Chapter 10

Other Public-Key Cryptosystems

Other PKCS

- In this module we will be looking in detail at other public key cryptosystems other than RSA
- We will look at
  - Diffie-Hellman Key exchange
  - Elliptic Curve Cryptography (ECC)
- For ECC we will look primarily at the basis of the algebraic structure that the algorithm is built on i.e. elliptic curve arithmetic (ECA)
- For ECA we will look at how
  - the operations work
  - we can create groups
  - we can create a one-way trapdoor function

Diffie-Hellman Key Exchange

- First published public-key algorithm
- A number of commercial products employ this key exchange technique
- Purpose is to enable two users to securely exchange a key that can then be used for subsequent symmetric encryption of messages
- The algorithm itself is limited to the exchange of secret values
- Its effectiveness depends on the difficulty of computing discrete logarithms

$p = 7 \rightarrow \{1, 2, 3, 4, 5, 6\}$

$a = 3$

$3^0 \overset{7}{\equiv} 1$

$3^1 \overset{7}{\equiv} 3$

$3^2 \equiv 2$

$3^3 \equiv 6$

$3^4 \equiv 4$

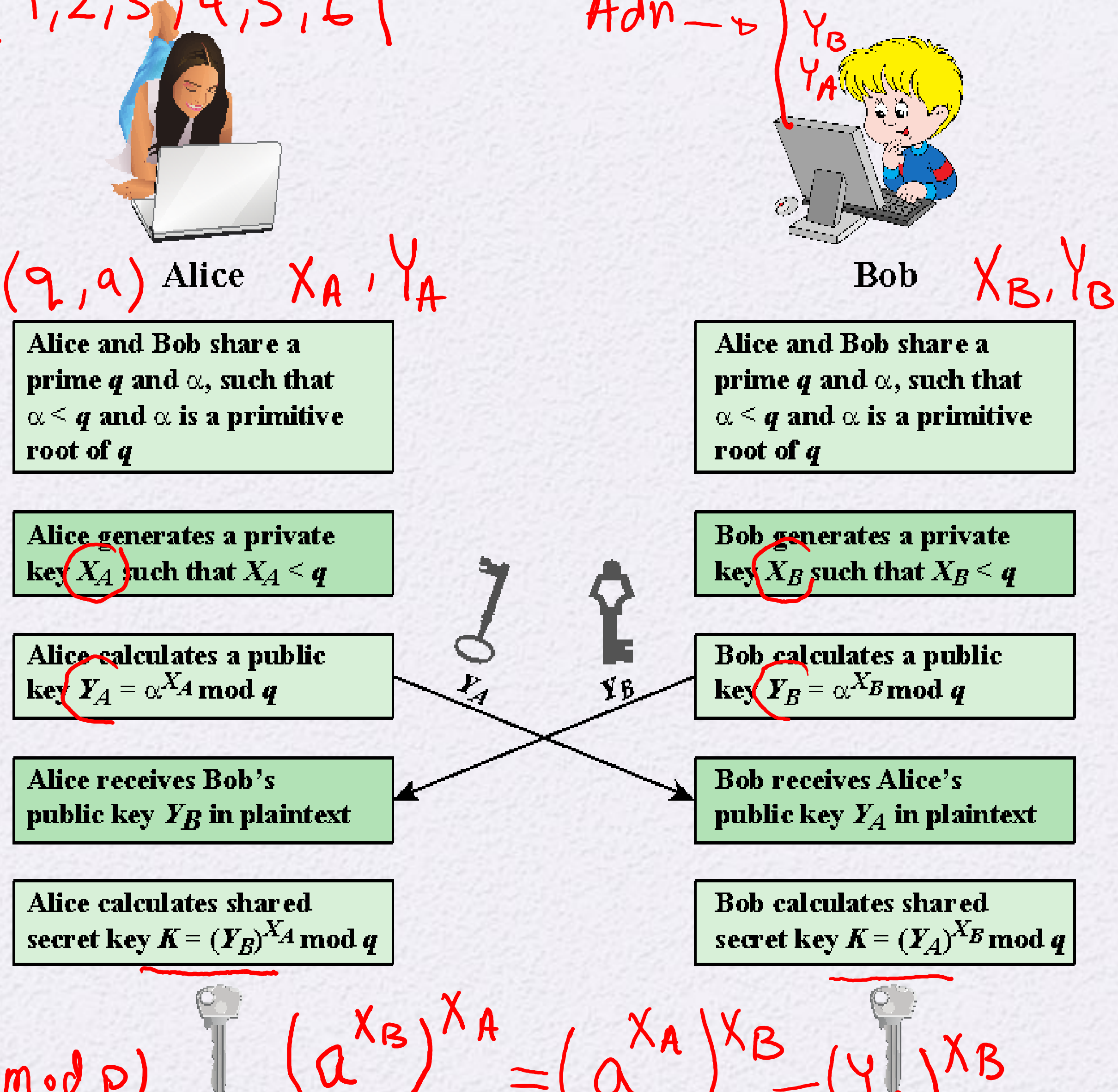
$3^5 \equiv 5$

$3^6 \equiv 1$

Adn $\rightarrow \{q, a, Y_B, Y_A\}$

(q, a) Alice X_A, Y_A

