Question 1: $M^X/M/1/4$ **Queue**

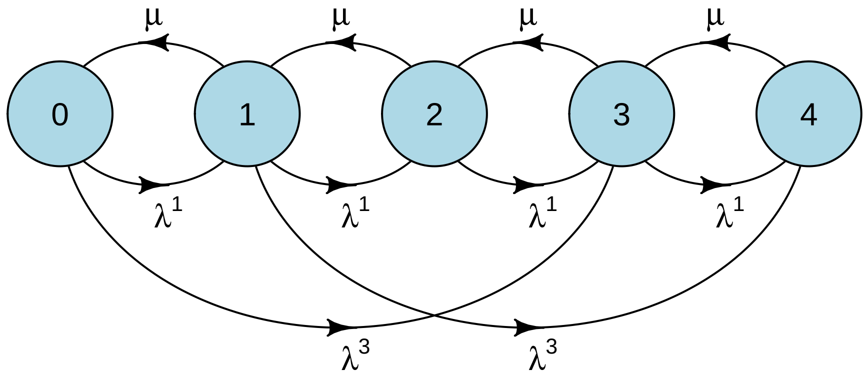

Figure 1: $M^X/M/1/4$, with λ^1 and λ^3 arrivals. See Annex A for R Code.

a) We can illustrate the steady state system, with λ^1 denoting the arrival of a batch with 1 customer, and λ^3 denoting a batch with 3 customers. *These are not indicies, but a limit of the software.* Now, from state space diagram, in Figure 1, we can form the steady state rate equations:

$$\mu P_1 = (\lambda_1 + \lambda_3)P_0, \qquad \mu P_2 = (\lambda_1 + \lambda_3)P_1, \qquad \mu P_3 = \lambda_1 P_2, \qquad \mu P_4 = \lambda_1 P_3$$

These rate equations lead us to an expression for each state:

$$P_1 = \left[\frac{\lambda_1 + \lambda_3}{\mu}\right] P_0, \qquad P_2 = \left[\frac{\lambda_1 + \lambda_3}{\mu}\right] P_1, \qquad P_3 = \frac{\lambda_1}{\mu}, \qquad P_4 = \frac{\lambda_1}{\mu} P_3$$

$$\lambda_1, \lambda_3 = 1, \quad \mu = 1 \implies P_1 = 2P_0, \quad P_2 = 2P_1, \quad P_3 = P_2, \quad P_4 = P_3$$

Now, since $\sum_{n=0}^{4} P_n = 1$, then

$$\begin{aligned} \sum_{n=0}^{4} P_n &= P_0 + 2P_0 + 4P_0 + 4P_0 + 4P_0 = 1 \\ \therefore P_0 &= \frac{1}{15} \end{aligned}$$

Hence, we obtain: $P_0 = \frac{1}{15}$, $P_1 = \frac{2}{15}$, $P_2 = \frac{4}{15}$, $P_3 = \frac{4}{15}$ and $P_4 = \frac{4}{15}$.

b) From the steady state probabilities, we can deduce the expected length of the system L_2:

$$\begin{aligned} L_s = E(N_s) &= \sum_{n=1}^{4} nP_n = \frac{1}{15}(0 + 1(2) + 2(4) + 3(4) + 4(4)) \\ &= \frac{38}{15} \quad \approx 2.53 \quad (3s.f.) \end{aligned}$$

c) For expected waiting time W_s in the system, we notice that the waiting time in the queue is $W_q = E(T)E(N)$, where $E(T) = 1/\mu$ denotes the expected time for a set of individuals to be served. Hence $W_q = L_s/\mu$. Now, we also have that the mean waiting time is the mean time spent in the queue plus the mean service time: $W_s = W_q + 1/\mu$.
Hence, $W_s = \frac{38}{15} + 1 = \frac{53}{15} \implies$ 3 mins 32 seconds, expected total waiting time.

