# ADVANCED COUNTING TECHNIQUES

* **Sterling number of second kind:**

" " " " " $m_n$ is the number of possible ways to assign 'm' distinct objects (unidentical) into 'n' identicle places (boxes) with no place (box) left empty, where $m \geq n$.

OR

Sterling number of 2nd kind is the number of partitions of set of size 'm' into 'n' non-empty subsets, where $m \geq n$.
And it is denoted by $S(m,n)$

→ Properties:

1. $S(m,1) = 1$, for $m \geq 1$
2. $S(m,m) = m$, for $m \geq 1$
3. If m distinct objects kept in n identicle boxes with $m \geq n$, the sterling number of 2nd kind can be written as

$$\boxed{S(m,n) = S(m-1,n-1) + nS(m-1,n)} \rightarrow (1)$$

→ The table consisting of possible sterling number of 2nd kind for the objects

$m=1$: $\overset{1}{S(1,1)}$

$m=2$: $\overset{1}{S(2,1)}$ $\overset{1}{S(2,2)}$

$m=3$: $\overset{1}{S(3,1)}$ $\overset{3}{S(3,2)}$ $\overset{1}{S(3,3)}$

$m=4$: $\overset{1}{S(4,1)}$ $\overset{7}{S(4,2)}$ $\overset{6}{S(4,3)}$ $\overset{1}{S(4,4)}$

$m=5$: $\overset{1}{S(5,1)}$ $\overset{15}{S(5,2)}$ $\overset{25}{S(5,3)}$ $\overset{10}{S(5,4)}$ $\overset{1}{S(5,5)}$

⋮

$S(4,2) = S(3,1) + 2S(3,2)$
$= 1 + 2(3) = 7$.

$S(4,3) = S(3,2) + 3S(3,3)$
$= 3 + 3(1)$
$= 6$

* General formula of sterling number of second kind:
If 'm' distinct objects kept in 'n' identical boxes then we have the following general formula to find sterling number of 2nd kind:

$$S(m,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \, {}^nC_k (n-k)^m \quad \text{---} ②$$

Imp
* NOTE: Let A and B be two finite sets with $n(A) = m$, and $n(B) = n$, where $m \geq n$ then the number of onto functions from A to B denoted by $P(m,n)$ and given by the formula:

$$P(m,n) = n! \, S(m,n)$$

or $$P(m,n) = \sum_{k=0}^{n} (-1)^k \, {}^nC_k (n-k)^m \quad \text{---} ③$$

* Examples:
① Evaluate i) $S(5,4)$, ii) $S(8,6)$
→ soln: General formula of sterling no. of 2nd kind:

$$S(m,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \, {}^nC_k (n-k)^m \longrightarrow ①$$

i) Put $m=5$, $n=4$ in Eqn ①

$$S(5,4) = \frac{1}{4!} \sum_{k=0}^{4} (-1)^k \, {}^4C_k (4-k)^5$$

$$= 10$$

OR.

We know that

$S(m,n) = S(m-1, n-1) + nS(m-1, n)$

Put $m=5$, $n=4$

$S(5,4) = S(4,3) + 4S(4,4)$

$S(5,4) = \{S(3,2) + 3S(3,3)\} + 4 \cdot S(4,4)$

$S(5,4) = S(3,2) + 3S(3,3) + 4S(4,4)$

$S(5,4) = S(2,1) + 2S(2,2) + 3S(3,3) + 4S(4,4)$

$S(5,4) = 1+2+3+4$

$S(5,4) = 10$

ii) $S(8,6)$

Put $m=8$ and $n=6$ in $Eq^n$ 1.

$$S(8,6) = \frac{1}{6!} \sum_{k=0}^{6} (-1)^k \, {}^6C_k \, (6-k)^8$$

$= 266$

2 Evaluate: $S(8,7)$ given that $S(7,6) = 21$

$\rightarrow$ Given: $S(7,6) = 21$

WKT,

$S(m,n) = S(m-1, n-1) + nS(m-1, n)$

Put $m=8$ $n=7$

$S(8,7) = S(7,6) + 7S(7,7)$

$S(8,7) = 21 + 7(1)$

$S(8,7) = 28$

3 Let $A = \{1,2,3,4,5,6,7\}$ & $B = \{w,x,y,z\}$. Find the number of onto functions from A to B.

~~Soln~~: Given $A = \{1,2,3,4,5,6,7\}$, $n(A) = m = 7$

$B = \{w,x,y,z\}$ $n(B) = n = 4$

No. of onto functions from A to B is given

$$P(m,n) = \sum_{k=0}^{n} (-1)^k \, {}^nC_k \, (n-k)^m$$

Put $m=7$ $n=4$.

$$P(7,4) = \sum_{k=0}^{4} (-1)^k \, {}^4C_k \, (4-k)^7$$

$$= {}^4C_0 (4)^7 - {}^4C_1 (3)^7 + {}^4C_2 (2)^7 + {}^4C_3 (1)^7 + {}^4C_4 \times 0.$$

$P(7,4) = 8400$

$\therefore$ 8400 onto functions from set A to B can be formed.

(Q) If A and B are two finite sets with $n(A)=5$ and $n(B)=3$ then find the no. of onto functions from A to B

→ soln: Given: $n(A)=5$, $n(B)=3$

No. of onto functions from A to B can be given as:

$$P(m,n) = \sum_{k=0}^{n} (-1)^k \, {}^nC_k (n-k)^m$$

Put $m = 5$ $n=3$

$$P(5,3) = \sum_{k=0}^{3} (-1)^k \, {}^3C_k (3-k)^5$$

$$= {}^3C_0 (3)^5 - {}^3C_1 (2)^5 + {}^3C_2 (1)^5 - 0$$

$P(5,3) = 150$

$\therefore$ 150 no. of onto func fun. A to B can be formed.

(5) There are 6 programmers who can assist 8 executives. In how many ways can the executive be assisted so that each programmer assists at least one executive.

→ soln: Given: $n(A) = m = 8$ (Executive)

$n(B) = n = 6$ (Programmer)

No. of ways, so that each p" assists atleast 1 executive.

$$P(m, n) = \sum_{k=0}^{n} (-1)^k \, {}^nC_k (n-k)^m$$

Put $m = 8$, $n = 6$

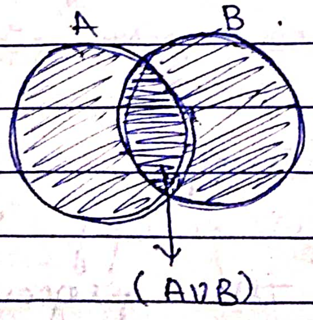
$$P(8, 6) = \sum_{k=0}^{6} (-1)^k \, {}^6C_k (6-k)^8$$

$= 191520.$

* Principle of inclusion and exclusion.

