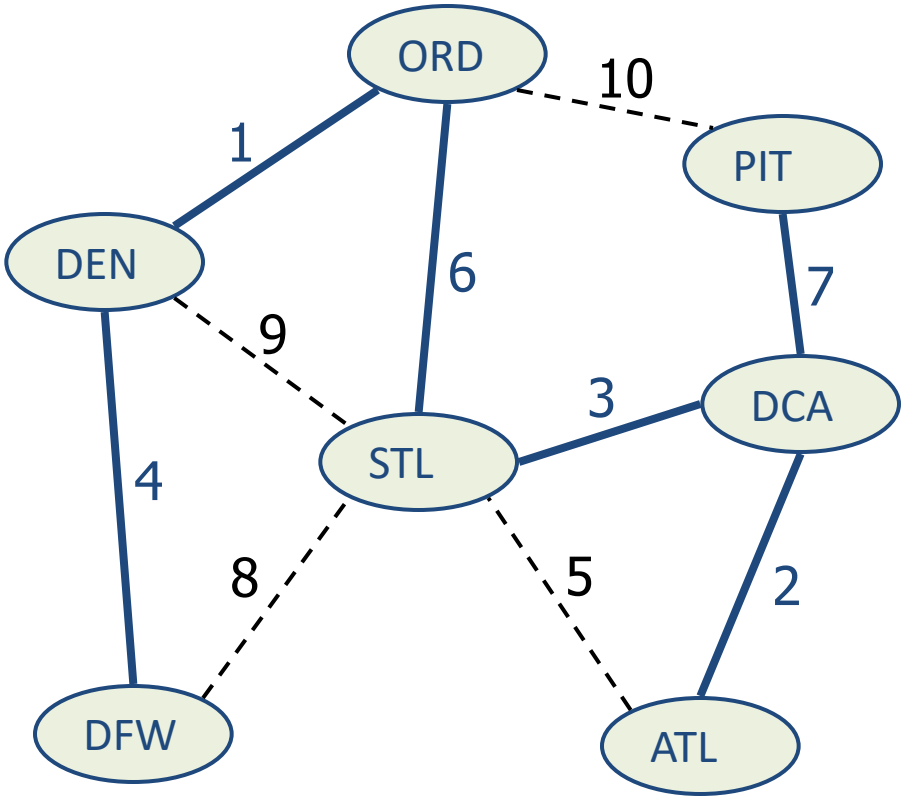
Spanning tree

- Spanning subgraph that is itself a (free) tree

Minimum spanning tree (MST)

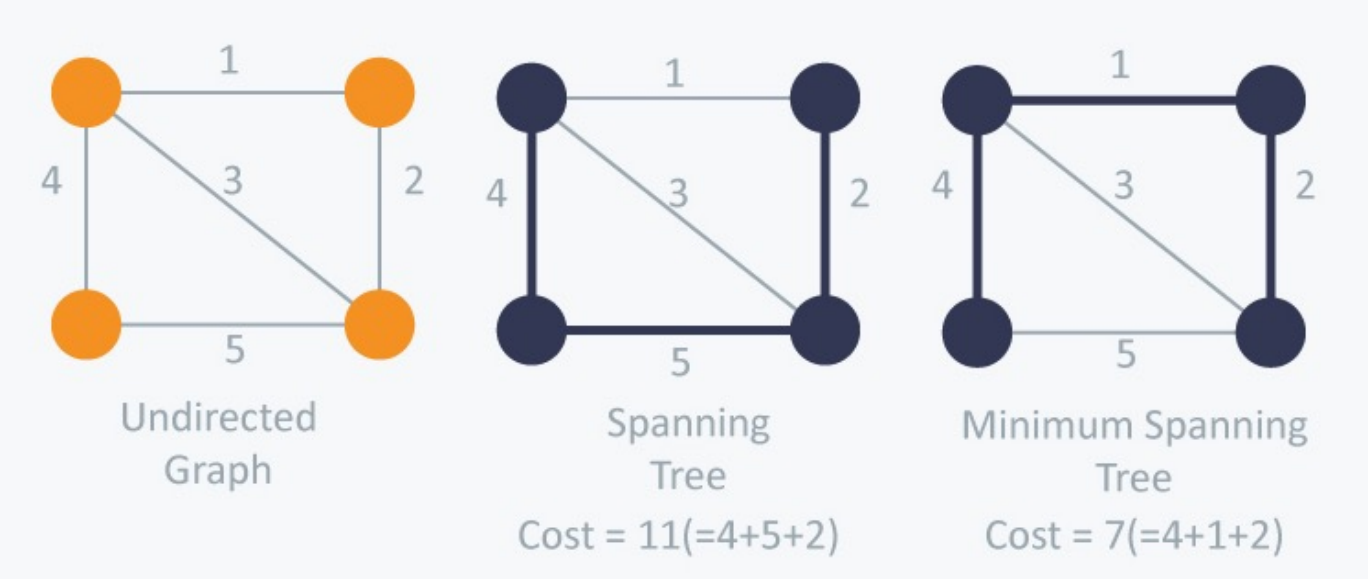
- Spanning tree of a weighted graph with minimum total edge weight

- Applications
  - Communications networks
  - Transportation networks

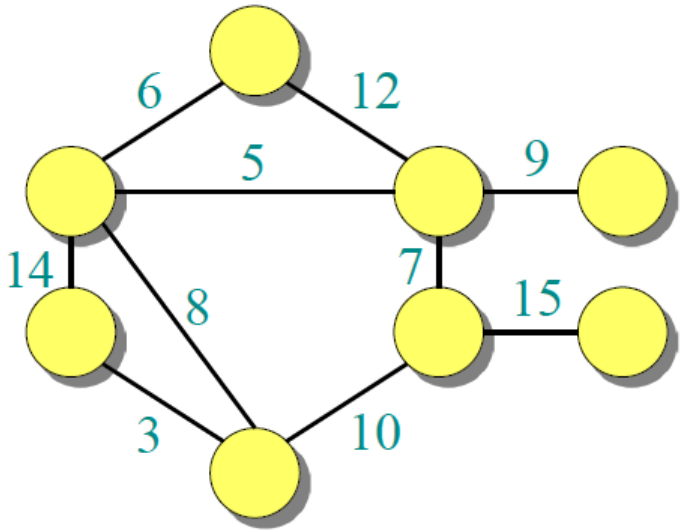


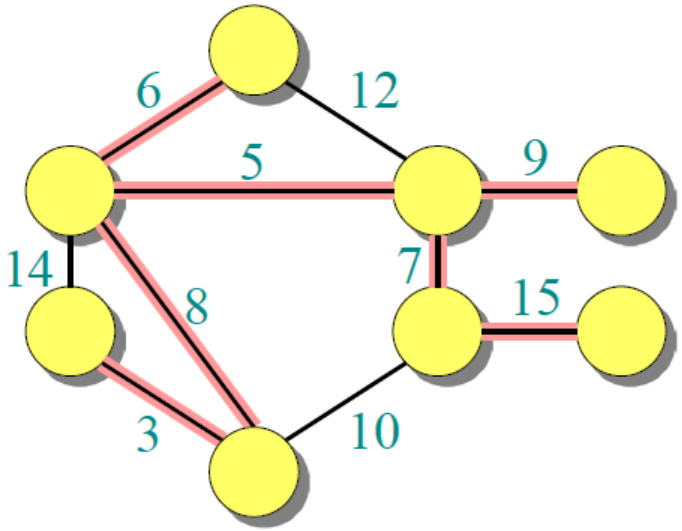
Minimum Spanning tree

- MST is a subset of edges of a connected edge-weighted undirected graph that connects all the vertices together without any cycles with the minimum possible edge weight



Minimum Spanning tree - Example





Minimum Spanning Tree

Input: A connected, undirected graph $G = (V, E)$ with weight function $w : E \to \mathsf{R}$.

Output: A ***spanning tree*** T — a tree that connects all vertices — of minimum weight:

$$w(T) = \sum_{(u,v) \in T} w(u,v) .$$

- The two algorithms we consider use a greedy approach to the problem.
  - **Prim's algorithm**
  - **Kruskal's algorithm**

Growing a minimum spanning tree

- This greedy strategy is captured by the following generic method, which **grows the minimum spanning tree one edge at a time**.
- The generic method manages a set of edges A, maintaining the following loop invariant: ***Prior to each iteration, A is a subset of some minimum spanning tree.***
- At each step, we determine an edge (u,v) that we can add to A without violating this invariant, in the sense that A Ụ {(u,v)} is also a subset of a minimum spanning tree
- We call such an **edge a safe edge for A**, since we can add it safely to A while maintaining the invariant

A loop invariant is a property of a program loop that is true before (and after) each iteration

Minimum Spanning Tree

- In Prim's algorithm, the set A forms a single tree. The safe edge added to A is always a least-weight edge connecting the tree to a vertex not in the tree.

- In Kruskal's algorithm, the set A is a forest whose vertices are all those of the given graph. The safe edge added to A is always a least-weight edge in the graph that connects two distinct components.

Prim's algorithm

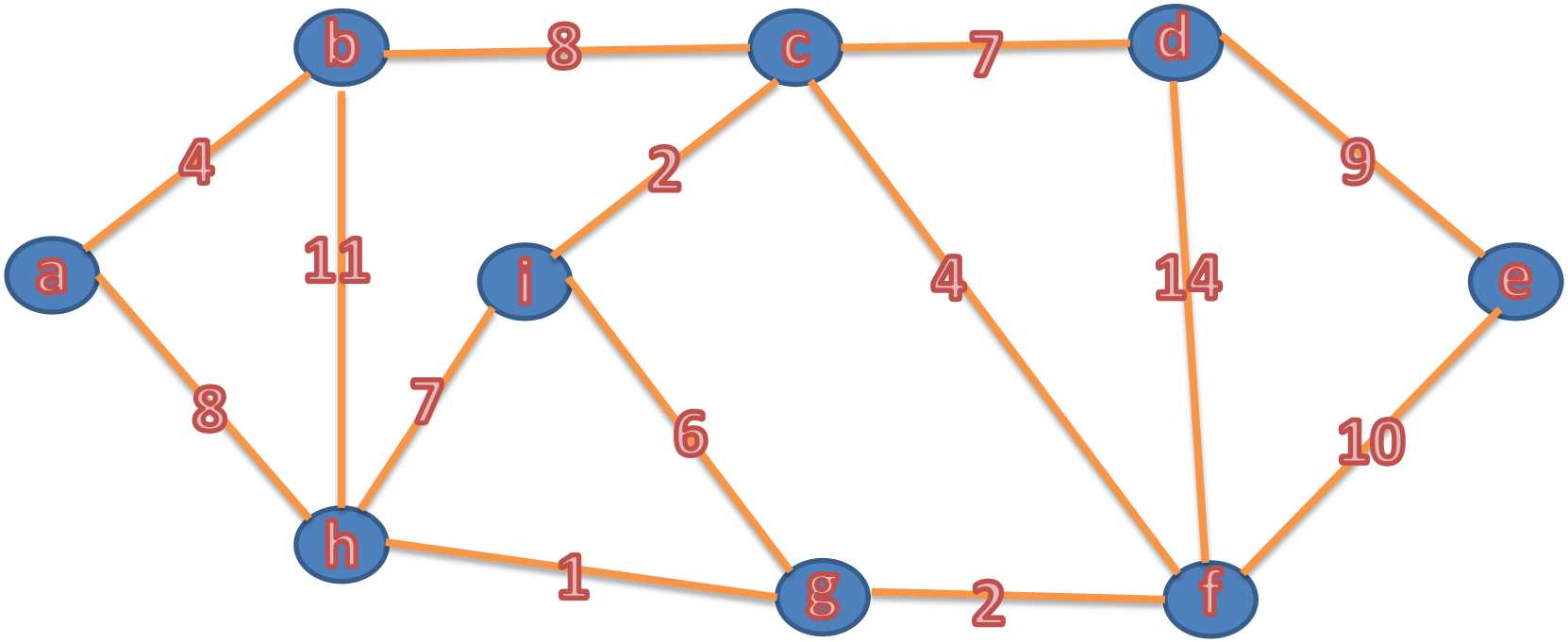
- Uses a greedy strategy
- You start with an arbitrary vertex v, choose the minimum edge that connects v to a new vertex w.
  - There's a extraction step and a relaxation step
- Apply the same process on {u,w} until you find the MST.

