

**Question 1:  $M^X/M/1/4$  Queue**

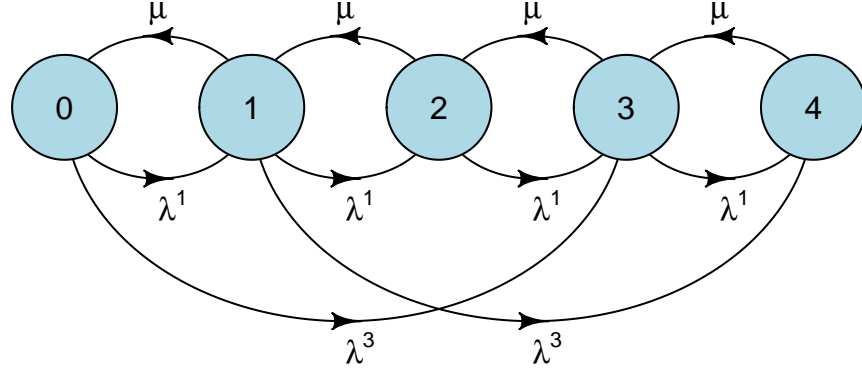


Figure 1:  $M^X/M/1/4$ , with  $\lambda^1$  and  $\lambda^3$  arrivals. See Annex A for R Code.

- a) We can illustrate the steady state system, with  $\lambda^1$  denoting the arrival of a batch with 1 customer, and  $\lambda^3$  denoting a batch with 3 customers. *These are not indices, but a limit of the software.* Now, from state space diagram, in Figure 1, we can form the steady state rate equations:

$$\mu P_1 = (\lambda_1 + \lambda_3) P_0, \quad \mu P_2 = (\lambda_1 + \lambda_3) P_1, \quad \mu P_3 = \lambda_1 P_2, \quad \mu P_4 = \lambda_1 P_3$$

These rate equations lead us to an expression for each state:

$$P_1 = \left[ \frac{\lambda_1 + \lambda_3}{\mu} \right] P_0, \quad P_2 = \left[ \frac{\lambda_1 + \lambda_3}{\mu} \right] P_1, \quad P_3 = \frac{\lambda_1}{\mu} P_2, \quad P_4 = \frac{\lambda_1}{\mu} P_3$$

$$\lambda_1, \lambda_3 = 1, \quad \mu = 1 \implies P_1 = 2P_0, \quad P_2 = 2P_1, \quad P_3 = P_2, \quad P_4 = P_3$$

Now, since  $\sum_{n=0}^4 P_n = 1$ , then

$$\begin{aligned} \sum_{n=0}^4 P_n &= P_0 + 2P_0 + 4P_0 + 4P_0 + 4P_0 = 1 \\ \therefore P_0 &= \frac{1}{15} \end{aligned}$$

Hence, we obtain:  $P_0 = \frac{1}{15}$ ,  $P_1 = \frac{2}{15}$ ,  $P_2 = \frac{4}{15}$ ,  $P_3 = \frac{4}{15}$  and  $P_4 = \frac{4}{15}$ .

- b) From the steady state probabilities, we can deduce the expected length of the system  $L_2$ :

$$\begin{aligned} L_s = E(N_s) &= \sum_{n=1}^4 n P_n = \frac{1}{15} (0 + 1(2) + 2(4) + 3(4) + 4(4)) \\ &= \frac{38}{15} \approx 2.53 \text{ (3s.f.)} \end{aligned}$$

- c) For expected waiting time  $W_s$  in the system, we notice that the waiting time in the queue is  $W_q = E(T)E(N)$ , where  $E(T) = 1/\mu$  denotes the expected time for a set of individuals to be served. Hence  $W_q = L_s/\mu$ . Now, we also have that the mean waiting time is the mean time spent in the queue plus the mean service time:  $W_s = W_q + 1/\mu$ .