Question 2: $M^X/M/1/\infty$ Queue

Queue Process $M/M/1$ with infinite capacity and variable arrival rate. An arrival rate of $\lambda_n \leq 1$, and service rate $\mu = 1$. Arrival and Service are independent Poisson processes.

a) $\lambda_n = \frac{n^4}{(n+1)^4}$, with $\lambda_0 = 1$
In a steady state, this system has: $\lambda_0 P_0 = P_1$, $\lambda_1 P_1 = P_2 \ldots$
This leads to $P_n = P_0 \prod_{m=0}^{n} \lambda_m$. Hence, if a steady state does exist then $\sum_{n\geq 0} P_n = 1$. This is only possible if $\sum_{n\geq 0} \prod_{m=0}^{n} \lambda_m$ is finite – converges. Importantly, we reduce the product to:

$$\begin{aligned} \prod_{m=0}^{n} \lambda_m &= 1 \cdot \frac{1^4}{2^4} \cdot \frac{2^4}{3^4} \cdot \ldots \cdot \frac{n^4}{(n+1)^4} = \left[\frac{n!}{(n+1)!}\right]^4 \\ &= \frac{1}{(n+1)^4} \end{aligned}$$

Hence, we obtain

$$\sum_{n\geq 0} \frac{1}{(n+1)^4} = \sum_{n\geq 1} \frac{1}{n^4}$$

Now, by observing the similar general problem, $\sum_{k=1}^{\infty} \frac{1}{k^\alpha} < \infty$ if $\alpha > 1$, we see that $\sum_{n\geq 0} \prod_{m=0}^{n} \lambda_m$ converges for this λ_n, where $\alpha = 4 > 1$. Hence, it is clear that a form for P_0 exists ($P_0 = \frac{90}{\pi^4}$), and a steady state probability distribution <u>does exist</u>.
Now, for the expected system size, $L_s = E(N_s)$

$$\begin{aligned} E(N_s) &= \sum_{n\geq 0} n P_n \\ &= P_0 \sum_{n\geq 0} n \cdot \frac{1}{(n+1)^4} = P_0 \sum_{n\geq 1} \frac{n-1}{n^4} \\ &= P_0 \sum_{n\geq 1} \left[\frac{1}{n^3} - \frac{1}{n^4}\right] \end{aligned}$$

Clearly, this converges and, since $\frac{1}{n^4} < \frac{1}{n^3}$ for all n, L_s is <u>finite</u> and positive.

b) $\lambda_n = \frac{\sqrt{n}}{\sqrt{n+1}}$, with $\lambda_0 = 1$
Similarly, as before, we have $P_n = P_0 \prod_{m=0}^{n} \lambda_m$, where the product decomposes to:

$$\begin{aligned} \prod_{m=0}^{n} \lambda_m &= 1 \cdot \frac{\sqrt{1}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{3}} \cdot \ldots \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = \sqrt{\frac{n!}{(n+1)!}} \\ &= \frac{1}{\sqrt{n+1}} \end{aligned}$$

This leads us to:

$$\sum_{n\geq 0} P_n = P_0 \sum_{n\geq 1} \frac{1}{\sqrt{n}}$$

Now, studying the same general form $\sum_{k=1}^{\infty} \frac{1}{k^\alpha} = \infty$ if $\alpha \leq 1$, we see that $\alpha = \frac{1}{2} \leq 1$. Hence, this does not converge and there is no viable form for P_n for a steady state. Therefore there is no steady state probability distribution.

c) $\lambda_n = \frac{n^{\frac{3}{2}}}{(n+1)^{\frac{3}{2}}}$, with $\lambda_0 = 1$.
Similar, as above, we have $\lambda_0 P_0 = P_1$, $\lambda_1 P_1 = P_2 \ldots$
Now we have a P_n with a product term of the form:

$$\begin{aligned}\prod_{m=0}^{n} \lambda_m &= 1 \cdot \frac{1^{\frac{3}{2}}}{2^{\frac{3}{2}}} \cdot \frac{2^{\frac{3}{2}}}{3^{\frac{3}{2}}} \cdot \ldots \cdot \frac{n^{\frac{3}{2}}}{(n+1)^{\frac{3}{2}}} = \left[\frac{n!}{(n+1)!}\right]^{\frac{3}{2}} \\ &= \frac{1}{(n+1)^{\frac{3}{2}}}\end{aligned}$$

This leads to $\sum_{n\geq 0} P_n$ of the form:

$$P_0 \sum_{n\geq 1} \frac{1}{n^{\frac{3}{2}}}$$

This clearly has $\alpha = 1.5 > 1$, hence this will converge and deliver a density function for P_n, so a steady state distribution does exist.
Now, for the form of the expected system size $L_s = E[N_s]$:

$$\begin{aligned}E[N_s] &= \sum_{n\geq 0} nP_n \\ &= P_0 \sum_{n\geq 0} n \cdot \frac{1}{(n+1)^{\frac{3}{2}}} \\ &= P_0 \sum_{n\geq 1} \left[\frac{1}{n^{\frac{1}{2}}} - \frac{1}{n^{\frac{3}{2}}}\right]\end{aligned}$$

Notice that we have one sum that converges and one that doenst. Hence the overall term does not converge, and there is no finite form for L_s. We can expect the system to have infinite size.

