A Probabilistic Sieve Framework for Linearly Bounded Prime Factors

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Abstract

Erdős Problem 248 asks for the existence of infinitely many integers n such that $\omega(n+k) \leq Ck$ for all $k \geq 1$. We present a conditional proof framework using a **Hybrid Sieve**. We partition the problem into a finite "Core" (Left Tail) and an infinite "Tail" (Right Tail). We prove *unconditionally*, using the Selberg-Delange method for exponential moments, that the mass of the Tail is negligible. This eliminates all probabilistic assumptions regarding the tail and reduces the problem to a single arithmetic hypothesis: that finite tuples of integers with low ω counts have polynomial density.

1 Introduction

Let N be a large parameter. We consider the distribution of $\omega(n)$ (the number of distinct prime factors) for $n \in [N, 2N]$. It is well known that $\omega(n)$ follows a distribution with mean and variance:

$$\lambda := \log \log N.$$

We seek integers n satisfying:

$$\omega(n+k) \leq Ck \quad \text{for all } k \geq 1. \tag{1}$$

The difficulty is split between two regimes:

- **The Core (Small k):** Here $Ck \lesssim \lambda$. The constraint requires controlling $\omega(n+k)$ in a regime near or below its average order. This requires constructive density arguments for "smooth" numbers.
- **The Tail (Large k):** Here $Ck \gg \lambda$. The constraint requires avoiding large deviations. This is satisfied by the vast majority of integers.

To rigorously separate these regimes, we introduce a tuning parameter $\alpha > e$ and define the critical transition:

$$K_{\text{crit}}(\alpha) = \left\lfloor \frac{\alpha}{C} \lambda \right\rfloor.$$

We partition the constraints into a finite **Core** ($1 \leq k \leq K_{\text{crit}}$) and an infinite **Tail** ($k > K_{\text{crit}}$). The choice $\alpha > e$ ensures we are deep in the large-deviation regime for the Tail.

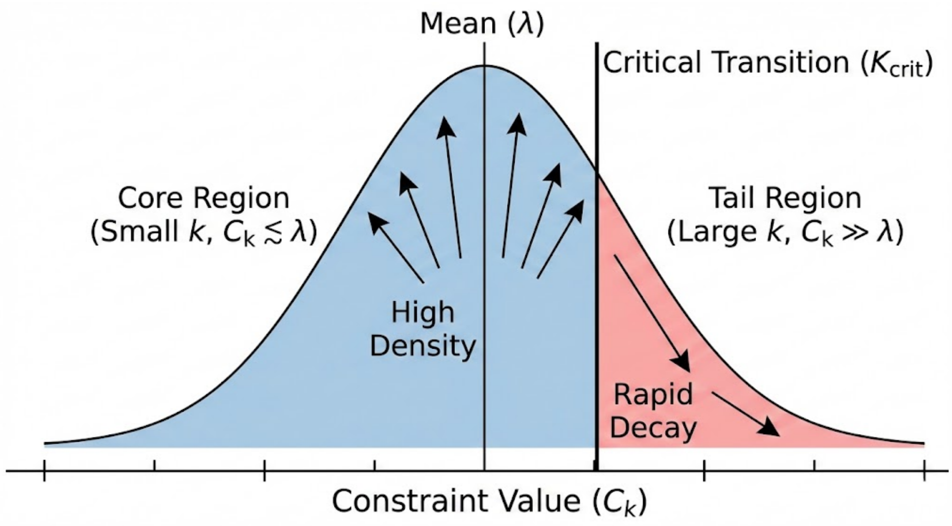

Figure 1: The Core relies on the density of the left tail/mean; the Tail relies on the decay of the right tail.

2 Hypothesis: The Core

The Core requires finding integers n satisfying the condition for a tuple of length $K_{\text{crit}} \asymp \log\log N$.