→ Statement: Let A and B be any two finite sets. Then the number of elements in union of A and B is its sum of the numbers of elements in these sets minus the " " " in their intersection.

i.e $\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$

→ 


Illy, if A, B and C are any three sets then number of elements in union of these three sets is given by:

$\cancel{n(A\cap B\cap C) = n[(A\cap B)\cap}$

$n(A\cup B\cup C) = n[(A\cup B)\cup C]$

$= n(A\cup B) + n(C) - n[(A\cup B)\cap C]$

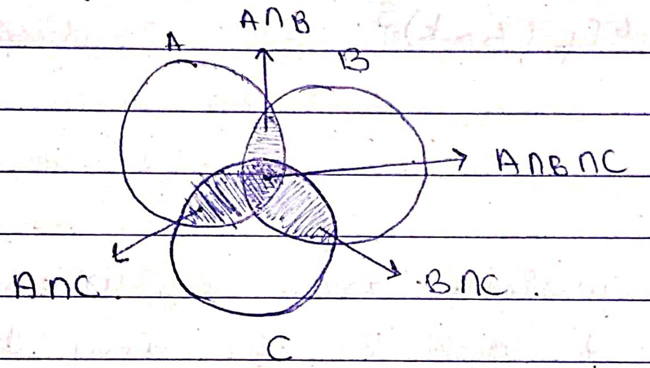
$= n(A) + n(B) - n(A\cap B) + n(C) - n[(A\cap C)\cup(B\cap C)]$

[∵ De-morgan's law]

$= n(A) + n(B) - n(A\cap B) + n(C) - \{n(A\cap C) +$
$n(B\cap C) - n[(A\cap C)\cap(B\cap C)]\}$

$= n(A) + n(B) - n(A\cap B) + n(C) - n(A\cap C) - n(B\cap C)$
$+ n(A\cap B\cap C)$

$$\boxed{n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(A\cap C) - n(B\cap C) + n(A\cap B\cap C)}$$




★ **Pigeonhole principle :**

Statement : If 'm' pigeons occupy 'n' pigeonholes and m>n, then two or more pigeons occupy the same pigeon hole.

OR

If 'm' ..., then atleast on of the pigeonhole must contain two or more pigeons in it.

★ **Generalised pigeonhole principle :**

Statement : If 'm' pigeons occupy 'n' pigeonholes then atleast one of the pigeonhole must contain $\left[\frac{m-1}{n}\right]+1$ or more pigeons.

where [ ] represents the greatest integer function

Proof : we prove this principle by the method of contradiction.

Assume that each pigeon hole contains no more than $\left[\frac{m-1}{n}\right]$ pigeons.

$\Rightarrow$ Total number of pigeons in n pigeonholes

$$\Rightarrow n\left[\frac{m-1}{n}\right] \le n \cdot \frac{m-1}{n}$$

$$n\left[\frac{m-1}{n}\right] \le m-1$$

It means that, total number of pigeons in 'n' pigeonholes is less than $m-1$.

This contradicts our fact that 'm' pigeons

$\therefore$ Atleast one pigeonhole contains $\left[\frac{m-1}{n}\right]+1$ or more pigeons.

Hence proved.

**$\star$ Examples:**

① Prove that in a set of 13 children at least two have the birthday during the same month.

Soln. Let us take 13 children as pigeons and 12 months as pigeonholes.

$\therefore$ $m=13$ and $n=12$

By generalised pigeonhole principle we have atleast 1 month has $\left[\frac{m-1}{n}\right]+1$ or more births

$$\left[\frac{m-1}{n}\right]+1 = \left[\frac{13-1}{12}\right]+1 = 2.$$

$\therefore$ Atleast 2 children ~~are~~ ~~to~~ have birthday in d same month.

② If 7 cars carry 26 passengers prove that atleast 1 car must ~~contain~~ have 4 or more passengers.

Soln → Let 26 passengers be pigeons and 7 cars be pigeonholes.

$\therefore$ $m = 26$ & $n = 7$.

According to generalized pigeonhole principle, at least one cars will have $\left[\frac{m-1}{n}\right]+1$ number of passangers in it.

i.e $\left[\frac{m-1}{n}\right]+1 = \left[\frac{26-1}{7}\right]+1 = [3.57]+1$.

$= 3+1 = 4$.

$\therefore$ Atleast 1 car must have 4 or more passangers.

③ What should be the minimum number of students, so that atleast 2 students have their last name with the same english letter.

soln → Let us take m students as pigeons and 26 english letters as pigeonholes.

$\therefore$ $m = ?$ $n = 26$

Given that at least 2 students have their last name beginning with same english letter.

$\therefore$ By generalised pigeonhole principle we have ~~m+1~~

$$\left[\frac{m-1}{n}\right]+1 = 2.$$

$$\Rightarrow \left[\frac{m-1}{26}\right]+1 = 2. \quad \Rightarrow \left\{\frac{m-1}{26}\right\}+1 \geq 2.$$

$$\Rightarrow \left\{\frac{m-1}{26}\right\} \geq 1 \quad \Rightarrow m-1 \geq 26.$$

$$\Rightarrow m \geq 27.$$

$\therefore$ Minimum number of students is 27.

Knives.

* 3 different is mentioned so combination

*(4) Find the least number of ways of choosing three different numbers from 1 to 10, so that all choices have the same sum.

sol$^n$ → No. of ways of choosing 3 different numbers from 1 to 10 is $^{10}C_3 = \frac{10!}{(10-3)!\,3!}$

$= \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!} \times 3!}$

$= 120$ ways.

Next, the smallest possible sum we get from choosing 1, 2, 3 is 6.
and the largest possible sum " " " choosing 8, 9, 10 is 27
∴ Sum of these 3 numbers vary from 6 to 27 and sums are 22 in number

$\begin{bmatrix} 6-1 \\ 7-2 \\ 8-3 \\ \vdots \\ 27-22 \end{bmatrix}$

Now Let us take 120 choices as pigeons & 22 sums as pigeon holes
∴ $m = 120$ & $n = 22$

By generalised pigeonhole principle, at least no. of choices are $\left[\frac{m-1}{n}\right]+1$ that have same sum.

i.e $\left[\frac{m-1}{n}\right]+1 = \left[\frac{120-1}{22}\right]+1$

$= [5.41] + 1 = 5 + 1 = 6.$

∴ Least no. of choices assigned to the same sum is 6.

* Recurrence relation / Difference equation :

A " " for the sequence $\{a_n\}$ $(a_0, a_1, a_2, \ldots a_n)$ is an equation that expresses $a_n$ in terms of one or more number of the previous terms, namely $a_0, a_1, a_2 \ldots a_{n-1}$ for all integers $n \geq 0$.

A sequence $a_n$ (or $a_0, a_1, a_2 \ldots a_n$) is called a solution of a recurrence relation if its terms satisfies the recurrence relation.