Hence,  $W_s = \frac{38}{15} + 1 = \frac{53}{15} \implies 3 \text{ mins } 32 \text{ seconds, expected total waiting time.}$

## Question 2: $M^X/M/1/\infty$ Queue

Queue Process  $M/M/1$  with infinite capacity and variable arrival rate. An arrival rate of  $\lambda_n \leq 1$ , and service rate  $\mu = 1$ . Arrival and Service are independent Poisson processes.

a)  $\lambda_n = \frac{n^4}{(n+1)^4}$ , with  $\lambda_0 = 1$

In a steady state, this system has:  $\lambda_0 P_0 = P_1$ ,  $\lambda_1 P_1 = P_2 \dots$

This leads to  $P_n = P_0 \prod_{m=0}^n \lambda_m$ . Hence, if a steady state does exist then  $\sum_{n \geq 0} P_n = 1$ . This is only possible if  $\sum_{n \geq 0} \prod_{m=0}^n \lambda_m$  is finite – converges. Importantly, we reduce the product to:

$$\begin{aligned} \prod_{m=0}^n \lambda_m &= 1 \cdot \frac{1^4}{2^4} \cdot \frac{2^4}{3^4} \cdot \dots \cdot \frac{n^4}{(n+1)^4} = \left[ \frac{n!}{(n+1)!} \right]^4 \\ &= \frac{1}{(n+1)^4} \end{aligned}$$

Hence, we obtain

$$\sum_{n \geq 0} \frac{1}{(n+1)^4} = \sum_{n \geq 1} \frac{1}{n^4}$$

Now, by observing the similar general problem,  $\sum_{k=1}^{\infty} \frac{1}{k^\alpha} < \infty$  if  $\alpha > 1$ , we see that  $\sum_{n \geq 0} \prod_{m=0}^n \lambda_m$  converges for this  $\lambda_n$ , where  $\alpha = 4 > 1$ . Hence, it is clear that a form for  $P_0$  exists ( $P_0 = \frac{90}{\pi^4}$ ), and a steady state probability distribution does exist.

Now, for the expected system size,  $L_s = E(N_s)$

$$\begin{aligned} E(N_s) &= \sum_{n \geq 0} n P_n \\ &= P_0 \sum_{n \geq 0} n \cdot \frac{1}{(n+1)^4} = P_0 \sum_{n \geq 1} \frac{n-1}{n^4} \\ &= P_0 \sum_{n \geq 1} \left[ \frac{1}{n^3} - \frac{1}{n^4} \right] \end{aligned}$$

Clearly, this converges and, since  $\frac{1}{n^4} < \frac{1}{n^3}$  for all  $n$ ,  $L_s$  is finite and positive.

b)  $\lambda_n = \frac{\sqrt{n}}{\sqrt{n+1}}$ , with  $\lambda_0 = 1$

Similarly, as before, we have  $P_n = P_0 \prod_{m=0}^n \lambda_m$ , where the product decomposes to:

$$\begin{aligned} \prod_{m=0}^n \lambda_m &= 1 \cdot \frac{\sqrt{1}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{3}} \cdot \dots \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = \sqrt{\frac{n!}{(n+1)!}} \\ &= \frac{1}{\sqrt{n+1}} \end{aligned}$$

This leads us to:

$$\sum_{n \geq 0} P_n = P_0 \sum_{n \geq 1} \frac{1}{\sqrt{n}}$$

Now, studying the same general form  $\sum_{k=1}^{\infty} \frac{1}{k^\alpha} = \infty$  if  $\alpha \leq 1$ , we see that  $\alpha = \frac{1}{2} \leq 1$ . Hence, this does not converge and there is no viable form for  $P_n$  for a steady state. Therefore there is no steady state probability distribution.

c)  $\lambda_n = \frac{n^{\frac{3}{2}}}{(n+1)^{\frac{3}{2}}}$ , with  $\lambda_0 = 1$ .