Bob X_B, Y_B



$b \equiv a^i \pmod{p}$

$(a^{X_B})^{X_A} = (a^{X_A})^{X_B} = (Y_A)^{X_B}$

Figure 10.1 Diffie-Hellman Key Exchange

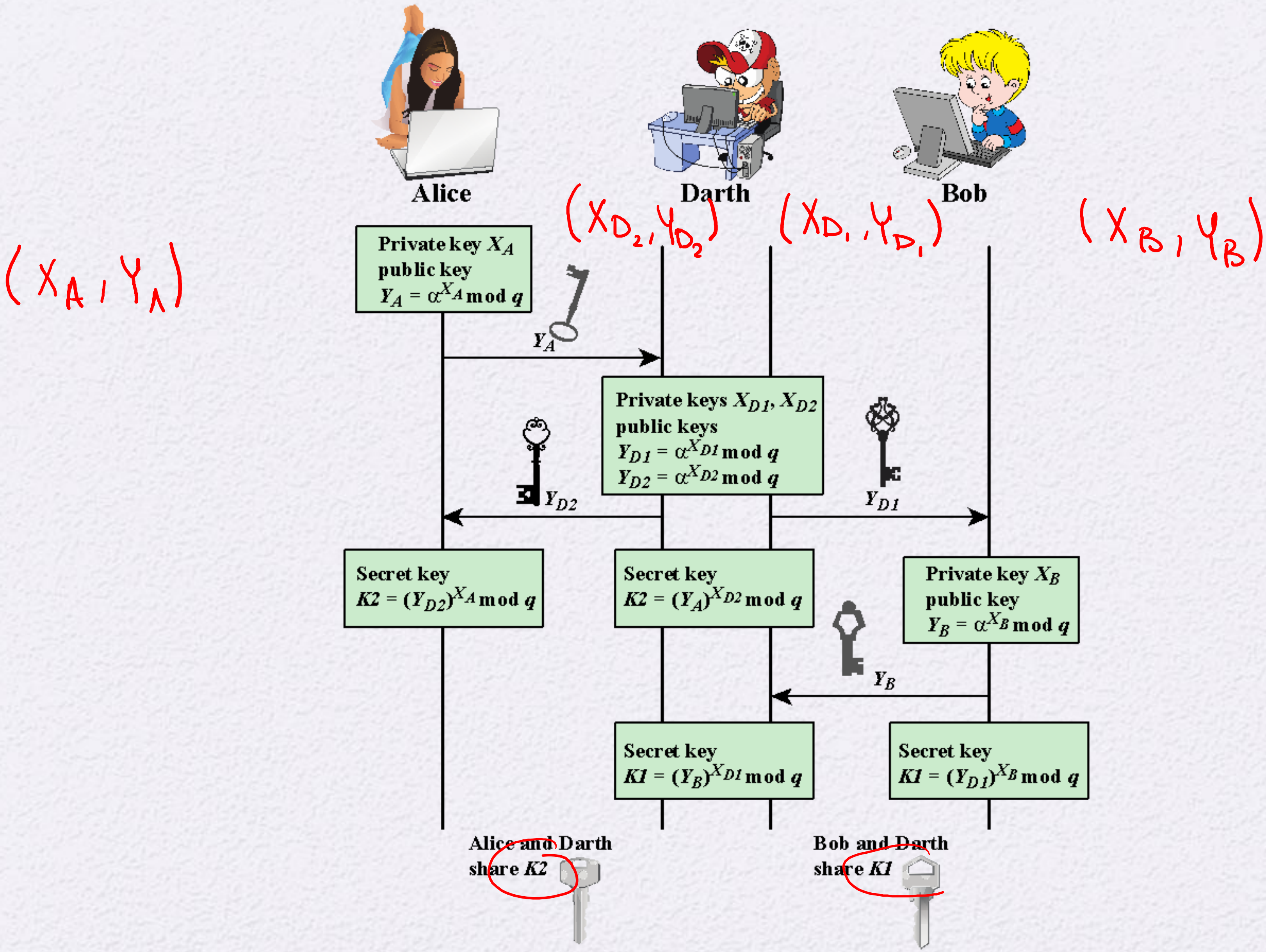


Figure 10.2 Man-in-the-Middle Attack

EIGAMAL: Based on DLP

APP: Digital signiture standard (DSS)

global elemets: (q, a)

Alice:

$1 \leqslant X_A \leqslant q-1$

$Y_A = a^{X_A} \pmod q$

$PR_A = X_A \qquad PU_A = \{q, a, Y_A\}$

Bob: uses Alice's public key Y_A to encrypt msg M. $E_{Y_A}(M) = (C_1, C_2)$

1. Convert M to some blocks of msgs : $0 \leqslant M \leqslant q-1$

2. Pick a random $1 \leqslant h < q-1$

3. Compute a one time key $K = (Y_A)^h \bmod q$

$c_1 = a^h \pmod q$

$c_2 = KM \pmod q \qquad (a^h, KM) = (c_1, c_2)$

Alice: $D_{X_A}(C_1, C_2) = M$

$K = c_1^{X_A} \pmod q$

$M = (c_2 K^{-1}) \pmod q$

Correctness:

$K = (Y_A)^h = (a^{X_A})^h = (a^h)^{X_A} = C_1^{X_A}$

$M = C_2 K^{-1} = M \cancel{K} \cancel{K^{-1}} = M$

$p = 7 \qquad a = 3$

$X_A = 5$

$Y_A = (a^{X_A}) \bmod 7 = 3^5 \overset{7}{\equiv} 5$

$m = 4$

$h = 2 \qquad k = (Y_A)^h \overset{p}{\equiv} 5^2 \overset{7}{\equiv} 4$

$c_1 = 3^2 \overset{7}{\equiv} 2$

$C_2 = kM = 4.4 \overset{7}{\equiv} 16 \overset{7}{\equiv} 2$

$(c_1, c_2) = (2, 2) \qquad m = ?$

