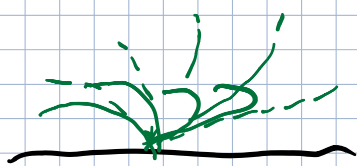
Goal: Model a single flagellum



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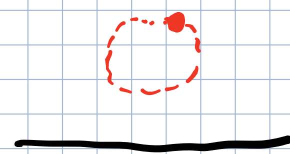


e.g. a sphere undergoing a circular trajectory.

A common assumption for self propelled things in a fluid is that the net force is 0

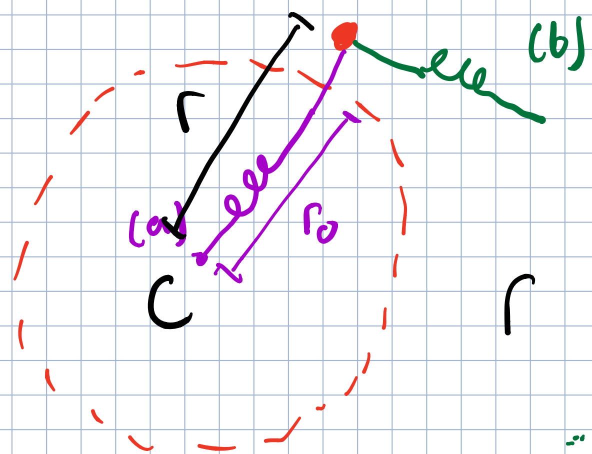
There are 2 forces acting on the sphere

1) A system of springs that drives the sphere and keeps it roughly on a circular trajectory

2) The sphere experiences resistance, e.g. drag, against its motion.

The forces in detail

Springs



(a) A radial spring encourages the sphere to stay on a circular trajectory with a set radius

$$F_{radial} = -\lambda(r - r_0)$$

r: the actual distance of the sphere from the center

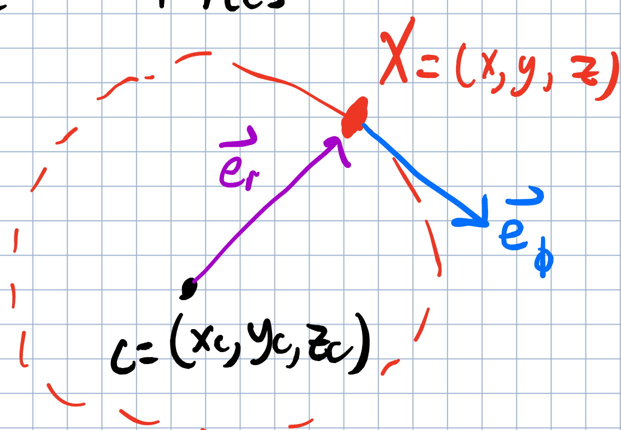
λ: the spring constant

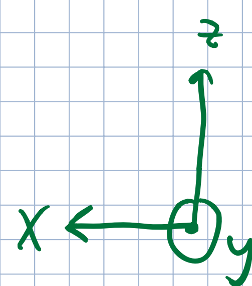
r_0: is the preferred radius

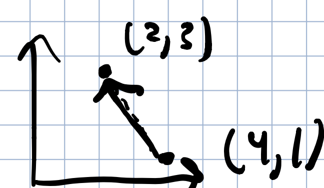
(b) The sphere is driven by assuming a constant tangential force is applied

$$F_{tangental} = f^{drive}$$

These are the magnitudes of the spring forces, but we also need to know the direction of the forces







$$\begin{bmatrix} 2-4 \\ 3-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\vec{e}_r = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} x - x_c \\ y - y_c \\ z - z_c \end{bmatrix}$$

$$r = ||\vec{e}_r||$$

The direction vectors for the forces need to be unit vectors

$$\hat{e}_r = \frac{\vec{e}_r}{\|\vec{e}_r\|} = \frac{\begin{bmatrix} x - x_c \\ y - y_c \\ z - z_c \end{bmatrix}}{\sqrt{(x-x_c)^2 + (y-y_c)^2 + (z-z_c)^2}}$$

$\vec{e}_\phi$ is the tangential direction

$$\vec{e}_\phi = \hat{e}_r \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{vmatrix} i & j & k \\ \hat{e}_{r_x} & \hat{e}_{r_y} & \hat{e}_{r_z} \\ 0 & 1 & 0 \end{vmatrix}$$

$$\vec{e}_\phi = \begin{bmatrix} -\hat{e}_{r_z} \\ 0 \\ \hat{e}_{r_x} \end{bmatrix}$$