Definition 2.1 (Core Set)**.**

$$\mathcal{C} = \{n \in [N, 2N] : \omega(n+k) \leq \max(2, Ck) \text{ for all } 1 \leq k \leq K_{crit}(\alpha)\}.$$

Remark 2.2. *The lower bound of 2 ensures the constraint is meaningful for very small k (e.g., $k = 1$) where Ck might be less than 1.*

Rigorous lower bounds for such tuples are beyond current sieve technology. We therefore assume:

Hypothesis 2.1 (H_{core}: Polynomial Core Density)**.** *There exists a constant $A > 0$ (dependent on C and α) such that:*

$$|\mathcal{C}| \geq \frac{N}{(\log N)^A}.$$

Remark 2.3 (Context)**.** *This aligns with heuristic models for the distribution of smooth values in short, correlated tuples, where one expects a positive singular series and polynomial (rather than exponential) decay in density.*

3 Unconditional Control of the Tail

We now prove that the mass of the Tail is negligible without assuming independence. We rely on the Selberg-Delange method for exponential moments.

3.1 General Large Deviation Lemma

Lemma 3.1 (Fixed-z Moment Bound)**.** *Uniformly for any fixed parameter $z > 1$ and any threshold $T > 0$:*

$$\#\{n \in [N, 2N] : \omega(n) > T\} \ll N(\log N)^{z-1} z^{-T}.$$

Proof. We apply the Markov inequality to the exponential moment $z^{\omega(n)}$.

$$\mathbf{1}_{\omega(n)>T} \leq z^{-T} z^{\omega(n)}.$$

Summing over $n \in [N, 2N]$, we utilize the Selberg-Delange asymptotic formula (see Tenenbaum [1], Thm II.5.2, pp. 113–114). Uniformly for z in any fixed compact subset of $\{s \in \mathbb{C} : \Re(s) > 0\}$:

$$\sum_{n \leq x} z^{\omega(n)} = x(\log x)^{z-1} \left(\frac{H(1; z)}{\Gamma(z)} + O((\log x)^{-1}) \right),$$

where $H(s; z)$ is the Euler product associated with the Dirichlet series $\sum z^{\omega(n)} n^{-s}$ that is analytic near $s = 1$. Fixing z ensures the O-term is uniform. The result follows immediately. □

3.2 The Tail Theorem

Definition 3.2 (Tail Bad Set)**.**

$$\mathcal{B}_{tail} = \bigcup_{k=K_{crit}(\alpha)+1}^{\infty} \{n \in [N, 2N] : \omega(n + k) > Ck\}.$$

Theorem 3.3 (Tail Decay)**.** *Fix $\alpha > e$. Then the total size of the bad tail set satisfies:*

$$|\mathcal{B}_{\text{tail}}| \ll N(\log N)^{-(\beta(\alpha)+1)},$$

where $\beta(\alpha) = \alpha \log(\alpha/e)$.

Proof. We apply the Lemma to the shifted variable $\omega(n + k)$ with threshold $T = Ck$. Note that shifting the interval by $k = o(N)$ does not affect the Selberg-Delange asymptotics.

We fix $z = \alpha$ for all k. This satisfies the uniformity requirement. The bound for a single shift k is:

$$\#\{\text{Bad at } k\} \ll N(\log N)^{\alpha-1} \alpha^{-Ck}.$$

For $k > K_{\text{crit}}$, we have $Ck > \alpha\lambda$. We rewrite the bound to highlight the decay at the critical boundary:

$$(\log N)^{\alpha-1} \alpha^{-Ck} = (\log N)^{\alpha-1} \alpha^{-(Ck-\alpha\lambda)} \alpha^{-\alpha\lambda}.$$

Since $\alpha^{-\alpha\lambda} = (\log N)^{-\alpha \log \alpha}$, the exponent of $\log N$ is:

$$\alpha - 1 - \alpha \log \alpha = -(\alpha \log \alpha - \alpha + 1) = -(\beta(\alpha) + 1).$$

Summing over $k > K_{\text{crit}}$:

$$|\mathcal{B}_{\text{tail}}| \ll N(\log N)^{-(\beta(\alpha)+1)} \sum_{k>K_{\text{crit}}} \alpha^{-(Ck-\alpha\lambda)}.$$