Question 3: Markov Chain

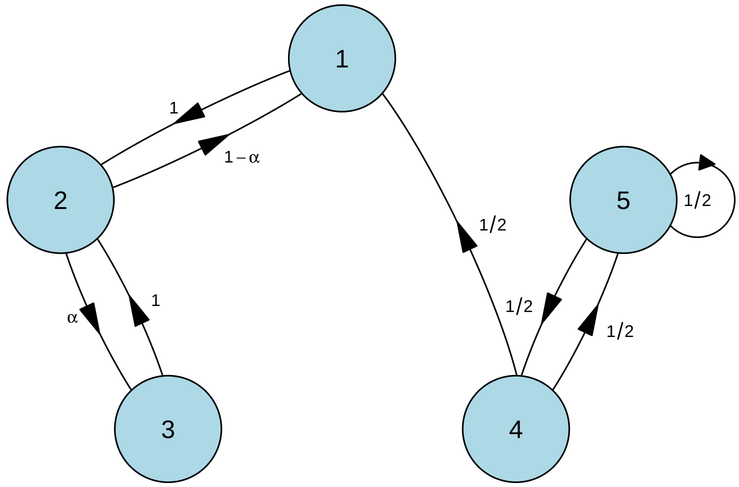

Figure 2: 5 State Markov Chain - See Annex A for R Code

a) The Markov system, with transition matrix T is illustrated in Figure 2. We can show state 1 is transient by studying the probabilities for first return to 1 in n steps, $f_1^{(n)}$:

$$f_1^1 = 0, \quad f_1^2 = 1 - \alpha, \quad f_1^3 = 0, \quad f_1^4 = \alpha(1-\alpha),$$

$$f_1^5 = 0, \quad f_1^6 = (1-\alpha)\alpha^2 \quad f_1^7 = 0, \quad f_1^8 = (1-\alpha)\alpha^3, \quad ...$$

Hence, we reach that $\sum_{n\geq 1} f_1^{(n)} = (1-\alpha)(1+\alpha+\alpha^2+\alpha^3+\ldots)$. Now observe that the expansion of $(1-x)^{-1} = 1+x+x^2+...$, hence, $\sum_{n\geq 1} f_1^{(n)} = (1-\alpha)(1-\alpha)^{-1} = 1$, Hence state 1 is a recurrent state, and will surely return to itself. Now to show positive recurrence – that state 1 will surely return to itself in finite time – we require that $\mu_1 = \sum_{n\geq 1} nf_1^{(n)} < \infty$, which can be shown, by considering only the non-zero terms, which arise from even values of $n \in \mathbb{N}$,

$$\begin{aligned} \mu_i = \sum_{n\geq 1} nf_1^{(n)} &= \sum_{n\geq 1}(2n)f_1^{(2n)} = 2\sum_{n\geq 1} n(-1\alpha)\alpha^{n-1} \\ &= 2(1-\alpha)\sum_{n\geq 1} n\alpha^{n-1} \end{aligned}$$

Now, since $\alpha \in (0,1)$, in particular $\alpha < 1$, then observe the general form of the sum on the RHS: $\sum_{k=1}^{\infty} kr^{k-1} = 1/(1-r^2)$, for $|r| < 1$. This leads to,

$$\sum_{n\geq 1} nf_1^{(n)} = \frac{2(1-\alpha)}{(1-\alpha)^2} = \frac{2}{1-\alpha}$$

Clearly, we have that $\mu_1 < \infty$, hence state 1 is positively recurrent as they can all return to themselves in finite time.
Similarly, we can show that state 4 is transient by forming $f_4^{(n)} = (2)^{-n}$, for $n \geq 2$, hence $\sum_{n\geq 2} f_4^{(n)} = \left[\sum_{n\geq 0} \frac{1}{2}^n\right] - \frac{3}{2}$, where we know that the solution to the infinite sum $\sum_{n\geq 0} \frac{1}{2}^n = 2$. Now we reach: $\sum_{n\geq 2} f_4^{(n)} = \frac{1}{2} < 1$. Hence, state 4 is clearly transient.

