Littlewood's Theorem on the Oscillation of $\pi(x) - \mathrm{li}(x)$

A Complete Proof of Infinite Sign Changes

Abstract

This document provides a complete exposition of J.E. Littlewood's 1914 proof that the difference $\pi(x) - \mathrm{li}(x)$ changes sign infinitely many times. The proof demonstrates that

$$\pi(x) - \mathrm{li}(x) = \Omega_{\pm}\left(\frac{x^{1/2}}{\log x} \log\log\log x\right),$$

meaning the error achieves this order of magnitude infinitely often in both positive and negative directions. This was a landmark result, as all computational evidence at the time suggested $\pi(x) < \mathrm{li}(x)$ for all x.

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1 Introduction and Background

In 1791, Gauss conjectured that the prime counting function $\pi(x)$ (the number of primes $\leq x$) is approximated by $x/\log x$. In a letter to Encke in 1849, Gauss refined this to the logarithmic integral:

$$\pi(x) \approx \mathrm{li}(x) = \int_2^x \frac{dt}{\log t}.$$

This was proved precisely in 1896 (the Prime Number Theorem) independently by Hadamard and de la Vallée Poussin.

Gauss observed that $\mathrm{li}(x)$ always exceeds $\pi(x)$ where he calculated up to $x = 3{,}000{,}000$. The conjecture that $\pi(x) < \mathrm{li}(x)$ for all x was widely believed until Littlewood proved it false in 1914.

Definition 1.1 (Notation). *We use the following standard definitions:*

- $\mathrm{Li}(x) = \lim_{\varepsilon\to 0^+}\left(\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t}\right)$
- $\mathrm{li}(x) = \int_2^x \frac{dt}{\log t} = \mathrm{Li}(x) - \mathrm{Li}(2)$ *where* $\mathrm{Li}(2) \approx 1.04$
- $\psi(x) = \sum_{n\leq x} \Lambda(n)$ *(Chebyshev's function)*
- $\vartheta(x) = \sum_{p\leq x} \log p$ *(sum over primes)*
- $\rho = \sigma + i\gamma$ *denotes a nontrivial zero of the Riemann zeta function*

Definition 1.2 (Omega Notation). *For real-valued functions f and positive functions g:*

$$f(x) = \Omega(g(x)) \quad \text{if} \quad \limsup_{x\to\infty} \frac{|f(x)|}{g(x)} > 0$$

$$f(x) = \Omega_+(g(x)) \quad \text{if} \quad \limsup_{x\to\infty} \frac{f(x)}{g(x)} > 0$$

$$f(x) = \Omega_-(g(x)) \quad \text{if} \quad \liminf_{x\to\infty} \frac{f(x)}{g(x)} < 0$$

$$f(x) = \Omega_\pm(g(x)) \quad \text{if both } \Omega_+ \text{ and } \Omega_- \text{ hold}$$

2 Preliminary Results

We first establish the tools needed for the main theorem.

Theorem 2.1 (Properties of Zeta Zeros). *Let $N(T)$ denote the number of zeros of $\zeta(s)$ with $0 < \gamma \leq T$. For $T \geq 4$:*

$$N(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T) \tag{1}$$

$$N(T+h) - N(T) \ll h\log T \tag{2}$$

Consequently:

$$\sum_{0<\gamma\le T} \frac{1}{\gamma} \ll (\log T)^2 \tag{3}$$

$$\sum_{\gamma>T} \frac{1}{\gamma^2} \ll \frac{\log T}{T} \tag{4}$$

$$\sum_{\gamma} \frac{1}{\gamma^\alpha} < \infty \quad \textit{for } \alpha > 1 \tag{5}$$

Theorem 2.2 (Explicit Formula for ψ)**.** *Let $\psi_0(x) = \frac{\psi(x^+)+\psi(x^-)}{2}$. Then for $x > 0$ and $c > 1$:*

$$\psi_0(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} \left(-\frac{\zeta'}{\zeta}(s) \right) ds$$

and

$$\psi_0(x) = x - \lim_{T\to\infty} \sum_{|\gamma|\le T} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) + \sum_{k\ge 1} \frac{x^{-2k}}{2k}.$$