Prim's algorithm

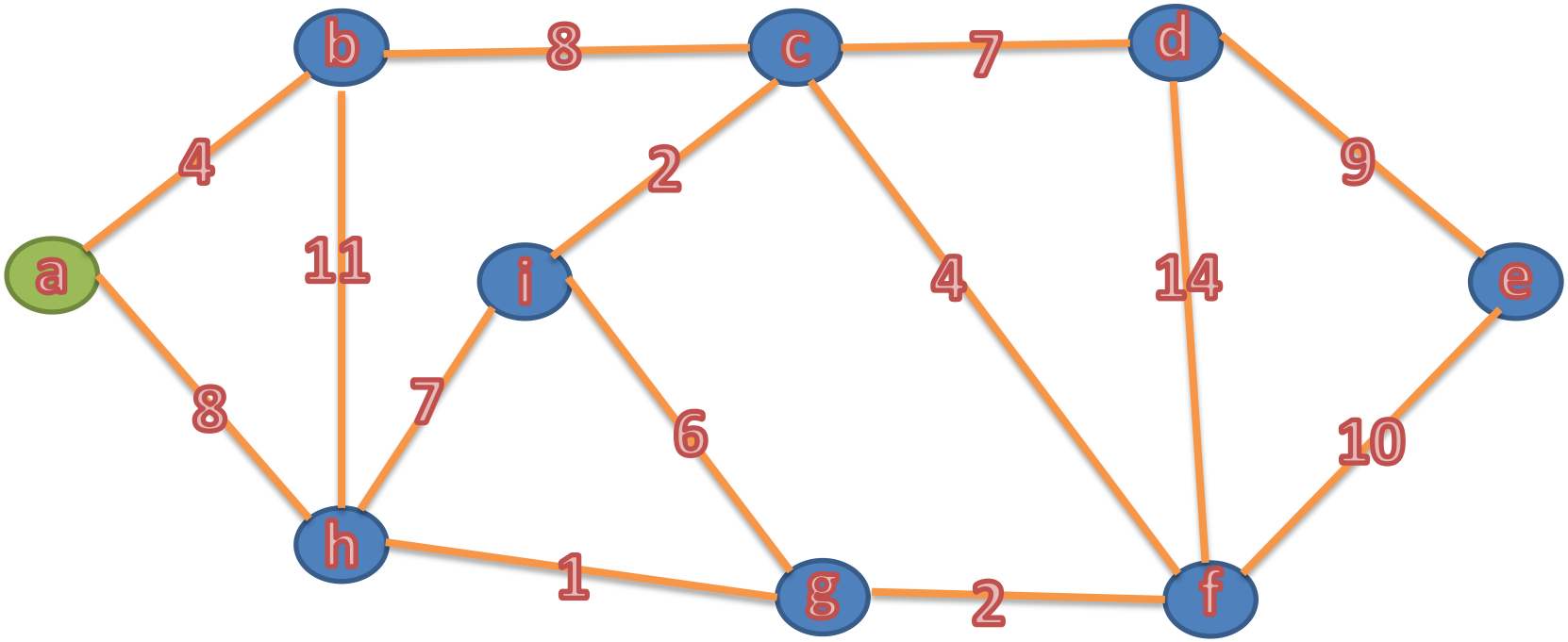
```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

- G – Graph
- *w – weight function*
- *r – arbitrary root node*
- *u.*π - parent of node u in the tree
- *v.key – minimum weight of edge connecting v to a vertex in the tree. This is infinity if no such edge exists*

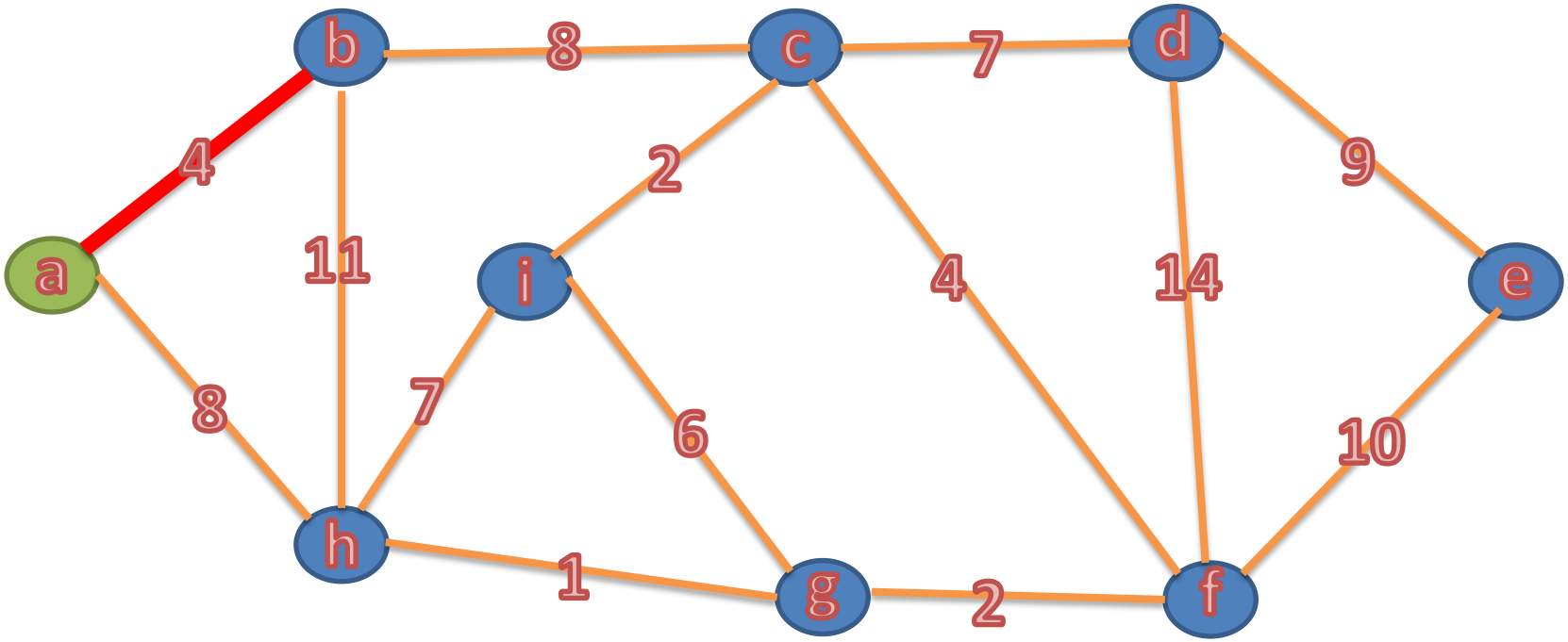
Prim’s algorithm



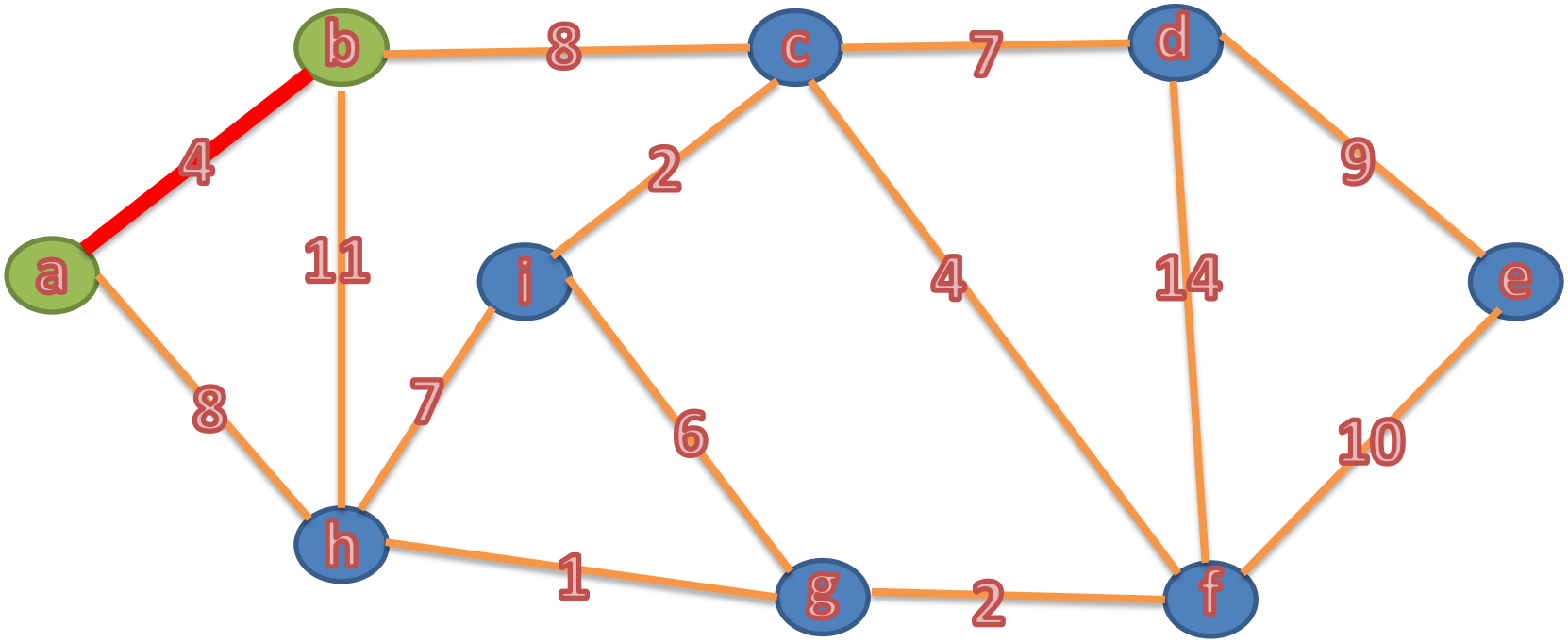
Prim's algorithm



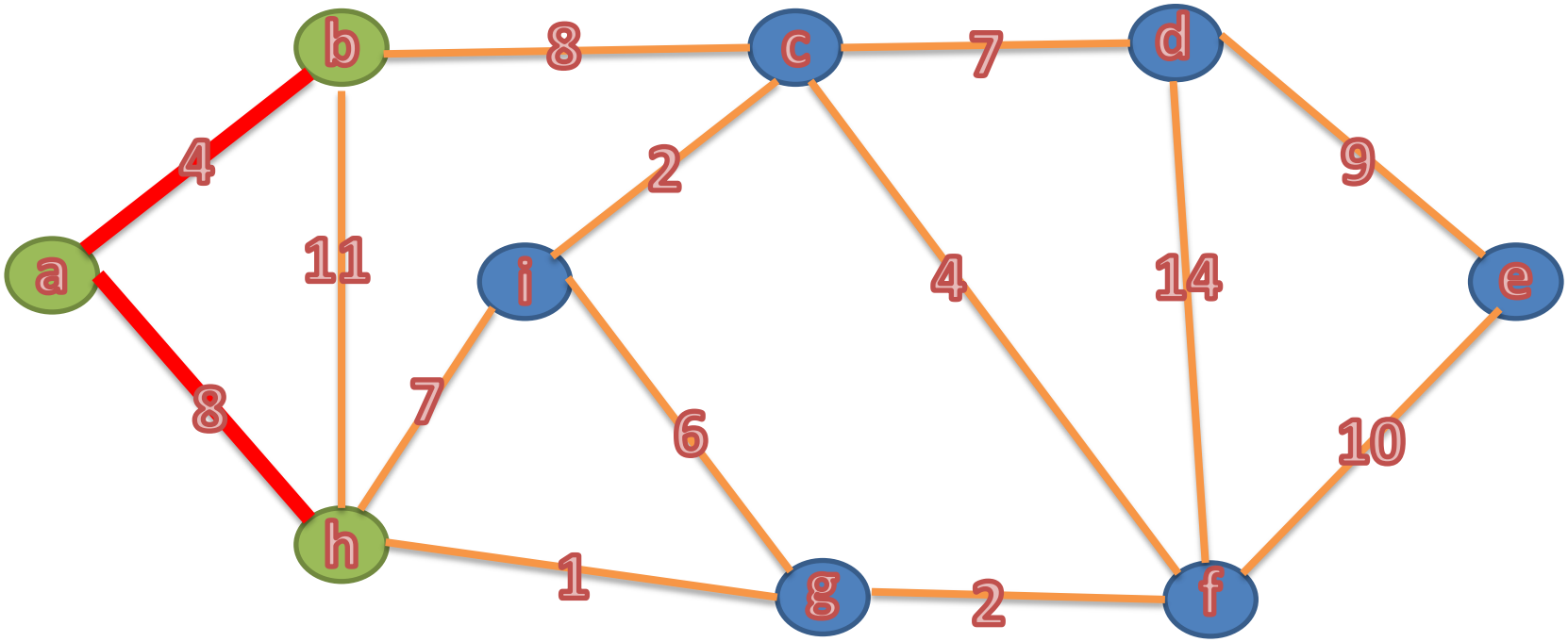
Prim’s algorithm



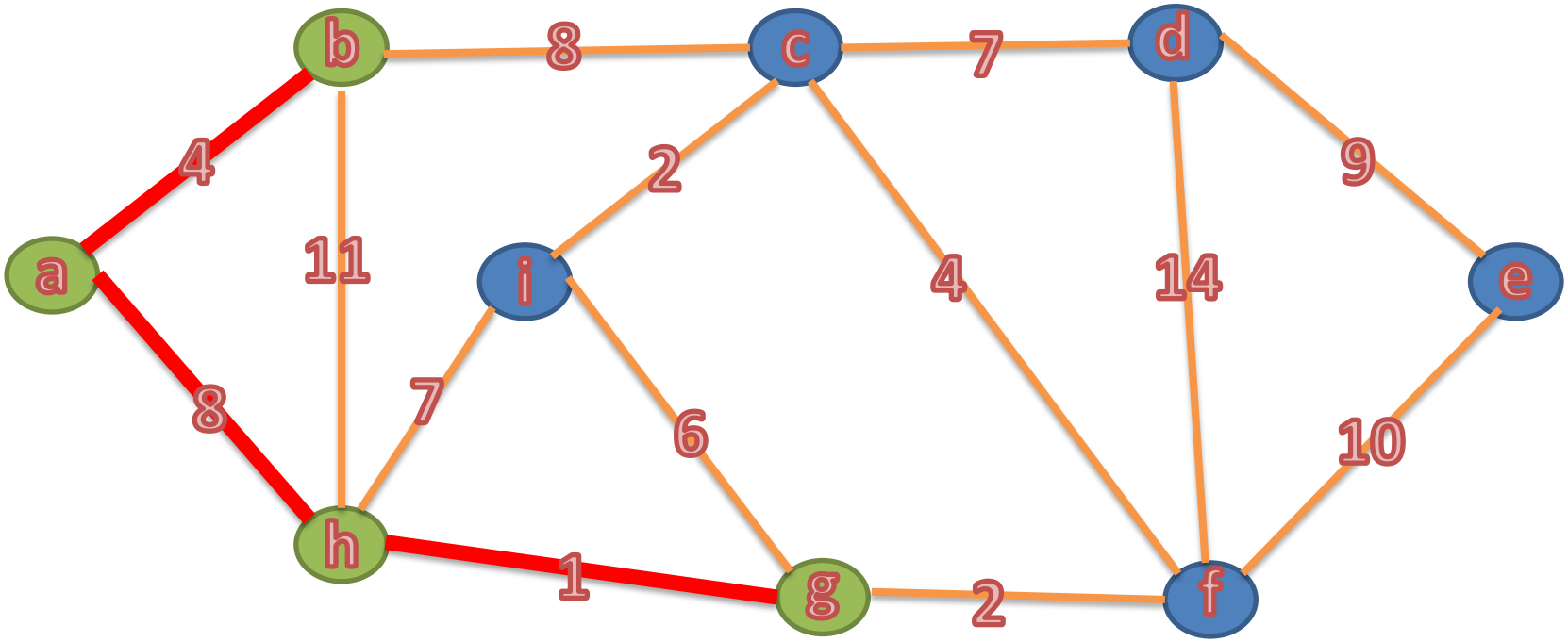
Prim’s algorithm



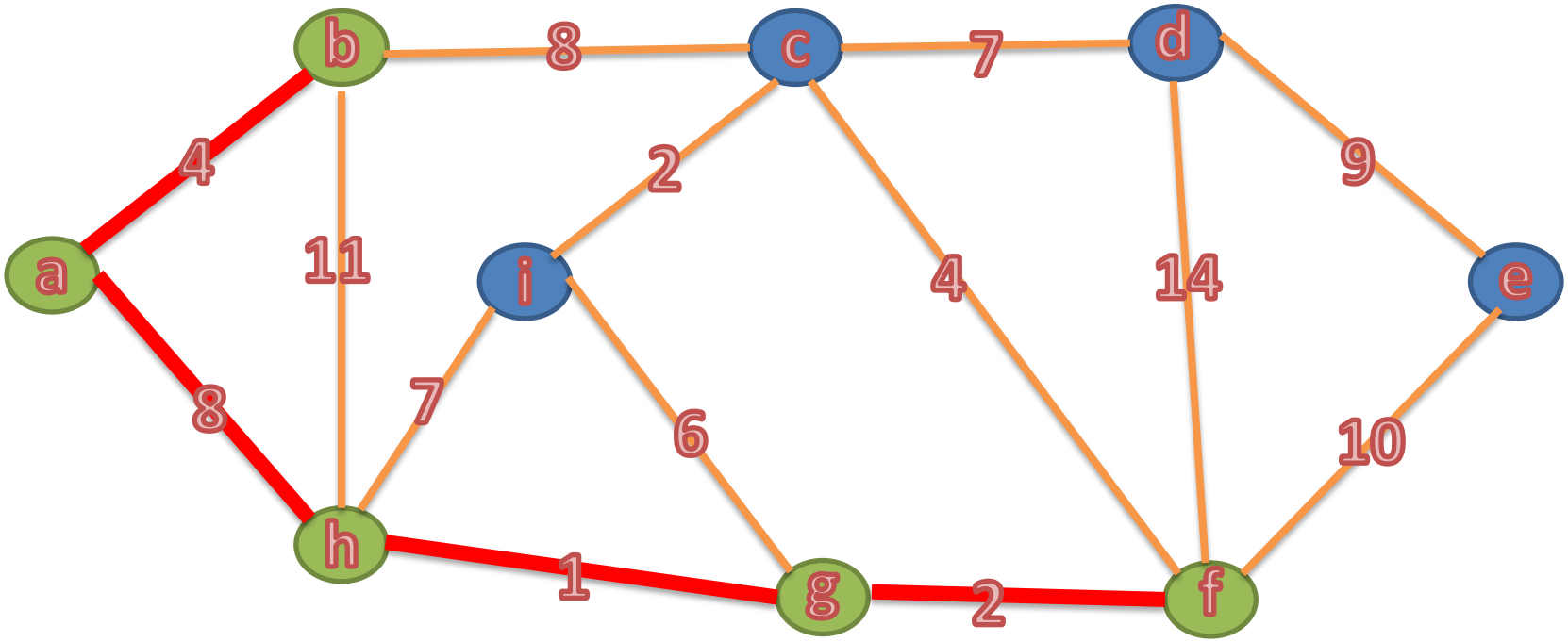
Prim's algorithm



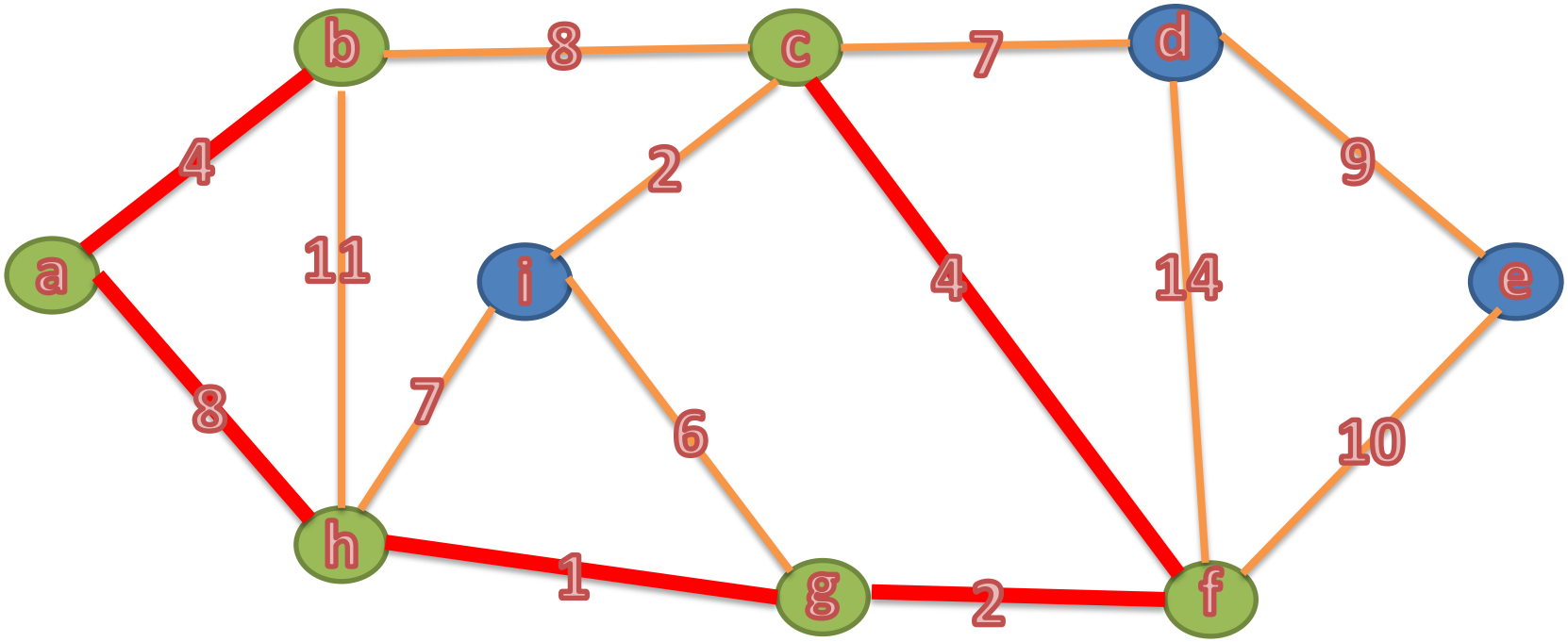
Prim’s algorithm



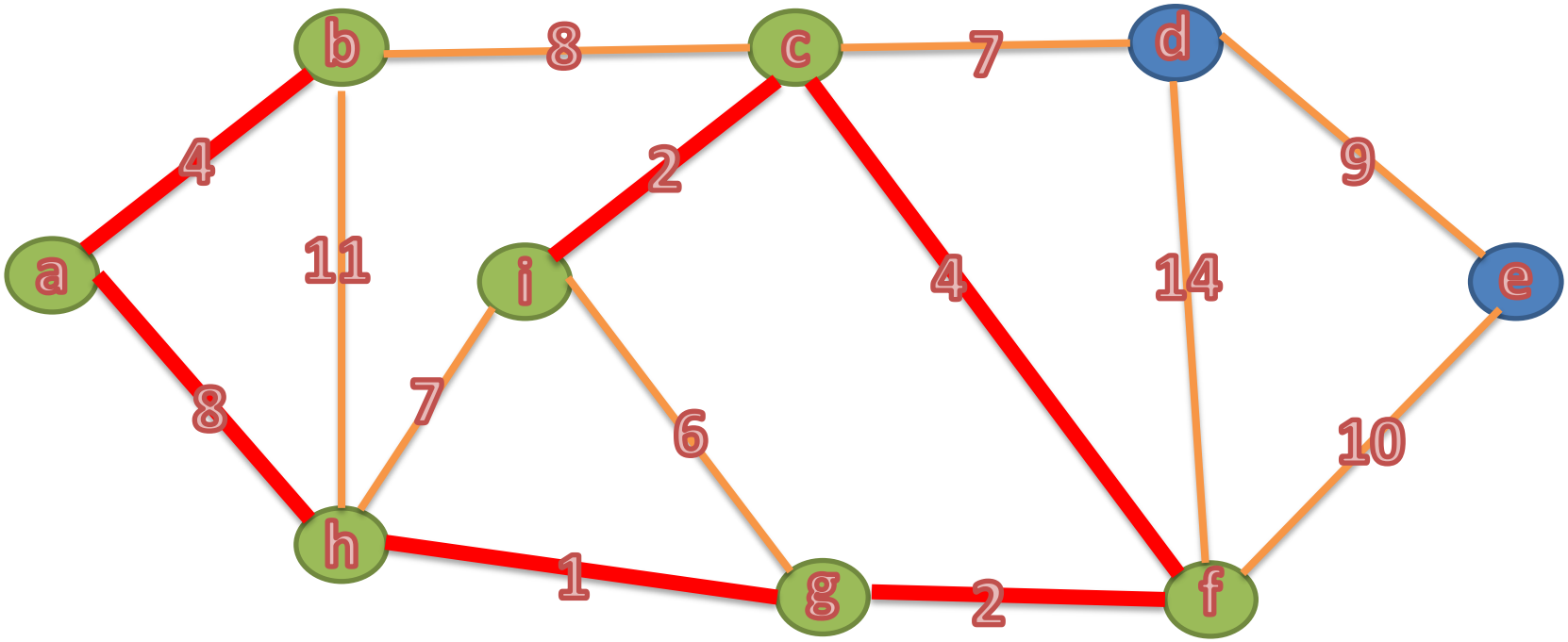
Prim’s algorithm



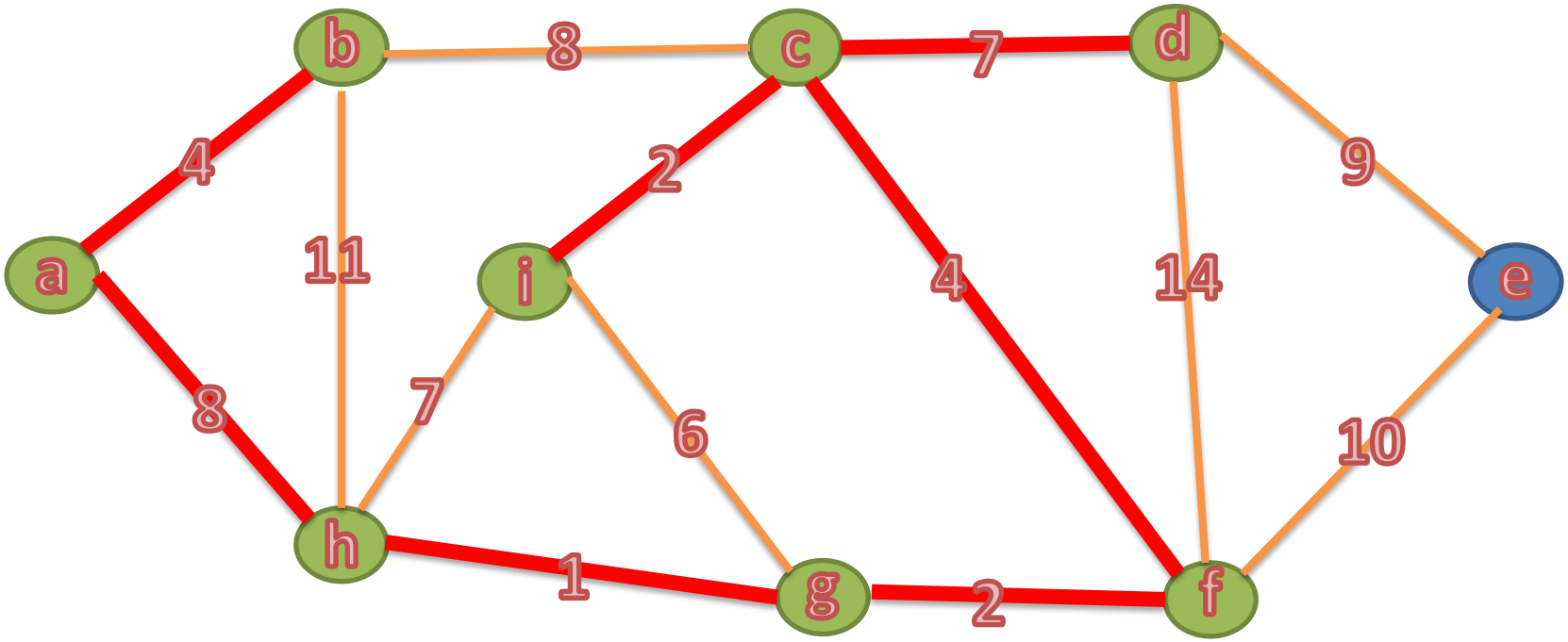
Prim’s algorithm



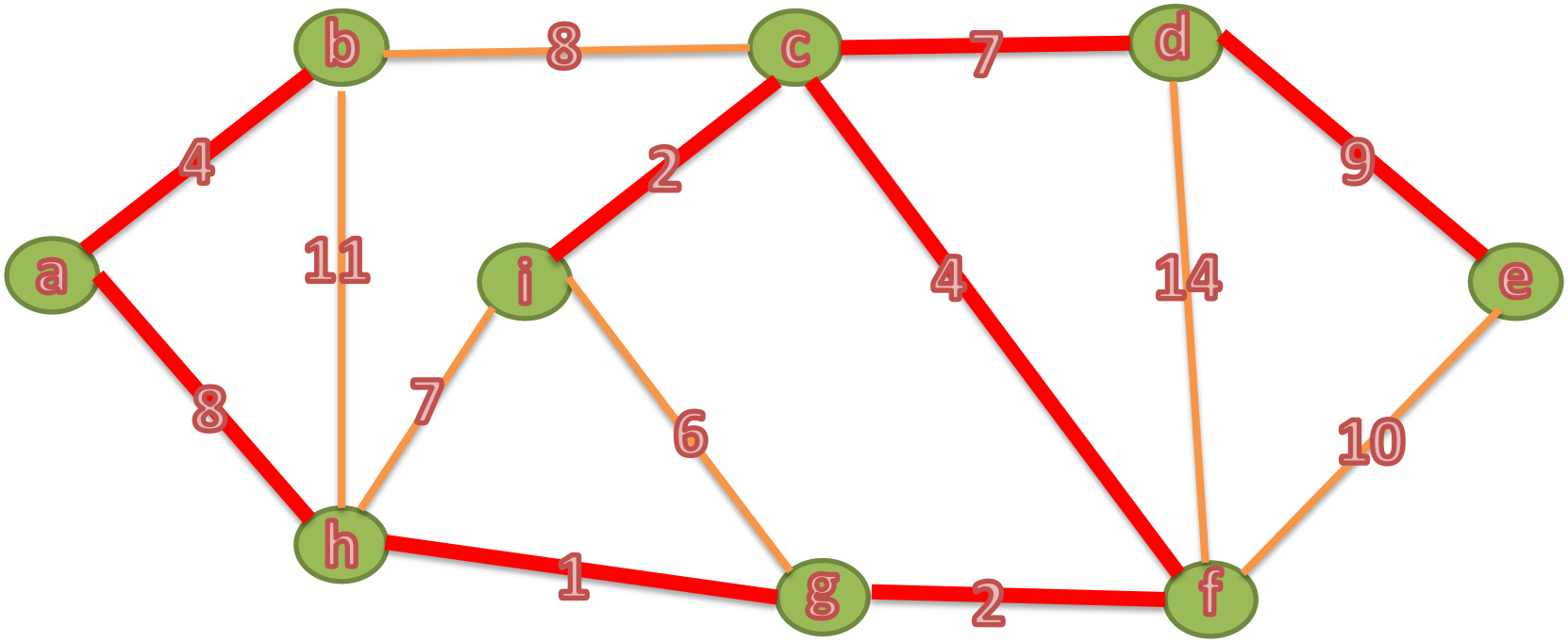
Prim’s algorithm



Prim’s algorithm



Prim’s algorithm



Prim's algorithm

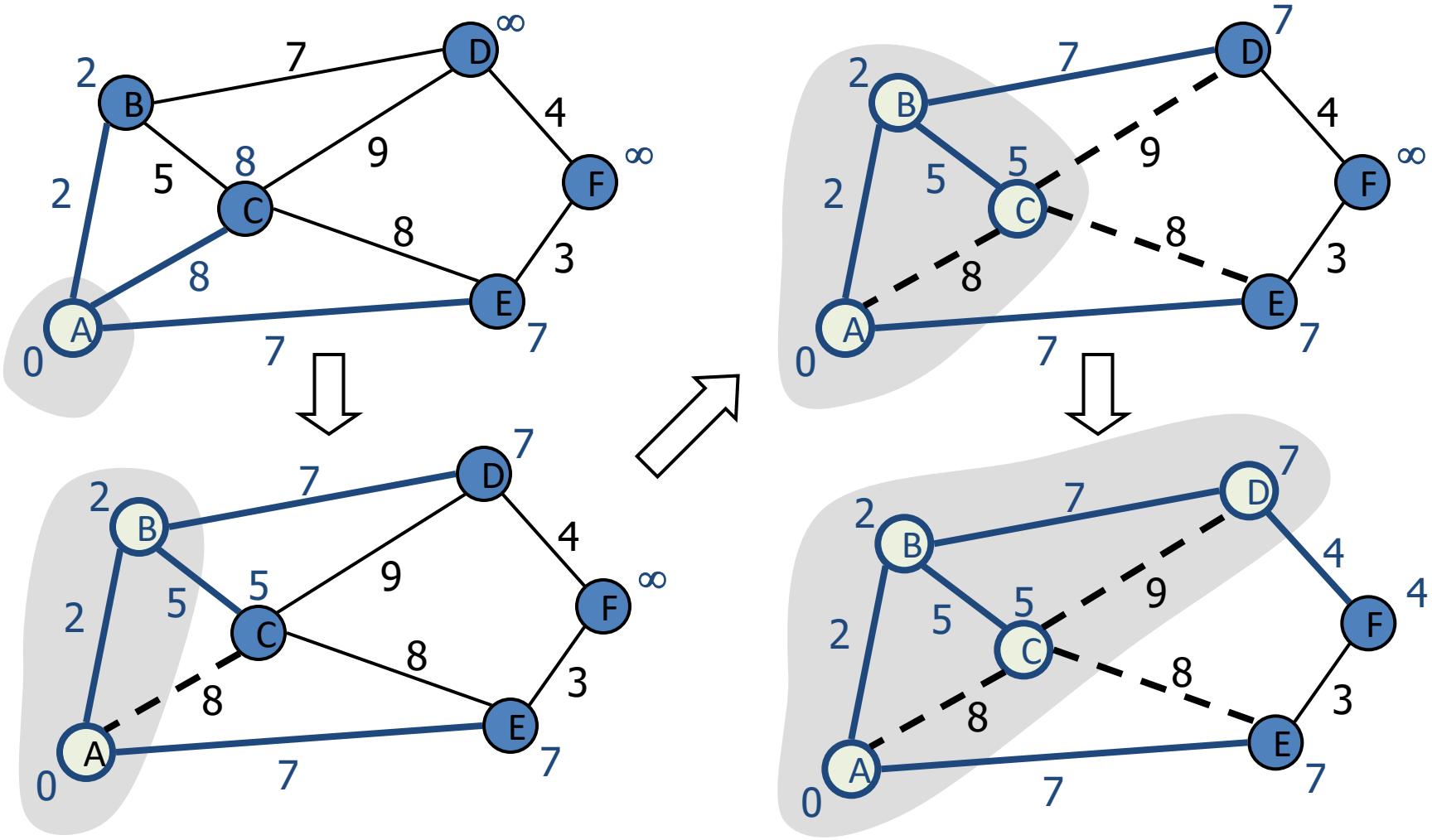
- Order:

Minimum edge weight data structure	Time complexity (total)
adjacency matrix, searching	$O(\|V\|^2)$
binary heap and adjacency list	$O((\|V\| + \|E\|) \log \|V\|) = O(\|E\| \log \|V\|)$
Fibonacci heap and adjacency list	$O(\|E\| + \|V\| \log \|V\|)$

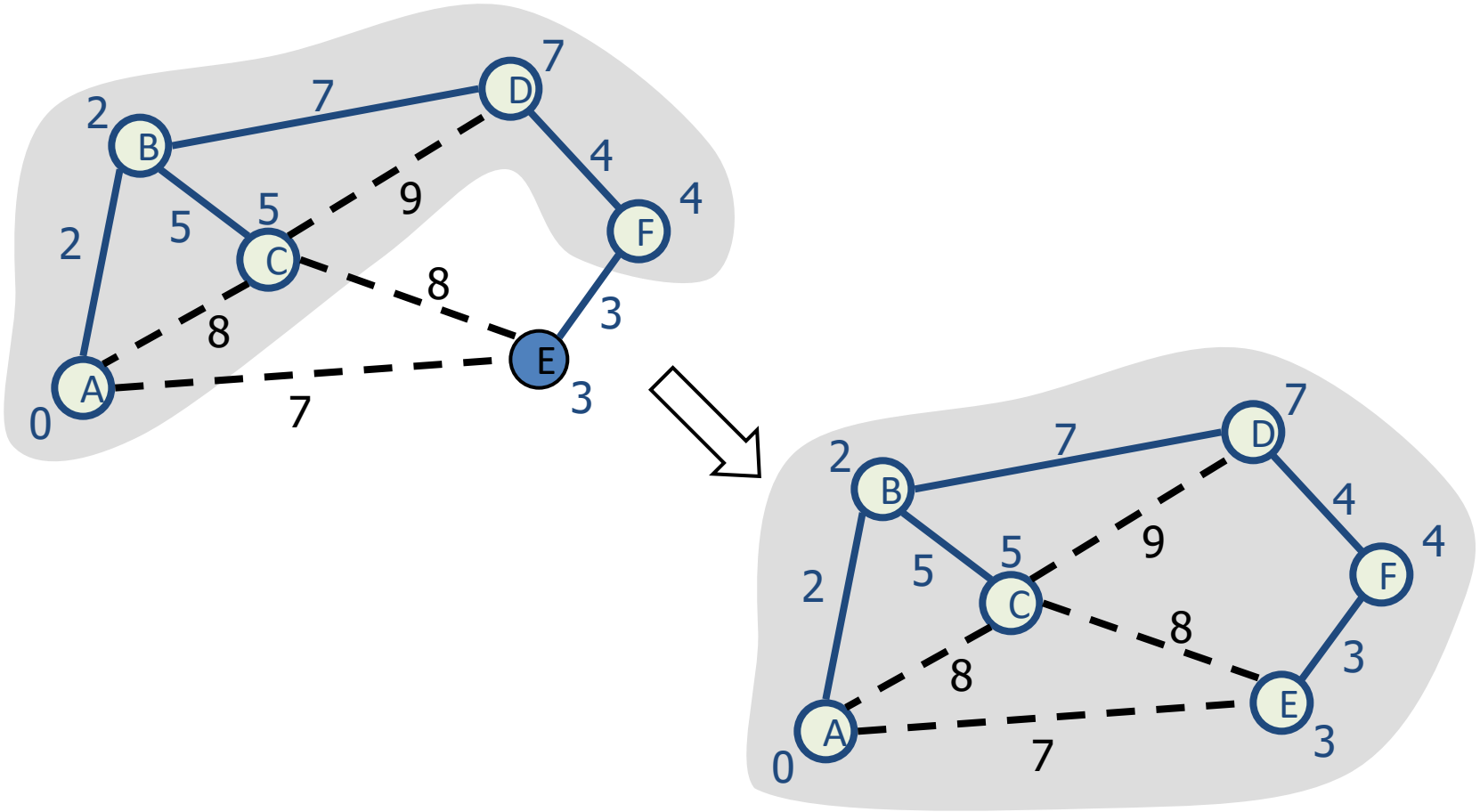
Prim's Algorithm Review

- Pick an arbitrary vertex ***s*** and grow the MST as a cloud of vertices, starting from ***s***