b) From Figure 2, we see that there are two unchains of groups of states that can communicate – mutually accessible. States 1, 2 & 3 make a subchain, and 4 & 5 make the second chain. This second chain contains transient states, because once the system moves from state 4 to state 1, it can no longer return to state 4, or 5. Hence, states 4 and 5 are aperiodic.
The subchain of states 1,2 & 3 contains positively recurrent states, because we showed that state 1 is positively recurrent and we know that all states in an irreducible subchain share this recurrence property; used without proof. Also, we observe that this subchain of recurrent states is aperiodic, hence it is ergodic.

c) For the steady state probability vector, we can resolve $T^t\tilde{P}^t = \tilde{P}^t$, where $\tilde{P}$ denotes the row matrix of the stationary distribution for the system:

$$\begin{pmatrix} 0 & 1-\alpha & 0 & 1/2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$\begin{pmatrix} (1-\alpha)x_2 + \frac{1}{2}x_4 \\ x_1 + x_3 \\ \alpha x_2 \\ \frac{1}{2}x_5 \\ \frac{1}{2}(x_4 + x_5) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

The final two rows require that $x_5 = x_4 = 0$. The second and third rows requrie that $x_1 = (1-\alpha)x_2$, which can be put into the equation in row 1, leading to : $x_2 = x_2$. We can now freely choose x_2 – let $x_2 = 1$. Hence, the steady state distribution $\tilde{P} = (1-\alpha, \quad 1, \quad \alpha, \quad 0, \quad 0)$.

d) Let $a_n = \frac{n}{n+1}$, such that the transition matrix, T, is a function of time: $T(n)$. In order to classify state 1 in the chain with time-dependent transition matrix T, we adjust our definition of recurrence: a state which we return to infinitely often.
Now, observe that every trajectory into state 1 must pass through through the link from state 2 to state 1, with probability $1 - \alpha_n$, where n is the time step. Hence, the probability of any such trajectory, will be multiplied by $\prod_{n\geq 1}[1-\alpha_n]$.
Now, since we are allowed to assume that $\prod_{n=1}^{\infty}[1-a_n] = 0$ if $a_n \in (0,1)$ and $\sum_{n=1}^{\infty} a_n$ diverges, then we seek to show that $\sum_{n\geq 1}\alpha_n$ diverges:

$$\begin{aligned} \sum_{n\geq 1}\alpha_n &= \sum_{n\geq 1}\frac{n}{n+1} \\ &= \sum_{n\geq 1}\frac{n-1}{n} \\ &= \sum_{n\geq 1}\left[1-\frac{1}{n}\right] = \sum_{n\geq 1}[1] - \sum_{n\geq 1}\frac{1}{n} \end{aligned}$$

This last summation expression clearly diverges, hence $\sum_{n\geq 1}\alpha_n$ diverges, and since $a_n \in (0,1)$, by definition, then $\prod_{n\geq 1}[1-\alpha_n] = 0$. This indicates that for very large trajectories (time steps, ∞), the probabilities are multiplied by zero. Hence, in the long run, trajectories do no visit state 1 infinitely often, hence state 1 is clearly transient.

Annex A: Queue System R Code

```
require('heemod')
require('shape')
require('diagram')
Tr2 <- define_transition( state_names = c('4', '3', '2', '1', '0'),
  0, mu, 0, 0, 0,
  lambda^1, 0, mu, 0, 0,
  0, lambda^1, 0, mu, 0,
  lambda^3, 0, lambda^1, 0, mu,
  0, lambda^3 , 0, lambda^1, 0)

curves <- matrix(nrow = 5, ncol = 5, 0.55)
plot(Tr2, pos=c(5),curve=curves, endhead=FALSE,
     arr.length=0.3, latex=FALSE,
     lwd = 1, box.lwd = 1, box.col = "lightblue",
     cex.txt = 1, box.size = 0.07)
```

Annex B: Markov Chain Plot in R

```
library('heemod')
library('shape')
library('diagram')

Tr <- define_transition(
  state_names = c('1', '2', '3', '4', '5'),
  0, 1, 0, 0, 0,
  1-alpha, 0, alpha, 0, 0,
  0, 1, 0, 0, 0,
  .5, 0, 0, 0, .5,
  0, 0, 0, .5, .5);

curves <- matrix(nrow = 5, ncol = 5, 0.065)

plot(Tr,
     curve=curves,
     self.shiftx = 0.125,
     self.shifty = 0,
     self.arrpos = 1.4,
     arr.type= "triangle",
     self.cex=0.7,
     latex = FALSE,
     lwd = 1,
     box.lwd = 1,
     box.col = "lightblue",
     cex.txt = 0.7,
     box.size = 0.09)
```