Condensed vector spaces

John MacQuarrie, UFMG

Dedicated to Said Sidki on the occasion of his 85th birthday.
Happy Birthday, Said!!!

Talk outline

1. Motivation and the main problem
2. Condensed mathematics
3. New work

Joint work with Lucas Henrique de Souza (UFMG) and Jeremy Rickard (University of Bristol)

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Part 1

Motivation and the main problem

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But actually this talk is about the underlying k-vector spaces, so we don't need to know anything about pseudocompact algebras in this talk (except a few slides of motivation at the start).

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with W indecomposable. If W is projective, the sequence splits, which isn't so interesting.

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So Auslander–Reiten theory relates almost all the indecomposable modules amongst each other. These relations are *fundamental* in modern representation theory.

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So to do AR-theory, it's very important that we can dualize finite-dimensional A-modules!

My algebras

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The underlying vector space of a pseudocompact k-algebra is LC.

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- A-**DMod** — discrete A-modules, whose underlying vector spaces are discrete (so just normal vector spaces: k-**Vec**).
- A-**PCMod** — pseudocompact A-modules, whose underlying vector spaces are in k-**LC**.
- These categories are abelian, and they are dual. At the level of vector spaces

$$\left(\bigoplus_I k\right)^* = \prod_I k \text{ and } \left(\prod_I k\right)^* = \bigoplus_I k.$$

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- abelian,
- with complete objects (a technical condition, but needed for results like Krull–Schmidt),
- and closed under duality.

Solution: first attempt

We want a category containing *both* A-**DMod** and A-**PCMod**.

So the underlying category of k-vector spaces contains both discrete spaces $\bigoplus k$ and LC spaces $\prod k$.

From now on we consider only vector spaces. Forget about A!

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  is continuous and bijective,
- but not an isomorphism! So

$$V^{\mathrm{dis}}/\mathrm{Ker}(\rho) = V^{\mathrm{dis}} \ncong V = \mathrm{Im}(\rho)$$

  —the first iso theorem fails: k-**LLC** is not abelian! ☹

Part 2

Condensed mathematics

The same problem for **TAb**

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I'll try and give the intuition, then we'll formalize what's going on!

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- Fix $C \subseteq \mathbb{R}$ as follows:

$$C = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\} \cup \{0\}.$$

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- Let's apply $(C, -)$ to $\rho : \mathbb{R}^{\mathrm{dis}} \to \mathbb{R}$:

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$(C, \mathbb{R}^{\mathsf{dis}})$: Given $\gamma : C \to \mathbb{R}^{\mathsf{dis}}$, $\gamma^{-1}(\gamma(0))$ is an open neighborhood of 0, so *cofinite* in C. Thus almost every point of C must be sent to the same value: $(C, \mathbb{R}^{\mathsf{dis}})$ consists of eventually constant sequences in $\mathbb{R}$ — $(C, \mathbb{R}^{\mathsf{dis}})$ is small.

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(C, ρ): the induced map $(C, \mathbb{R}^{\mathrm{dis}}) \to (C, \mathbb{R})$ is

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So (C, ρ) is injective, but has a huge cokernel! The functor $(C, -)$ can "see", via the cokernel, that ρ is not an isomorphism!

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Let's formalize what just happened:

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Example

Let
$$C_n = \left\{\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}\right\} \cup \{0\}.$$

For $n > m$, define
$$\varphi_{mn} : \left\{\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{m}, \cdots, \frac{1}{n}\right\} \cup \{0\} \to \left\{\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{m}\right\} \cup \{0\}$$

by sending $1/i$ $(i \leq m)$ to itself, and all other points to 0. Then $\varprojlim C_n = C$, so $C \in \mathbf{Prof}$.

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- Given $U \in \mathcal{O}(X)$ and an open cover $U = \bigcup_{i \in I} U_i$, we obtain

$$T(U) \xrightarrow{\alpha} \prod_{i \in I} T(U_i) \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{i,j \in I} T(U_i \cap U_j)$$

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- T is a sheaf if α is the equalizer of β and γ (for every open U and every open cover of U).

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Definition (deliberately vague!)

A site is a category $\mathcal{C}$ together with a collection of "covers" of each object — a cover of an object c being a collection of morphisms $\{\gamma_i : x_i \to c\}$ satisfying axioms analogous to those of ordinary open covers.

Key example

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Prof, with covers of X given by finite collections of "jointly surjective" maps: that is,

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- That is, in the *site* **Prof**, we have projective covers!

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$$T(S_1 \sqcup S_2) = T(S_1) \times T(S_2),$$
3. given a surjection $\rho : S' \twoheadrightarrow S$ in $\mathbf{Prof}$, $T(\rho)$ is the equalizer of
$$T(S) \xrightarrow{T(\rho)} T(S') \rightrightarrows T(S' \times_S S').$$

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- Advantage: when defined on projectives, Condition 3 follows from Conditions 1 and 2! That is, a condensed abelian group is a functor $T : \mathbf{Proj} \to \mathbf{Ab}$ such that $T(\varnothing) = 0$ and $T(S_1 \sqcup S_2) = T(S_1) \times T(S_2)$.

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- Disadvantage: the category **Proj** is very fragile; for example, the product $P \times Q$ of projectives is almost never projective!

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- *The functor* $\mathbf{CGAb} \to \mathbf{Cond}(\mathbf{Ab})$ *given by* $G \mapsto \underline{G}$ *is fully faithful!*
- *The category* $\mathbf{Cond}(\mathbf{Ab})$ *is abelian!*
- *In other words, we've embedded our category of topological objects, in the best possible way, into an abelian category!*

A technical point

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The category of solid objects is also abelian, and (following Clausen) this is where the rich theory really happens.

Part 3

New work

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- It has complete objects;
- It is closed under duality.

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In particular, we cannot take $k = \overline{k}$.

This looks bad. In my view, the issue is that **Prof** can see inverse limits, but the complexity of fields comes from direct limits ($\overline{k}$ is the *union* of the finite subextensions of k).

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But recall that the category of inverse limits of finite-dimensional k-vector spaces is k-**LC**. Thus:

Main definition

Definition (main one of the talk, with a small lie)

A *linearly condensed k-vector space* is a sheaf

$$k\text{-}\mathbf{LC} \to k\text{-}\mathbf{Vec}.$$

Their category, with morphisms given by natural transformations, is $\mathbf{LCond}(k\text{-}\mathbf{Vec})$.

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- *The functor* $k\text{-}\mathbf{LLC} \to \mathbf{LCond}(k\text{-}\mathbf{Vec})$ *given by* $V \mapsto \underline{V}$ *is fully faithful.*
- *The category* $\mathbf{LCond}(k\text{-}\mathbf{Vec})$ *is abelian!*

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- *It also follows that every object* $T \in \mathbf{LCond}(k\text{-}\mathbf{Vec})$ *is solid (= complete)!*

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- *On the subcategory* k-$\mathbf{LLC}$ *it is perfect:* $\underline{V}^{**} \cong \underline{V}$.
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We will see the consequences of this mystery when we start applying the theory to representations!

Thank you!