①

[redacted]

Prove $\mathcal{E}(XY) = \mathcal{E}(X)\cdot\mathcal{E}(Y)$

$$\mathcal{E}(x\cdot y) = \sum_i \sum_j x_i y_j \cdot f_{xy}(x_i, y_j)$$

$$= \sum_i \sum_j x_i y_j f_x(x_i) f_y(y_j)$$

$$= \Big(\sum_i x_i f_x(x_i)\Big)\Big(\sum_j y_j f_y(y_j)\Big)$$

$$= \mathcal{E}(x)\cdot\mathcal{E}(y)$$

~~$\mathrm{Cov}(x,y) = \mathcal{E}$~~

Prove $\mathrm{cov}(x,y) = \underset{\uparrow \atop = \mathcal{E}(x)\cdot\mathcal{E}(y)}{\mathcal{E}(xy)} - \mathcal{E}(x)\cdot\mathcal{E}(y) = 0$

$$\cancel{\mathcal{E}(x)\cdot\mathcal{E}(y)} - \cancel{\mathcal{E}(x)\cdot\mathcal{E}(y)}$$

$$\boxed{= 0}$$

②

$$\text{pdf}(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(\frac{-(\mu - x)^2}{2\sigma^2}\right)$$

$f(x), g(x)$

Convolution of $f(x)$ and $g(x)$: $\int f(z) f(z-t)\, dt$

$$h(z) = \int_{-\infty}^{\infty} \overset{\mu_1, \sigma_1^2}{f(z)} \cdot \overset{\mu_2, \sigma_2^2}{g(z-t)}\, dt$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$x \sim X \qquad y \sim Y$$

$Z = X + Y$

$z = \text{conv}(x, y)$ or $x + y$

$\mu_z = \mu_x + \mu_y$

$\sigma_z^2 = \sigma_x^2 + \sigma_y^2$

$h(z)$

$\underline{P(x+y)} \longrightarrow \int p(x+y)x + p(x+y)y \, dz$

$\hookrightarrow \int p(z)x + p(z)y$

$\downarrow \qquad\qquad \Downarrow$

$P(x) + P(y)$

$dz = dx + dy \qquad \int p(x)x\overset{dx}{dz} + \int p(y)y\, \overset{dy}{dz}$

$= x + y$

③

$p(x=1) = q$ $\quad$ $p(x|q) = \theta^{x}(1-\theta)^{1-x}$

$p(x=0) = p$

$P(D|\theta) = \theta^{N_1} \cdot (1-\theta)^{N_0}$ $\quad$ $N_1 + N_0 = N$

$= \theta^{1} \cdot (1-\theta)^{0}$

$= \theta \cdot \cancel{(1-\theta)^{0}}^{1}$

$$\boxed{= \theta = E(x)}$$

$Var(x) = \cancel{E(x^2) - (E(x))^2}$ $p(1-p) = pq$

$=$ [illegible]

$E(x^2) = \cancel{p}\, q \cdot 1^2 + q \cdot 0^2 = q + 0 = q = p($

$Var(x) = E(x^2) - (E(x))^2$

$= q - q^2 = q(1-q)$

$Var(\theta) \boxed{= \theta(1-\theta)}$

④

$$B(v; a, b) = \left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot v^{a-1}(1-v)^{b-1} \right] = \circ$$

$$\mu = a(b-a)\frac{a}{a+\beta}$$

$$\sigma^2 = \frac{(b-a)^2 \alpha \cdot \beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$X mode = a + (b-a)\left\{ \frac{\alpha-1}{\alpha+\beta-2} \right\}$$

$$(v-0)^{a-1} \cdot (1-v)^{b-1} \to ((\cancel{\ \ \ })^{a-1}$$

$$f(v') = v \to \int_s^t (v'-s)^n \cdot (t-v')^n \, dv \to \int_s^{\frac{1}{t-s}} \alpha(v'-s)^n (t-v')^n d$$

$$dv = \frac{v'-s}{t-s}$$

$$\boxed{PDF = \frac{(v-s)^{a-1} \cdot (t-v)^{b-1}}{(t-s)^{a+b+1} \cdot B(a,b)}}$$