- Store with each vertex ***v*** label ***d***(***v***) representing the smallest weight of an edge connecting ***v*** to a vertex in the cloud
- At each step:
  - We add to the cloud the vertex ***u*** outside the cloud with the smallest distance label
  - Make sure the result is a tree that is no cycle.
  - We update the labels of the vertices adjacent to ***u***

Prim's Algorithm for MST - Example



Example (contd.)



Analysis

- Graph operations
  - We cycle through the incident edges once for each vertex
- Label operations
  - We set/get the distance, parent and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex w in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time

- Prim's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure. Recall that sum of all degrees equals $2m$. The running time is $O(m \log n)$ since the graph is connected

Kruskal's Algorithm

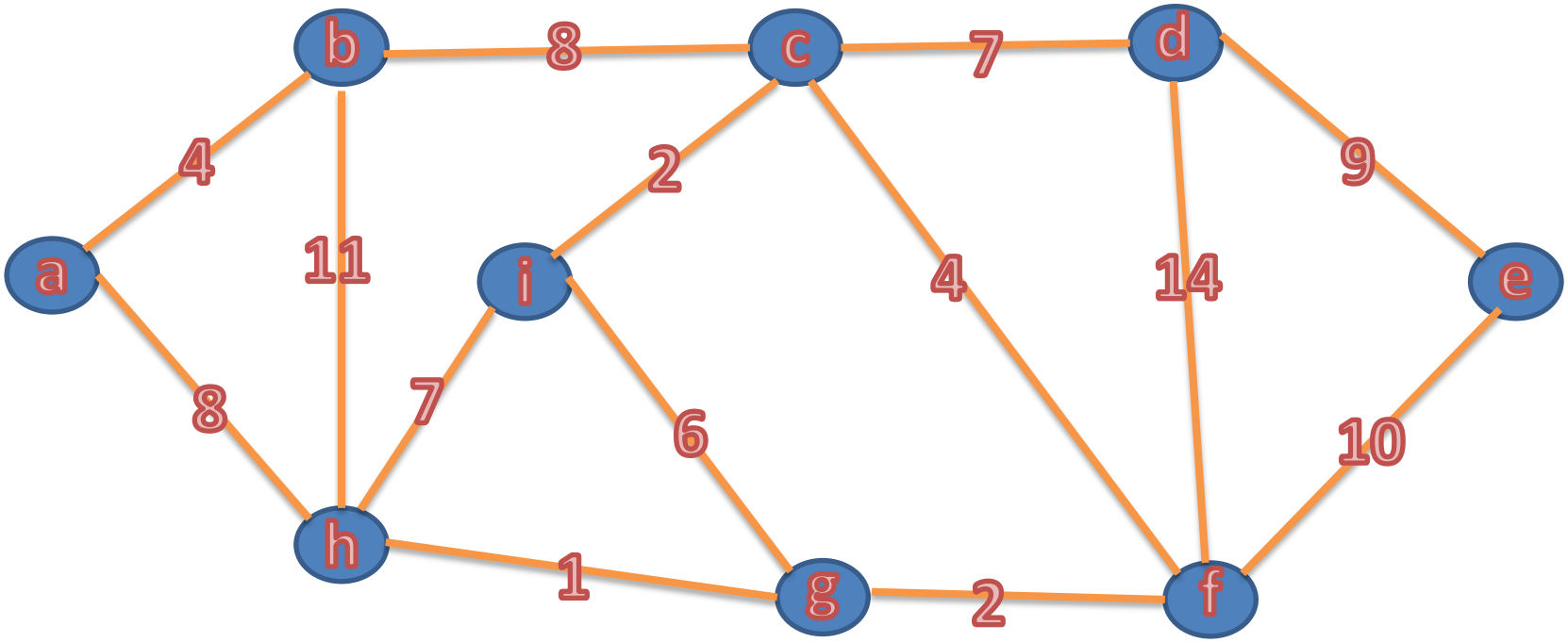
- In Kruskal's algorithm, edges are scanned once
- An edge with least weight is picked and added to vertices
- If an edge creates a cycle, it is deleted

Kruskal's Algorithm

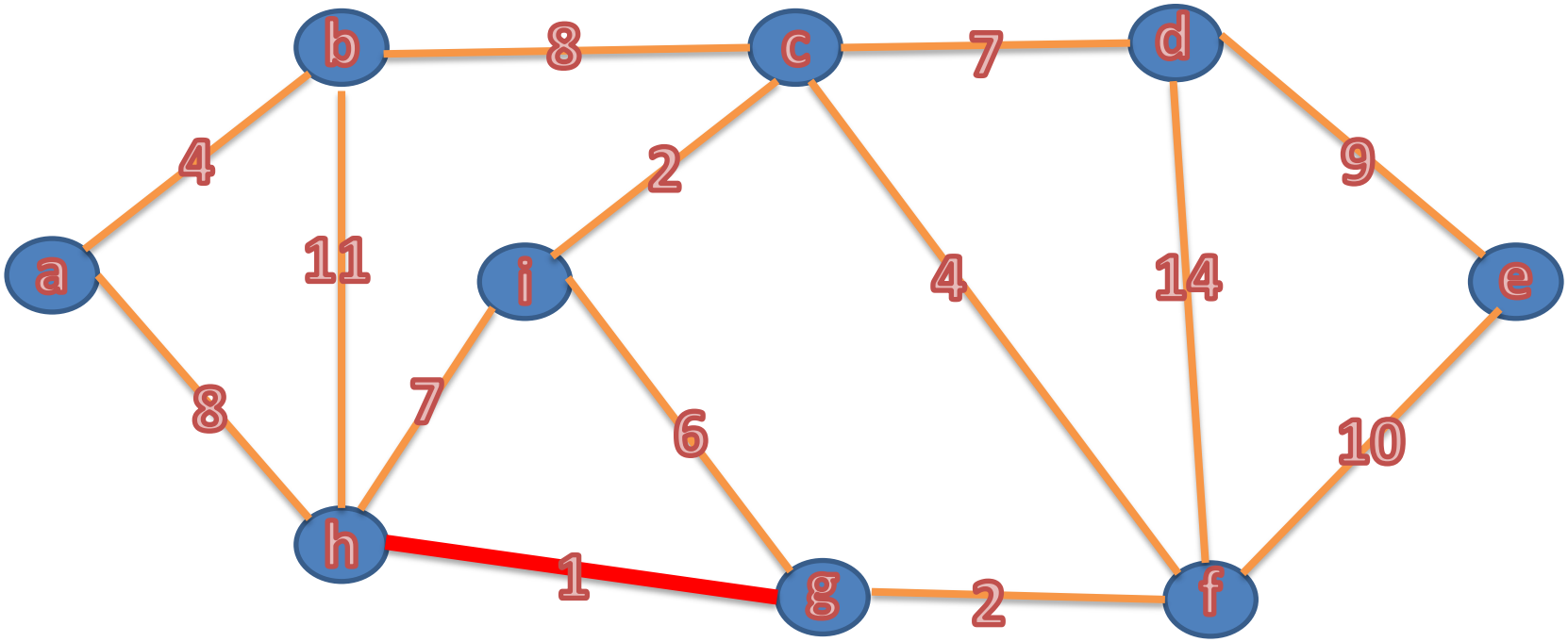
- G – Graph
- *w – weight function*
- *MAKE-SET(v) – Create a set containing node v*
- *FIND-SET(v) – Return a unique identifier for the set containing node v*
- *UNION(u,v) – merge the sets containing nodes u and v*