→ Example :

1) A recurrence relation $a_n = a_{n-1} - a_{n-2}$ $\forall n = 2, 3 \ldots$ with initial conditions $a_0 = 0$ and $a_1 = 2$. Satisfy the sequence $\{a_n\} = \{0, 2, 2, 0, -2, \ldots\}$.

$$a_2 = a_1 - a_0 = 2 - 0 = 2$$
$$a_3 = a_2 - a_1 = 2 - 2 = 0.$$
$$a_4 = a_3 - a_2 = 0 - 2 = -2$$

2) A recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ $\forall n = 2, 3, 4 \ldots$ Satisfies the sequence $\{a_n\}$, where $a_n = 3n$.

→ RHS : $2a_{n-1} - a_{n-2}$
$$= 2\{3(n-1)\} - 3(n-2)$$
$$= 6n - 6 - 3n + 6$$
$$= 3n$$
$$= a_n = \text{LHS}$$

Note : The recurrence relation $4a_{n+3} - a_{n+2} + 11a_{n+1} - 6a_n = 0$ can also be written as

$4y_{n+3} - y_{n+2} + 11y_{n+1} - 6y_n = 0$ or $4U_{n+3} - U_{n+2} + 11U_{n+1} - 6U_n = 0$

or $4f(n+3) - f(n+2) + 11f(n+1) - 6f(n) = 0$

★ Order of a recurrence relation:

The " " " " is the difference between the largest and the smallest subscript appearing in the relation.

→ Example: The recurrence relation given below

$4a_{n+3} - a_{n+2} + 11a_{n+1} - 6a_n = 0$

The order of this recurrence relation is:

$(n+3) - n = 3$.

★ Linear Recurrence relation with the constant coefficient

The " " " " " "

of order 'k' is of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \dots + c_k a_{n-k} = f(n) \longrightarrow (1)$$

where $c_0, c_1, c_2 \dots c_k$ are constants and $f(n)$ is function of 'n' only.

★ Homogeneous and non-homogenous linear recurrence relation with constant coefficients:

The recurrence relation (1) is said to be homogeneous linear " " of order 'k' if $f(n) = 0$. otherwise it is called non homogenous linear recurrence relation. ($f(n) \neq 0$)

(HLRR)

IMP. ★ Solution of homogeneous linear recurrence relation

We have two methods to find the soln.

(I) Characteristic root method:

Consider HLRR, $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = 0 \longrightarrow (1)$

characteristic eq$^n$ (auxillary equation) of degree 'k' can be formed by putting, $a_n = \alpha^k$, $a_{n-1} = \alpha^{k-1}$, $a_{n-2} = \alpha^{k-2}$ ... $a_{n-k} = \alpha^0 = 1$ in (1), we get

$c_0\alpha^k + c_1\alpha^{k-1} + c_2\alpha^{k-2} + \dots c_k = 0 \longrightarrow$ (2)

Eq$^n$ (2) is polynomial of degree k, ∴ we get k number of roots for eq$^n$ (2), namely given by:

$\alpha_1, \alpha_2, \alpha_3 \dots \alpha_k$.

For the sol$^n$ of equation (1), we have following 4 cases:

→ case 1: The roots are real and unequal ($\alpha_1 \neq \alpha_2 \dots \neq \alpha_k \in R$)

∴ Sol$^n$s of recurrence relation (1) is given by

$a_n = A_1\alpha_1^n$, $a_n = A_2\alpha_2^n$, $a_n = A_3\alpha_3^n \dots a_n = A_k\alpha_k^n$

General sol$^n$ of recurrence relation (1) is:

$$\boxed{a_n = A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n + \dots + A_k\alpha_k^n} \longrightarrow (A)$$

where $A_1, A_2, A_3, \dots A_k$ are constant

→ case 2: The roots are real and equal.

If two roots $\alpha_1$ and $\alpha_2$ are equal then general sol$^n$ is

$$\boxed{a_n = (A_1 + nA_2)\alpha_1^n + A_3\alpha_3^n + \dots + A_k\alpha_k^n} \longrightarrow (B_1)$$

If 3 roots $\alpha_1, \alpha_2$ and $\alpha_3$ are equal, then general sol$^n$ is:

$$\boxed{a_n = (A_1 + nA_2 + n^2A_3)\alpha_1^n + A_4\alpha_4^n + \dots + A_k\alpha_k^n} \longrightarrow (B_2)$$

⋮

If all k roots are equal ($\alpha_1 = \alpha_2 = \dots = \alpha_k$) then general sol$^n$ of (1) is

$$\boxed{a_n = (A_1 + nA_2 + n^2A_3 + \dots + n^{k-1}A_k)\alpha_1^n} \longrightarrow (B_3)$$

where $A_1, A_2 \dots A_k$ are consts.

→ case 3: The roots are complex and distinct.

If $\alpha_1 \pm i\beta_1$, $\alpha_2 \pm i\beta_2, \ldots$ are the " root, then general soln of ① is

$$\boxed{a_n = A_1(\alpha_1 + i\beta_1)^n + A_2(\alpha_1 - i\beta_1)^n + A_3(\alpha_2 + i\beta_2)^n + A_4(\alpha_2 - i\beta_2)^n + \ldots}$$

→ Ⓒ

→ case 4: The roots are complex and equal.

If $\alpha_1 \pm i\beta = \alpha_2 \pm i\beta_2$ and so on, then general solution of ① is

$$\boxed{a_n = (A_1 + nA_2)(\alpha_1 + i\beta_1)^n + (A_3 + nA_4)(\alpha_1 - i\beta_1)^n + \ldots} \quad Ⓓ$$

*Examples:

① Solve the ~~linear~~ HLRR $a_{n+2} - 3a_{n+1} + 2a_n = 0$ by characteristic roots method.

Solⁿ → Let $a_{n+2} - 3a_{n+1} + 2a_n = 0 \longrightarrow ①$

Order of the relation ① is $n+2-n = 2$.

∴ characteristic root eqⁿ (auxillary equation) of ① is given by putting $a_{n+2} = \alpha^2$, $a_{n+1} = \alpha^1$, $a_n = \alpha^0 = 1$

$$\therefore \alpha^2 - 3\alpha + 2(1) = 0$$

$$\alpha^2 - 3\alpha + 2 = 0.$$

$$\alpha^2 - 2\alpha - \alpha + 2 = 0$$

$$\alpha(\alpha - 2) - (\alpha - 2) = 0$$

$$\alpha = 2 \text{ or } \alpha = 1.$$

∴ $\alpha_1 = 1$ and $\alpha_2 = 2$

∴ Roots are real and unequal

∴ Solution of eqⁿ ① is given by

$a_n = A_1\alpha_1^n + A_2\alpha_2^n$.

$a_n = A_1 1^n + A_2 2^n$.

$\boxed{a_n = A_1 + A_2 2^n}$ where $A_1$ and $A_2$ are constants

$\frac{2+\sqrt{4-16}}{2}$ = $\frac{2 \pm 2i}{2}$ $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= 1 \pm i$

② Solve a HLRR $a_{n+2} - 2a_{n+1} + 4a_n = 0$ by characteristic root meth.

