```
require(tidyverse); set.seed(3028)
```

**Introduction:** The file `supernovae_mth3028.csv` contains velocity (Mpc/Ga) and distance (Mpc) data about 36 Type 1A supernovae. Let $y_i$ and $x_i$ denote the velocity and distance of a supernova, respectively. The velocity is to be modelled as: $Y_i = \beta x_i + \epsilon_i$, with density function:

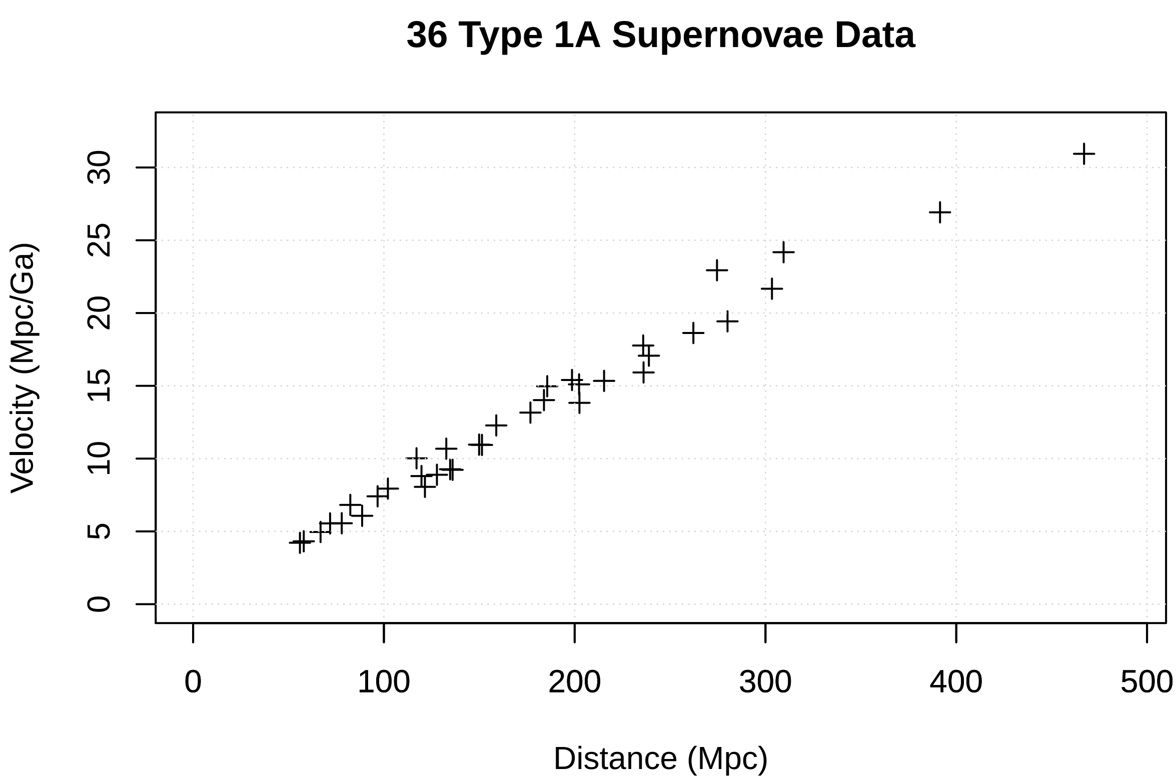
$$f(y_i; \beta, \tau) = (2\pi e^{\tau})^{-\frac{1}{2}} e^{-\frac{1}{2}e^{-\tau}(y_i - \beta x_i)^2}$$

where $\beta$ and $\tau$ are parameters. Let $\gamma = 1/\beta$.