```
MST-KRUSKAL(G, w)
1  A = ∅
2  for each vertex v ∈ G.V
3       MAKE-SET(v)
4  sort the edges of G.E into nondecreasing order by weight w
5  for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
6       if FIND-SET(u) ≠ FIND-SET(v)
7            A = A ∪ {(u, v)}
8            UNION(u, v)
9  return A
```

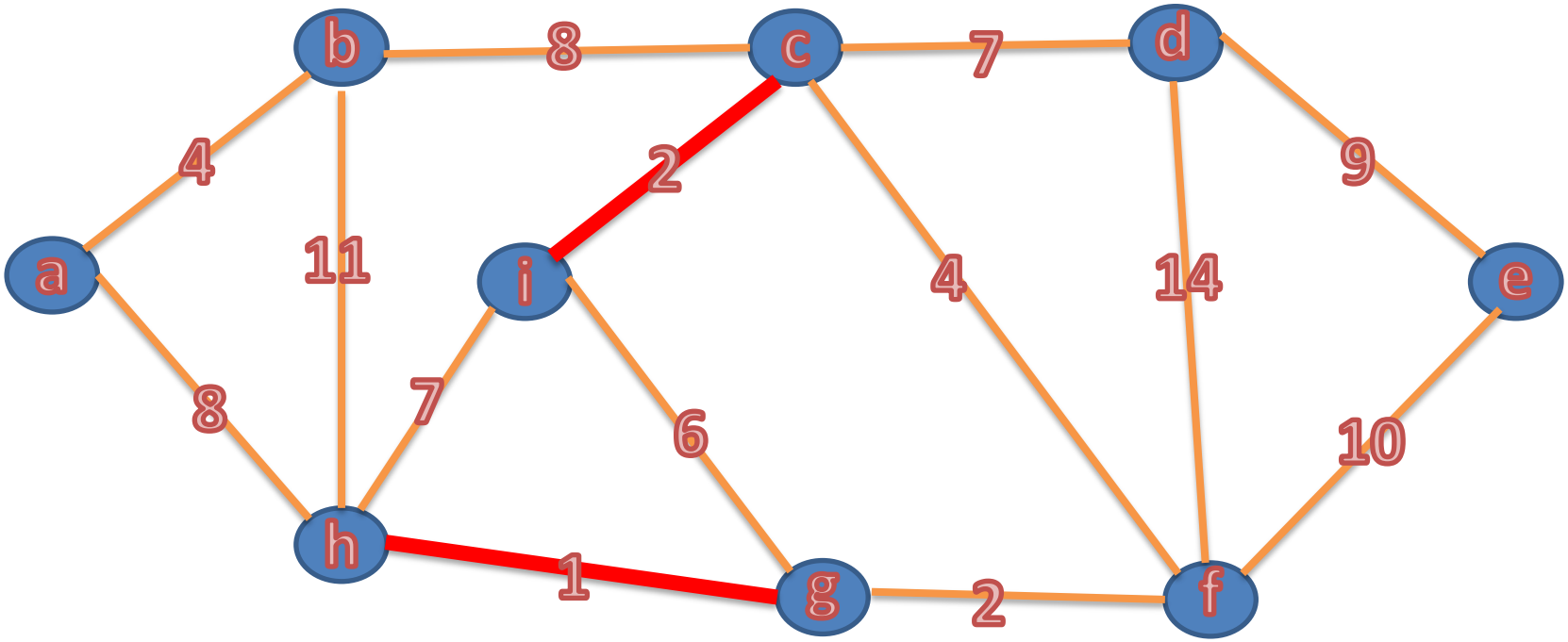
Kruskal's Algorithm



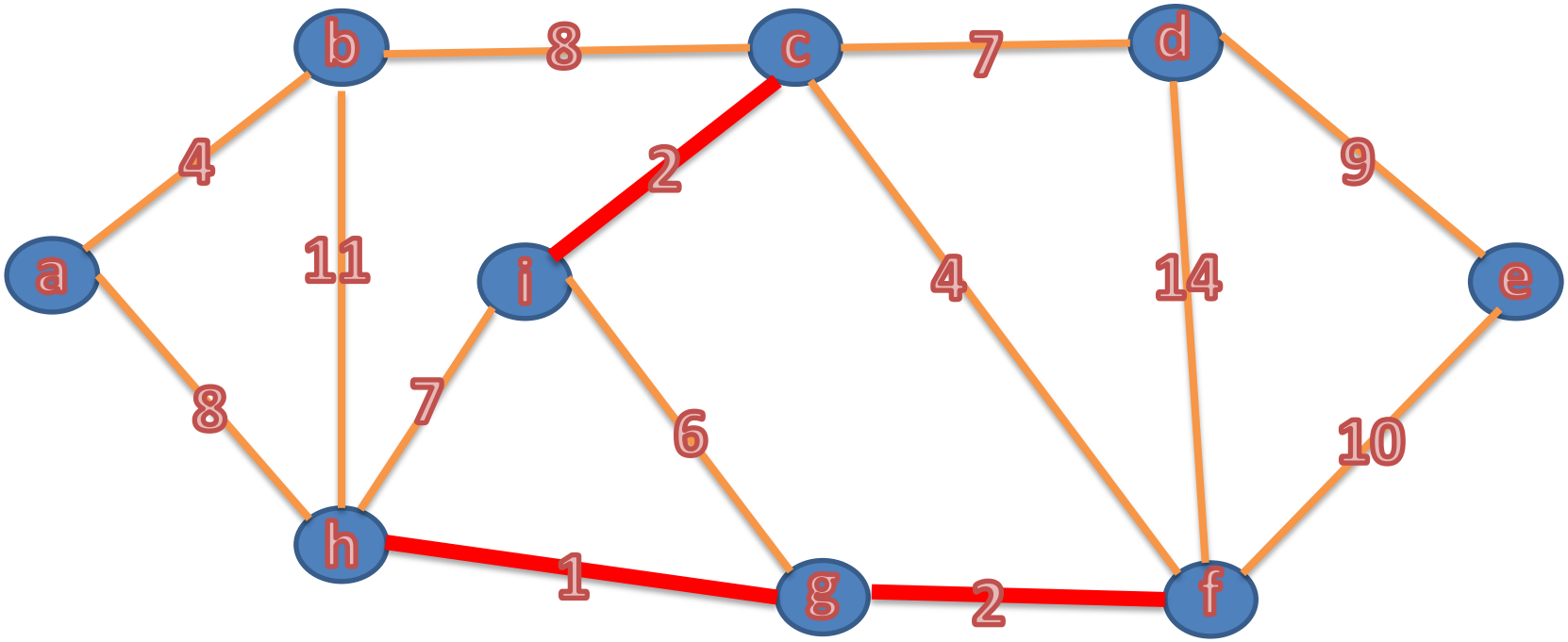
Kruskal's Algorithm



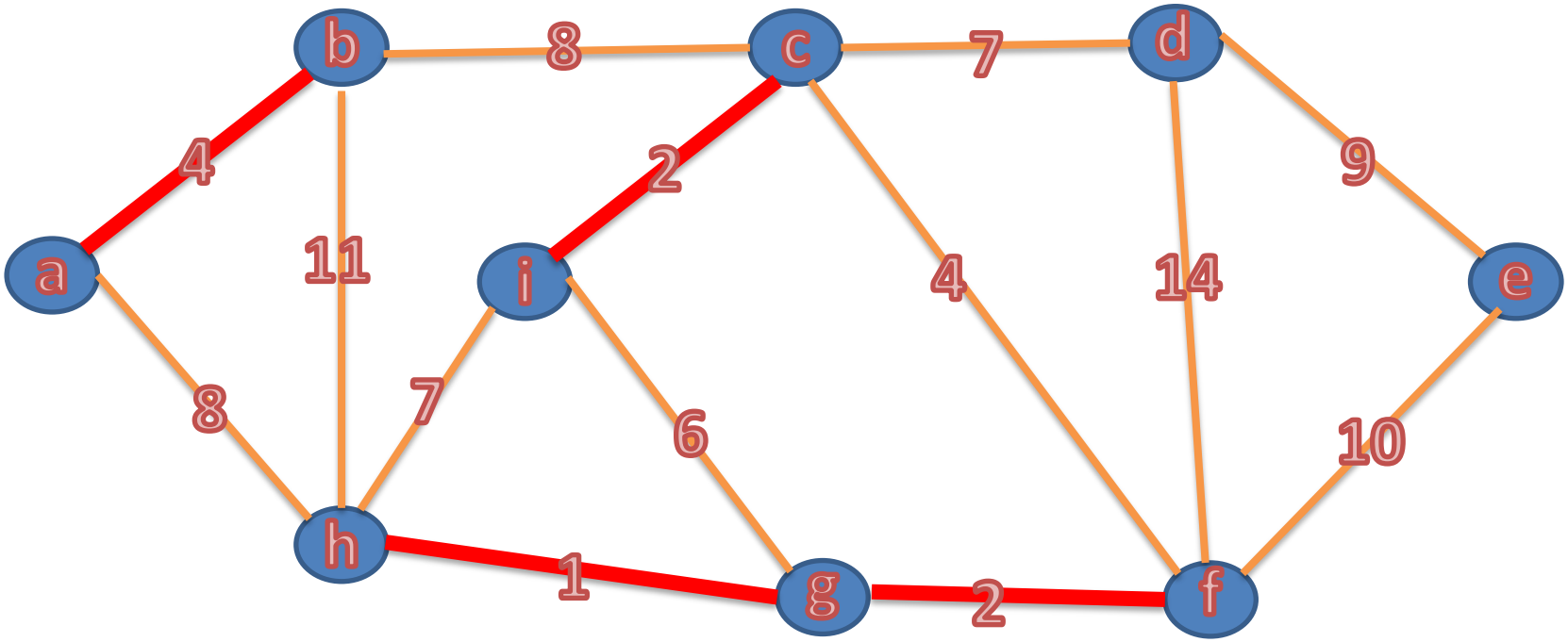
Kruskal's Algorithm



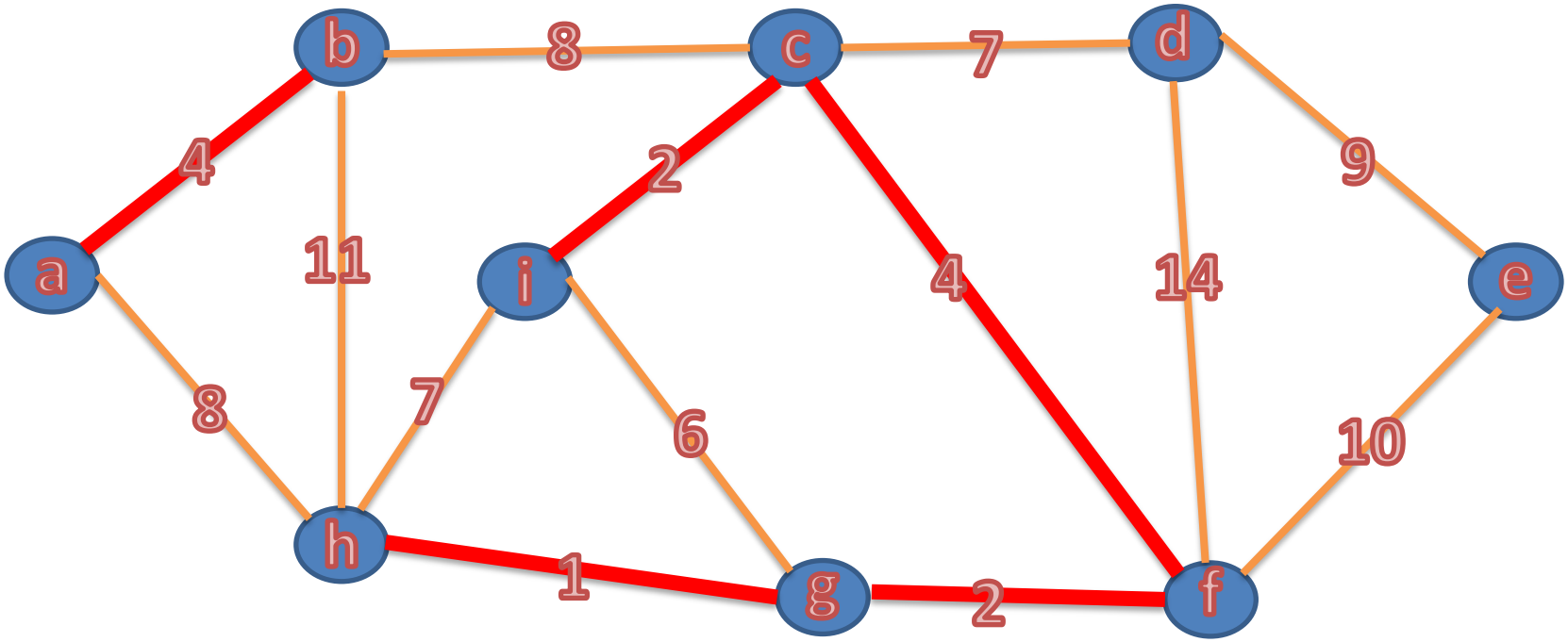
Kruskal's Algorithm



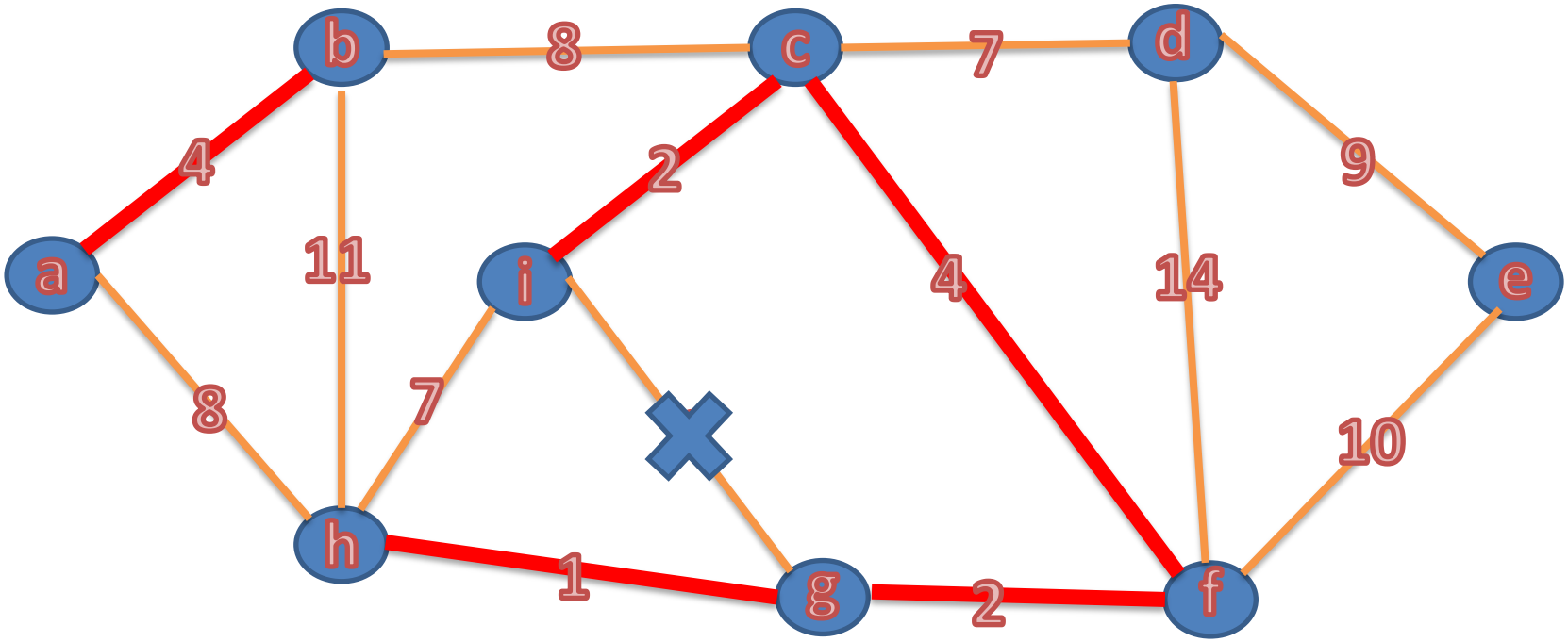
Kruskal's Algorithm



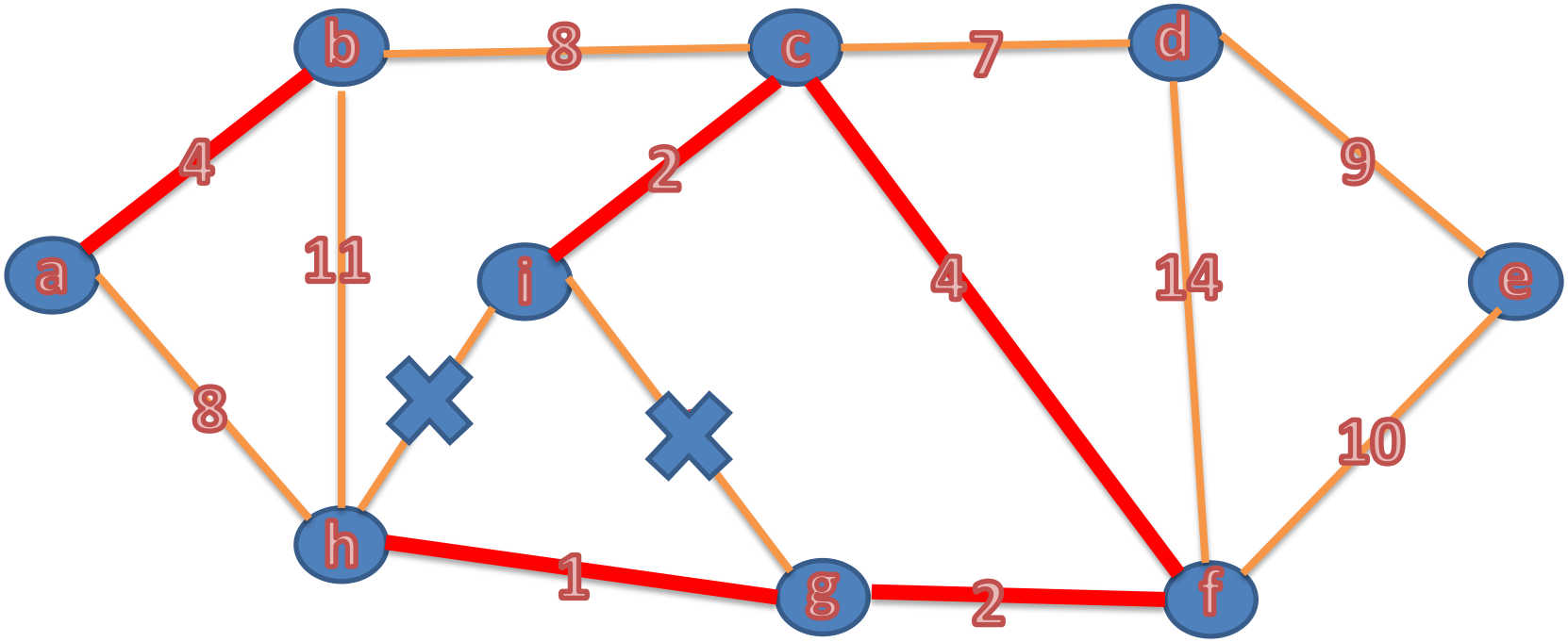
Kruskal's Algorithm



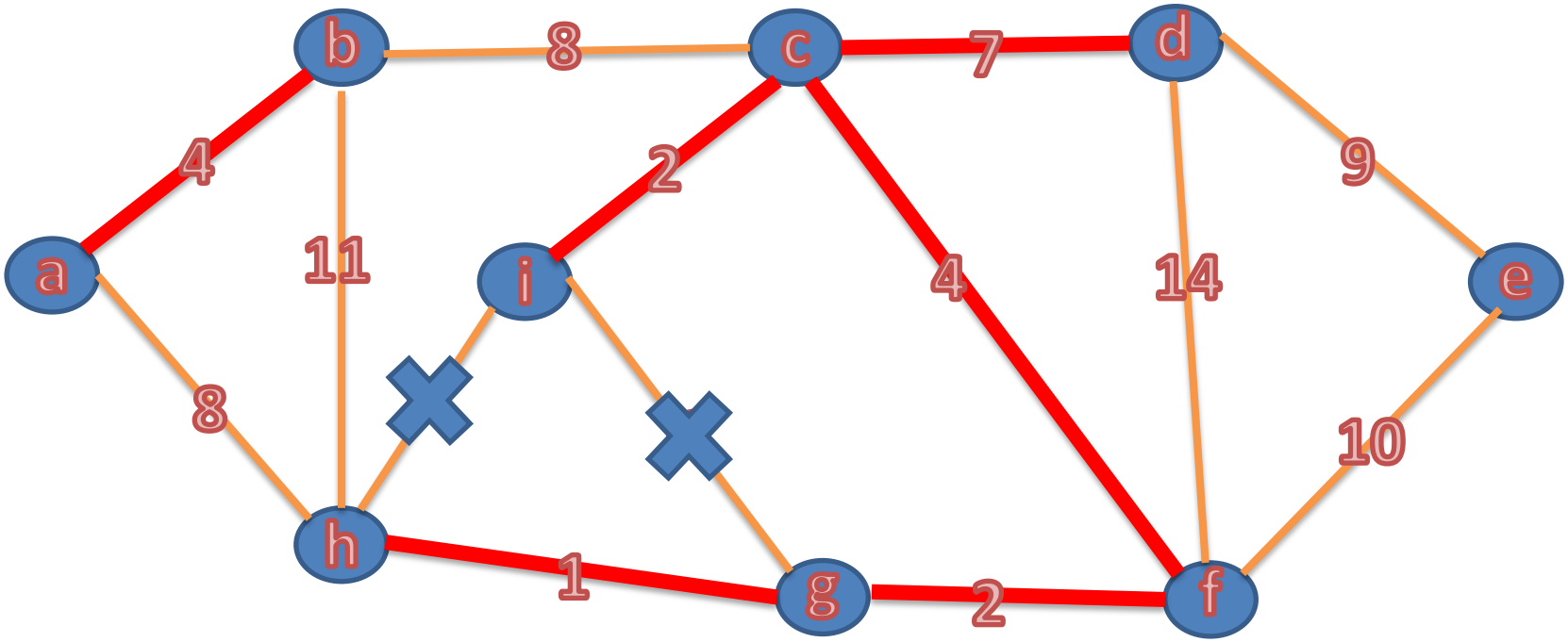
Kruskal's Algorithm



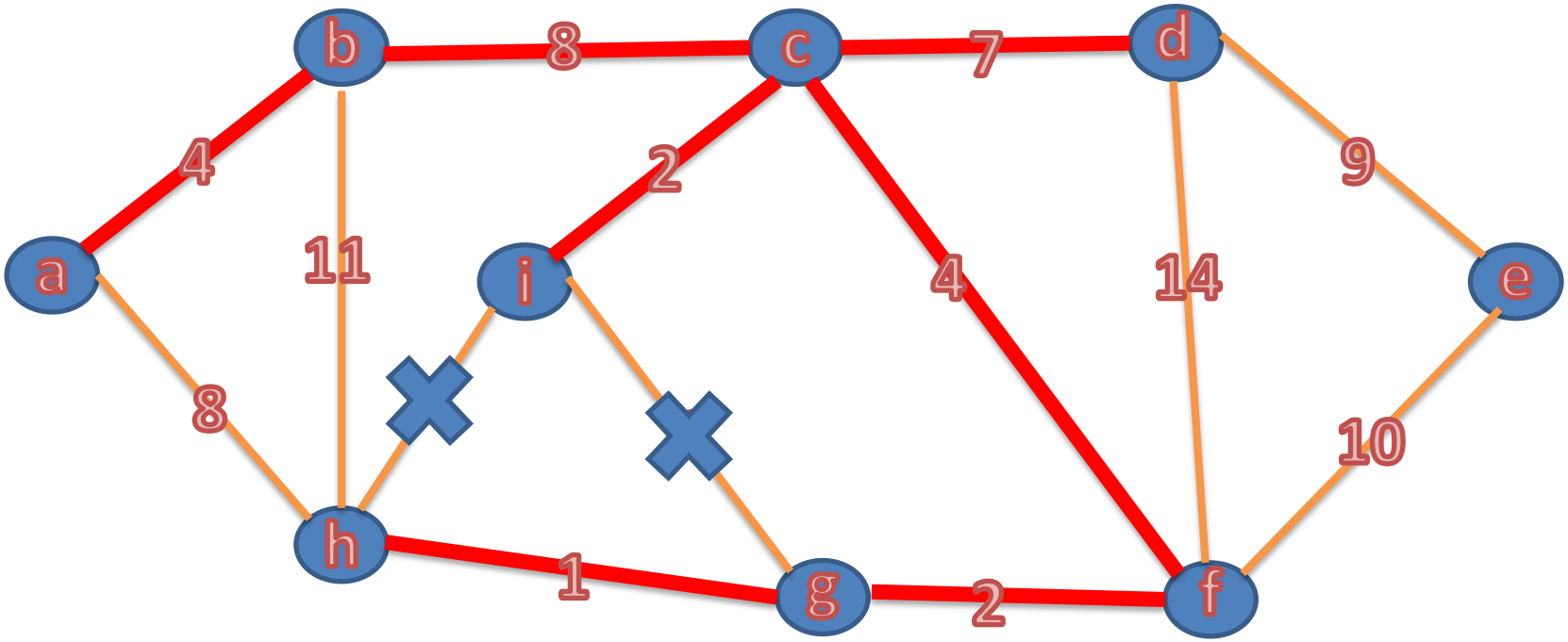
Kruskal's Algorithm



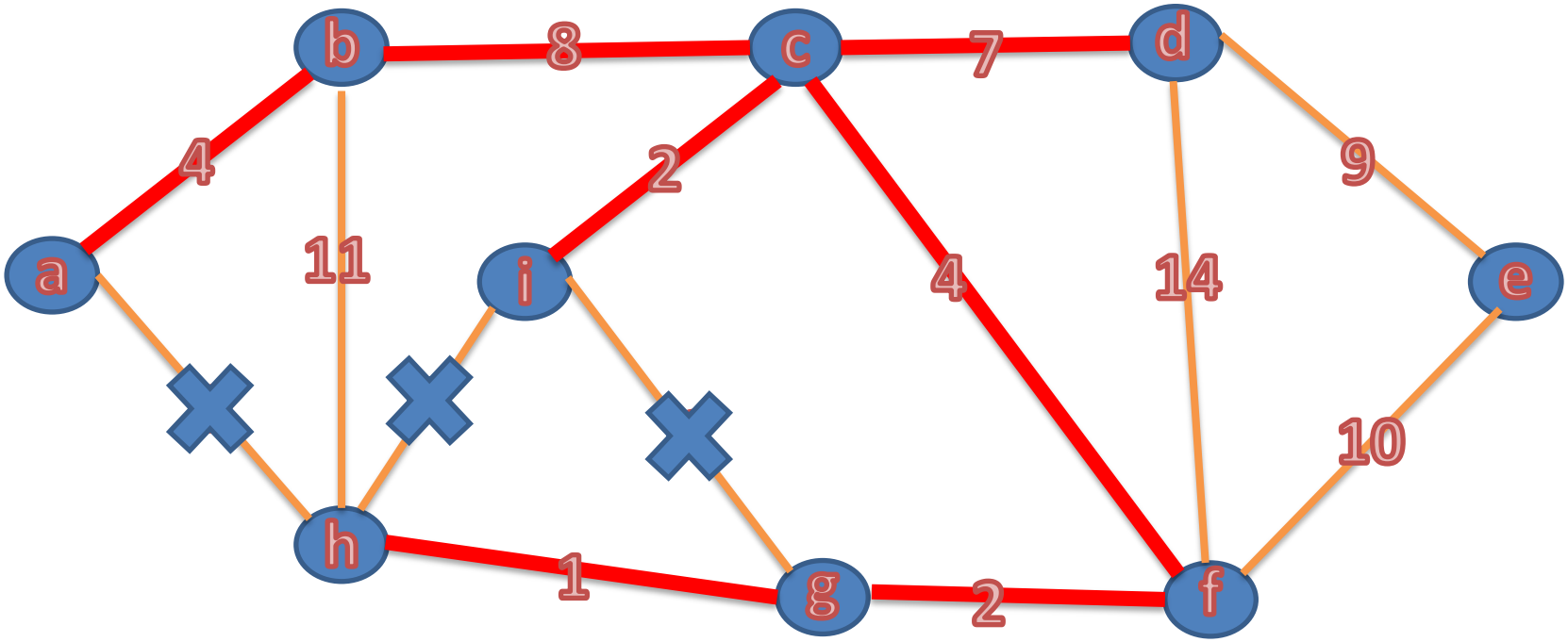
Kruskal's Algorithm



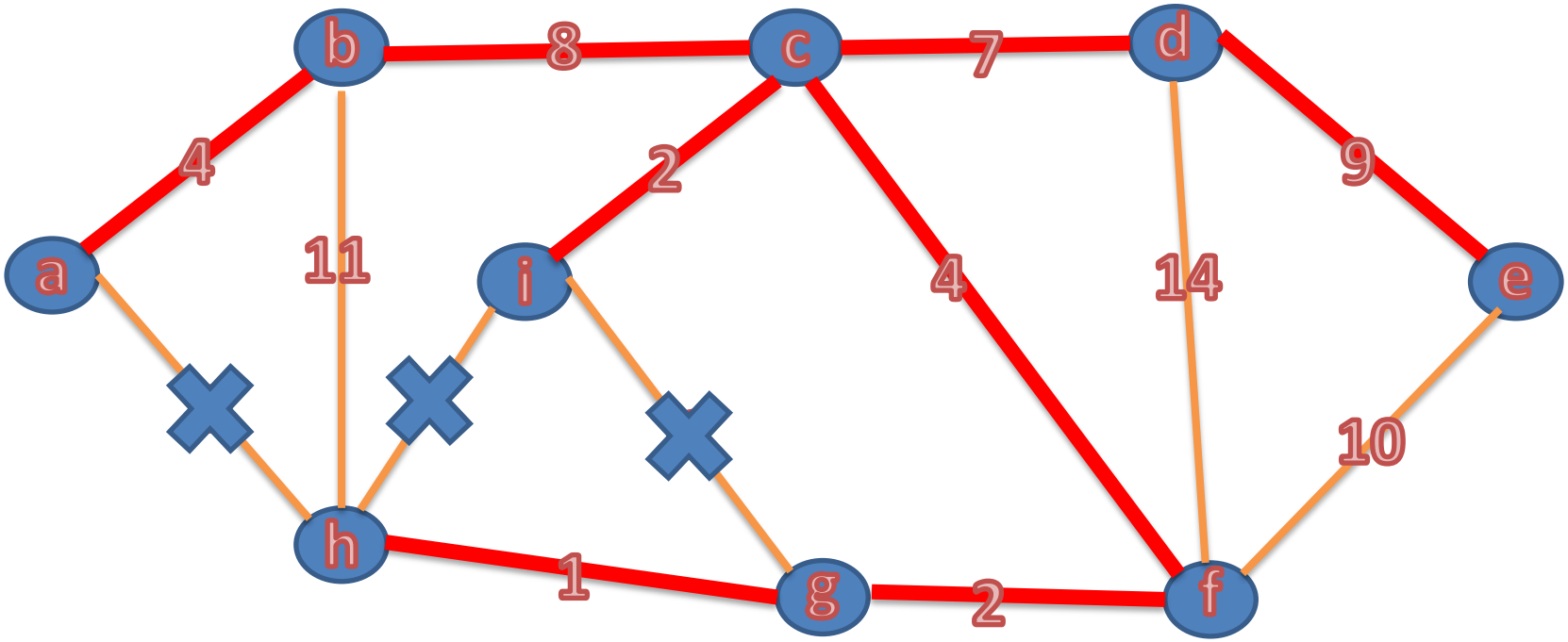
Kruskal's Algorithm



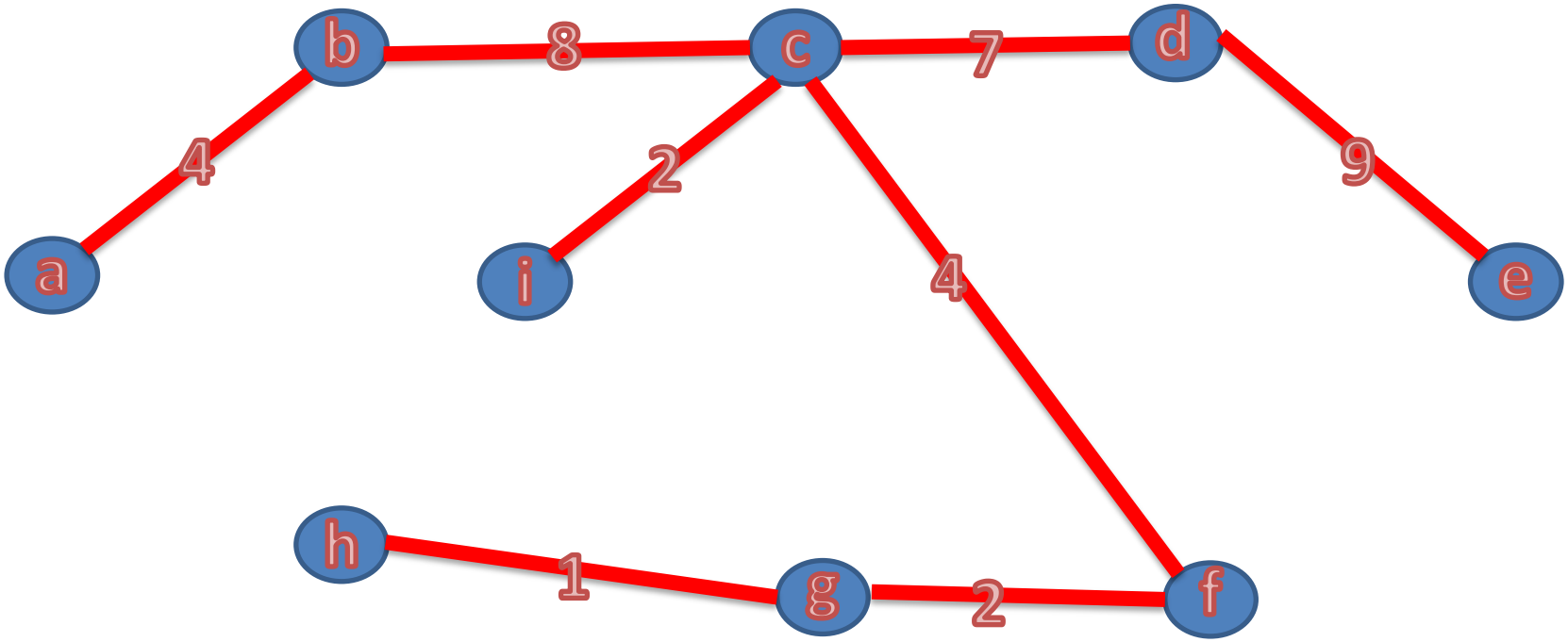
Kruskal's Algorithm



Kruskal's Algorithm



Kruskal's Algorithm



Kruskal's Algorithm

- Order: $\theta(ELogE)$

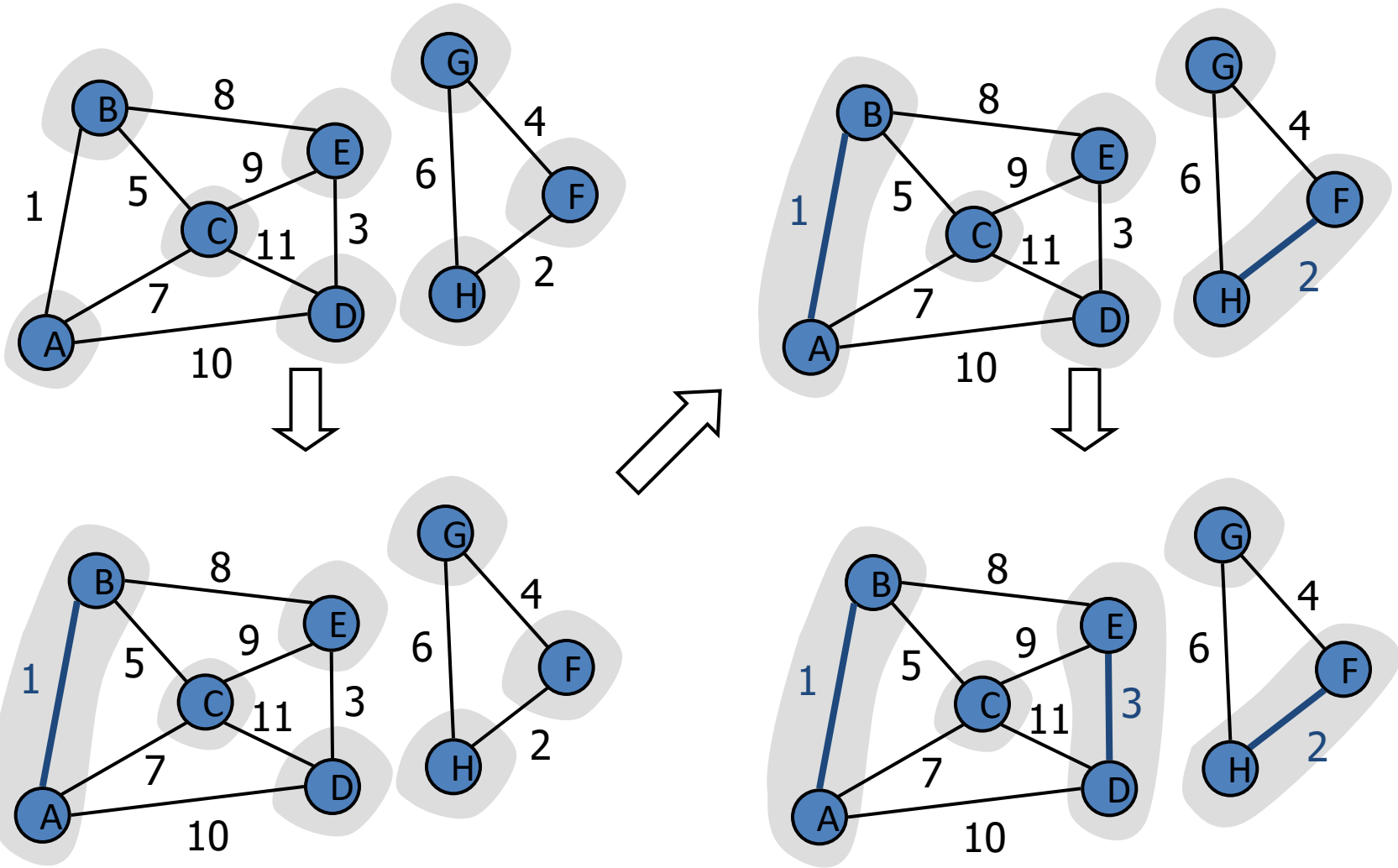
Kruskal's Algorithm Review

- Maintain a partition of the vertices into clusters
  - Initially, single-vertex clusters
  - Keep an MST for each cluster
  - Merge "closest" clusters and their MSTs
- A priority queue stores the edges outside clusters
  - Key: weight
  - Element: edge
- At the end of the algorithm
  - One cluster and one MST

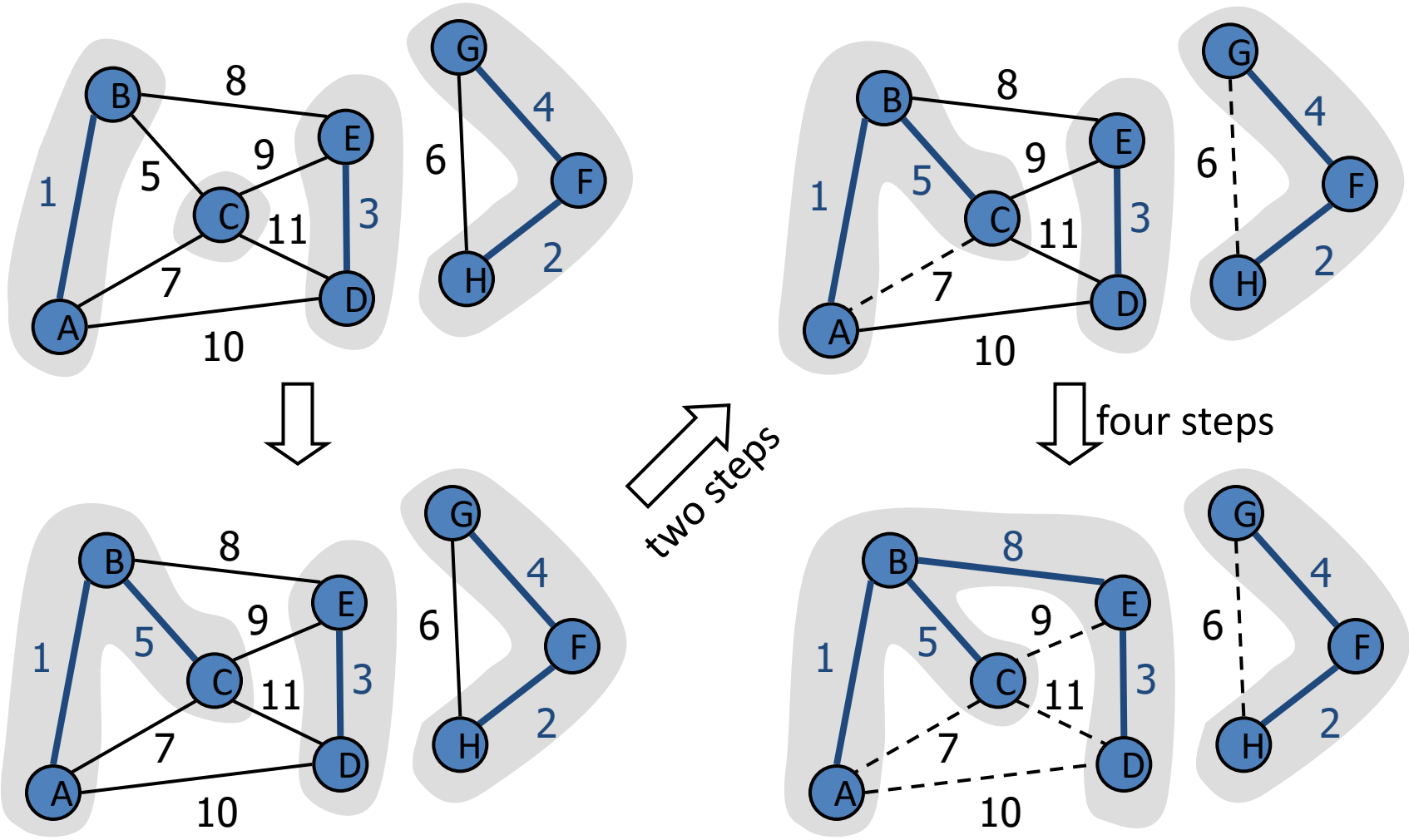
❑ The algorithm maintains a forest of trees

❑ A priority queue extracts the edges by increasing weight

Kruskal's Algorithm for MST - Example



Example (contd.)



Graph Traversal

- Depth First Search

- Breadth First Search

Graph Traversal - DFS

- Exploration of a vertex *v* is suspended as soon as a new vertex *u* is reached.
- Exploration of the new vertex *u* begins.
- Exploration of *v* continues after *u* has been explored.
- This can be expressed as a recursive algorithm

Depth-First Search (DFS)

- A general technique for traversing a graph G that
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G

- For a graph with n nodes and m edges it takes $O(n + m)$

- DFS can be extended to solve other graph problems
  - Find a path between two given vertices
  - Find a cycle in the graph

DFS Algorithm from a Vertex

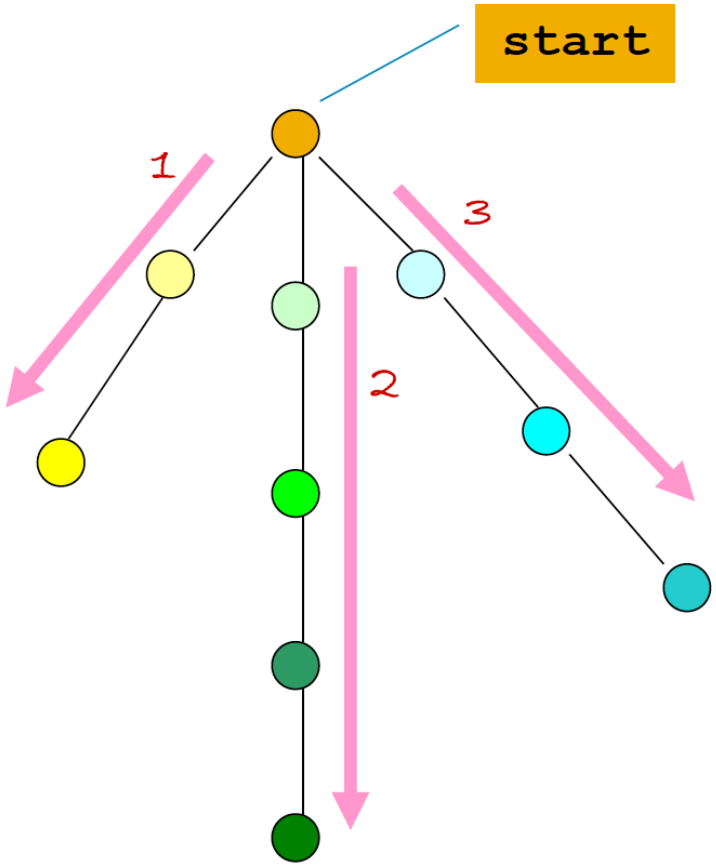
Algorithm DFS(G, u):

Input: A graph G and a vertex u of G

Output: A collection of vertices reachable from u, with their discovery edges

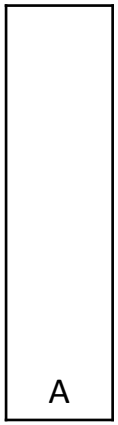
Mark vertex u as visited.
for each of u's outgoing edges, $e = (u, v)$ **do**
  if vertex v has not been visited **then**