Sol^n → Given eq^n : $a_{n+2} - 2a_{n+1} + 4a_n = 0$.

The order of the relation is $n+2-n = 2$

∴ characteristic root eq^n of ① is given by putting $a_{n+2} = \alpha^2$ ; $a_{n-1} = \alpha^1$ , $a_n = 1$

i.e. $\alpha^2 - 2\alpha + 4 = 0$

$$a = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{4-16}}{2}$$

$$= \frac{2 \pm 2i\sqrt{3}}{2}$$

$$= 1 \pm i\sqrt{3}$$

∴ Roots are unequal and complex

$\alpha_1 + i\beta_1 = 1 + i\sqrt{3}$ and $\alpha_1 - i\beta_1 = 1 - i\sqrt{3}$

→ soln of Eq^n ① is given by :

$$a_n = A_1(1+i\sqrt{3})^n + A_2(1-i\sqrt{3})^n$$

where $A_1$ and $A_2$ are constants

③ Solve the linear HRR $a_n - 8a_{n-1} + 16a_{n-2} = 0$ with initial condition $a_2 = 6$, $a_3 = 80$ by characteristics root method.

Soln → Given $a_n - 8a_{n-1} + 16a_{n-2} = 0 \longrightarrow ①$.

Order of the relation ① : $n - (n-2) = 2$

∴ Characteristic Eqⁿ can be obtained by putting

$a_n = \alpha^2$, $a_{n-1} = \alpha^1$, $a_{n-2} = 1$

i.e.

$$\alpha^2 - 8\alpha + 16 = 0$$

$$\alpha^2 - 4\alpha - 4\alpha + 16 = 0$$

$$\alpha(\alpha - 4) - 4(\alpha - 4) = 0$$

$$(\alpha - 4)(\alpha - 4) = 0$$

$$\alpha = 4 \text{ and } \alpha = 4$$

$$\alpha_1 = \alpha_2 = 4$$

∴ Roots are real and equal

∴ Soln of ① is given by:

$$a_n = (A_1 + nA_2)\alpha_1^n$$

$$a_n = (A_1 + nA_2)4^n \longrightarrow ②$$

Given: $a_2 = 6$ and $a_3 = 80$

∴ Put $n = 2$ in Eqⁿ ② ~~and~~

$$a_2 = (A_1 + 2A_2)4^2$$

$$6 = (A_1 + 2A_2)16$$

$$A_1 + 2A_2 = 3/8 \longrightarrow ③$$

Put $n = 3$ in eqⁿ ②

$$a_3 = (A_1 + 3A_2)4^3$$

$$\frac{80}{16 \times 4} = A_1 + 3A_2$$

$$A_1 + 3A_2 = \frac{5}{4} \longrightarrow ④$$

$\frac{5}{80}$ Solving ③ and ④, (check by putting $n = 2$ or 3)

$\frac{16 \times 4}{4}$ $A_1 = -\frac{11}{8}$ $A_2 = \frac{7}{8}$

∴ Solⁿ is : $\boxed{a_n = \left(-\frac{11}{8} + n\frac{7}{8}\right)4^n}$ or $\boxed{a_n = \left(\frac{7n-4}{2}\right)4^{n-1}}$

(4) Solve LHRR $a_n - 7a_{n-2} + 6a_{n-3} = 0$ with initial cond's $a_0 = 8$, $a_1 = 6$, $a_2 = 22$ by char. root meth.

Soln → Given : $a_n - 7a_{n-2} + 6a_{n-3} = 0 \longrightarrow (1)$

The order of (1) is $n-(n-3) = 3$

∴ To get char. eqn put

$a_n = \alpha^3$ $a_{n-2} = \alpha^1$ $a_{n-3} = 1$

in Eqn (1)

$$\therefore \quad \alpha^3 - 7\alpha + 6 = 0$$

$$\alpha_1 = -3 \qquad \alpha_2 = 2 \qquad \alpha_3 = 1$$

or

by synthetic division method.

root (in inspection) 1; coeff.

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 0 | -7 | 6 |
| | + 0 | 1 | 1 | -6 |
| | 1 | 1 | -6 | 0 |

$$\alpha^2 + \alpha - 6 = 0$$

$$\alpha_1 = 1 \qquad \alpha_2 = 2 \qquad \alpha_3 = -3$$

∴ The roots are real & unequal

∴ Soln of (1) is :

$$a_n = A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n$$

$$a_n = A_1 + A_2 2^n + A_3(-3)^n \longrightarrow (2)$$

Given: $a_0 = 8$ $a_1 = 6$ $a_2 = 22$

∴ Put $n=0$ : $a_0 = A_1 + A_2 2^0 + A_3(-3)^0$

$8 = A_1 + A_2 + A_3 \longrightarrow (3)$

Put $n=1$ : $a_1 = A_1 + A_2 2^1 + A_3(-3)^1$

$6 = A_1 + 2A_2 - 3A_3 \longrightarrow (4)$

Put $n=2$ : $a_2 = A_1 + 2^2 A_2 + A_3(-3)^2$

$22 = A_1 + 4A_2 + 9A_3 \longrightarrow (5)$

Solving (3), (4) and (5), we get

$$A_1 = 5 \qquad A_2 = 2 \qquad A_3 = 1$$

∴ Soln is : $a_n = 5 + 2\cdot 2^n + 1\cdot(-3)^n$

$$\boxed{a_n = 5 + 2^{n+1} + (-3)^n}$$

★ Generating functions (G.F)

Let $a_0, a_1, a_2 \dots a_n \dots \infty$ or $\{a_n\}$ be a sequence of real nos. The G.F of this funct is denoted by $G(a, z)$ & defined by

$$G(a,z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots \infty$$

OR

$$\boxed{G(a,z) = \sum_{n=0}^{\infty} a_n z^n} \longrightarrow (1)$$

where $z$ is a variable.

★ Generating functions of some std sequences.

1) If $\{a_n\}$ is a sequence & $a_n = c$, where 'c' is a const. then G.f of this seqence is given by:

$$G(a,z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} c z^n = c \sum_{n=0}^{\infty} z^n$$

$$= c(1 + z + z^2 + \dots)$$

$$= c \cdot \frac{1}{1-z} \quad \left(\because f(x) = \frac{1}{1-x} = 1 + x + x^2 + \dots\right)$$

$$\therefore \boxed{G(a,z) = \frac{c}{1-z}} \quad \forall\, n \geq 0.$$

In particular, if $c = 1$.

$$G(a,z) = \frac{1}{1-z}$$

if $c = 2$, $G(a,z) = \frac{2}{1-z}$.