For $k \ge 1$, define the smoothed version:

$$\psi_k(x) = \frac{1}{k!} \sum_{n\le x} (x-n)^k \Lambda(n).$$

Then:

$$\psi_k(x) = \frac{x^2}{(k+1)!} - \lim_{T\to\infty} \sum_{|\gamma|\le T} \frac{x^{\rho+1}}{\rho(\rho+1)\cdots(\rho+k)} - \frac{x}{k!}\frac{\zeta'}{\zeta}(0) + O(1) - \sum_{r=1}^{\infty} \frac{x^{-2r+1}}{(-2r)(-2r+1)\cdots(-2r+k)}.$$

Theorem 2.3 (Conversion Formulas)**.** *Assuming the Riemann Hypothesis (RH):*

$$\vartheta(x) = \psi(x) - x^{1/2} + O(x^{1/3}) \tag{6}$$

$$\pi(x) - \mathrm{li}(x) = \frac{\vartheta(x) - x}{\log x} + O\left(\frac{x^{1/2}}{\log^2 x} \right) \tag{7}$$

3 Landau's Lemma

The key tool for establishing oscillation results is Landau's lemma.

Lemma 3.1 (Landau)**.** *Suppose $A(x)$ is a bounded and Riemann integrable function on any finite interval $[1, x]$ and $A(x) \ge 0$ for all sufficiently large x. Let σ_c be the infimum of σ for which $\int_1^\infty A(x)x^{-\sigma}\,dx < \infty$. Then the function*

$$F(s) = \int_1^\infty A(x)x^{-s}\,dx$$

is analytic on $\mathrm{Re}(s) > \sigma_c$ *but is* not *analytic at* $s = \sigma_c$*.*

Proof. It follows from the definition of σ_c that $\int_1^\infty A(x)x^{-s}\,dx$ is absolutely convergent for $\mathrm{Re}(s) > \sigma_c$. Hence $\int_1^N A(x)x^{-s}\,dx \to \int_1^\infty A(x)x^{-s}\,dx$ uniformly in $\{s : \mathrm{Re}(s) \geq \sigma_c + \varepsilon\}$ for any $\varepsilon > 0$, so $\int_1^\infty A(x)x^{-s}\,dx$ is analytic on $\{s : \mathrm{Re}(s) > \sigma_c\}$.

Now assume on the contrary that $F(s)$ is analytic at $s = \sigma_c$. Since the integral on any finite interval $[1, x]$ is entire, we may assume $A(x) \geq 0$ for $x \geq 1$, and by replacing $A(x)$ with $A(x)x^\sigma$ for $\sigma \geq 0$, we may assume $\sigma_c = 0$.

Now F would be analytic on a neighborhood of 0, say $\{z \in \mathbb{C} : |z| < \delta\}$. Let $\Omega = \{s : \sigma > 0\} \cup \{s : |s| < \delta\}$. Then F is analytic in Ω.

Write $F(s) = \sum_{k\geq 0} c_k(s-1)^k$. Since the nearest points to 1 that are not in Ω are $\pm i\delta$, the radius of convergence is $\geq \sqrt{1+\delta^2} = 1 + \delta'$ for some $\delta' > 0$.

For s near 1, $\left|\int_1^\infty A(x)x^{-s}\,dx\right| \leq \int_1^\infty |A(x)|\,dx < \infty$, so we can differentiate under the integral sign:

$$c_k = \frac{1}{k!}F^{(k)}(1) = \frac{1}{k!}\int_1^\infty A(x)(-\log x)^k x^{-1}\,dx.$$

So:

$$F(s) = \sum_{k\geq 0}\frac{1}{k!}\int_1^\infty A(x)(-\log x)^k x^{-1}(s-1)^k\,dx.$$

Now if $-\delta' < s < 1$, the integrand is nonnegative, so we may switch the order of summation and integration:

$$\begin{aligned} F(s) &= \int_1^\infty \sum_{k\geq 0}\frac{1}{k!}A(x)(-\log x)^k x^{-1}(s-1)^k\,dx \\ &= \int_1^\infty \exp(\log(x)(1-s))A(x)x^{-1}\,dx \\ &= \int_1^\infty A(x)x^{-s}\,dx. \end{aligned}$$