    Record edge e as the discovery edge for vertex v.
    Recursively call DFS(G, v).

- Visit a vertex, *v*, move from *v* as deeply as possible
- Use a stack to store nodes
  - Stacks are LIFO
- DFS:
  - visit and push start
  - while (*s* not empty)
  - peek at node, *nd*, at top of *s*
  - if *nd* has an unvisited neighbour visit it and push it onto *s*
  - else pop *nd* from *s*

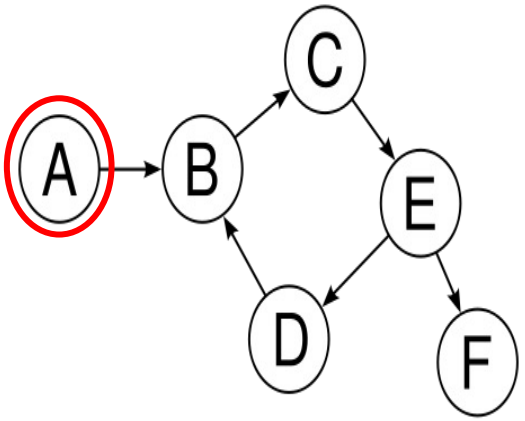


Graph Traversal - DFS

Stack

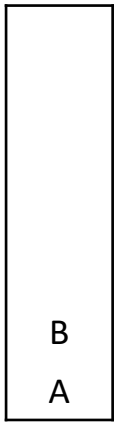


Output: A

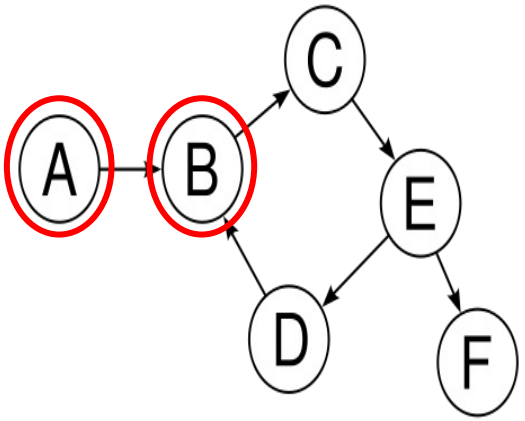


Graph Traversal - DFS

Stack

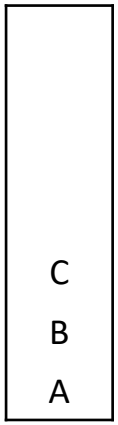


Output: A, B

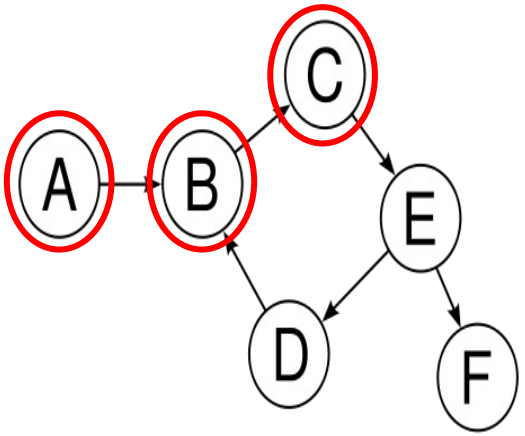


Graph Traversal - DFS

Stack

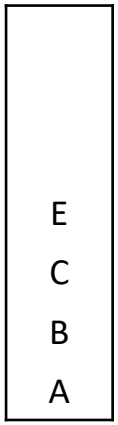


Output: A, B, C

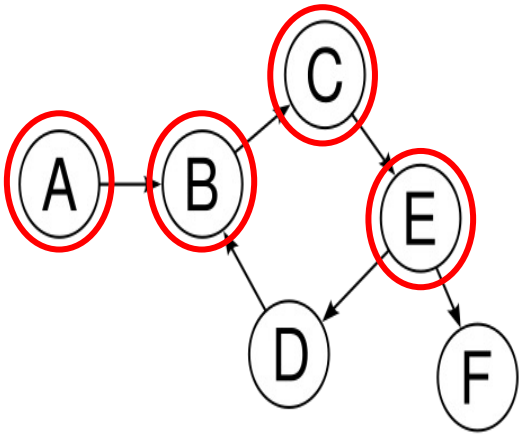


Graph Traversal - DFS

Stack

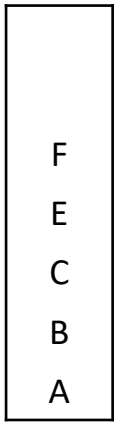


Output: A, B, C, E

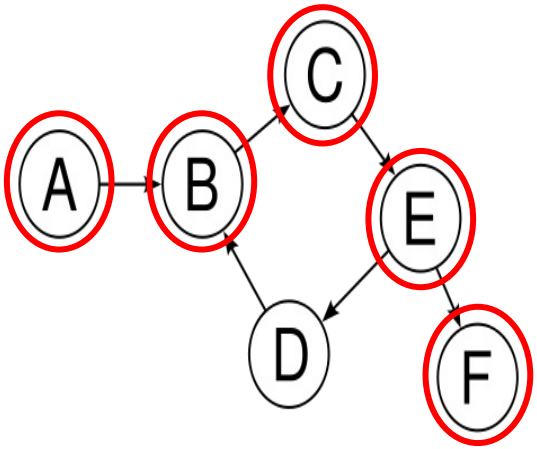


Graph Traversal - DFS

Stack

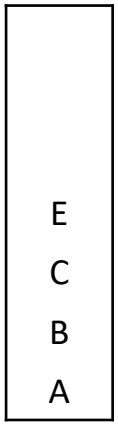


Output: A, B, C, E, F



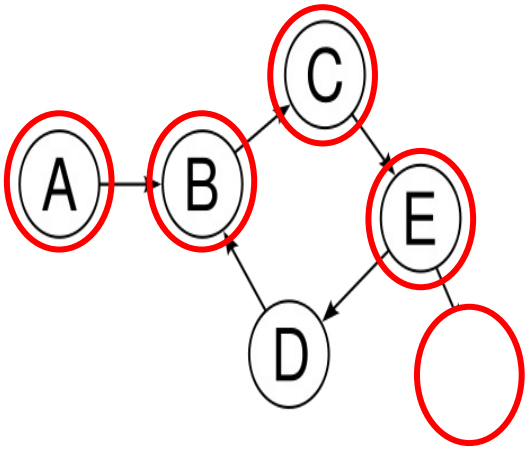
Graph Traversal - DFS

Stack



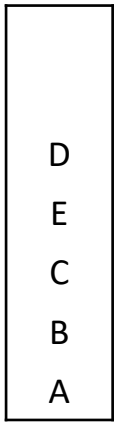
Pop

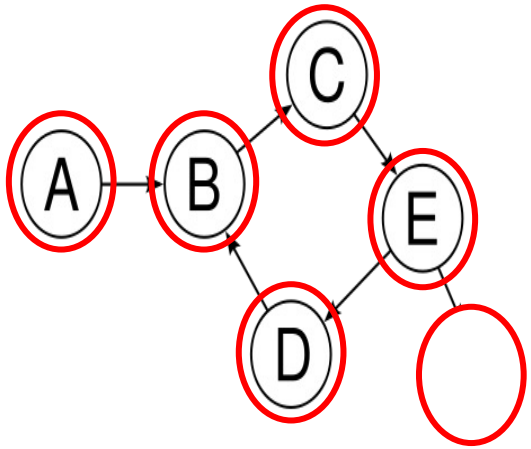
Output: A, B, C, E, F



Graph Traversal - DFS

Stack

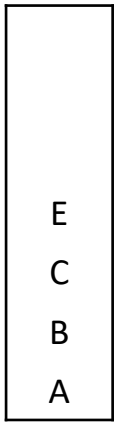




Output: A, B, C, E, F, D

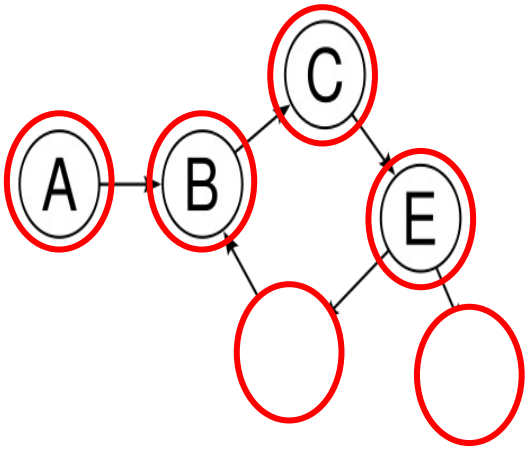
Graph Traversal - DFS

Stack



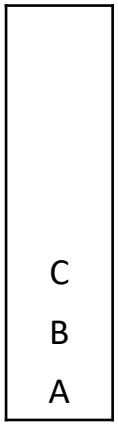
Pop

Output: A, B, C, E, F, D



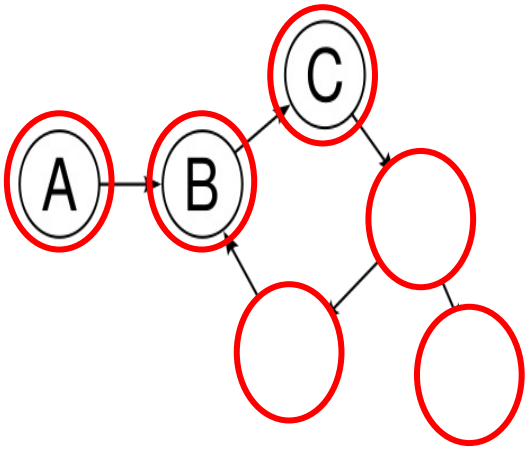
Graph Traversal - DFS

Stack



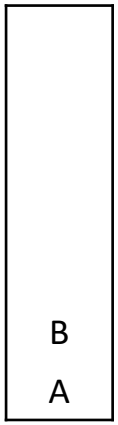
Pop

Output: A, B, C, E, F, D



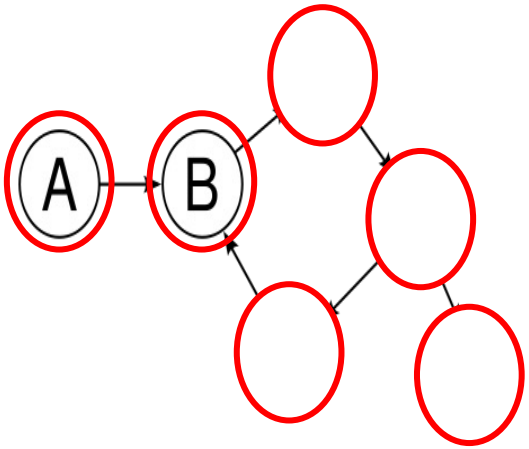
Graph Traversal - DFS

Stack



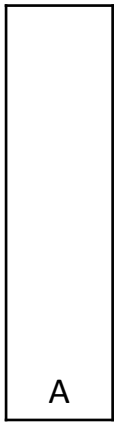
Pop

Output: A, B, C, E, F, D



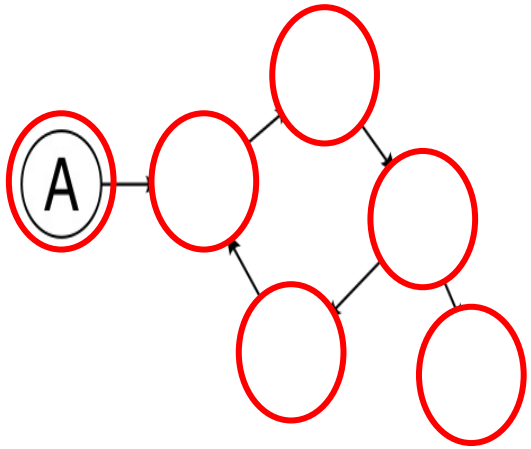
Graph Traversal - DFS

Stack



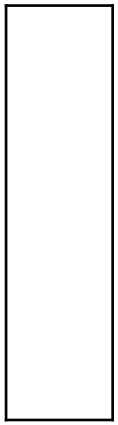
Pop

Output: A, B, C, E, F, D



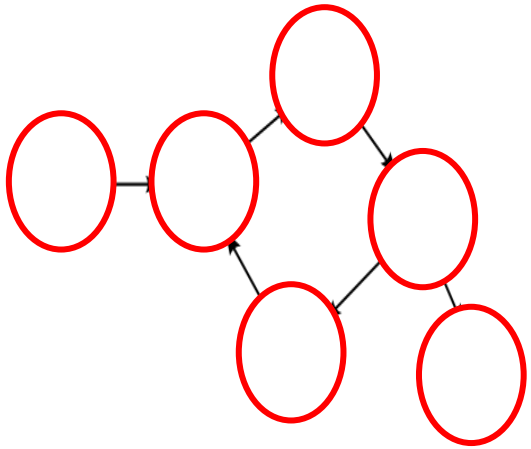
Graph Traversal - DFS

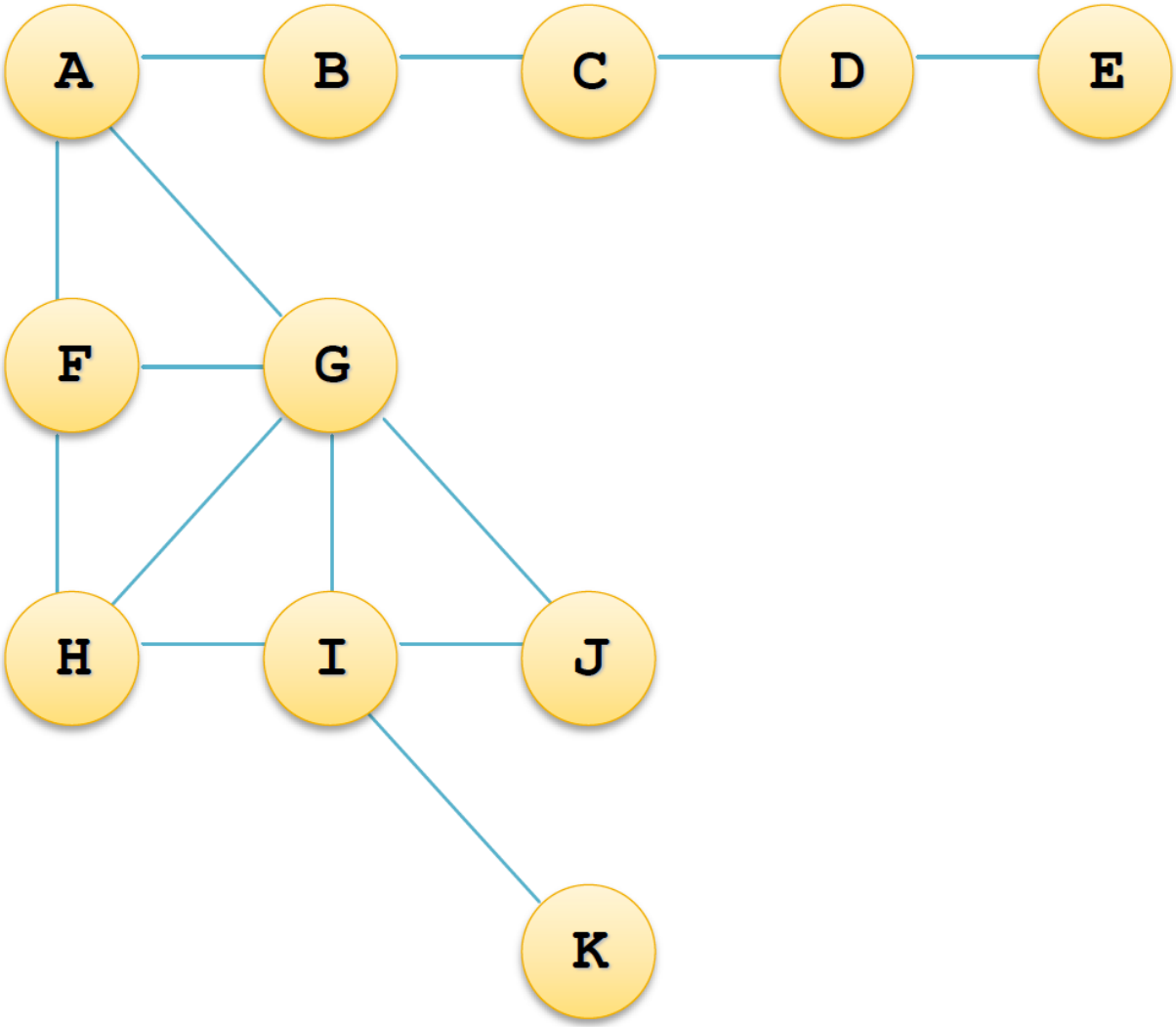
Stack



Done!

Output: A, B, C, E, F, D

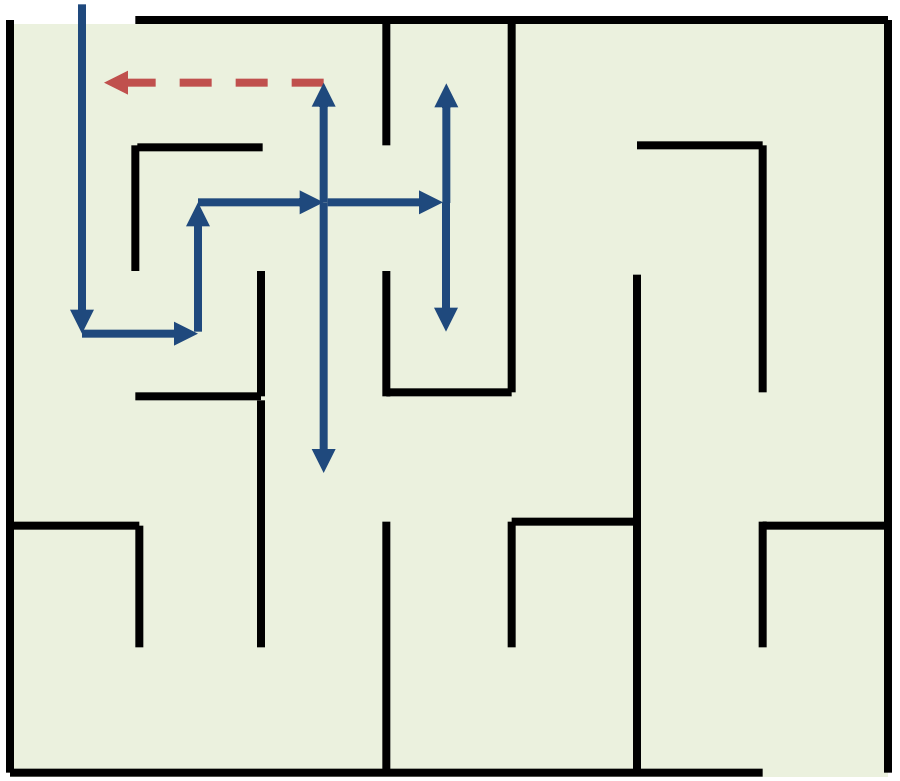




stack	visited
J K	A
E I	B
D H	C
C G	D
B F	E
A	F
	G
	H
	I
	J
	K

DFS and Maze Traversal

- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



Analysis of DFS

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\sum_v \deg(v) = 2m$

Path Finding

- We can specialize the DFS algorithm to find a path between two given vertices ***u*** and ***z*** using the template method pattern
- We call ***DFS*(*G, u*)** with ***u*** as the start vertex
- We use a stack ***S*** to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex ***z*** is encountered, we return the path as the contents of the stack