$k = c_1^{X_A} \bmod 7 = 2^5 \overset{7}{\equiv} 4$

$k k^{-1} \overset{7}{\equiv} 1$

$4 k^{-1} \overset{7}{\equiv} 1 \rightarrow k^{-1} = 2$

$M = c_2 \cdot k^{-1} \underset{7}{\equiv} 2 \cdot 2 = 4$

Elliptic Curves

- Elliptic curves are not ellipses. They are so named because they are described by **cubic equations,** similar to those used for calculating the circumference of an ellipse
- They are defined by the equation
  $y^2 = x^3 + ax + b$ or $y = \sqrt{x^3 + ax + b}$

$\forall a, b;\ 4a^3 + 27b^2 \neq 0 \pmod{p}$

Elliptic Curves

- For given values of a and b, the plot consists of positive and negative values of y for each value of x. Thus, each curve is symmetric about $y = 0$
- Also included in the definition of an elliptic curve is a single element denoted O and called the ***point at infinity*** or the **zero point**

Elliptic Curves

- The set of points E(*a*,*b*) consists of all of the points (*x*,*y*) that satisfy the equation of the curve together with the element *O*.
- The shape of each curve is determined by the pair (*a*,*b*).
- So, an elliptic curve can be denoted as E(*a*,*b*), where **a** and **b** are the values used in the equation that defines the curve