In particular, $\int_1^\infty A(x)x^{-s}\,dx$ converges at $s = -\delta'/2$, contradicting the definition of σ_c. $\square$

4 Schmidt's Result: First Oscillation Theorem

Theorem 4.1 (E. Schmidt, 1903)**.** *Let Θ denote the supremum of real parts of the zeros of $\zeta(s)$. Then for any $\varepsilon > 0$:*

$$\psi(x) - x = \Omega_\pm(x^{\Theta-\varepsilon}).$$

Proof. By the Mellin transform formula, for $\sigma > 1$:

$$-\frac{\zeta'}{\zeta}(s) = s\int_1^\infty \psi(x)x^{-s-1}\,dx.$$

Hence for $\sigma > 1$:

$$\frac{1}{s-\Theta+\varepsilon} + \frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} = \int_1^\infty (x^{\Theta-\varepsilon} + \psi(x) - x)x^{-s-1}\,dx.$$

Now assume that $\psi(x) - x \geq -x^{\Theta-\varepsilon}$ for x large enough. The left-hand side has a pole at $\Theta - \varepsilon$, and $\zeta(s)$ is nonzero for real $s \in (0,1)$. We see that the left-hand side is analytic for real $s > \Theta - \varepsilon$, i.e., no such point is the abscissa σ_c.

Applying Landau's lemma, $\int_1^\infty (x^{\Theta-\varepsilon} + \psi(x) - x)x^{-s-1}\,dx$ is analytic for $\mathrm{Re}(s) > \Theta - \varepsilon$, so the equation holds for $\mathrm{Re}(s) > \Theta - \varepsilon$. This is a contradiction since $\frac{\zeta'}{\zeta}$ has a pole with real part $> \Theta - \varepsilon$ by definition of Θ.

It follows that $\psi(x) - x = \Omega_-(x^{\Theta-\varepsilon})$.

Similarly, if $\psi(x) - x < x^{\Theta-\varepsilon}$ for large enough x, consider:

$$\frac{1}{s - \Theta + \varepsilon} + \frac{\zeta'(s)}{s\zeta(s)} + \frac{1}{s-1} = \int_1^\infty (x^{\Theta-\varepsilon} - \psi(x) + x)x^{-s-1}\,dx.$$

The same argument gives $\psi(x) - x = \Omega_+(x^{\Theta-\varepsilon})$. □

Theorem 4.2. *Let Θ denote the supremum of real parts of zeros of $\zeta(s)$. For any $\varepsilon > 0$:*

$$\Pi(x) - \mathrm{li}(x) = \Omega_\pm(x^{\Theta-\varepsilon}),$$

and assuming RH is false (i.e., $\Theta > 1/2$):

$$\pi(x) - \mathrm{li}(x) = \Omega_\pm(x^{\Theta-\varepsilon}).$$

Corollary 4.3. *As $x \to \infty$:*

$$\begin{aligned} \psi(x) - x &= \Omega_\pm(x^{1/2}) \\ \vartheta(x) - x &= \Omega_-(x^{1/2}) \\ \pi(x) - \mathrm{li}(x) &= \Omega_-\left(\frac{x^{1/2}}{\log x}\right) \end{aligned}$$

Proof. If RH is false, then Theorem 4.2 gives a stronger result, so assume $\Theta = 1/2$. By the refinement involving zeros on the critical line, we have $\psi(x) - x = \Omega_\pm(x^{1/2})$.

By Theorem 2.3, $\vartheta(x) - x = \psi(x) - x - x^{1/2} + O(x^{1/3}) = \Omega_-(x^{1/2})$. Hence:

$$\pi(x) - \mathrm{li}(x) = \frac{\vartheta(x) - x}{\log x} + O\left(\frac{x^{1/2}}{\log^2 x}\right) = \Omega_-\left(\frac{x^{1/2}}{\log x}\right).$$

□

Remark 4.4. *Note that in case RH is true (i.e., $\Theta = 1/2$), since $\Pi(x) > \pi(x)$ we have $\pi(x) - \mathrm{li}(x) = \Omega_-(x^{1/2-\varepsilon})$, but the Ω_+ result is not obtainable by this method. This is where Littlewood's deeper analysis becomes essential.*

5 Littlewood's Main Theorem

We now present Littlewood's 1914 theorem, which establishes that $\pi(x) - \mathrm{li}(x)$ oscillates in *both* directions.