```
Algorithm pathDFS(G, v, z)
  setLabel(v, VISITED)
  S.push(v)
  if v = z
    return S.elements()
  for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        S.push(e)
        pathDFS(G, w, z)
        S.pop(e)
      else
        setLabel(e, BACK)
  S.pop(v)
```

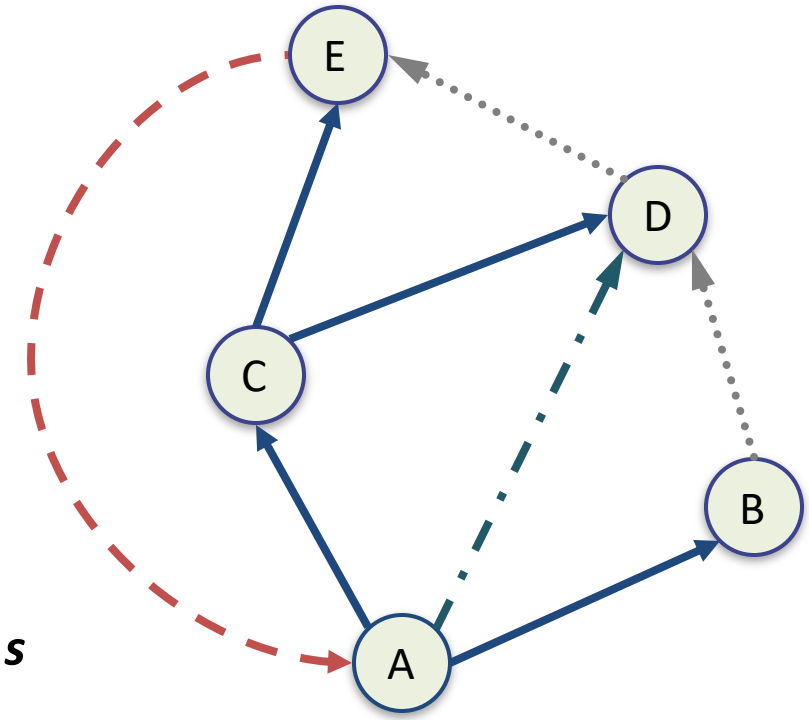
Cycle Finding

- We can use DFS to find a simple cycle
- We use a stack ***S*** to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (***v, w***) is encountered, we return the cycle as the portion of the stack from the top to vertex ***w***