Elliptic Curve Arithmetic

- Most of the products and standards that use public-key cryptography for encryption and digital signatures use RSA
  - The key length for secure RSA use has increased over recent years and this has put a heavier processing load on applications using RSA
- Elliptic curve cryptography (ECC) is showing up in standardization efforts including the IEEE P1363 Standard for Public-Key Cryptography
- Principal attraction of ECC is that it appears to offer equal security for a far smaller key size

Elliptic Curve Arithmetic

- The fundamental operation in elliptic curve arithmetic is addition (of two points on the curve)
  - This is very different from addition in regular arithmetic
- Multiplication is defined as multiple addition
- Identity element is the *point at infinity* or the ***zero point***

Elliptic Curve Arithmetic

- If three points on an elliptic curve lie on a straight line their sum is taken as **O**
  - P + Q + R = **O**
  - P + Q = -R (for two points with different x coordinates)

$P+Q+R = P+Q-(P+Q) = O$

- For a point P = (x,y) we have that -P =(x,-y)

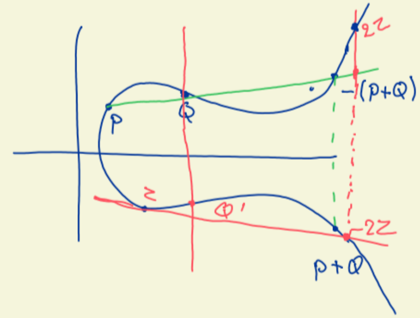
- P + (-P) = **O** (for two points with the same x coordinates)

Elliptic Curve Arithmetic

- To add a point to itself
  - $Q + Q = 2Q = -S$
  - Where S is the point of intersection of the line tangent to Q and the curve
- You can only multiply a point with an integer value. You cannot multiply two points
  - $3P = P + P + P$

Addition:

$P + Q$



$Z + Z \rightarrow$ point doubling

$Q' + Q = 0$ $\quad$ $Q = (x, y)$

$(-Q) + Q = 0$ $\quad$ $-Q = (x, -y)$

Inverse:

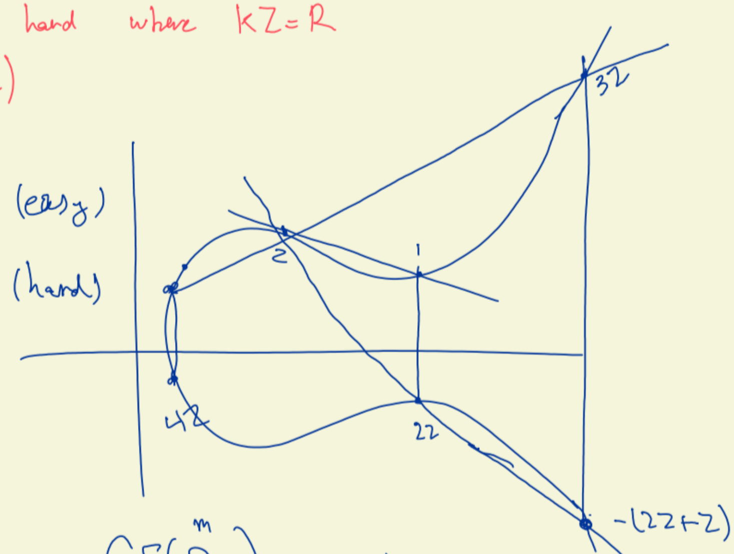
$Q = (x, y)$ $\quad$ $-Q = (x, -y)$ $\quad$ $Q + (-Q) = 0$

$Z + Z + Z + \dots + Z = kZ = R$ $\quad$ $k \in \mathbb{N}$ $\quad$ $1, 2, 3, 4, \dots$

Finding k is hard where $kZ = R$
(having R, Z)

$f(k) = kZ$ (easy)

$f^{-1}(kZ) = k$ (hard)