Theorem 5.1 (Littlewood, 1914)**.**

$$\psi(x) - x = \Omega_\pm(x^{1/2} \log\log\log x).$$

$$\pi(x) - \mathrm{li}(x) = \Omega_\pm\left(\frac{x^{1/2}}{\log x} \log\log\log x\right).$$

The proof requires two key lemmas.

Lemma 5.2 (Weighted Average Formula)**.** *Assume RH. Then:*

$$\frac{1}{x(e^\delta - e^{-\delta})} \int_{e^{-\delta}x}^{e^\delta x} (\psi(u) - u)\, du = -2x^{1/2} \sum_{\gamma>0} \frac{\sin(\gamma\delta)}{\gamma\delta} \cdot \frac{\sin(\gamma \log x)}{\gamma} + O(x^{1/2}),$$

where $\frac{1}{2x} \le \delta \le \frac{1}{2}$ *and the* O*-constant is uniform for* $x \ge 4$*.*

Proof. By the explicit formula:

$$\int_0^x (\psi(u) - u)\, du = -\sum_\rho \frac{x^\rho}{\rho(\rho+1)} - cx + O(1).$$

Taking the average:

$$\frac{1}{(e^\delta - e^{-\delta})x} \int_{e^{-\delta}x}^{e^\delta x} (\psi(u) - u)\, du = \frac{-\delta}{\sinh(\delta)} \sum_\rho \frac{e^{\delta(\rho+1)} - e^{-\delta(\rho+1)}}{2\delta\rho(\rho+1)} x^\rho + O(1).$$

Now observe that $e^{\pm\delta(\rho+1)} = e^{\pm\delta(\mathrm{Re}(\rho)+1+i\gamma)} = e^{\pm\delta i\gamma}(1 + O(\delta)) = e^{\pm\delta i\gamma} + O(\delta)$.
Since $\sum_\rho \frac{1}{|\rho(\rho+1)|} \le \sum \gamma^{-2} < \infty$, and $\frac{\delta}{\sinh(\delta)} = \frac{\delta}{\delta+O(\delta^3)} = 1 + O(\delta^2) \asymp 1$ as $\delta \to 0$.
Assuming RH, $|x^\rho| = x^{1/2}$. Replacing $e^{\delta(\rho+1)}$ by $e^{i\delta\gamma}$ gives:

$$\begin{aligned}\frac{-\delta}{\sinh(\delta)} \sum_\rho \frac{e^{\delta(\rho+1)} - e^{-\delta(\rho+1)}}{2\delta\rho(\rho+1)} x^\rho &= -ix^{1/2} \frac{\delta}{\sinh(\delta)} \sum_\rho \frac{\sin(\gamma\delta)}{\delta} \cdot \frac{x^{i\gamma}}{\rho(\rho+1)} + O\left(x^{1/2} \frac{\delta}{\sinh(\delta)} \sum_\rho \frac{1}{\rho(\rho+1)}\right) \\ &= -ix^{1/2} \frac{\delta}{\sinh(\delta)} \sum_\rho \frac{\sin(\gamma\delta)}{\delta} \cdot \frac{x^{i\gamma}}{\rho(\rho+1)} + O(x^{1/2}).\end{aligned}$$

Now:

$$\sum_\rho \frac{\sin(\gamma\delta)}{\delta} \cdot \frac{x^{i\gamma}}{\rho(\rho+1)} \ll \sum_\rho \frac{\gamma}{|\rho(\rho+1)|} \ll \sum_{0<\gamma<\delta^{-1}} \frac{1}{\gamma} + \delta^{-1} \sum_{\gamma>\delta^{-1}} \frac{1}{\gamma^2} \ll \log^2(\delta^{-1}) + \frac{\delta^{-1}\delta \log \delta^{-1}}{1} \ll \log^2(\delta^{-1}).$$