```
Algorithm cycleDFS(G, v, z)
    setLabel(v, VISITED)
    S.push(v)
    for all e ∈ G.incidentEdges(v)
        if getLabel(e) = UNEXPLORED
            w ← opposite(v,e)
            S.push(e)
            if getLabel(w) = UNEXPLORED
                setLabel(e, DISCOVERY)
                pathDFS(G, w, z)
                S.pop(e)
            else
                T ← new empty stack
                repeat
                    o ← S.pop()
                    T.push(o)
                until o = w
                return T.elements()
    S.pop(v)
```

Directed DFS

- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction
- In the directed DFS algorithm, we have four types of edges
  - discovery edges
  - back edges
  - forward edges
  - cross edges
- A directed DFS starting at a vertex ***s*** determines the vertices reachable from ***s***



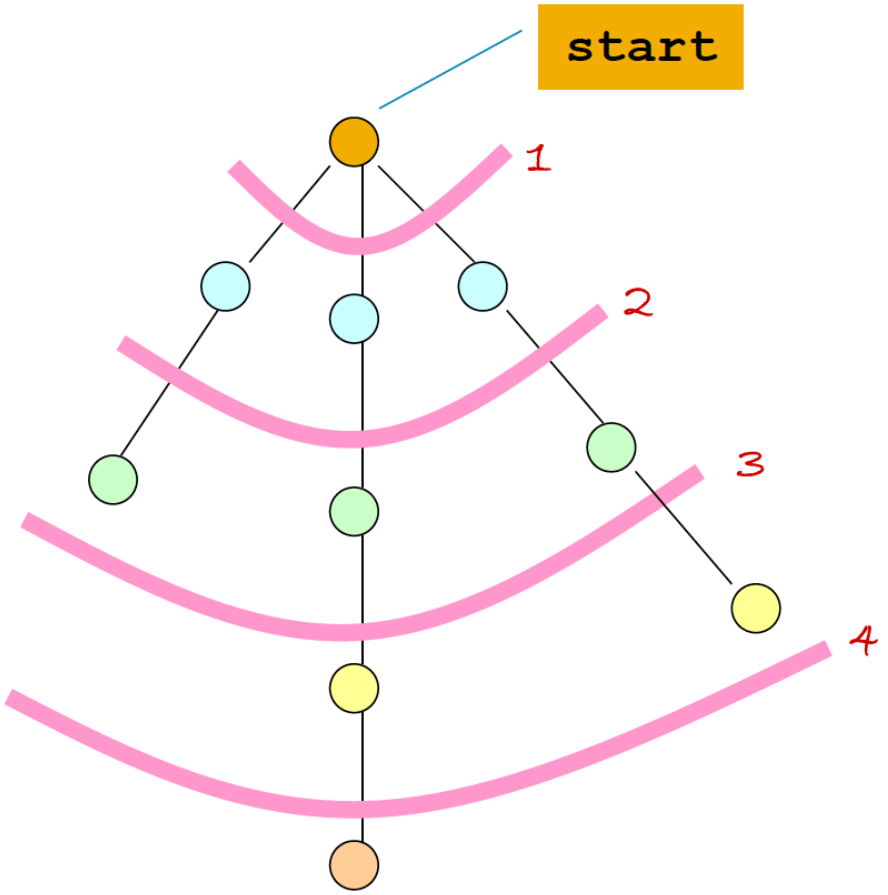
Breadth-First Search

- A general technique for traversing a graph G that
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G

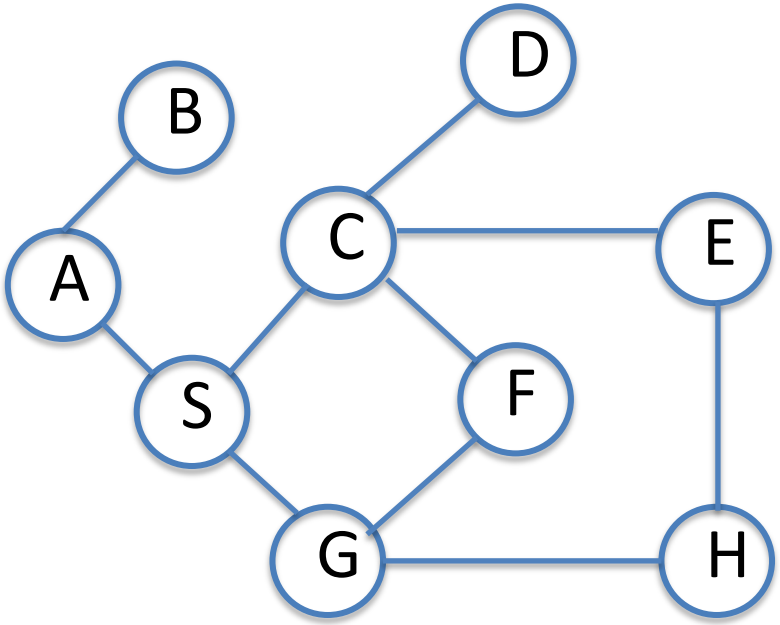
- For a graph with n nodes and m edges it takes $O(n + m)$

- BFS can be extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

- After visiting a vertex, *v*, visit every vertex adjacent to *v* before moving on
- Use a queue to store nodes
  - Queues are FIFO
- BFS:
  - visit and insert start
  - while (*q* not empty)
  - remove node from *q* and make it *current*
  - visit and insert the unvisited nodes adjacent to *current*



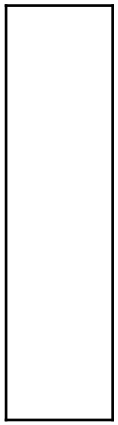
Graph Traversal - BFS

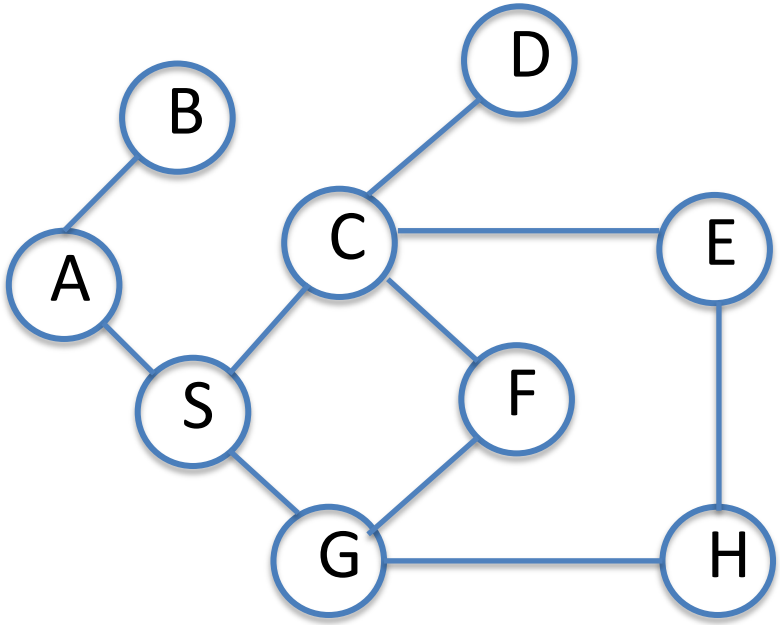


A node is fully explored before any other can begin.

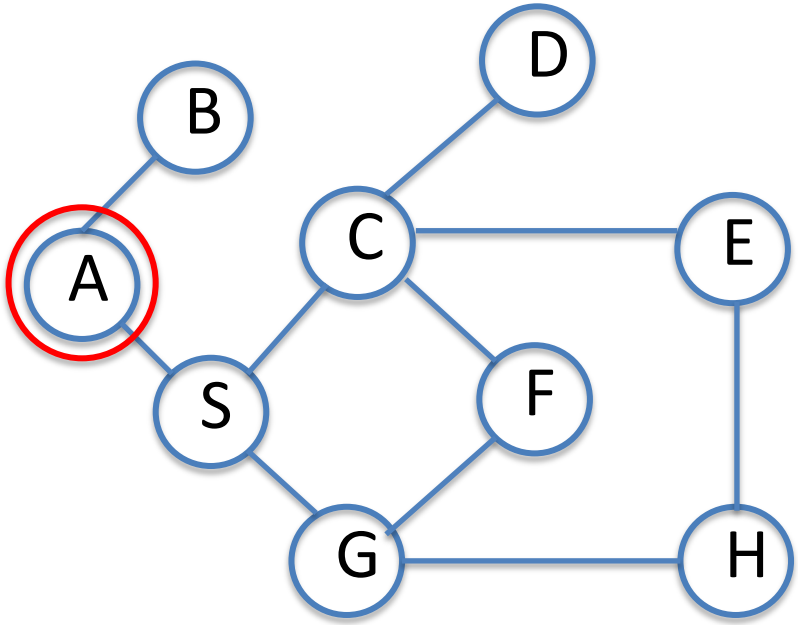
Graph Traversal - BFS

Queue

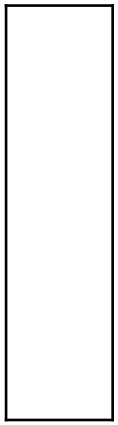




Graph Traversal - BFS

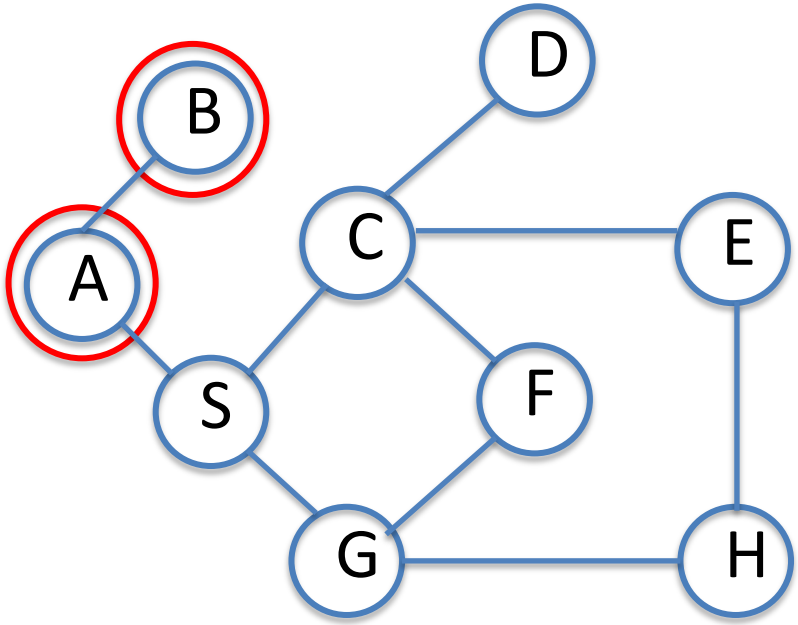


Queue



Output: A

Graph Traversal - BFS



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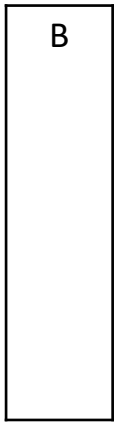
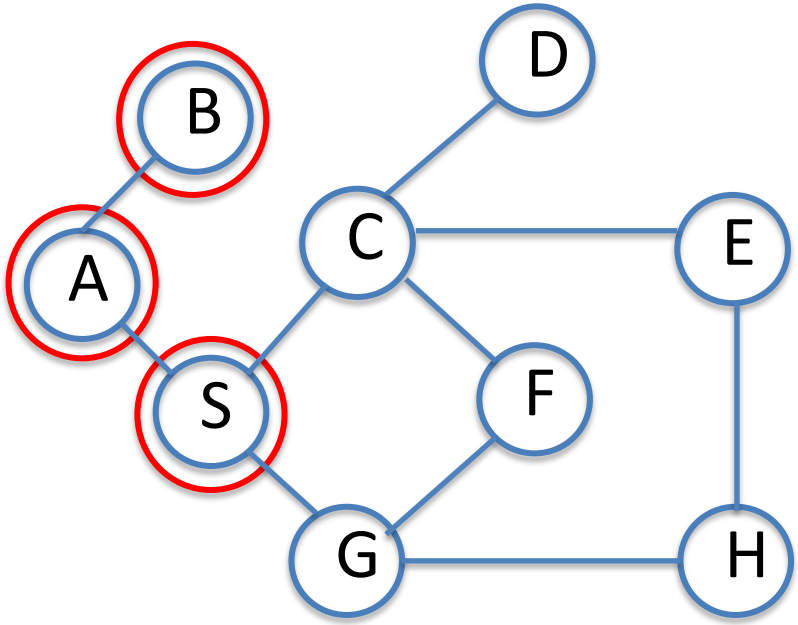


Output: A, B,

Graph Traversal - BFS



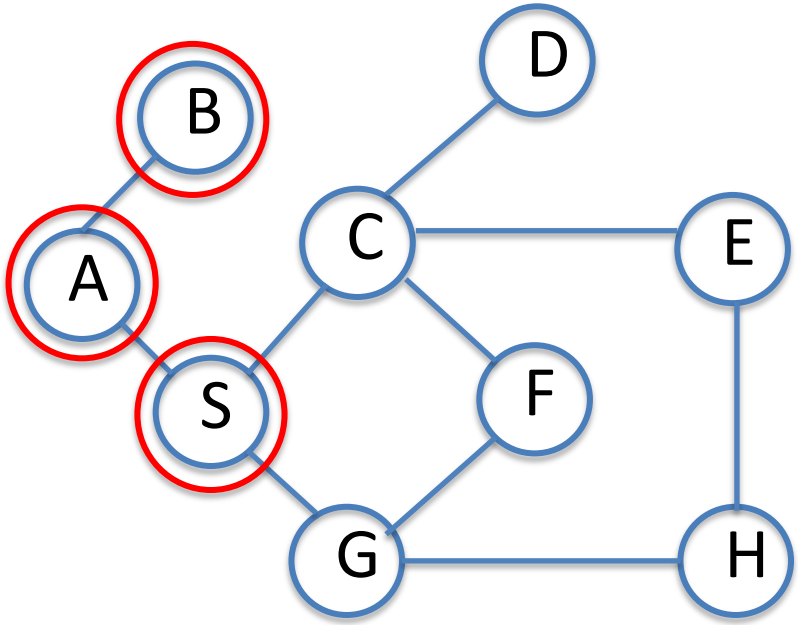
Queue

B

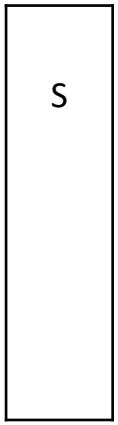
S

Output: A, B, S

Graph Traversal - BFS

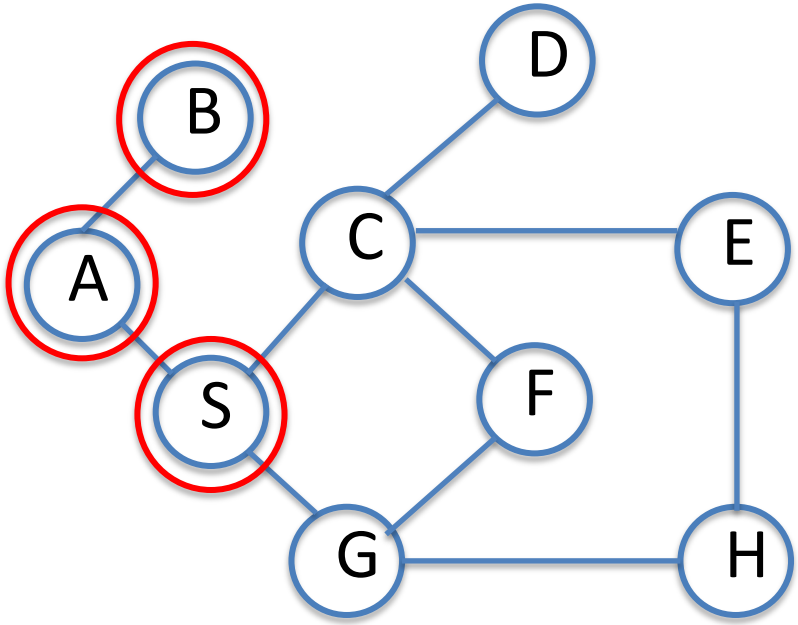


Queue



Output: A, B, S

Graph Traversal - BFS



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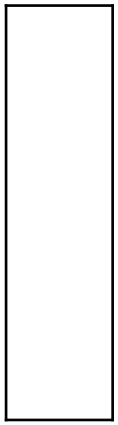
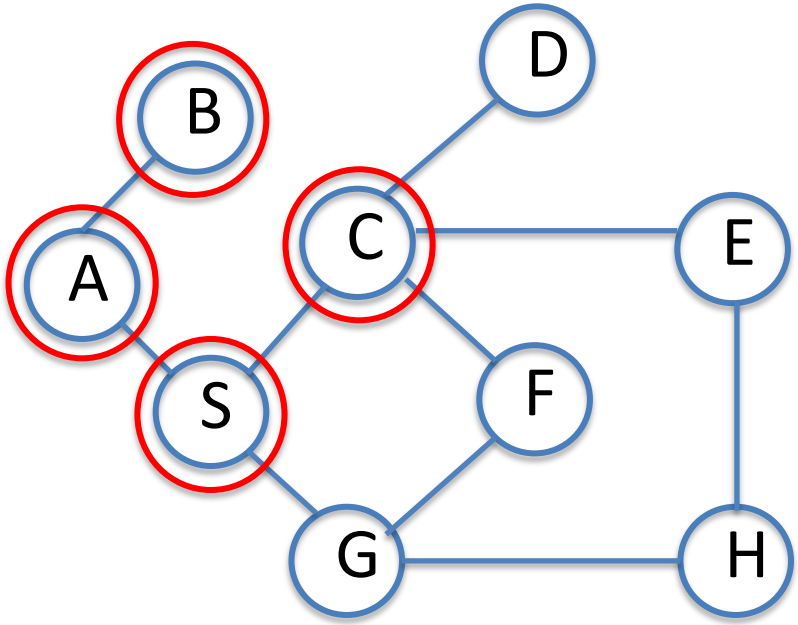
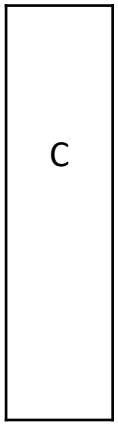


Output: A, B, S

Graph Traversal - BFS

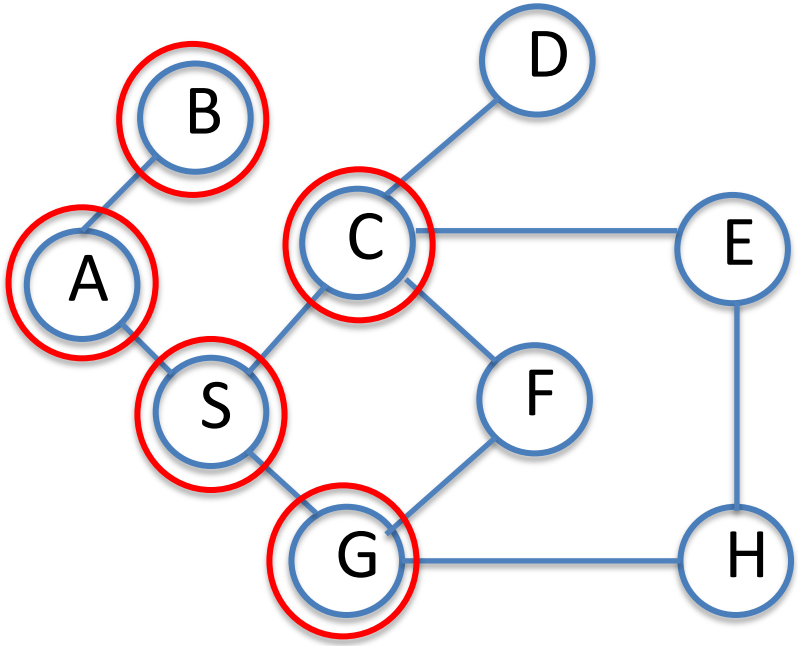


Queue



Output: A, B, S, C

Graph Traversal - BFS



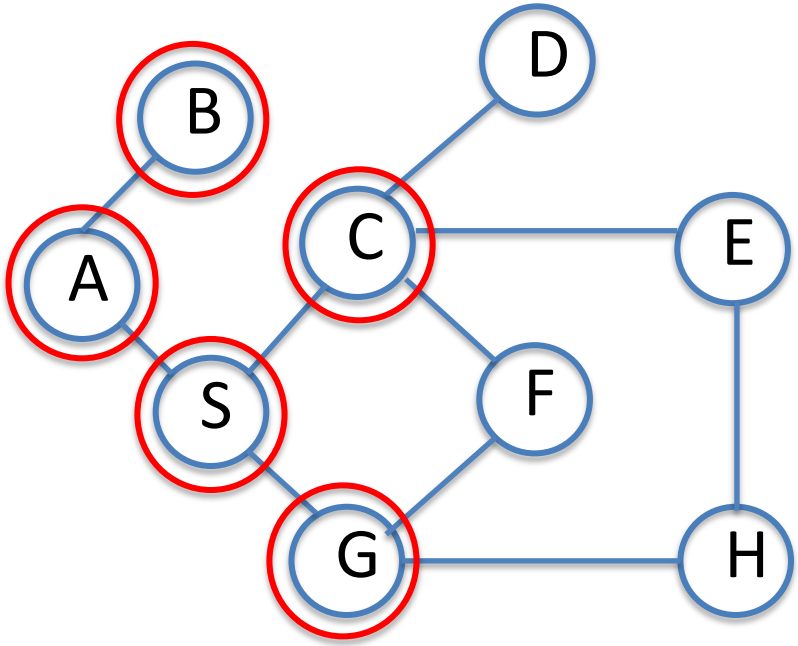
Queue

C

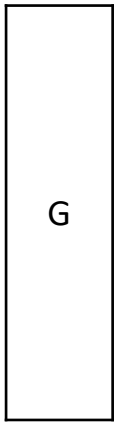
G

Output: A, B, S, C, G

Graph Traversal - BFS

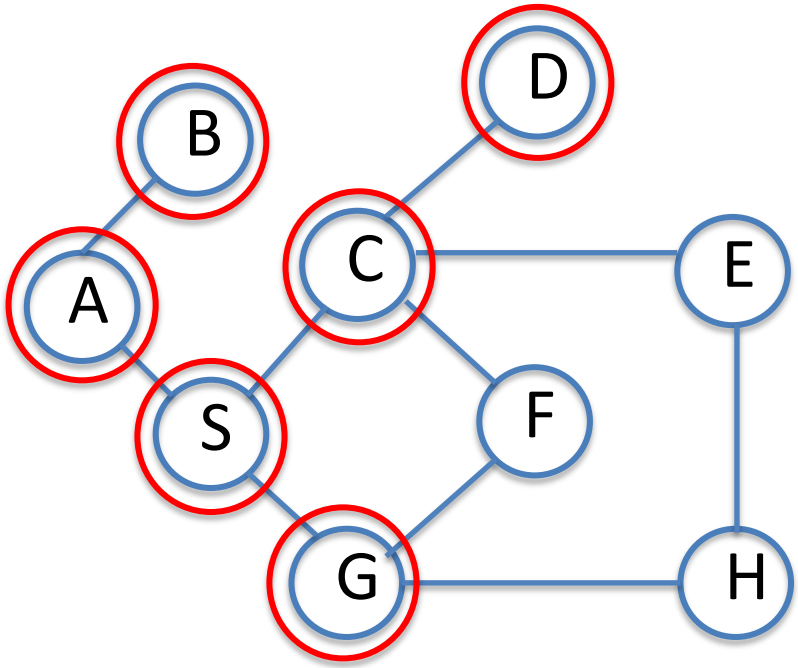


Queue



Output: A, B, S, C, G

Graph Traversal - BFS



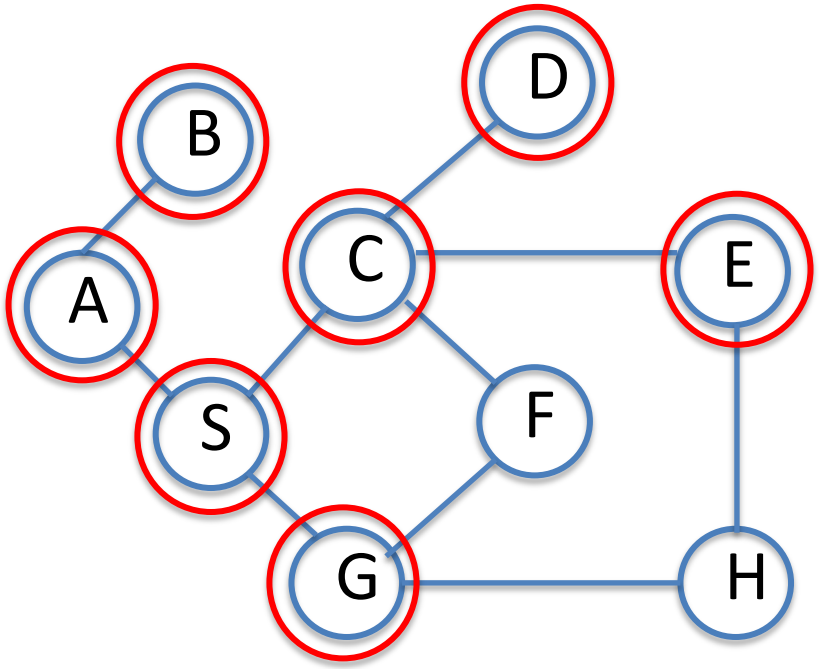
Queue

G

D

Output: A, B, S, C, G, D

Graph Traversal - BFS

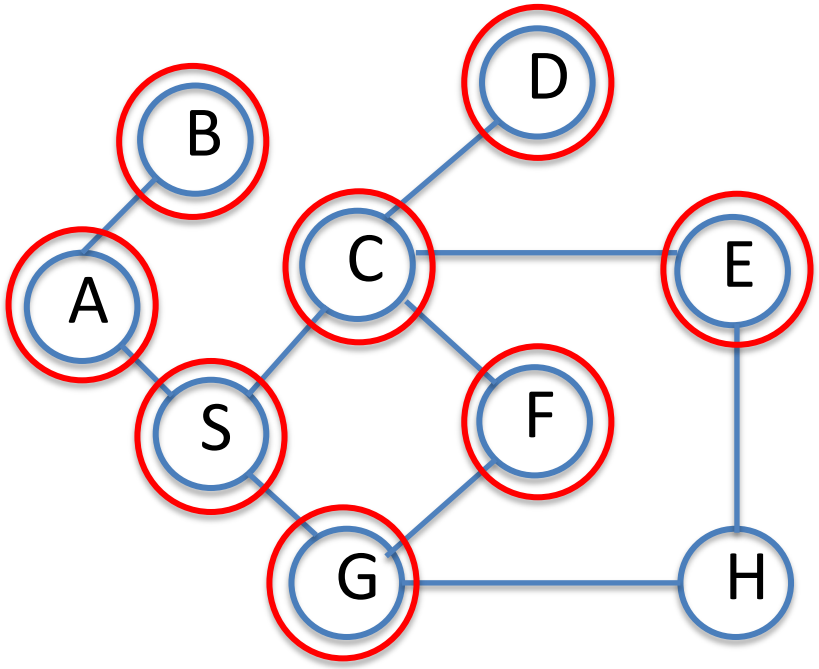


Queue

G
D
E

Output: A, B, S, C, G, D, E

Graph Traversal - BFS

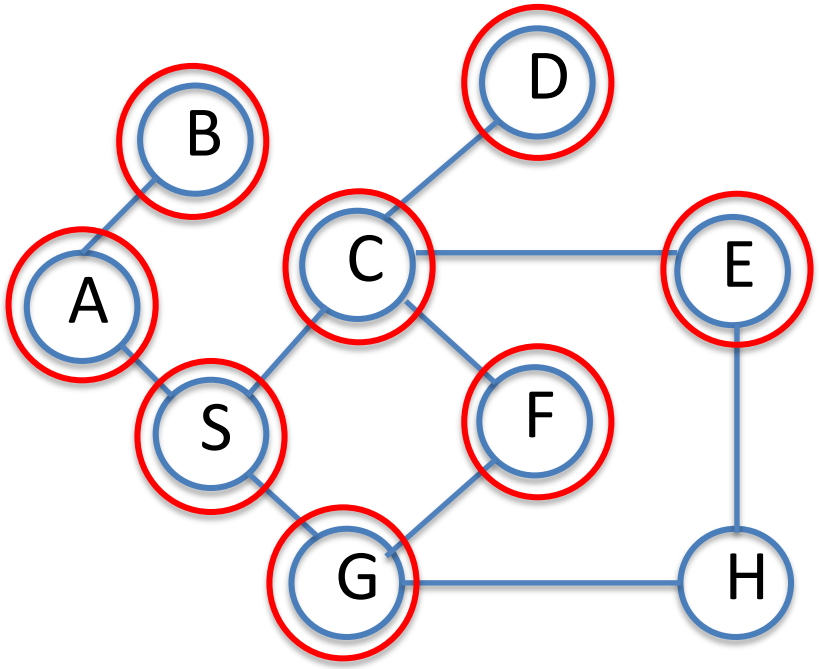


Queue

G
D
E
F

Output: A, B, S, C, G, D, E, F

Graph Traversal - BFS



Queue

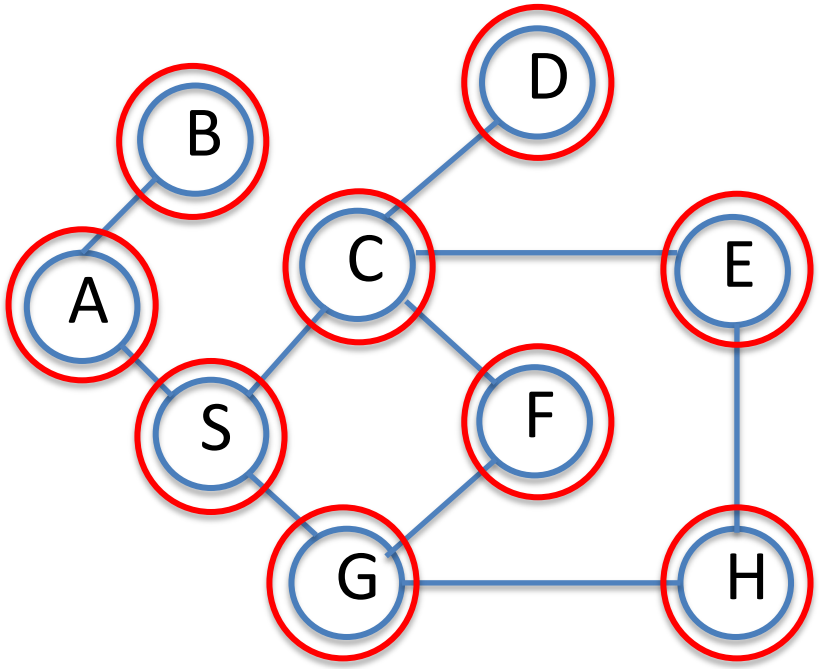
D

E

F

Output: A, B, S, C, G, D, E, F

Graph Traversal - BFS

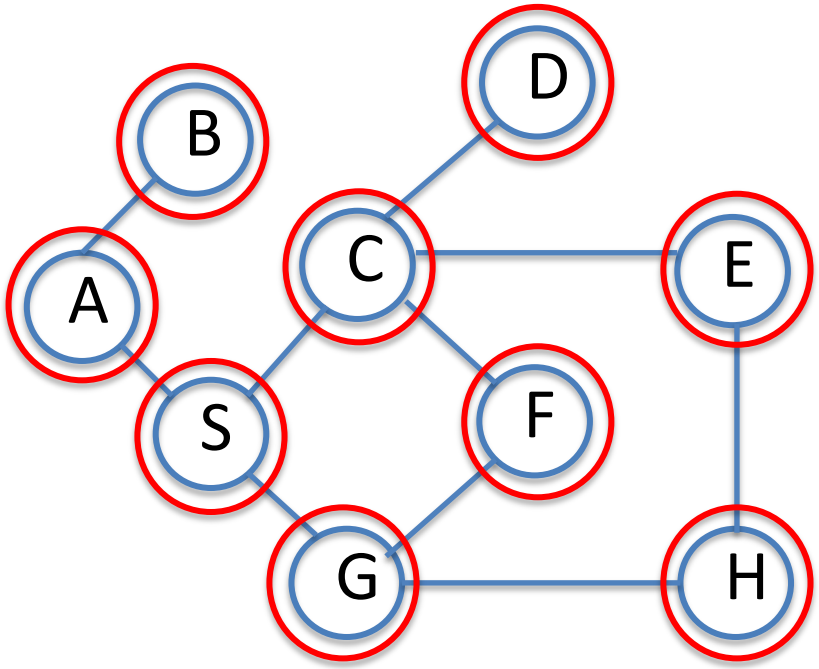


Queue

D
E
F
H

Output: A, B, S, C, G, D, E, F, H

Graph Traversal - BFS



Queue

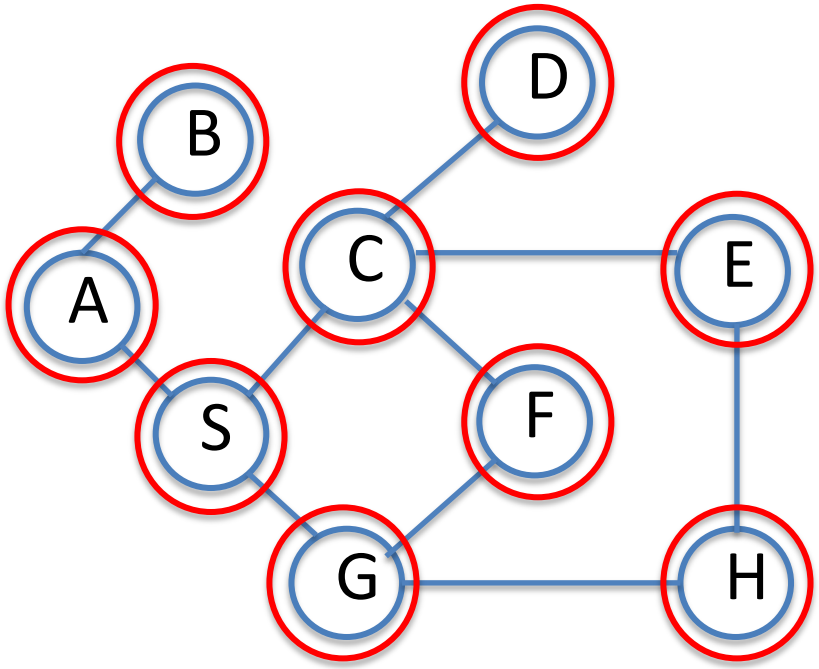
E

F

H

Output: A, B, S, C, G, D, E, F, H

Graph Traversal - BFS

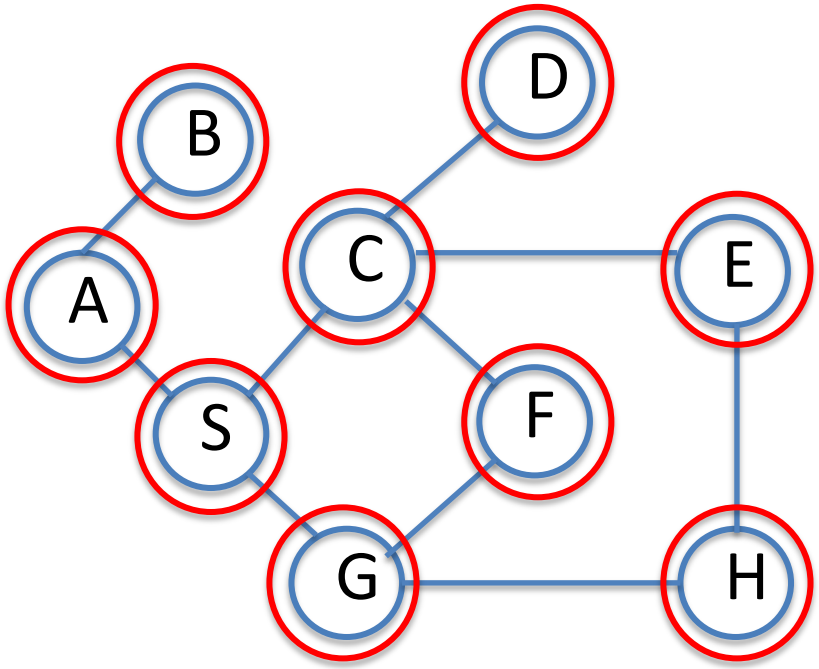


Queue

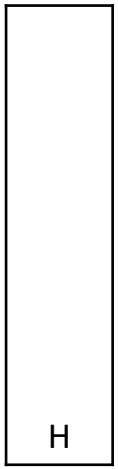
F
H

Output: A, B, S, C, G, D, E, F, H

Graph Traversal - BFS

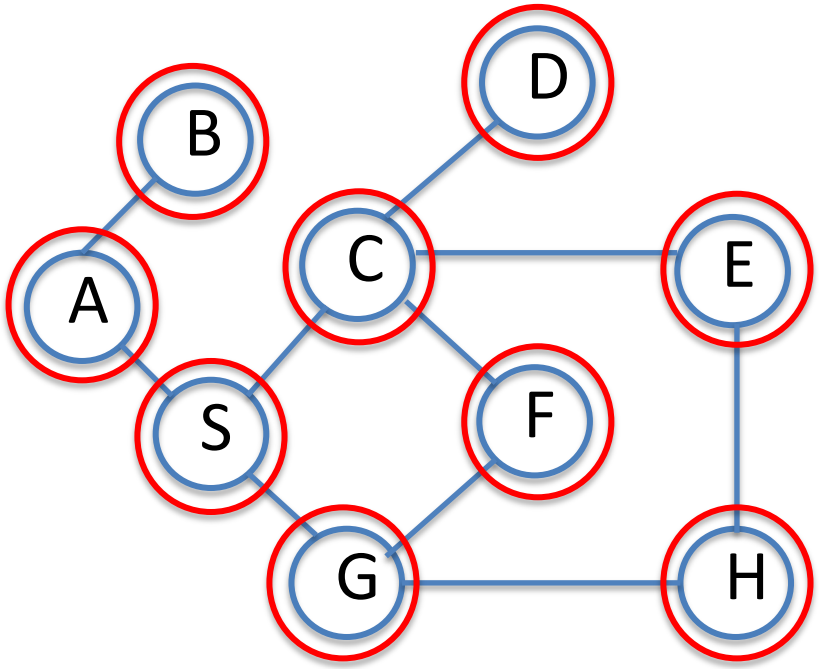


Queue

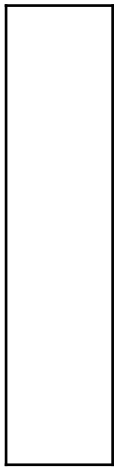


Output: A, B, S, C, G, D, E, F, H

Graph Traversal - BFS

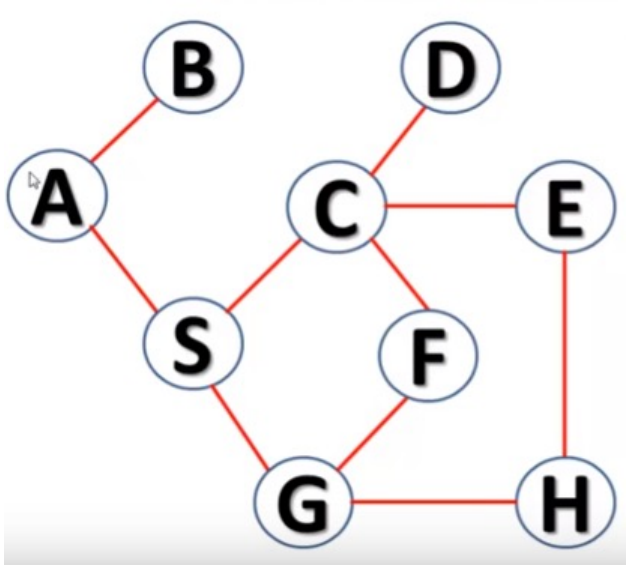


Queue



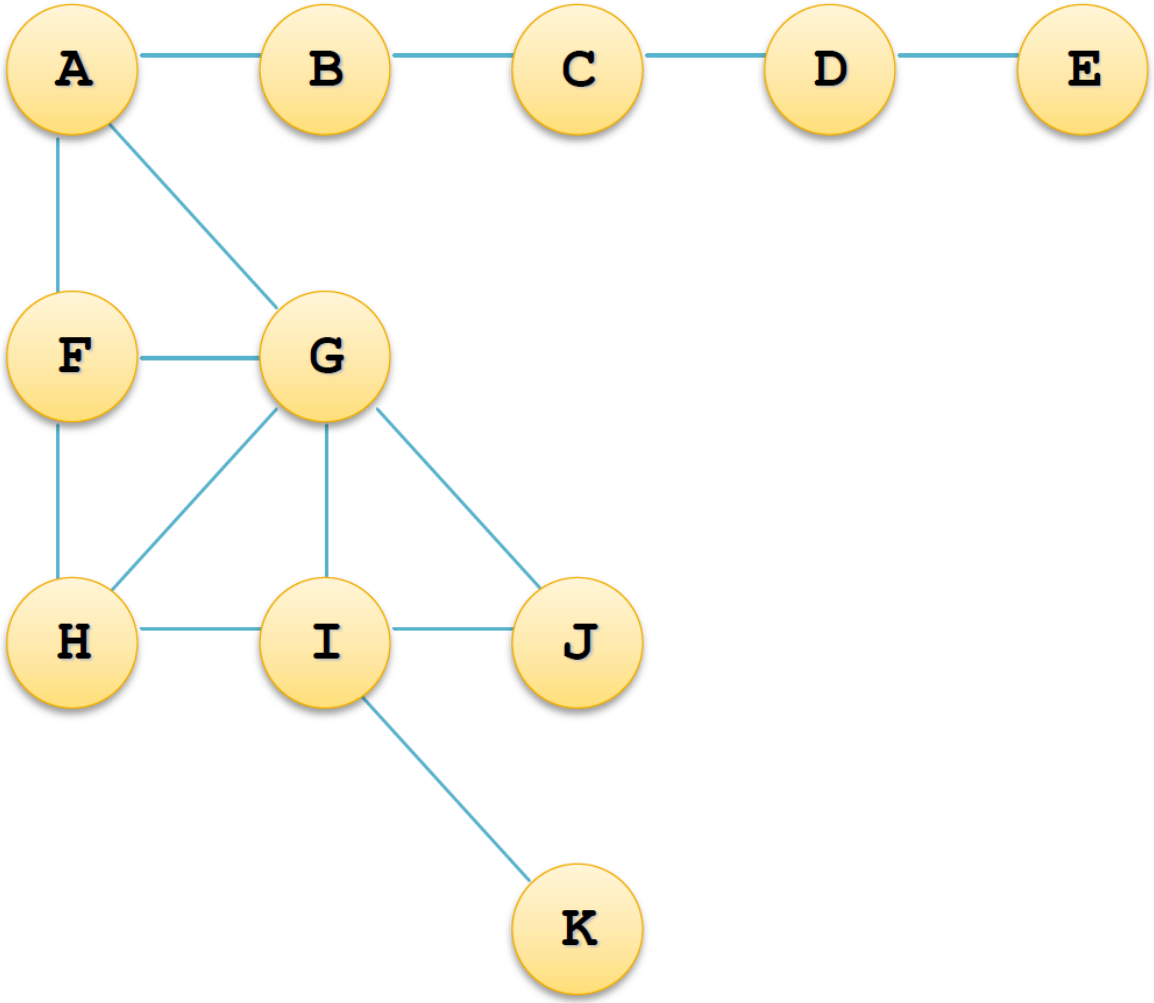
Output: A, B, S, C, G, D, E, F, H

DFS and BFS



DFS: A B S C D E H G F

BFS: A B S C G D E F H



queue	visited
A	A
B	B
F	F
G	G
C	C
H	H
I	I
J	J
D	D
K	K
E	E

BFS Algorithm