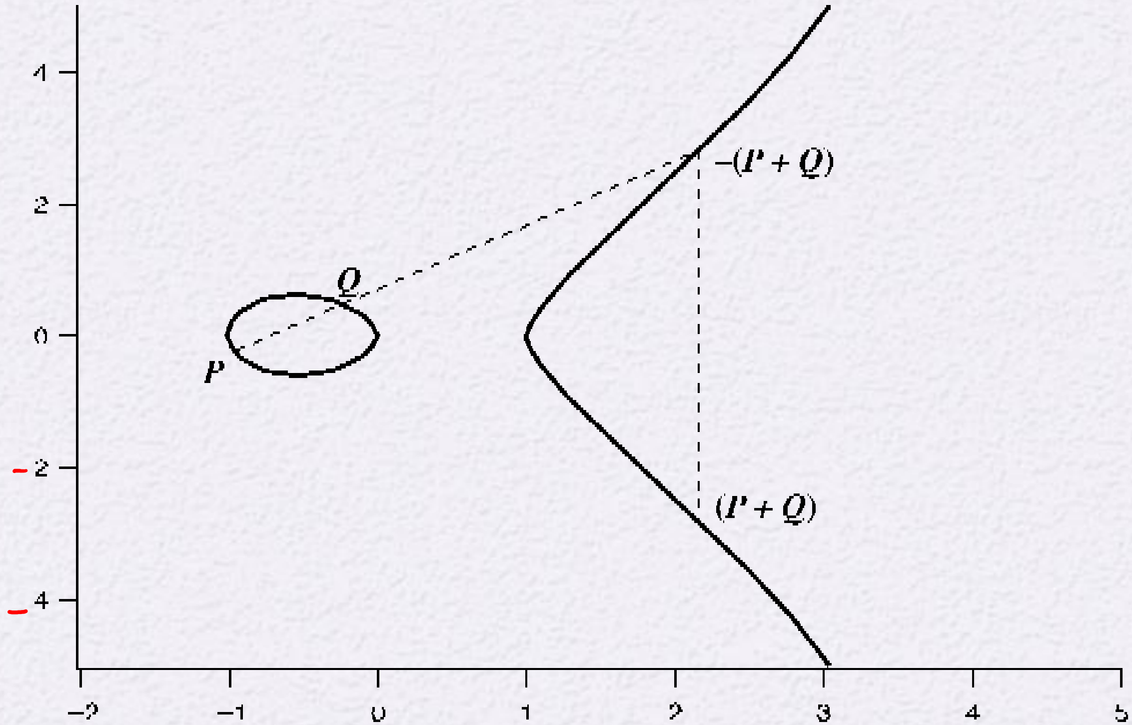
Prime curve

$\uparrow$

over Z_p

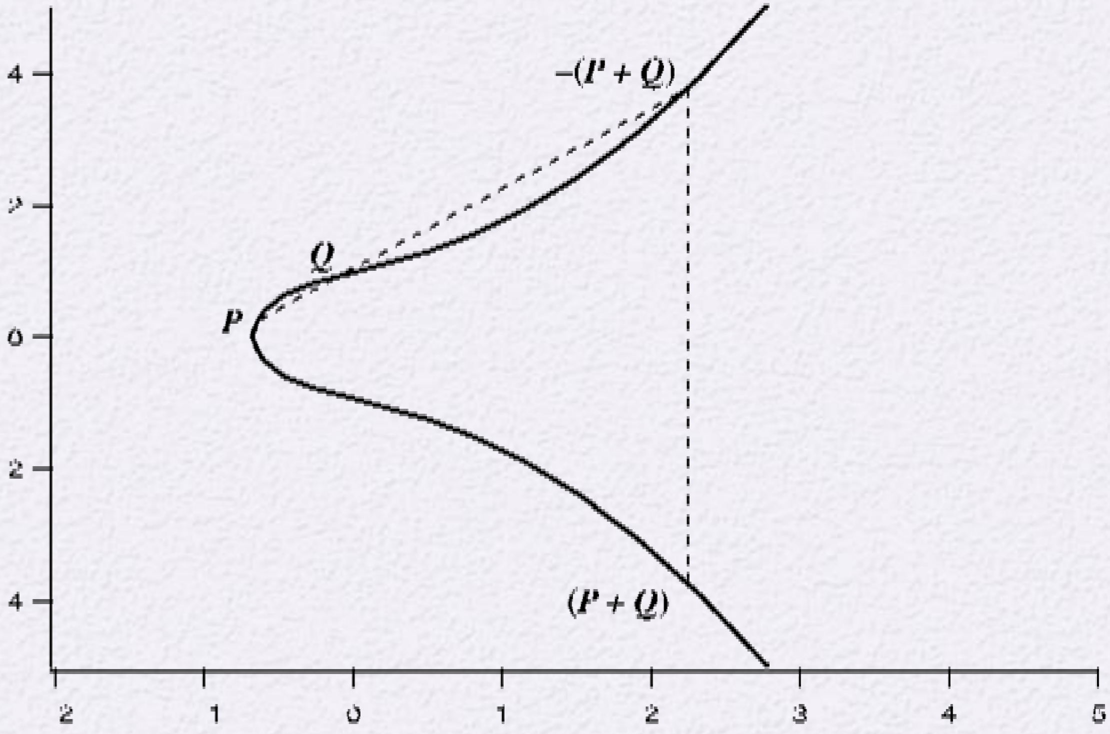
$GF(2^m) \rightarrow$ Binary curve

Addition in ECA



(a) $y^2 = x^3 - x$

Addition in ECA



(b) $y^2 = x^3 + x + 1$

Abelian Group

- A set of elements with a binary operation, denoted by •, that associates to each ordered pair (a, b) of elements in G an element $(a \bullet b)$ in G, such that the following axioms are obeyed:

(A1) Closure: If a and b belong to G, then $a \bullet b$ is also in G

(A2) Associative: $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ for all a, b, c in G

(A3) Identity element: There is an element e in G such that $a \bullet e = e \bullet a = a$ for all a in G

(A4) Inverse element: For each a in G there is an element a' in G such that $a \bullet a' = a' \bullet a = e$

(A5) Commutative: $a \bullet b = b \bullet a$ for all a, b in G

Abelian Group

- For an elliptic curve to form an abelian group, its values of a and b must fulfill the inequality
  - $4a^3 + 27b^2 \neq 0 \pmod{p}$

Finite Fields

- Creating elliptic curves that are defined over finite fields require additional steps
  - These will not be covered here. You are however encouraged to look this up
- There are two kinds of elliptic curves defined over finite fields
  - **Prime Curves** – with variables and coefficients taking values between *(0 – p-1)* and all calculations are done mod p
  - **Binary Curves** - with variables and coefficients taking values in GF(2^m) and calculations performed over GF(2^m)