Substituting $\frac{\delta}{\sinh(\delta)} = 1 + O(\delta^2)$:

$$-ix^{1/2} \sum_\rho \frac{\sin \gamma\delta}{\delta} \cdot \frac{x^{i\gamma}}{\rho(\rho+1)} + O(x^{1/2}).$$

Assuming RH, $\frac{1}{\rho} = \frac{1}{i\gamma} + O\left(\frac{1}{\gamma^2}\right)$, and note that $\frac{\sin\gamma\delta}{\delta} \le |\gamma|$. Replacing $\frac{1}{\rho}$ by $\frac{1}{i\gamma}$ gives error $\ll x^{1/2}\sum \gamma^{-2} \ll x^{1/2}$. Similarly for $\frac{1}{\rho+1}$.

Our expression becomes:

$$-x^{1/2}\sum_{\rho}\frac{\sin\gamma\delta}{\gamma\delta}\cdot\frac{x^{i\gamma}}{i\gamma} + O(x^{1/2}) = -2x^{1/2}\sum_{\gamma>0}\frac{\sin\gamma\delta}{\gamma\delta}\cdot\frac{e^{i\gamma\log x} - e^{-i\gamma\log x}}{2i\gamma} + O(x^{1/2})$$
$$= -2x^{1/2}\sum_{\gamma>0}\frac{\sin(\gamma\delta)}{\gamma\delta}\cdot\frac{\sin(\gamma\log x)}{\gamma} + O(x^{1/2}).$$

□

Lemma 5.3 (Dirichlet's Approximation Theorem)**.** *For $x \in \mathbb{R}$, define $\|x\|$ to be the distance to the nearest integer. Let $\theta_1, \ldots, \theta_K$ be real numbers and N be a positive integer. Then there exists a positive integer n with $1 \le n \le N^K$ such that:*

$$\|\theta_i n\| < \frac{1}{N} \quad \textit{for all } i = 1, \ldots, K.$$

Proof. Partition $[0,1)^K$ into N^K equal subcubes. Then there exist $0 \le n_1 < n_2 \le N^K$ such that $(\theta_1 n_1, \ldots, \theta_K n_1)$ and $(\theta_1 n_2, \ldots, \theta_K n_2)$ lie in the same subcube.

Let $n = n_2 - n_1 \in [1, N^K]$. Then for $1 \le i \le K$:

$$\|n\theta_i\| = \|n_2\theta_i - n_1\theta_i\| \le |n_2\theta_i - n_1\theta_i| < \frac{1}{N}.$$

□

6 Proof of Littlewood's Theorem

Proof of Theorem 5.1. If RH is false, then Theorem 4.1 gives a stronger result, so assume RH.

If we can prove that

$$\psi(x) - x = \Omega_\pm(x^{1/2}\log\log\log x),$$

then since $\psi(x) - \vartheta(x) = O(x^{1/2})$, we have

$$\vartheta(x) - x = \Omega_\pm(x^{1/2}\log\log\log x).$$

Under RH, using Theorem 2.3:

$$\pi(x) - \mathrm{li}(x) = \frac{\vartheta(x) - x}{\log x} + O\left(\frac{x^{1/2}}{\log^2 x}\right),$$

which gives:

$$\pi(x) - \mathrm{li}(x) = \Omega_\pm\left(\frac{x^{1/2}}{\log x}\log\log\log x\right).$$

Main Argument:

Let N be a large integer. Apply Dirichlet's Lemma 5.3 to the numbers $\frac{\gamma \log N}{2\pi}$ for $0 < \gamma \leq T = N \log N$.

Here K, the number of elements in that set, is $N(T) \asymp T \log T$, and there exists n with $1 \leq n \leq N^K$ such that:

$$\left\| \frac{\gamma n}{2\pi} \log N \right\| < \frac{1}{N}, \quad 0 < \gamma \leq T. \tag{8}$$

Note the inequality:

$$|\sin(\pi x)| \leq \pi \|x\|. \tag{9}$$

This can be verified directly for $x \in [0,1]$ and extends by periodicity.

