A Proof that Equal Consecutive Prime Gaps Have Density Zero

John N. Dvorak
Independent Researcher
john.n.dvorak@gmail.com
www.linkedin.com/in/john-n-dvorak
https://x.com/DvorakJohnN
ORCID: 0009-0001-3691-2066

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Abstract

Let p_n denote the n-th prime and $d_n = p_{n+1} - p_n$ the n-th prime gap. We prove that the set of indices where consecutive gaps are equal has asymptotic density zero:

$$\lim_{x\to\infty} \frac{|\{n \le \pi(x) : d_{n+1} = d_n\}|}{\pi(x)} = 0.$$

This result, suggested by Terence Tao, reduces Erdős Problem #218 to the study of strict inequalities between consecutive gaps.

MSC 2020: Primary 11N05; Secondary 11B25

1 Introduction

Erdős Problem #218 concerns the distribution of consecutive prime gaps $d_n = p_{n+1} - p_n$. The conjecture states that the sets

$$\{n : d_{n+1} > d_n\} \quad \text{and} \quad \{n : d_{n+1} < d_n\}$$

each have asymptotic density 1/2. A natural subproblem is to show that equal consecutive gaps occur rarely.

Under Cramér's random model for the primes, the probability that two successive gaps are equal is asymptotically $\sim 1/(\log p_n)^2$, so the expected number of equal consecutive gaps up to x is $O(\pi(x)/\log x)$, which is $o(\pi(x))$. We make this rigorous:

Theorem 1.1. *Let* $T(x) = |\{n \le \pi(x) : d_{n+1} = d_n\}|$. *Then*

$$\lim_{x\to\infty} \frac{T(x)}{\pi(x)} = 0.$$

Historical Note. The fact that equal consecutive prime gaps have density zero has been considered folklore since at least the 1990s and is mentioned informally on erdosproblems.com. To the author's knowledge, the complete rigorous proof presented here—combining uniform Brun sieve bounds with the optimal threshold $y = (\log x)^{4/3}$—has not previously appeared in the literature.

2 Preliminaries

2.1 Sieve Theory for k-Tuples

We require the following standard result from sieve theory:

Theorem 2.1 (Uniform k-tuple Sieve)**.** *For fixed k and integers $a_1 < \cdots < a_k$ forming an admissible k-tuple, the number of $n \leq x$ with all $n + a_i$ prime satisfies:*

$$\ll_{k,A} \frac{x}{(\log x)^k}$$

uniformly for $|a_i| \leq (\log x)^A$ for any fixed A, where the implied constant may depend on k and A.[1]

References: Halberstam & Richert, *Sieve Methods*, Theorem 2.5; Friedlander & Iwaniec, *Opera de Cribro*, Theorem 6.9; Kedlaya, *Prime k-tuples*, 2017.

2.2 Partitioning Gap Sizes

Define:

$$y = (\log x)^{4/3}.$$

We partition $T(x) = T_{\text{small}}(x) + T_{\text{large}}(x)$ where:

- $T_{\text{small}}(x)$ counts indices with $d_{n+1} = d_n \leq y$, and
- $T_{\text{large}}(x)$ counts indices with $d_{n+1} = d_n > y$.

We need only consider **even** values of d, as three odd primes cannot form an arithmetic progression with odd common difference d.

3 Bounding Small Gaps

For even $d \leq y$, let $U_d(x)$ denote the number of prime triples $(p, p+d, p+2d)$ with $p \leq x$ (not necessarily consecutive primes). Let $T_d(x)$ denote the number of such triples where p, $p+d$, and $p+2d$ are consecutive primes.

Proposition 3.1. $T_{\text{small}}(x) = o(\pi(x))$.

Proof. For every even d we have

$$T_d(x) \leq U_d(x) \ll \frac{x}{(\log x)^3}$$

by Theorem 2.1 applied to the admissible 3-tuple $\{0, d, 2d\}$ with $A = 2$. (A tuple is admissible when $d \equiv 0 \pmod 6$; non-admissible d contribute $O(1)$ each, for a total of $O(y) = o(\pi(x))$.)

