CS 234

Module 8

November 15, 2018

ADT Priority Queue

Data: (key, element pairs) where

- keys are orderable but not necessarily distinct, and
- elements are any data.

Preconditions: For all P is a priority queue, k is a key, and e is an element; for LookUpMin and DeleteMin, P is not empty.

Postconditions: Mutation by Add (add item with key k) and DeleteMin (delete an item with minimum key).

Name	Returns
Create()	a new empty priority queue
IsEmpty(P)	true if empty, else false
LookUpMin(P)	item with minimum key k
Add(P, k, e)	
DeleteMin(P)	item with minimum key k

Array implementations of priority queues

Unsorted array:

- store full then empty (plus variable to locate first empty)
- $O(n)$ LookUpMin
- $O(1)$ Add (put in first empty, update variable)
- $O(n)$ DeleteMin (look through all elements, swap with last full, update variable)

Unsorted array with a variable storing the location of the min:

- store full then empty (plus variable to locate first empty)
- $O(1)$ LookUpMin
- $O(1)$ Add (like above but also compare to min variable)
- $O(n)$ DeleteMin (modify constant but linear to update min variable)

Sorted array:

- Store index of first empty in variable.
- $O(1)$ LookUpMin
- $O(n)$ Add (find location by binary search or linear, shift all)
- $O(1)$ DeleteMin (end of array)

Linked implementations of priority queues

Unsorted linked list:

- store items in arbitrary order
- $O(n)$ LookUpMin
- $O(1)$ Add (put front of linked list)
- $O(n)$ DeleteMin (look through all elements to find smallest)

Unsorted linked list with a pointer to min:

- $O(1)$ LookUpMin
- $O(1)$ Add (put at beginning of list, compare to min, update if needed)
- $O(n)$ DeleteMin (modify constant but linear to update min variable)

Sorted linked list:

- $O(1)$ LookUpMin
- $O(n)$ Add (scan in linked list)
- $O(1)$ DeleteMin

An improved implementation

Goal:

- Maybe not as fast as $\Theta(1)$ for all operations
- Never as slow as $\Theta(n)$

Using a tree:

- Modify BST to allow duplicate keys (e.g. values in left subtree $\leq$ value at node).
- Where is minimum element?
- What is cost of Add in general?
- $\Theta(\log n)$ time for balanced tree

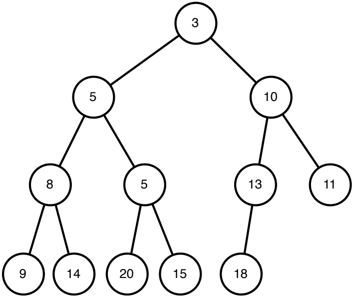
Keeping FindMin cheap

- Store smallest at root.
- Bound on height of tree.
- Come up with less stringest condition than binary search order in order to make the condition cheap to maintain.

Data structure: heap

A binary tree satisfies the **heap-order property** if for each node, the value stored at the node is no greater than that stored in either child (if any).

A **heap** is a complete binary tree that satisfies the heap-order property.



Using a heap to implement the ADT Dictionary: store (key, element) pairs at each node, ensuring that the keys satisfy the heap-order property.

Observations about example of previous slide

Consequences of heap-order property:

- Path from root to leaf in nondecreasing order (remember that equal values are possible.)
- Relative values of left and right children unknown.
- Where are biggest and smallest?

Consequences of complete:

- Nice array implementation
- Logarithmic bound on height

Customizing ADT Binary Tree to implement a heap

Additional data: Store both keys and elements at nodes.

Note: Only keys are ordered using the heap order property.