Similar, as above, we have  $\lambda_0 P_0 = P_1$ ,  $\lambda_1 P_1 = P_2 \dots$

Now we have a  $P_n$  with a product term of the form:

$$\begin{aligned} \prod_{m=0}^n \lambda_m &= 1 \cdot \frac{1^{\frac{3}{2}}}{2^{\frac{3}{2}}} \cdot \frac{2^{\frac{3}{2}}}{3^{\frac{3}{2}}} \cdot \dots \cdot \frac{n^{\frac{3}{2}}}{(n+1)^{\frac{3}{2}}} = \left[ \frac{n!}{(n+1)!} \right]^{\frac{3}{2}} \\ &= \frac{1}{(n+1)^{\frac{3}{2}}} \end{aligned}$$

This leads to  $\sum_{n \geq 0} P_n$  of the form:

$$P_0 \sum_{n \geq 1} \frac{1}{n^{\frac{3}{2}}}$$

This clearly has  $\alpha = 1.5 > 1$ , hence this will converge and deliver a density function for  $P_n$ , so a steady state distribution does exist.

Now, for the form of the expected system size  $L_s = E[N_s]$ :

$$\begin{aligned} E[N_s] &= \sum_{n \geq 0} n P_n \\ &= P_0 \sum_{n \geq 0} n \cdot \frac{1}{(n+1)^{\frac{3}{2}}} \\ &= P_0 \sum_{n \geq 1} \left[ \frac{1}{n^{\frac{1}{2}}} - \frac{1}{n^{\frac{3}{2}}} \right] \end{aligned}$$

Notice that we have one sum that converges and one that doesn't. Hence the overall term does not converge, and there is no finite form for  $L_s$ . We can expect the system to have infinite size.

### Question 3: Markov Chain

a) With the given transition matrix, we can illustrate the markov chain, with 'a' denoting  $\alpha$ :

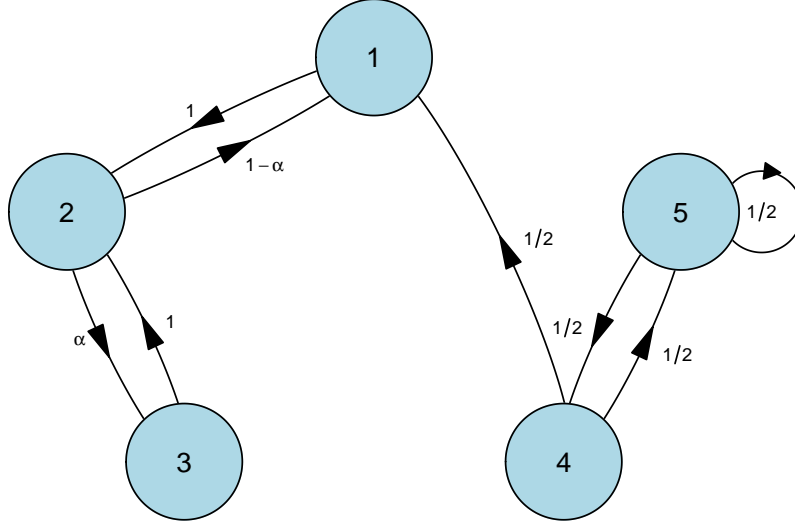


Figure 2: 5 State Markov Chain - See Annex A for R Code

$$f_1^1 = 0, \quad f_1^2 = 1 - \alpha, \quad f_1^3 = 0, \quad f_1^4 = \alpha(1 - \alpha),$$

$$f_1^5 = 0, \quad f_1^6 = (1 - \alpha)\alpha^2, \quad f_1^7 = 0, \quad f_1^8 = (1 - \alpha)\alpha^3, \quad \dots$$

Hence, we reach that  $\sum_{n \geq 1} f_1^{(n)} = (1 - \alpha)(1 + \alpha + \alpha^2 + \alpha^3 + \dots)$ . Now observe that the expansion of  $(1 - x)^{-1} = 1 + x + x^2 + \dots$ , hence,  $\sum_{n \geq 1} f_1^{(n)} = (1 - \alpha)(1 - \alpha)^{-1} = 1$ , Hence state 1 is a recurrent state, and will surely return to itself. Now to show positive recurrence – that state 1 will surely return to itself in finite time – we require that  $\mu_1 = \sum_{n \geq 1} n f_1^{(n)} < \infty$ , which can be shown, by considering only the non-zero terms, which arise from even values of  $n \in \mathbb{N}$ ,

$$\begin{aligned} \mu_i &= \sum_{n \geq 1} n f_1^{(n)} = \sum_{n \geq 1} (2n) f_1^{(2n)} \\ &= 2 \sum_{n \geq 1} n (-1\alpha) \alpha^{n-1} \\ &= 2(1 - \alpha) \sum_{n \geq 1} n \alpha^{n-1} \end{aligned}$$

