A Proof that Equal Consecutive Prime Gaps Have Density Zero

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Abstract

Let p_n denote the n-th prime and $d_n = p_{n+1} - p_n$ the n-th prime gap. We prove that the set of indices where consecutive gaps are equal has asymptotic density zero:

$$\lim_{x\to\infty} \frac{|\{n \le \pi(x) : d_{n+1} = d_n\}|}{\pi(x)} = 0.$$

This result, suggested by Terence Tao, reduces Erdős Problem #218 to the study of strict inequalities between consecutive gaps.

1 Introduction

Erdős Problem #218 concerns the distribution of consecutive prime gaps $d_n = p_{n+1} - p_n$. The conjecture states that the sets

$$\{n : d_{n+1} > d_n\} \quad \text{and} \quad \{n : d_{n+1} < d_n\}$$

each have asymptotic density 1/2. A natural subproblem is to show that equal consecutive gaps occur rarely:

Theorem 1.1. *Let $T(x) = |\{n \le \pi(x) : d_{n+1} = d_n\}|$. Then*

$$\lim_{x\to\infty} \frac{T(x)}{\pi(x)} = 0.$$

Historical Note. The density-zero assertion for equal consecutive gaps has been informally suggested in the literature since at least the 1990s, with experts noting that Brun's sieve method should suffice (see comments on erdosproblems.com). However, the complete rigorous argument presented here—including the singular series averaging and optimal parameter selection—appears not to have been formally published previously.

2 Preliminaries

2.1 Sieve Theory for k-Tuples

We require the following standard result from sieve theory:

Theorem 2.1 (Uniform k-tuple Sieve). *For fixed k and integers $a_1 < \cdots < a_k$ forming an admissible k-tuple, the number of $n \le x$ with all $n + a_i$ prime satisfies:*

$$\ll_{k,A} \frac{x}{(\log x)^k}$$

uniformly for $|a_i| \le (\log x)^A$ for any fixed A, where the implied constant may depend on k and A.[1]

References: Halberstam & Richert, *Sieve Methods*, Theorem 2.5; Friedlander & Iwaniec, *Opera de Cribro*, Theorem 6.9; Kedlaya, *Prime k-tuples*, 2017.

2.2 Partitioning Gap Sizes

Define:

$$y = (\log x)^{4/3}.$$

We partition $T(x) = T_{\text{small}}(x) + T_{\text{large}}(x)$ where:

- $T_{\text{small}}(x)$ counts indices with $d_{n+1} = d_n \le y$, and
- $T_{\text{large}}(x)$ counts indices with $d_{n+1} = d_n > y$.

We need only consider **even** values of d, as three odd primes cannot form an arithmetic progression with odd common difference d.

3 The Consecutive Constraint Lemma

Lemma 3.1. *For even $d \le y$, let $T_d(x)$ count consecutive prime triples $(p, p+d, p+2d)$ and let $U_d(x)$ count all prime triples $(p, p+d, p+2d)$. Then:*

$$U_d(x) - T_d(x) \ll \frac{dx}{(\log x)^4}.$$

Proof. If $(p, p+d, p+2d)$ are primes but not consecutive, there exists at least one "interloper" prime q in either $(p, p+d)$ or $(p+d, p+2d)$. For a fixed even d, there are exactly $d-1$ possible integer positions for such q in the first gap and $d-1$ in the second, yielding $2(d-1) = O(d)$ potential 4-tuples of the form $(p, p+d, p+2d, q)$ (or any permutation thereof). By Theorem 2.1 with $k = 4$ and $A = 2$ (since $2d \le 2(\log x)^{4/3}$), each fixed 4-tuple configuration contributes $O(x/(\log x)^4)$. Multiplying by the $O(d)$ choices for q gives the error term $O(dx/(\log x)^4)$. Summing over all $d \le y$ yields the stated aggregate bound. □

4 Bounding Small Gaps

Proposition 4.1. $T_{small}(x) = o(\pi(x))$.

[1] This follows from the fundamental lemma of Brun's sieve applied to the linear forms $n + a_i$, combined with Siegel–Walfisz uniformity for primes in arithmetic progressions to moduli $(\log x)^A$ and uniform boundedness of the singular series $\mathfrak{S}(a_1, \ldots, a_k)$ in this range. For non-admissible tuples the count is 0, which is trivially $\ll_{k,A} x/(\log x)^k$. See Kedlaya, *Prime k-tuples* (2017) or Pollack, *Gaps between primes: the story so far* (2014) for expositions.

Proof. Using Lemma 3.1 and Theorem 2.1 with $k = 3$ and $A = 2$:

$$T_{\text{small}}(x) = \sum_{\substack{d \le y \\ d \text{ even}}} T_d(x) \ll \sum_{\substack{d \le y \\ d \text{ even}}} \left(\frac{x}{(\log x)^3} + \frac{dx}{(\log x)^4} \right).$$

(Only $d \equiv 0 \pmod 6$ give admissible tuples, as otherwise 2 or 3 would always divide one term. Non-admissible d contribute $O(1)$ each, for total $O(y)$ absorbed by the $\ll$ notation. The factor 1/2 from summing over even d is also absorbed.)

The first term sums to $\frac{yx}{(\log x)^3} = \frac{x}{(\log x)^{5/3}}$. The second term sums to $\frac{y^2 x}{(\log x)^4} = O\left(\frac{x}{(\log x)^{4/3}}\right)$. Both are $o(x/\log x) = o(\pi(x))$. □

5 Bounding Large Gaps

Lemma 5.1. *$T_{large}(x) \ll x/y$.*

Proof. For large gaps, we use a sum-of-gaps argument. The injection

$$n \in \{d_{n+1} = d_n > y\} \mapsto m = n + 1 \in \{m : d_m > y\}$$

gives $T_{\text{large}}(x) \le \#\{m : d_m > y\}$. Using the telescoping identity:

$$\sum_{m \le \pi(x)} d_m = p_{\pi(x)+1} - 2 \ll x,$$

we obtain:

$$\#\{m : d_m > y\} \cdot y \le \sum_{m : d_m > y} d_m \le \sum_{m \le \pi(x)} d_m \ll x.$$

Hence $T_{\text{large}}(x) \ll x/y$. This argument automatically handles chains of equal large gaps. □

Proposition 5.2. *$T_{large}(x) = o(\pi(x))$.*

Proof. With $y = (\log x)^{4/3}$:

$$\frac{T_{\text{large}}(x)}{\pi(x)} \ll \frac{x/y}{x/\log x} = \frac{1}{(\log x)^{1/3}} \to 0.$$

□

6 Completion and Consequences

Proof of Main Theorem. Combining Propositions 4.1 and 5.2:

$$\frac{T(x)}{\pi(x)} = \frac{T_{\text{small}}(x)}{\pi(x)} + \frac{T_{\text{large}}(x)}{\pi(x)} \to 0.$$

□

Corollary 6.1. *This establishes:*

$$\delta(\{n : d_{n+1} \ge d_n\}) = \delta(\{n : d_{n+1} > d_n\}),$$

reducing Erdős Problem #218 to proving the strict inequalities each have density 1/2.

Remark. The same argument shows that for any fixed $k \ge 2$, the number of $n \le \pi(x)$ with $d_{n+1} = d_{n+2} = \cdots = d_{n+k}$ is $o(\pi(x))$. One applies Theorem 2.1 with k replaced by $k + 2$ to bound non-consecutive $(k + 2)$-term prime arithmetic progressions (not necessarily consecutive primes), and the same large-gap argument applies.

Acknowledgments

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References

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