Thermal phase transition of 2D classical Ising model

PHYS3999 Directed Studies in Physics

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Outline

Main objectives

- Investigate phase transition of 2D classical Ising model using Monte Carlo method

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  - Find the critical temperature T_c of 2D classical Ising model
- Explore finite scaling effect of physical observables
  - Find the critical exponents $\eta, \alpha, \beta, \gamma, \nu$ of 2D classical Ising model

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$\rightarrow$ Solve complex models with large degrees of freedom

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Heisenberg model without external field

$$H = -\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j \tag{1}$$

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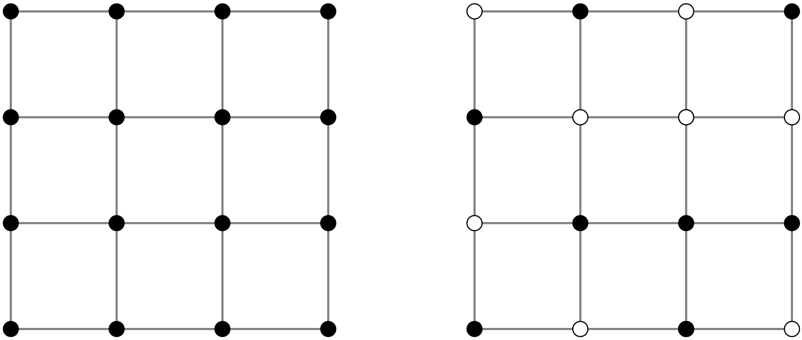
Onsager (1944) exact solutions: $T_c \approx 2.269$

2D Ising Model



(a) Ordered phase $T < T_c$ (b) Disordered phase $T > T_c$

Figure: Ising model with system size $L = 4$

Solid node: up spin $+1$
Open node: down spin -1

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  - Metropolis algorithm - Single-spin update (Metropolis et al., 1953)
  - Swendsen-Wang algorithm - Cluster update (Swendsen & Wang, 1987)

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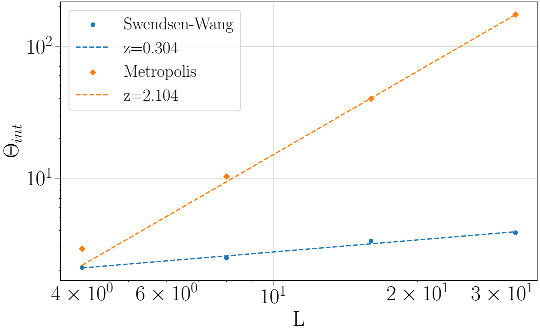

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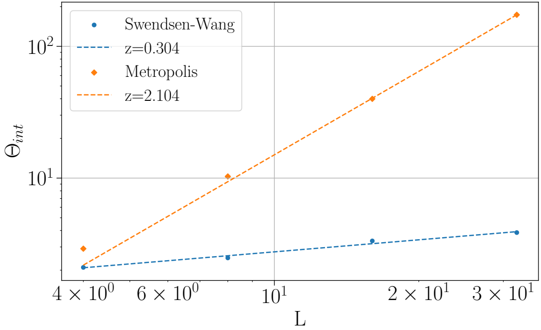

- Metropolis algorithm takes longer steps to generate statistically independent configurations than Swendsen-Wang algorithm.

Physical observables and critical phenomena

Useful variables:

Reduced temperature:

$$t = \frac{|T - T_c|}{T_c} \tag{3}$$

Correlation length:

$$\xi \sim t^{-\nu} \tag{4}$$

ν: one of the critical exponents

Physical observables: spin-spin correlation

Spin-spin correlation:

$$G(\vec{r}) \sim \begin{cases} \dfrac{\mathrm{e}^{-r/\xi}}{r^{(d-2)/2}} & \text{when } T \neq T_c \\ \dfrac{1}{r^{d-2+\eta}} & \text{when } T = T_c,\ \xi \to \infty \end{cases} \tag{5}$$

d: dimension of the system

η: one of the critical exponents

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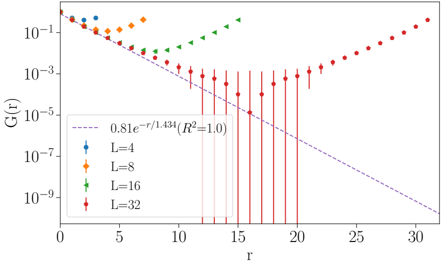

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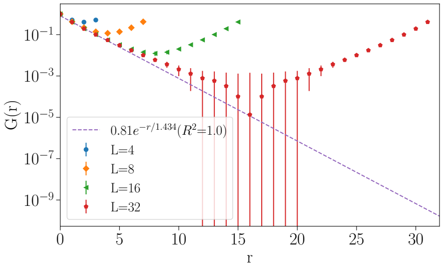