Modified operations:

- AddNode(B, parent, key, element, side)

Additional operations (also for BST):

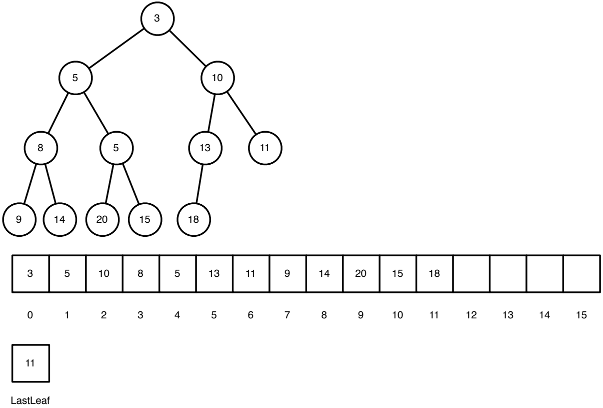
- KeyAtNode(B, n)
- ElementAtNode(B, n)
- StoreInNode(B, n, key, element) - changes what is stored in a node

Additional operations (just for heap):

- LastLeaf(B) - node that is the last leaf in complete tree
- PreviousLeaf(B) - leaf before last leaf in complete tree
- NextLeaf(B) - node that will be the next leaf in complete tree
- SwapValues(B, node1, node2) - exchanges both key and element

Store af variable lastleaf in either implementation.

Implementing a heap using an array implementation of ADT Binary Tree



- Illustrations show keys only, not elements.
- lastleaf is a pointer for a linked implementation of ADT Binary Tree.

Implementing ADT Priority Queue using a heap

LookUpMin(P)

- Return the data item in the root of tree.
- $\Theta(1)$ array implementation of ADT Binary Tree
- $\Theta(1)$ linked implementation of ADT Binary Tree

Add(P, k, e)

- To preserve completeness, add at next leaf position.
- NextLeaf can be found in time $\Theta(1)$ for the array implementation of ADT Binary Tree.
- NextLeaf can be found in time $\Theta(\log n)$ for the linked implementation of ADT Binary Tree.
- Problem: Heap-order property violated.

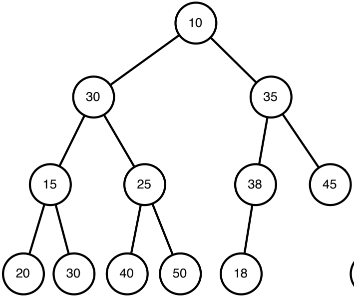
DeleteMin(P)

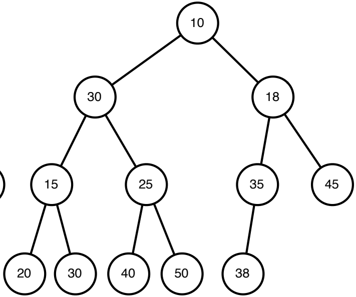
- To preserve heap-order property, remove the root.
- Problem: What remains is not a tree.

Implementing Add(P, k, e) in a heap

"Bubble up", fixing heap-order property on path from leaf to root by using Swap operation.

Here 18 has just been added.





Pseudocode for Add(P, k, e)

```
ℓ ← NextLeaf(B)
lastleaf ← NextLeaf(B)
StoreInNode(B, ℓ, k, e)
curr ← ℓ
par ← Parent(B, curr)
while par ≠ false and KeyAtNode(B, curr) < KeyAtNode(B, par)
    SwapValues(B, curr, par)
    curr ← par
    par ← Parent(B, curr)
```

Observations about Add

Q: Why can we stop tracing up the path once we find a child that has a greater key value than its parent?

A: We verify that the heap property is satisfied everywhere.
When we swap a child C with its parent P:

- C ¡ P
- P ¡ O (for O the other child)
- Thus C ¡ O (heap property OK at C)

Running time of Add:

- Number of iterations is height of heap in worst case.
- $\Theta(\log n)$ time due to height.

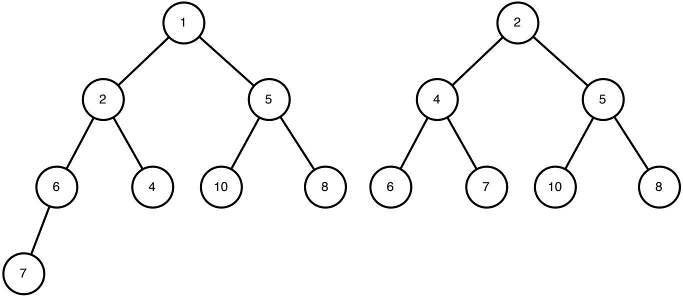
Various names:

- bubble-up
- sift
- percolate

Implementing DeleteMin(P) in a heap

Delete value in root, move value in last leaf to root.