## Question 1: Prelims

a) We begin by visualizing the the speed and distance of the 36 supernovae. We also conduct some data wrangling to suit the needs of the study; transforming the string list to numeric data.

```
supern <- read.csv(file="supernovae_mth3028.csv", header=FALSE)[2:37,] # Only Relevant Data
super <- data.frame(matrix(as.numeric(unlist(supern)), nrow=36, ncol=2)) # Numeric data frame
colnames(super) <- c("velocity", "distance")
plot(super$distance, super$velocity, main="36 Type 1A Supernovae Data",
     xlab="Distance (Mpc)",ylab="Velocity (Mpc/Ga)",
     xlim=c(0, max(super$distance)*1.05), ylim=c(0,max(super$velocity)*1.05), pch=3)
grid(lwd = 0.75); axis(1)
```




b) The quantity $\gamma$ can be interpreted as the age of the universe: The velocity of the i$^{th}$ star ($y_i$) is linearly related to the distance of teh star ($x_i$). When travelling at constant velocity yi, the time it takes to travel distance xi is equal to $x_i/y_i$ . Due to the linear relationship, this fraction is the same for each star, namely $\gamma$ . Thus, the observations suggest that all stars have have moved from an original point ($x = 0$) at the same initial time, $\gamma$ years ago.

## Question 2: Likelihood Function

Here we derive the likelihood function, and its relevant log derivatives.

a) For the likelihood function, $L(\beta, \tau; \mathbf{y})$, we suppose all $y_i$ are independently and identically distributed, with density function $f(y_i; \beta, \tau)$:

$$L(\beta, \tau; \mathbf{y}) = \prod_{i=1}^{n} (2\pi e^{\tau})^{-\frac{1}{2}} e^{-\frac{1}{2} e^{-\tau} (Y_i - \beta x_i)^2}$$

b) Log-Likelihood, $l(\beta, \tau) = \log [L(\beta, \tau; \mathbf{y})]$

$$\begin{aligned} l(\beta, \tau) &= \sum_{i=1}^{n} \log \left[ (2\pi e^{\tau})^{-\frac{1}{2}} e^{-\frac{1}{2} e^{-\tau} (Y_i - \beta x_i)^2} \right] \\ &= -\frac{n}{2} [\log(2\pi) + \tau] - \frac{e^{-\tau}}{2} \sum_{i=1}^{n} \left[ Y_i^2 - 2\beta x_i Y_i + \beta^2 x_i^2 \right] \end{aligned}$$

c) Score function $U(\beta, \tau) = \left( \frac{\partial l(\beta,\tau)}{\partial \beta}, \frac{\partial l(\beta,\tau)}{\partial \tau} \right)$, where

$$\begin{aligned} \frac{\partial}{\partial \beta} l(\beta, \tau) &= -\frac{e^{-\tau}}{2} \sum_{i=1}^{n} \left[ 2\beta x_i^2 - 2x_i Y_i \right] \\ \frac{\partial}{\partial \tau} l(\beta, \tau) &= -\frac{n}{2} + \frac{e^{-\tau}}{2} \sum_{i=1}^{n} \left[ Y_i^2 - 2\beta x_i Y_i + \beta^2 x_i^2 \right] \end{aligned}$$

d) Observed Information $J(\beta, \tau)$, where

$$\begin{aligned} J(\beta, \tau) &= \begin{bmatrix} -\frac{\partial^2 l}{\partial \beta^2} & -\frac{\partial^2 l}{\partial \beta \partial \tau} \\ -\frac{\partial^2 l}{\partial \beta \partial \tau} & -\frac{\partial^2 l}{\partial \tau^2} \end{bmatrix} \\ \frac{\partial^2 l}{\partial \beta^2} &= -e^{-\tau} \sum_{i=1}^{n} x_i^2 \\ \frac{\partial^2 l}{\partial \beta \partial \tau} &= \frac{e^{-\tau}}{2} \sum_{i=1}^{n} \left[ 2\beta x_i^2 - 2x_i Y_i \right] \\ \frac{\partial^2 l}{\partial \tau^2} &= -\frac{e^{-\tau}}{2} \sum_{i=1}^{n} \left[ Y_i^2 - 2\beta x_i Y_i + \beta^2 Y_i^2 \right] \end{aligned}$$

e) Expected Information $I(\beta, \tau) = E[J(\beta, \tau)]$:

$$\begin{aligned} I(\beta, \tau) &= \begin{bmatrix} -E\left[\frac{\partial^2 l}{\partial \beta^2}\right] & -E\left[\frac{\partial^2 l}{\partial \beta \partial \tau}\right] \\ -E\left[\frac{\partial^2 l}{\partial \beta \partial \tau}\right] & -E\left[\frac{\partial^2 l}{\partial \tau^2}\right] \end{bmatrix} \\ &= \end{aligned}$$