⋮ ⋮

2) If $\{a_n\}$ is a sequence & $a_n = b^n$, then G.F of this sequence is:

$$G(a,z) = \sum_{n=0}^{\infty} b^n z^n$$

$$= 1 + bz + b^2 z^2 + \ldots$$

$$= 1 + (bz) + (bz)^2 + (bz)^3 + \ldots$$

$$\boxed{G(a,z) = \frac{1}{1-bz}} \quad \forall n \geq 0.$$

3) If $\{a_n\}$ is a sequence and $a_n = cb^n$, the GF is

$$G(a,z) = \sum_{n=0}^{\infty} cb^n z^n = c \sum_{n=0}^{\infty} b^n z^n$$

$$= c \cdot \frac{1}{1-bz}$$

$$\boxed{G(a,z) = \frac{c}{1-bz}} \quad \forall n \geq 0$$

4) If $\{a_n\}$ is a sequence & $a_n = n \; \forall n \geq 0$, then GF is:

$$G(a,z) = \sum_{n=0}^{\infty} nz^n$$

$$= 0 + z + 2z^2 + 3z^3 + \ldots$$

$$= z(1 + 2z + 3z^2 + \ldots)$$

$$= z\left(\frac{1}{(1-z)^2}\right)$$

$$\boxed{G(a,z) = \frac{z}{(1-z)^2}}$$

( Maclurian series expansion:

$$f(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots$$ )

~~imp~~

| Sl. No. | Sequence, $\{a_n\}$ | Generating function. |
|---|---|---|
| 1. | $a_n = c$, $\forall n \geq 0$ | $G(a,z) = \frac{c}{1-z}$ |
| 2. | $a_n = b^n$, $\forall n \geq 0$. | $G(a,z) = \frac{1}{1-bz}$. |
| 3. | $a_n = cb^n$, $\forall n \geq 0$ | $G(a,z) = \frac{c}{1-bz}$. |
| 4. | $a_n = n$, $\forall n \geq 0$. | $G(a,z) = \frac{z}{(1-z)^2}$. |

★ Solution of HLRR by GF.

→ Consider a HLRR,

$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = 0 \longrightarrow (1)$. $\forall n \geq k$.

We have following steps to solve (1).

- **Step I**: Multiply both sides by $z^n$ and take summation from $n=k$ to $n=\infty$.
- **Step II**: Write each term of (1) in terms of $G(a,z)$.
- **Step III**: Solve $G(a,z)$ by using the table of standard G.f. for the given sequence, so that $a_n$ can be found.

⟹ Examples:

1) Solve homogenous LRR $c_n = 3c_{n-1} - 2c_{n-2}$ $\forall n \geq 2$ given initial cond$^n$ are $c_1 = 5$, $c_2 = 3$ by using G.f.

→ Let $c_n = 3c_{n-1} - 2c_{n-2}$, $\forall n \geq 2$.

Put $n=2$ in above eq$^n$ & use $c_1 = 5$ & $c_2 = 3$,

$c_2 = 3c_1 - 2c_0$

$3 = 3(5) - 2c_0 \Rightarrow 2c_0 = 15 - 3$

$\boxed{c_0 = 6}$.

Rewrite the given RR is

$c_n - 3c_{n-1} + 2c_{n-2} = 0 \quad \forall\ n \geq 2 \longrightarrow (1)$

$\Rightarrow \Sigma$

$\Rightarrow G(c,z) - c_0 - c_1 z - 3z\{G(c,z) - c_0\} + 2z^2 G(c,z) = 0$

Put $c_0 = 6$ and $c_1 = 5$

$\therefore G(c,z) - 6 - 5z - 3z\,G(c,z) + 18z + 2z^2 G(c,z) = 0$

$\therefore G(c,z)(1 - 3z + 2z^2) = 6 - 13z$

$$G(c,z) = \frac{6-13z}{1-3z+2z^2} = \frac{6-13z}{2z^2-3z+1}$$

$$G(c,z) = \frac{6-13z}{(2z-1)(z-1)} = \frac{A}{(z-1)} + \frac{B}{(2z-1)} \longrightarrow (2)$$

($\because$ By partial fraction)

$$\frac{6-13z}{(z-1)(2z-1)} = \frac{A}{(z-1)} + \frac{B}{2z-1}$$

$6 - 13z = A(2z-1) + B(z-1) \longrightarrow (3)$

Put $z = 1$ in (3)

$6 - 13 = A(2-1)$

$\boxed{A = -7}$ ~~Put A = -7 in (3)~~

Put $z = \frac{1}{2}$

$$6 - 13\left(\frac{1}{2}\right) = 0 + B\left(\frac{1}{2} - 1\right)$$

$$\frac{-1}{2} = B\left(\frac{-1}{2}\right)$$

$$\boxed{B = 1}$$

$\therefore$ Eq$^n$ (2) becomes $G(c,z) = \frac{-7}{(z-1)} + \frac{1}{2z-1}$

$$G(c,z) = \frac{7}{1-z} - \frac{1}{1-2z}$$

$\therefore$ By std G.f, we have

$$\boxed{c_n = 7 - 2^n}$$

(2) Solve the homogeneous LRR $u_n = u_{n-1} + u_{n-2}$ for, $\forall\, n \geq 2$, wid initial conditions $u_1 = 1$, $u_2 = 3$ by G.f.

$\longrightarrow$ Given $u_n = u_{n-1} + u_{n-2}$, $\forall\, n \geq 2$

Put $n = 2$ (order of d relation).

$$u_2 = u_1 + u_0$$

$$3 = 1 + u_0$$

$$\boxed{u_0 = 2}$$

rewriting given RR:

$$u_n - u_{n-1} - u_{n-2} = 0 \longrightarrow (1)$$

multiply both sides by $z^n$ and take summation from $n = 2$ to $\infty$.

$$\sum_{n=2}^{\infty} u_n z^n - \sum_{n=2}^{\infty} u_{n-1} z^n - \sum_{n=2}^{\infty} u_{n-2} z^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} u_n z^n - \sum_{n=2}^{\infty} u_{n-1} z^{n-1} \cdot z - \sum_{n=2}^{\infty} u_{n-2} z^{n-2} \cdot z^2 = 0$$

$$\sum_{n=2}^{\infty} u_n z^n - z \sum_{n=2}^{\infty} u_{n-1} z^{n-1} - z^2 \sum_{n=2}^{\infty} u_{n-2} z^{n-2} = 0$$

$$\sum_{n=0}^{\infty} u_n z^n - u_0 - u_1 z - z \left\{ \sum_{n=0}^{\infty} u_n z^{n} - u_0 \right\} - z^2 \sum_{n=0}^{\infty} u_n z^n = 0$$

$$G(u,z) - u_0 - u_1 z - z\,G(u,z) + u_0 z - z^2 G(u,z) = 0$$

Put $u_0 = 2,\ u_1 = 1$ ~~and~~ in above eqn.