Now, since  $\alpha \in (0, 1)$ , in particular  $\alpha < 1$ , then observe the general form of the sum on the RHS:  $\sum_{k=1}^{\infty} k r^{k-1} = 1/(1 - r^2)$ , for  $|r| < 1$ . This leads to,

$$\sum_{n \geq 1} n f_1^{(n)} = \frac{2(1 - \alpha)}{(1 - \alpha)^2} = \frac{2}{1 - \alpha}$$

Clearly, we have that  $\mu_1 < \infty$ , hence state 1 is positively recurrent as they can all return to themselves.

- b) From Figure 2, we see that there are two unchains of groups of states that can communicate – mutually accessible. States 1, 2 & 3 make a subchain, and 4 & 5 make the second chain. This second chain contains transient states, because once the system moves from state 4 to state 1, it can no longer return to state 4, or 5. Hence, states 4 and 5 are aperiodic.

The subchain of states 1,2 & 3/ contains positively recurrent states, because we showed that state 1 is positively recurrent and we know that all states in an irreducible subchain share this recurrence property; used without proof. Also, we observe that this subchain of recurrent states is aperiodic, hence it is ergodic.

- c) For the steady state probability vector, we can resolve  $T^t \tilde{P}^t = \tilde{P}^t$ , where  $\tilde{P}$  denotes the row matrix of the stationary distribution for the system:

$$\begin{pmatrix} 0 & 1-\alpha & 0 & 1/2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$\begin{pmatrix} (1-\alpha)x_2 + \frac{1}{2}x_4 \\ x_1 + x_3 \\ \alpha x_2 \\ \frac{1}{2}x_5 \\ \frac{1}{2}(x_4 + x_5) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

The final two rows require that  $x_5 = x_4 = 0$ . The second and third rows require that  $x_1 = (1-\alpha)x_2$ , which can be put into the equation in row 1, leading to :  $x_2 = x_2$ . We can now freely choose  $x_2 = 1$ . Hence, the steady state distribution  $\tilde{P} = (1-\alpha, 1, \alpha, 0, 0)$

## Annex A: Queue System R Code

```
require('heemod')
require('shape')
require('diagram')
Tr2 <- define_transition( state_names = c('4', '3', '2', '1', '0'),
  0, mu, 0, 0, 0,
  lambda^1, 0, mu, 0, 0,
  0, lambda^1, 0, mu, 0,
  lambda^3, 0, lambda^1, 0, mu,
  0, lambda^3, 0, lambda^1, 0)

curves <- matrix(nrow = 5, ncol = 5, 0.55)
plot(Tr2, pos=c(5),curve=curves, endhead=FALSE,
  arr.length=0.3, latex=FALSE,
  lwd = 1, box.lwd = 1, box.col = "lightblue",
  cex.txt = 1, box.size = 0.07)
```

## Annex B: Markov Chain Plot in R

```
library('heemod')
library('shape')
library('diagram')

Tr <- define_transition(
  state_names = c('1', '2', '3', '4', '5'),
  0, 1, 0, 0, 0,
  1-alpha, 0, alpha, 0, 0,
  0, 1, 0, 0, 0,
  .5, 0, 0, 0, .5,
  0, 0, 0, .5, .5);

curves <- matrix(nrow = 5, ncol = 5, 0.065)

plot(Tr,
  curve=curves,
  self.shiftx = 0.125,
  self.shifty = 0,
  self.arrpos = 1.4,
  arr.type= "triangle",
  self.cex=0.7,
  latex = FALSE,
  lwd = 1,
  box.lwd = 1,
  box.col = "lightblue",
  cex.txt = 0.7,
  box.size = 0.09)
```