Elliptic Curve Cryptography (ECC)

- Addition operation in ECC is the counterpart of modular multiplication in RSA
- Multiple addition is the counterpart of modular exponentiation

To form a cryptographic system using elliptic curves, we need to find a "hard problem" corresponding to factoring the product of two primes or taking the discrete logarithm

- $Q=kP$, where Q, P belong to a prime curve
- Is "easy" to compute Q given k and P
- But "hard" to find k given Q, and P
- Known as the elliptic curve logarithm problem

Security of Elliptic Curve Cryptography

- Depends on the difficulty of the elliptic curve logarithm problem
- Fastest known technique is “Pollard rho method”
- Compared to factoring, can use much smaller key sizes than with RSA
- For equivalent key lengths computations are roughly equivalent
- Hence, for similar security ECC offers significant computational advantages

Table 10.3

Comparable Key Sizes in Terms of Computational Effort for Cryptanalysis (NIST SP-800-57)

Symmetric key algorithms	Diffie-Hellman, Digital Signature Algorithm	RSA (size of n in bits)	ECC (modulus size in bits)
80	$L = 1024$ $N = 160$	1024	160–223
112	$L = 2048$ $N = 224$	2048	224–255
128	$L = 3072$ $N = 256$	3072	256–383
192	$L = 7680$ $N = 384$	7680	384–511
256	$L = 15{,}360$ $N = 512$	15,360	512+

Note: L = size of public key, N = size of private key

Summary

- Define Diffie-Hellman Key Exchange
- Understand the Man-in-the-middle attack

- Understand Elliptic curve arithmetic
- Present an overview of elliptic curve cryptography