“Bubble down”, fixing heap-order property on path from root to leaf using Swap operation



DeleteMin pseudocode

```
min ← Root(B)
SwapValues(B, Root(B), lastleaf)
DeleteNode(B, lastleaf)
lastleaf ← PreviousLeaf(B)
curr ← Root(B)
left ← LeftChild(B, curr)
right ← Rightchild(B, curr)
key ← KeyAtNode(curr)
stop ← false
if left ≠ false and right = false
    lkey ← KeyAtNode(B, left)
    if lkey < key
        SwapValues(B, curr, left)
        stop ← true
```

DeleteMin pseudocode, continued

```
while left ≠ false and right ≠ false and not stop)
    lkey ← KeyAtNode(B, left)
    rkey ← KeyAtNode(B, right)
    if key ≤ lkey and key ≤ rkey
        stop ← true
    else
        if lkey < rkey
            SwapValues(B, curr, left)
            curr ← left
    else
            SwapValues(B, curr, right)
            curr ← right
        left ← LeftChild(B, curr)
        right ← RightChild(B, curr)

return min
```

Observations about DeleteMin

PreviousLeaf:

- array-based implementation: $\Theta(1)$ time
- linked implementation: $\Theta(\log n)$ time

Running time of DeleteMin:

- Number of iterations is height of heap in worst case.
- $\Theta(\log n)$ time due to height.

Various names:

- bubble-down
- trickle
- percolate (again!)

Moral of story: careful about names
Why do we need to look at both children?
Did we need to look at both siblings in bubble-up?

Sorting using a priority queue

User view

Algorithm:

- Repeatedly use Add until all values entered.
- Repeatedly use DeleteMin.

Analysis:

- $\Theta(n \log n)$ for n Add operations
- $\Theta(n \log n)$ for n DeleteMin operations

Provider view

Can we find a faster way to make a heap out of n elements?

Customizing ADT Priority Queue

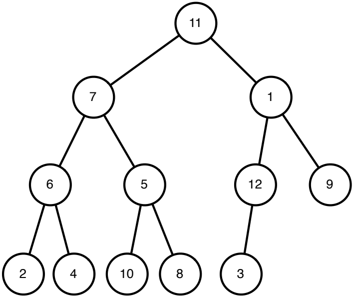
Form heap out of a bunch of (key, element) pairs.

The **heapify** operation forms a heap out of an array of items.

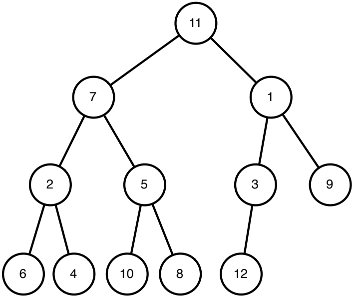
Idea:

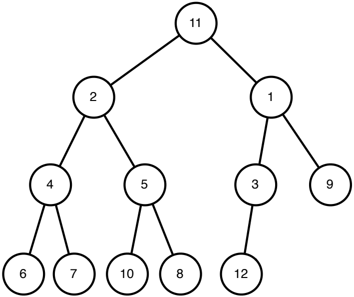
- Place all items into the structure.
- Fix heap-order property from bottom up.
- Observe that leaves are heaps of height 0.
- At phase i, form heaps of height at most i from two heaps of height at most $i-1$.

Heapify example, phase 1

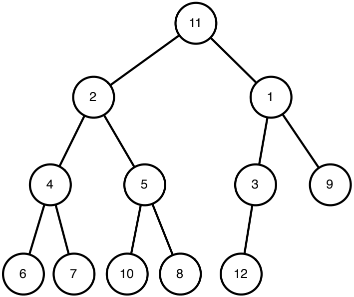


Heapify example, phase 2





Heapify example, phase 3



Linear-time heapify analysis

- Placement into structure takes $\Theta(n)$ time total.
- Logarithmic number of phases, in phase i forming at most $n/2^{i+1}$ heaps of height i each.
- Cost of making one heap of height i is in $\Theta(i)$, bounded above by some ci.
- Total cost of phases is at most $\sum_{i=1}^{\lfloor \log n \rfloor} ci \cdot \frac{n}{2^{i+1}} \leq cn(\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \cdots)$, which is linear in n.
- Total cost in $O(n)$.