From (9):

$$|\sin(2\pi\alpha) - \sin(2\pi\beta)| = |2\sin(\pi(\alpha - \beta))\cos(\pi(\alpha+\beta))| \leq 2\pi\|\alpha - \beta\|.$$

Take $x = N^n e^{\pm 1/N}$ and $\delta = \frac{1}{N}$. Then:

$$|\sin(\gamma \log x) \mp \sin(\gamma/N)| \leq 2\pi \left\| \frac{\gamma(\log x \mp 1/N)}{2\pi} \right\| = 2\pi \left\| \frac{n\gamma \log N}{2\pi} \right\| \leq \frac{2\pi}{N}$$

by Dirichlet's lemma.

Estimating the tail:

$$\sum_{\gamma > N \log N} \frac{\sin(\gamma/N)}{\gamma/N} \cdot \frac{\sin(\gamma \log x)}{\gamma} \ll N \sum_{\gamma > N \log N} \frac{1}{\gamma^2} \ll N(N \log N)^{-1} \log(N \log N) \ll 1.$$

Main sum estimate:

The right-hand side of Lemma 5.2 becomes:

$$\begin{aligned} -2x^{1/2} \sum_{\gamma>0} \frac{\sin(\gamma/N)}{\gamma/N} \cdot \frac{\sin(\gamma \log x)}{\gamma} &= \mp 2x^{1/2} \sum_{\gamma>0} \frac{\sin(\gamma/N)}{\gamma/N} \cdot \frac{\sin(\gamma/N)}{\gamma} \\ &\quad + O\left(\frac{1}{N} x^{1/2} \sum_{\gamma>0} \left| \frac{\sin(\gamma/N)}{\gamma/N} \cdot \frac{1}{\gamma} \right| \right) \\ &= \mp \frac{2x^{1/2}}{N} \sum_{\gamma>0} \left(\frac{\sin(\gamma/N)}{\gamma/N} \right)^2 \mp 2x^{1/2} N \sum_{\gamma > N \log N} \left(\frac{\sin(\gamma/N)}{\gamma} \right)^2 + O(x^{1/2}). \end{aligned}$$

Now:

$$\sum_{0<\gamma \leq N \log N} 1 = N(N \log N) \asymp N \log N,$$

and for $0 < \gamma \leq N \log N$, we have $\left(\frac{\sin(\gamma/N)}{\gamma/N}\right)^2 \asymp 1$ (bounded away from 0 and bounded above).

Therefore:

$$\sum_{0<\gamma \leq N \log N} \left(\frac{\sin(\gamma/N)}{\gamma/N} \right)^2 \asymp N \log N,$$

so the main sum is:

$$\asymp \mp \frac{2x^{1/2}}{N} \cdot N \log N \mp 2x^{1/2} N \cdot \frac{\log(N \log N)}{N \log N} + O(x^{1/2}) = \mp 2x^{1/2} \log N + O(x^{1/2}).$$

Relating N to $\log\log\log x$:

Since $x \leq N^{N^K} e^{1/N}$ where $K = N(T) \asymp T \log T \asymp N(\log N)^2$, we have:

$$\log\log x \leq K \log N + \log\log N \asymp N(\log N)^3.$$

Then for some constant C:

$$\log N \geq \log\log\log x - \log C - 3\log\log N.$$

Since $x \geq N^{e^{\pm(1/N)}} \gg N$, we have $\log\log N = o(\log\log\log x)$, so:

$$\log N \geq (1 + o(1)) \log\log\log x.$$

Conclusion:

By Lemma 5.2, the quantity

$$\frac{1}{x(e^{\delta} - e^{-\delta})} \int_{e^{-\delta}x}^{e^{\delta}x} (\psi(u) - u)\, du$$

is an average of $\psi(u) - u$ over a neighborhood of x, where $x \asymp N$ and N can be arbitrarily large.