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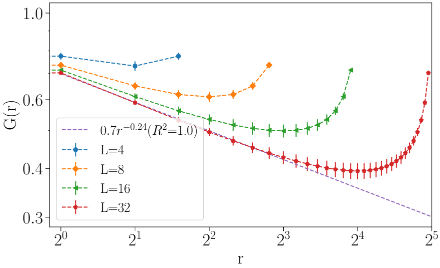

(b) Log-log plot of G(r) at T=2.285

Physical observables in infinite lattice

Net Magnetization:

$$\langle |m| \rangle \sim \begin{cases} (T_c - T)^{\beta} & \text{when } T < T_c \\ 0 & \text{when } T > T_c \end{cases} \tag{6}$$

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Specific Heat Capacity:

$$C \sim t^{-\alpha} \tag{8}$$

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Scaling hypotheses (Kadanoff, 1966):

$$|m|(T_c - T, L) = L^{-\beta/\nu} f_{|m|}\left((T_c - T)L^{1/\nu}\right) \tag{9}$$

$$\chi(t, L) = L^{\gamma/\nu} f_{\chi}\left(tL^{1/\nu}\right) \tag{10}$$

$$C(t, L) = \ln(L) f_C\left(tL^{1/\nu}\right) \tag{11}$$

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In equation (11), $\ln(L)$ is used because $\alpha = 0$

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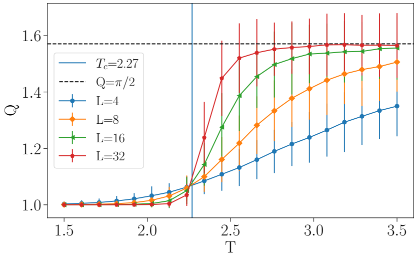

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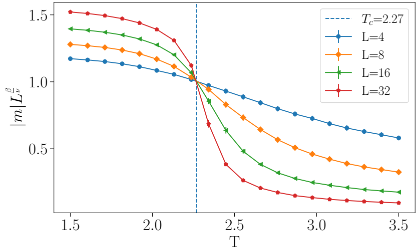

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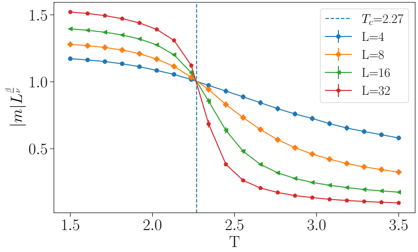
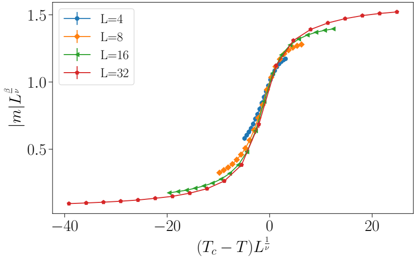
3. Plot scaling function f_m against its argument $(T_c - T)L^{1/\nu}$ to find out $1/\nu$ and hence β





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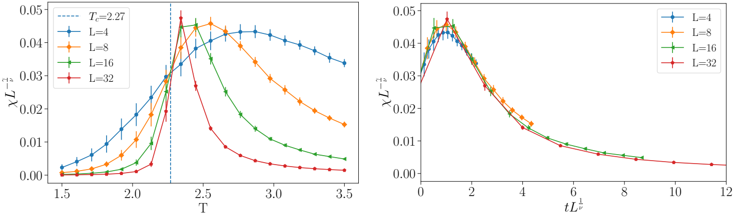

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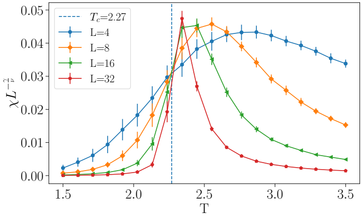
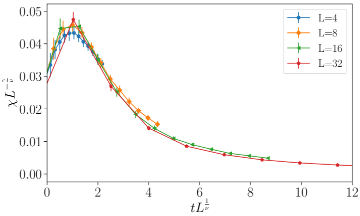
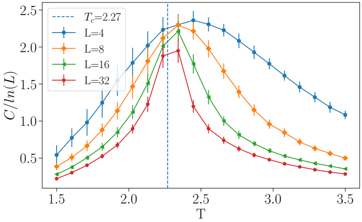
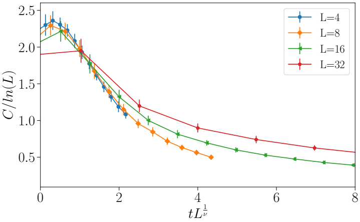




Specific heat capacity C:





Results

- Spin-spin correlation
  - $\eta \approx -0.243$
- Binder ratio
  - $T_c = 2.269$
- Data collapse
  - $\alpha = 0$
  - $\beta = 1/8$
  - $\nu = 1$
  - $\gamma = 7/4$

All results above are in reasonable accord with the exact solutions.

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4. Obtained critical exponents using data collapse

References

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