```
Algorithm BFS(G, s)
    L0 ← new empty sequence
    L0.addLast(s)
    setLabel(s, VISITED)
    i ← 0
    while Li.isNotEmpty()
        Li+1 ← new empty sequence
        for all v ∈ Li.elements()
            for all e ∈ G.incidentEdges(v)
                if getLabel(e) = UNEXPLORED
                    w ← opposite(v,e)
                    if getLabel(w) = UNEXPLORED
                        setLabel(e, DISCOVERY)
                        setLabel(w, VISITED)
                        Li+1.addLast(w)
                    else
                        setLabel(e, CROSS)
        i ← i +1
```

Analysis

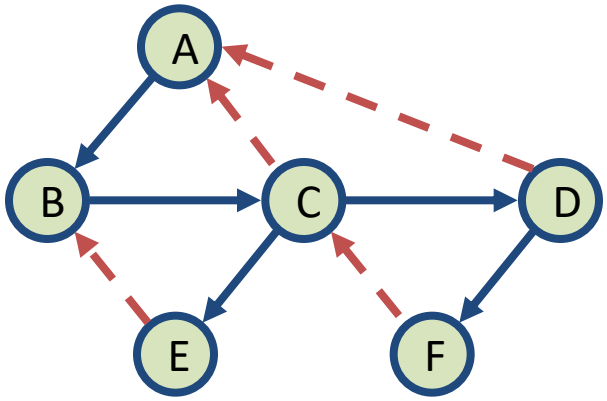
- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence L_i
- Method incidentEdges is called once for each vertex
- BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\sum_v \deg(v) = 2m$

Applications

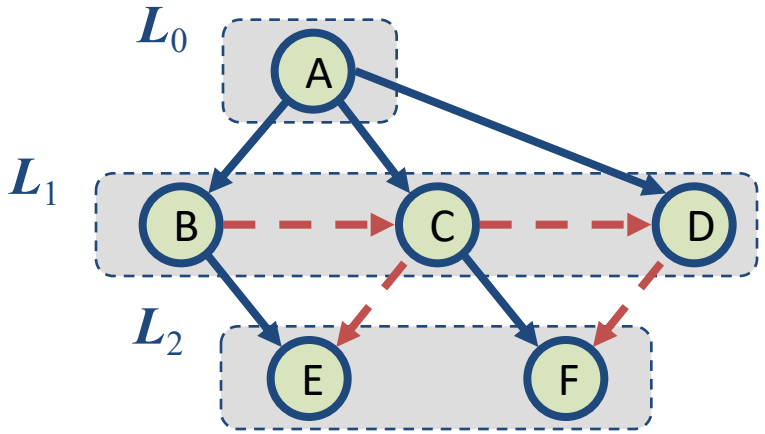
- Using the template method pattern, we can specialize the BFS traversal of a graph G to solve the following problems in $O(n + m)$ time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

DFS vs. BFS

Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	√	√
Shortest paths		√
Biconnected components	√	



DFS



BFS

BFS – Algorithm

1. Store source vertex S in a queue and mark as processed
2. while queue is not empty
   Read vertex v from the queue
   for all neighbors w:
   If w is not processed
      Mark as processed
      Append in the queue
      Record the parent of w to be v (necessary only if we need the shortest path tree)

Depth-First Search with Stack

Initialization:
mark all vertices as unvisited,
visit(s)

while the stack is not empty:
pop **(v,w)**
if w is not visited
add **(v,w)** to tree T
visit(w)

Procedure visit(v)
mark **v** as visited
for each edge **(v,w)**
push **(v,w)** in the stack