This average achieves order $\pm x^{1/2} \log N \asymp \pm x^{1/2} \log\log\log x$ infinitely often in both signs. Since this is an average, the function $\psi(u) - u$ must achieve values of at least this magnitude (in both directions) infinitely often.

Therefore:

$$\psi(x) - x = \Omega_{\pm}(x^{1/2} \log\log\log x).$$

□

7 Quantitative Refinements

The proof actually gives us quantitative information:

Theorem 7.1.

$$\limsup_{x\to\infty} \frac{\psi(x) - x}{x^{1/2} \log\log\log x} \geq \frac{1}{2}, \qquad \liminf_{x\to\infty} \frac{\psi(x) - x}{x^{1/2} \log\log\log x} \leq -\frac{1}{2}.$$

The same bounds hold for $\pi(x) - \mathrm{li}(x)$ in place of $\psi(x) - x$.

Theorem 7.2 (Sign Changes in Bounded Intervals)**.** *Let Θ denote the supremum of the real parts of zeros of $\zeta(s)$. If ζ has a zero with real part Θ, then there exists a constant $C > 0$ such that $\psi(x) - x$ changes sign in every interval $[x, Cx]$ for $x \geq 2$.*

The proof uses the explicit formula and a mean value theorem argument on the smoothed functions $R_k(y)$.

8 Historical Remarks and Subsequent Developments

1. **Littlewood's Original Proof (1914):** The proof is inherently non-constructive—it uses a case distinction on whether RH is true or false, and employs Dirichlet's approximation theorem which is also non-effective.

2. **Skewes' Numbers:** Littlewood's proof gave no information about where the first sign change occurs. Skewes (1933), assuming RH, showed there exists $x < e^{e^{e^{79}}} < 10^{10^{10^{34}}}$ with $\pi(x) > \mathrm{li}(x)$. Without assuming RH, Skewes (1955) gave $x < 10^{10^{10^{963}}}$.

3. **Modern Bounds:** These bounds have been dramatically reduced:

   - Lehman (1966): Between 1.53×10^{1165} and 1.65×10^{1165}, there are more than 10^{500} consecutive integers x with $\pi(x) > \mathrm{li}(x)$.
   - te Riele (1987): $x < 6.69 \times 10^{370}$
   - Bays–Hudson (2000): $x < 1.40 \times 10^{316}$
   - Chao–Plymen (2010): Slight further improvements

4. **Logarithmic Density:** Wintner (1941) showed that the logarithmic density of integers with $\pi(x) > \mathrm{li}(x)$ is positive. Rubinstein–Sarnak (1994), under GRH and a linear independence hypothesis for zeta zeros, showed this proportion is approximately 2.6×10^{-7}.

5. **Verified Range:** It has been computed that $\pi(x) < \mathrm{li}(x)$ for all $x \leq 10^{14}$.

9 Conclusion

Littlewood's theorem stands as one of the great achievements of analytic number theory. It showed that our intuition about $\pi(x) < \mathrm{li}(x)$—based on all available numerical evidence—was fundamentally misleading about the long-term behavior of these functions.

The key insight of the proof is that while the "random" contributions from the zeta zeros usually cancel, Dirichlet's approximation theorem guarantees that sometimes many of them align constructively, overwhelming the systematic bias from the $-\frac{1}{2}\,\mathrm{li}(x^{1/2})$ term.

The proof beautifully combines:

- The explicit formula connecting primes to zeta zeros
- Landau's lemma on Dirichlet series with non-negative coefficients
- Dirichlet's approximation theorem from Diophantine analysis
- Careful estimates on sums over zeros

References

[1] J.E. Littlewood, "Sur la distribution des nombres premiers," *C.R. Acad. Sci. Paris*, **158** (1914), 1869–1872.

[2] H.L. Montgomery and R.C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge Studies in Advanced Mathematics **97**, Cambridge University Press, 2007.

[3] A.E. Ingham, *The Distribution of Prime Numbers*, Stechert-Hafner, 1964.

[4] H.M. Edwards, *Riemann's Zeta Function*, Academic Press, 1974.

[5] M. Rubinstein and P. Sarnak, "Chebyshev's bias," *Experiment. Math.*, **3** (1994), 173–197.