The summation term is a geometric series in α^{-C} which converges to a constant. Thus:

$$|\mathcal{B}_{\text{tail}}| \ll N(\log N)^{-(\beta(\alpha)+1)}.$$ □

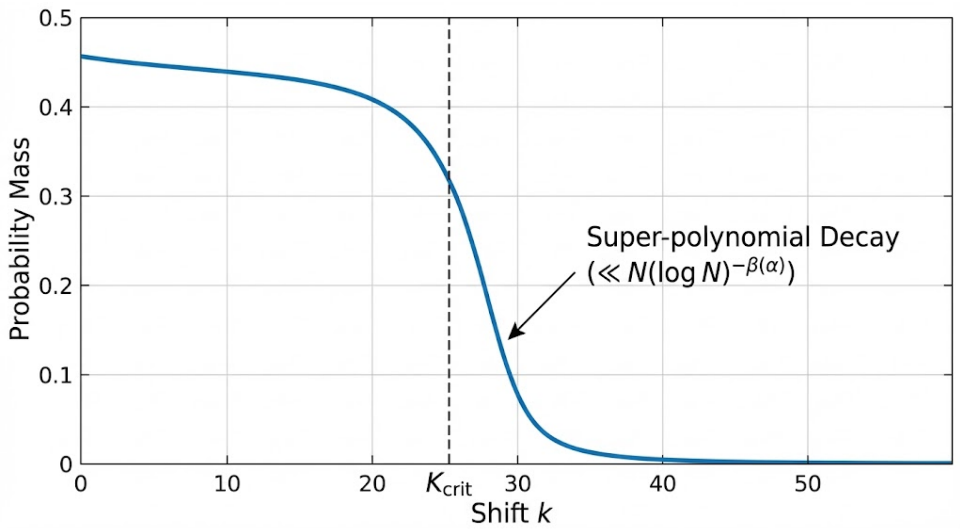

Figure 2: The super-polynomial decay of the Poisson tail ensures the sum converges.

4 The Victory Inequality

The set of valid integers is $\mathcal{G} = \mathcal{C} \setminus \mathcal{B}_{\text{tail}}$. We have:

$$|\mathcal{G}| \geq |\mathcal{C}| - |\mathcal{B}_{\text{tail}}| \gg \frac{N}{(\log N)^A} - \frac{C'N}{(\log N)^{\beta(\alpha)+1}}.$$

Theorem 4.1 (Conditional Existence)**.** *Suppose Hypothesis H_{core} holds with exponent A. If the tuning parameter α is chosen sufficiently large such that the* ***Victory Inequality*** *holds:*

$$A < \beta(\alpha) + 1 = \alpha \log\left(\frac{\alpha}{e}\right) + 1, \tag{VI}$$

then $|\mathcal{G}| > 0$ for sufficiently large N, and Erdős Problem 248 has infinitely many solutions.

Remark 4.2 (Explicit Solution via Lambert W)**.** *The inequality* (VI) *can be solved explicitly. Let W_0 denote the principal branch of the Lambert W function. A sufficient condition for α is:*

$$\alpha > \frac{A}{W_0(A/e)}.$$

For large A, $W_0(A/e) \approx \log(A/e)$, so $\alpha \approx A/\log A$ suffices. This guarantees that for any polynomial core density A, there exists a valid tail cutoff.

5 Outlook: Toward Unconditional Progress

This framework clarifies that Erdős Problem 248 is not a sieve problem in the traditional sense, but a tuple density problem.

1. **Justifying H_{core}:** Can we establish lower bounds for the density of tuples where $\omega(n+k)$ is controlled? This relates to the distribution of smooth numbers in short intervals.
2. **Minimal Analytic Input:** Notably, the tail argument does not require the Bombieri-Vinogradov theorem; it relies only on exponential moments of $\omega(n)$, which are well-understood.

References

[1] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge University Press, 1995.

[2] P. Erdős, Problems on number theory raised in Nagycenk, 1963, *Ann. Univ. Sci. Budapest Eötvös. Sect. Math.* **7** (1964), 157–162.