⑤

$$\ln(L(s,t)|X) = \sum_{i=1}^{N} \ln(L_i(s,t)X_i))$$

$$= \sum_{i=1}^{N} \ln\left(\frac{X_i^{s-1}(1-X_i)^{\beta-1}}{B(s,t)}\right)$$

$$= (s-1)\sum_{i=1}^{N}\ln(X_i) + (t-1)\sum_{i=1}^{N}\ln(1-x_i$$

$$- N\cdot\ln B(s,t)$$

$$\frac{\partial \ln L(s,t|x)}{\partial s} = \sum_{i=1}^{N}\ln(X_i) - N\frac{\partial \ln B(s,t)}{\partial s} = 0$$

$$\frac{\partial \ln L(s,t|X)}{\partial t} = \sum_{i=1}^{N}\ln(1-X_i) - N\frac{\partial \ln B(s,t}{\partial t}$$

$$\psi(s) = \frac{\partial \ln \Gamma(s}{\partial s}$$

$$\frac{\partial \ln B(s,t)}{\partial s} = -\psi(s+t) + \psi(s) + 0$$

$$\frac{\partial \ln B(s,t)}{\partial t} = -\psi(s+t) + 0 + \psi(t)$$

⑥ ⓐ $\sigma(x) = \frac{1}{1+e^{(-x)}}$ $\quad y^{-1} = 1+e^{-x}$

$$f(x) = pdf(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$f(y) = f_x(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y\right.$$

$$g^{-1}(y) = x = -\ln(y^{-1}-1)$$

$$f_x(g^{-1}(y)) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(-\ln(y^{-1}-1)-}{2\sigma^2}}$$

$$\left|\frac{d}{dy}g^{-1}(y)\right| = \left|\frac{d-\ln(y^{-1}-1)}{dy}\right| = \left|\frac{-y^{-}}{y^{-1}}\right.$$

$$= \frac{y^{-2}}{y^{-1}-1}$$

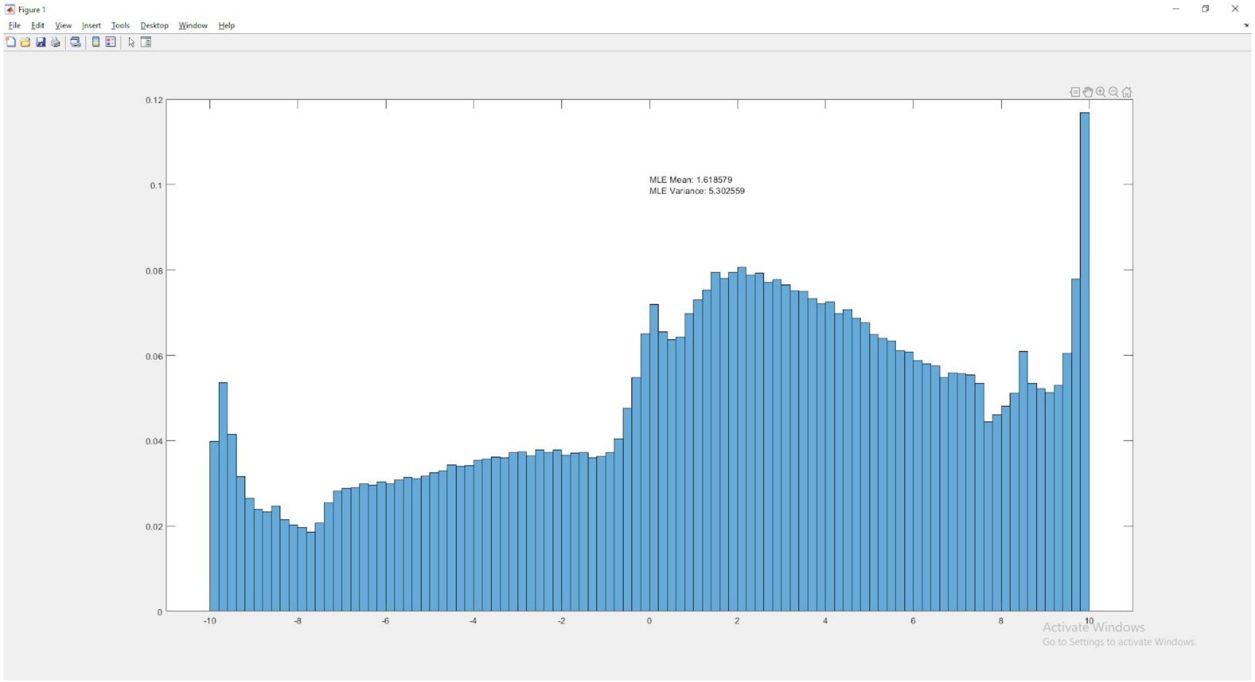
$$pdf = f_x(y) = \frac{y^{-2}}{y^{-1}-1\sqrt{2\pi}\,\sigma} e^{\frac{-(-\ln(y^{-1}-1)-\mu)^2}{2\sigma^2}}$$