$$G(u,z)(1 - z - z^2) - 2 - z + 2z = 0$$

$$G(u,z)(1 - z - z^2) = 2 - z$$

$$\therefore G(u,z) = \frac{2 - z}{1 - z - z^2} = \frac{z - 2}{z^2 + z - 1}.$$

$$G(u,z) = \frac{z-2}{z^2 + z + \frac{1}{4} - \frac{1}{4} - 1} = \frac{z-2}{\left(z+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2}$$

$$(\because a^2 - b^2 = (a+b)(a-b))$$

$$= \frac{z-2}{\left(z + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(z + \frac{1}{2} - \frac{\sqrt{5}}{2}\right)}$$

$$= \frac{z-2}{\left(z + \frac{1+\sqrt{5}}{2}\right)\left(z + \frac{1-\sqrt{5}}{2}\right)} = \frac{A}{z + \frac{1+\sqrt{5}}{2}} + \frac{B}{z + \frac{1-\sqrt{5}}{2}} \longrightarrow (2)$$

By partial fraction

$$z - 2 = A\left(z + \frac{1-\sqrt{5}}{2}\right) + B\left(z + \frac{1+\sqrt{5}}{2}\right) \longrightarrow (B)$$

Put $z = -\left(\frac{1+\sqrt{5}}{2}\right)$ in (B).

$$\frac{-1-\sqrt{5}}{2} - 2 = A\left(\frac{-1-\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}\right) + B(0)$$

$$\frac{-1-\sqrt{5}-4}{2} = A\left(\frac{-2\sqrt{5}}{2}\right)$$

$$-5 - \sqrt{5} = A(-2\sqrt{5})$$

$$A = \frac{-5-\sqrt{5}}{-2\sqrt{5}}$$

$$A = \frac{5+\sqrt{5}}{2\sqrt{5}} = \frac{\sqrt{5}(\sqrt{5}+1)}{2\sqrt{5}}$$

$$\boxed{A = \frac{\sqrt{5}+1}{2}}$$

Put $z = \frac{-1+\sqrt{5}}{2}$ in (3).

$$\boxed{B = \frac{1-\sqrt{5}}{2}}$$

$\therefore$ Eqⁿ (2) becomes

$$G(u,z) = \frac{(1+\sqrt{5})/2}{z + (1+\sqrt{5})/2} + \frac{(1-\sqrt{5})/2}{z + (1-\sqrt{5})/2}$$

$$= \frac{(1+\sqrt{5})/2}{\left(\frac{1+\sqrt{5}}{2}\right)\left(z \cdot \frac{2}{1+\sqrt{5}} + 1\right)} + \frac{(1-\sqrt{5})/2}{\left(\frac{1-\sqrt{5}}{2}\right)\left(z \cdot \frac{2}{1-\sqrt{5}} + 1\right)}$$

$$= \frac{1}{1 - \left(\frac{-2}{1+\sqrt{5}}\right)z} + \frac{1}{1 - \left(\frac{-2}{1-\sqrt{5}}\right)z}$$

By std G.F.

$$\boxed{u_n = \left(\frac{-2}{1+\sqrt{5}}\right)^n + \left(\frac{-2}{1-\sqrt{5}}\right)^n}$$ //.

optionally.

$$u_n = (-2)^n \left(\frac{1}{(1+\sqrt{5})^n} + \frac{1}{(1-\sqrt{5})^n}\right)$$

$$= (-2)^n \left(\frac{(1-\sqrt{5})^n + (1+\sqrt{5})^n}{(1+\sqrt{5})^n (1-\sqrt{5})^n}\right)$$

$$u_n = (-2)^n$$

Examples:

① Solve the non-HLRR $a_{n+2}-2a_{n+1}-a_n=2^n, \forall n\geq 0$ $a_0=2, a_1=1$ by G.F

→ Soln: Let $a_{n+2}-2a_{n+1}+a_n=2^n \longrightarrow$ ①.

Multiplying by $z^n$, taking summation from $n=0$ to $\infty$.

$$\sum_{n=0}^{\infty} a_{n+2} z^n - 2\sum_{n=0}^{\infty} a_{n+1} z^n + \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} 2^n z^n.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2} \frac{z^{n+2}}{z^2} - 2\sum_{n=0}^{\infty} a_{n+1}\frac{z^{n+1}}{z} + \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z}.$$

$$\Rightarrow \frac{1}{z^2}\left\{(a_2 z^2 + a_3 z^3 + \dots) + a_0 + a_1 z - a_0 - a_1 z\right\} - \frac{2}{z}\left\{(a_1 z + a_2 z^2 + \dots) + a_0 - a_0\right\} + \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z}$$

$$\frac{1}{z^2}\left\{\sum_{n=0}^{\infty} a_n z^n - a_0 - a_1 z\right\} - \frac{2}{z}\left\{\sum_{n=0}^{\infty} a_n z^n - a_0\right\} + \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z}.$$

$$\frac{1}{z^2}\left\{G(a,z) - a_0 - a_1 z\right\} - \frac{2}{z}\left\{G(a,z) - a_0\right\} + G(a,z) = \frac{1}{1-2z}$$

Given $a_0=2$ $a_1=1$

$$\therefore \frac{1}{z^2}\left\{G(a,z) - 2 - z\right\} - \frac{2}{z}\left\{G(a,z) - 2\right\} + G(a,z) = \frac{1}{1-2z}.$$

$$G(a,z)\left\{\frac{1}{z^2} - \cancel{\frac{2}{z^2}} - \frac{1}{z}\right\} - \frac{2}{z}\frac{2}{z} + \frac{4}{z} - 1 = \frac{1}{1-2z}$$

$$G(a,z)\left\{\frac{1}{z^2} - \frac{2}{z} + 1\right\} - \frac{2}{z^2} - \frac{1}{z} + \frac{4}{z} = \frac{1}{1-2z}.$$

$$\therefore G(a,z)\left\{\frac{1-2z+z^2}{z^2}\right\} - \frac{(2-3z)}{z^2} = \frac{1}{1-2z}.$$

$$G(a,z)\left\{\frac{1-2z+z^2}{z^2}\right\} = \frac{1}{1-2z} + \frac{(2-3z)}{z^2}.$$

$$G(a,z)(1-2z+z^2) = \frac{z^2+(2-3z)(1-2z)}{(1-2z)}$$

$$G(a,z) = \frac{z^2 + 2 - 4z - 3z + 6z^2}{(1-2z)(1-2z+z^2)} \quad (1)$$

$$= \frac{7z^2 - 7z + 2}{(1-2z)(z-1)^2} = \frac{A}{(1-2z)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2} \quad (2)$$

(∵ By partial fraction)

$$7z^2 - 7z + 2 = A(z-1)^2 + B(1-2z)(z-1) + C(1-2z) \longrightarrow (3)$$

Put $z = \frac{1}{2}$

$$\frac{7}{4} - \frac{7}{2} + 2 = A\left(\frac{1}{2} - 1\right)^2$$

$$\frac{1}{4} = A\left(\frac{1}{4}\right)$$

$$\boxed{A = 1}$$

Put $z = 1$ in (3)

$$2 = 0 + 0 + (-C)$$

$$\boxed{C = -2}$$

Put $z=0$ in (3), $2 = A + (-B) + C$

$$2 = 1 - B - 2$$

$$B = 1 - 2 - 2$$

$$\boxed{B = -3}$$

∴ Eqⁿ (2) becomes:

$$G(a,z) = \frac{1}{1-2z} - \frac{3}{(z-1)} - \frac{2}{(z-1)^2}$$

$$= \frac{1}{1-2z} + \frac{3}{1-z} - \frac{2}{(1-z)^2}$$

By std. G.F.