The unit vector is

$$\hat{e}_\phi = \frac{\vec{e}_\phi}{\|\vec{e}_\phi\|}$$

So the total spring forces are

$$F_{spring} = -\lambda(r-r_0)\hat{e}_r + f^{drive}\hat{e}_\phi$$

Now the drag force:

$$F_{drag} = 6\pi\mu a\left[I + \frac{9a}{16z}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right]\vec{v}$$

where

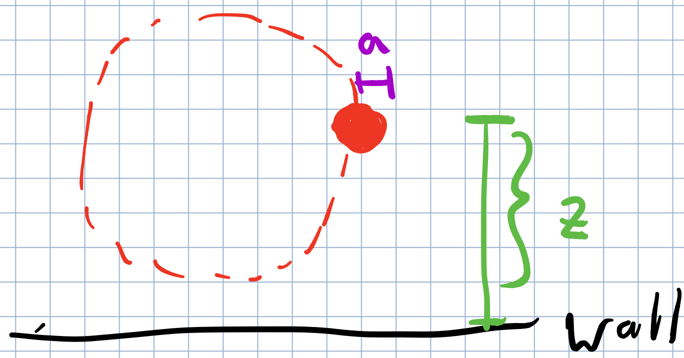
a: the radius of the sphere

μ: the fluid viscosity (How thick the fluid is)

I: 3x3 identity matrix

z: The height of the sphere above the wall

$\vec{v}$: the velocity of the sphere



So if the net force is 0 then

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

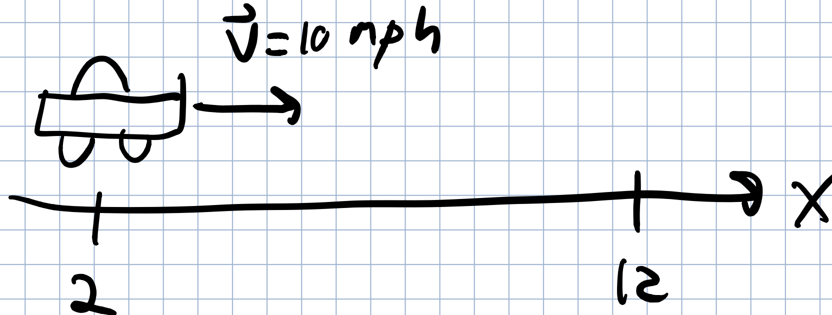
$$F_{springs} + F_{drag} = 0$$

$$-\lambda(r-r_0)\hat{e}_r + f^{drive}\vec{e}_\phi + 6\pi\mu a\left(I + \frac{9a}{16z}\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}\right)\vec{v} = \vec{0}$$

$$\underbrace{6\pi\mu a\begin{pmatrix}1+\frac{9a}{16z}&0&0\\0&1+\frac{9a}{16z}&0\\0&0&2+\frac{9a}{16z}\end{pmatrix}}_{A}\vec{v} = \underbrace{\lambda(r-r_0)\hat{e}_r - f^{drive}\hat{e}_\phi}_{\vec{b}}$$

$$A\vec{v} = \vec{b}$$

At every time step we solve for $\vec{v}$ from $A\vec{v} = \vec{b}$ and then use the velocity to update the position and then repeat.



$$X_1 = X_0 + \vec{v}\,\underbrace{dt}_{1\text{ hour}}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Forward Euler

$$\vec{X}_1 = \vec{X}_0 + \vec{v}\,\underbrace{dt}_{\text{another parameter}}$$

Parameters

λ: radial spring stiffness

r_0: preferred radius of orbit

a: sphere's radius

μ: The viscosity

dt: the time step

f^{drive}: the driving force

Final time:

Location of sphere is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Locations in time

$$3 \left\{ \overbrace{\begin{bmatrix} x_0 & x_1 & \cdots & x_f \\ y_0 & y_1 & & y_f \\ z_0 & z_1 & & z_f \end{bmatrix}}^{\#\text{ of time steps}} \right.$$

$$[x_0\ y_0\ z_0\ \ x_1\ y_1\ z_1\ \ \cdots\ \ x_f\ y_f\ z_f]$$