Ⓑ

$$\hat{\mu}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i \rightarrow \sigma_\mu(x) = \frac{1}{1+e^{(-\hat{\mu}_{MLE})}}$$

$$\hat{\sigma}_{MLE} = \sqrt{1/n \cdot \sum_{i=1}^{n}(x-\mu)^2} \rightarrow \sigma_\sigma(x) = \frac{1}{1+e^{(-\hat{\mu}}}$$

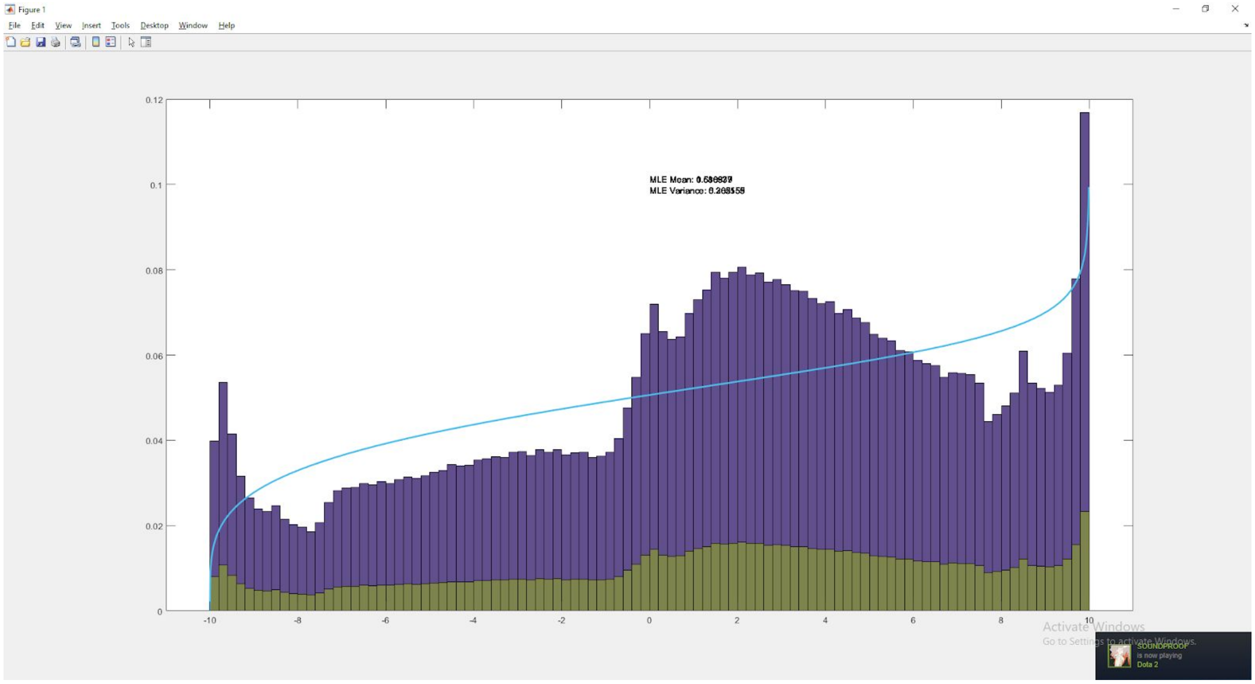
Problem 7 and 8: The data appears to be multi-modal, and the peaks do not change with a change in the number of bins



Problem 9:

Beta: Alpha/beta values [1.2404,0.09265]

Lognormal values: [-0.76396,0.89287]



Problem 10: Beta seemed to perform the best, code will show results of log probabiliies