~~$G(a,z) = a$~~

$$a_n = 2^n + 3 - 2(n+1)$$

$$a_n = 2^n + 3 - 2n - 2 = 2^n - 2n + 1$$

$$\boxed{a_n = 2^n - 2n + 1}$$

Q. Solve the non-HLRR as given below: by GF

(1) $a_{n+1} - a_n = 3^n$, $\forall n \geq 0$, with initial cond$^n$ $a_0 = 1$

(2) $a_{n+2} - 5a_{n+1} + 6a_n = 2$, $\forall n \geq 0$, wid in. con $a_0 = 3, a_1 = 7$

(Ans) 2: $a_n = 2(3^n) + 1$  1: $a_n = \frac{1 + 3^n}{2}$

## ★ Divide and conquer algo

" " " " works recursively breaking down a problem (P) into two or more sub problems (P₁, P₂ ...) of d same or related type until these becomes simple enough to be solved directly. The solutions to the sub problems are then combined to give a solution to the original problem.

A typical d&c algo solves a problem using the following 3 steps

1. Divide : Break or divide the given problem into sub problems of the same type.
2. Conquer : Recursively solve the subproblems.
3. Combine : Combine the solutions of " " to get d soln or answer for d given original problem.

★ The idea of divide & c algo:

Given a problem P of size $n = 2^k$

Algorithm DAC(P)

- If 'n' is small, solve it
- else
  - divide P into two sub-problems $P_1$ and $P_2$ of size $\frac{n}{2} = 2^{k-1}$
  - DAC($P_1$)
  - DAC($P_2$)

$\Big\}$ soln of $P_1$ and $P_2$ can be found

(not in else)

- combine the solⁿs of subproblems $P_1$ and $P_2$ to get soln of P.

Problem | Size
Level-1: P | $n = 2^k$
Level-2: $P_1$ $P_2$ | $\frac{n}{2} = 2^{k-1}$
Level-3: $P_3$ $P_4$ $P_5$ $P_6$ | $n/4 = 2^{k-2}$
Level-4: $P_7$ $P_8$ $P_9$ $P_{10}$ $P_{11}$ $P_{12}$ $P_{13}$ $P_{14}$ | $n/8 = 2^{k-3}$

⋮

Level-k-1 | $n/n = 1$.

Time to solve problem P

$$\underset{P}{T(n)} = \underset{P_1}{T(n/2)} + \underset{P_2}{T(n/2)} + f(n)$$

(f(n): additional time to combine d soln)

$$\boxed{T(n) = 2T(n/2) + f(n)}$$

where f(n) is additional cost of combining.

In general

$$\boxed{T(n) = a\,T(n/b) + f(n)}$$

(b: subproblems, a: no. of " we actually add)

is a recurrence relation for a divide and conquer algorithm.

→ Examples:

① Solve $T(n) = 2T(n/2) + n$ using iteration/substitution method ($n = 2^k$)

$\rightarrow$ soln: Let $T(n) = 2T(n/2) + n \longrightarrow (1)$

$$T(n) = 2\left\{2T\left(\frac{n}{4}\right) + \frac{n}{2}\right\} + n \quad (\because \text{by } (1))$$

$$T(n) = \cancel{2}\,4T\left(\frac{n}{4}\right) + 2n$$

$$T(n) = 4\left\{2T\left(\frac{n}{8}\right) + \frac{n}{4}\right\} + 2n$$

$$T(n) = 8T(n/8) + 3n = 2^3 T\left(\frac{n}{2^3}\right) + 3n$$

$\vdots$

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + kn$$

$$T(n) = nT\left(\frac{n}{n}\right) + kn \quad (\because n = 2^k$$

$$T(n) = \underbrace{nT(1)}_{\text{v. small}} + kn$$

$T(n) = kn$ ($\because nT(1)$ is small, so we neglect it)

But $n = 2^k \Rightarrow \log n = k \log 2$

$$k = \frac{\log n}{\log 2}$$

$$\boxed{k = \log_2 n}$$

$\therefore T(n) = n \log_2 n$

$\therefore$ complexity is $O(n \log_2 n)$

(2) In an algorithm, we divide large problems into 3 equal parts and discard two of them in const. time, what is the complexity of this algo. for the size $n = 3^k$

Soln

$$p^n \longrightarrow \underset{P_1}{n/3} \quad \underset{P_2}{n/3} \quad \underset{P_3}{n/3}$$

$$T(n) = T(n/3) + c \longrightarrow (1)$$

where c is a const.

$$T(n) = \{T(n/9) + c\} + c \quad (\because (1))$$

$$T(n) = T(n/9) + 2c \longrightarrow (2)$$

$$T(n) = \{T(n/27) + c\} + 2c$$

$$= T(n/3^3) + 3c$$

$$\vdots$$

$$T(n) = T(n/3^k) + kc$$

$$T(n) = T(n/n) + kc \quad [\because n = 3^k]$$

$$T(n) = T(1) + kc$$

$T(n) = kc$ [$\because$ T(1) is small, so we neglect it]