Summing over even $d \leq y$ gives

$$T_{\text{small}}(x) \ll \frac{y \cdot x}{(\log x)^3} = \frac{x}{(\log x)^{5/3}} = o\left(\frac{x}{\log x}\right) = o(\pi(x)). \qquad \square$$

[1]This follows from the fundamental lemma of Brun's sieve applied to the linear forms $n + a_i$, combined with Siegel–Walfisz uniformity for primes in arithmetic progressions to moduli $(\log x)^A$ and uniform boundedness of the singular series $\mathfrak{S}(a_1, \ldots, a_k)$ in this range. For non-admissible tuples the count is 0, which is trivially $\ll_{k,A} x/(\log x)^k$. See Kedlaya, *Prime k-tuples* (2017) or Pollack, *Gaps between primes: the story so far* (2014) for expositions.

Remark (Consecutive vs. All Prime Triples)**.** One can refine the relationship between $T_d(x)$ and $U_d(x)$. If $(p, p+d, p+2d)$ are primes but not consecutive, there exists at least one "interloper" prime q in either $(p, p+d)$ or $(p+d, p+2d)$. For fixed even d, there are $O(d)$ possible positions for such q, yielding $O(d)$ potential 4-tuple configurations. By Theorem 2.1 with $k=4$, each contributes $O(x/(\log x)^4)$, so

$$U_d(x) - T_d(x) \ll \frac{d \cdot x}{(\log x)^4}.$$

Thus $T_d(x) = U_d(x) + O(dx/(\log x)^4)$, showing that the number of consecutive prime triples is asymptotically the same as the total number of prime triples with common difference d whenever $d \ll (\log x)^{4/3}$. This refinement is not needed for the main theorem but may be useful for obtaining sharper asymptotics. In particular, it becomes essential if one wishes to obtain the sharper bound $T(x) \ll \pi(x)/\log\log x$ using modern multidimensional sieves.

4 Bounding Large Gaps

Lemma 4.1. *$T_{\text{large}}(x) \ll x/y$.*

Proof. The injection

$$n \in \{d_{n+1} = d_n > y\} \mapsto m = n+1 \in \{m : d_m > y\}$$

gives $T_{\text{large}}(x) \le \#\{m \le \pi(x) : d_m > y\}$. Using the telescoping identity:

$$\sum_{m \le \pi(x)} d_m = p_{\pi(x)+1} - 2 \ll x,$$

we obtain:

$$\#\{m : d_m > y\} \cdot y \le \sum_{m : d_m > y} d_m \le \sum_{m \le \pi(x)} d_m \ll x.$$

Hence $T_{\text{large}}(x) \ll x/y$. This argument automatically handles chains of equal large gaps. □

Proposition 4.2. *$T_{\text{large}}(x) = o(\pi(x))$.*

Proof. With $y = (\log x)^{4/3}$:

$$\frac{T_{\text{large}}(x)}{\pi(x)} \ll \frac{x/y}{x/\log x} = \frac{1}{(\log x)^{1/3}} \to 0.$$ □

5 Completion and Consequences

Proof of Main Theorem. Combining Propositions 3.1 and 4.2:

$$\frac{T(x)}{\pi(x)} = \frac{T_{\text{small}}(x)}{\pi(x)} + \frac{T_{\text{large}}(x)}{\pi(x)} \to 0.$$ □

Corollary 5.1. *This establishes:*

$$\delta(\{n : d_{n+1} \ge d_n\}) = \delta(\{n : d_{n+1} > d_n\}),$$

reducing Erdős Problem #218 to proving the strict inequalities each have density $1/2$.

Remark (Longer Chains of Equal Gaps)**.** The same argument shows that, for any fixed $\ell \geq 1$, the number of $n \leq \pi(x)$ such that

$$d_{n+1} = d_{n+2} = \cdots = d_{n+\ell}$$

is $o(\pi(x))$. Indeed, such an n produces an arithmetic progression of $\ell+1$ primes $p_{n+1}, p_{n+2}, \ldots, p_{n+\ell+1}$ with common difference $d = d_{n+1}$. Applying Theorem 2.1 to $(\ell+1)$-tuples (with shifts bounded by ℓy) and using the same large-gap telescoping argument as in Lemma 4.1 yields the desired bound.

Acknowledgments

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