But $n = 3^k \Rightarrow \log n = k \log 3$

$$k = \frac{\log n}{\log 3}$$

$$\boxed{k = \log_3 n}$$

$$\boxed{T(n) = c \log_3 n}$$

$\therefore$ complexity is $O(c \log_3 n)$

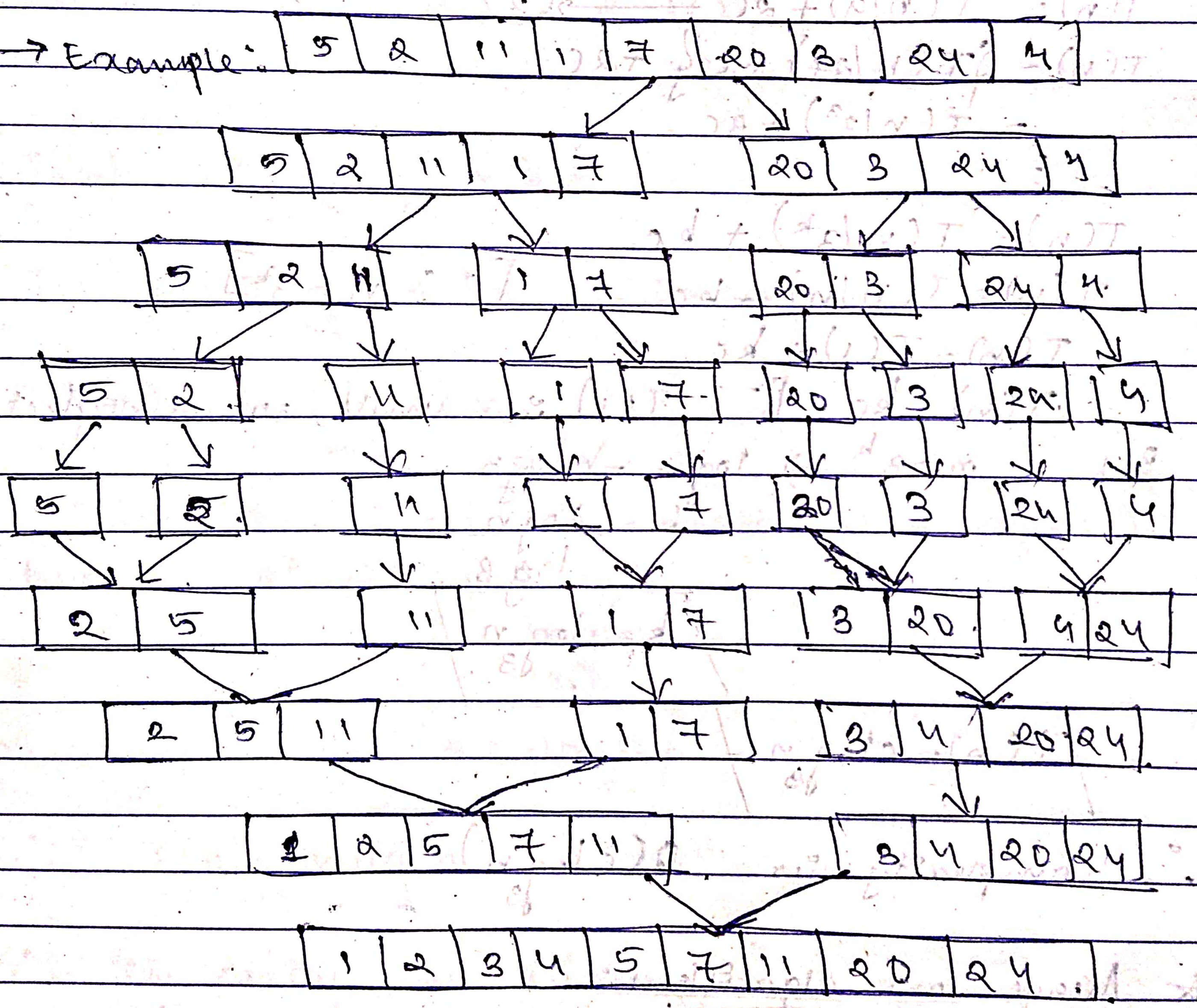
★ **Merge sort algorithm:**

" " keeps on dividing lists into equal halves, until it can no more be divided. Then merge sort combines the smaller sorted list keeping the new list sorted.

## Algorithm:

- Step 1: If it is only one element in the list, it is sorted, return.
- Step 2: Divide the list recursively into 2 halves until it can no more be divided
- Step 3: Merge the smaller lists into new list in sorted order.

→ Example:



UNIT-IV

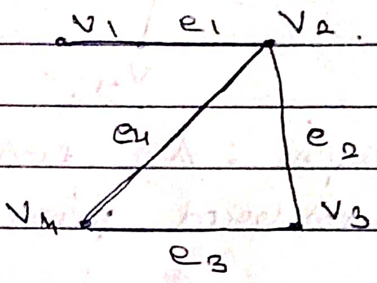
# BASIC GRAPH THEORY

* Graph:

→ defn: A graph consists of a finite set of vertices V, a finite set of edges E, and a function that assigns to each edge an ordered or unordered pair (v1, v2), where v1 & v2 are vertices. And graph is denoted by G or G(V, E)

→ Example: Consider a vertex set $V=\{v_1, v_2, v_3, v_4\}$ and edge set $E=\{e_1, e_2, e_3, e_4\}$ where $e_1, e_2, e_3$ & $e_4$ assigns to the pairs $(v_1,v_2)$, $(v_2,v_3)$, $(v_3,v_4)$ and $(v_2,v_4)$ respectively.

$G = G(V,E)$:

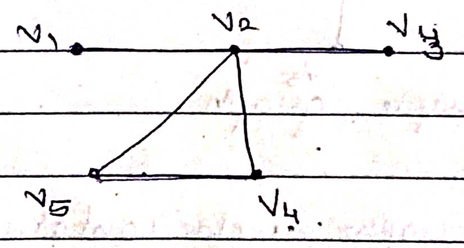



→ Degree of vertex:

The " " v is denoted by deg(v) or d(v) is the number of edges connected to it.

For example:

G:

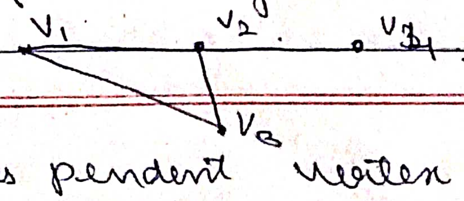



∴ $\deg(v_1)=1$  $\deg(v_2)=4$,  $\deg(v_3)=1$

$d(v_4)=2$.  $d(v_5)=2$.

* Pendent vertex: A vertex is said to be PV or an end " if its degree is one.

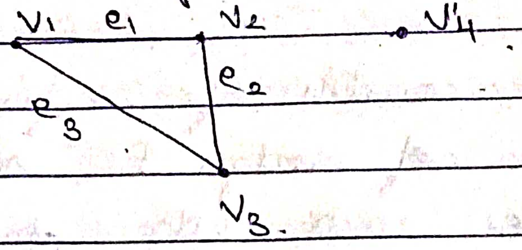
Example: G:




∴ Vertex $v_4$ is pendent vertex as degree of $v_4$ is 1 ($\deg(v_4)=1$)"

★ **Isolated vertex**: A vertex is said to be isolated vertex if its deg is 0.

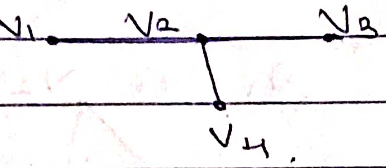
Ex: G:




vertex $v_4$ is isolated as $\deg(v_4) = 0$

★ **Adjacent vertices or nodes**: Two vertices are said to be " " if they are connected by an edge.

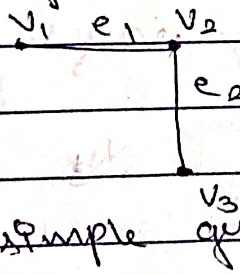
Ex. G:




★ **Undirected graph**: An u.g is a graph whose edges are unordered pairs of distinct vertices.

★ **Simple graph**: A graph which has neither loops nor multiple edges is called a.

→ Ex: G:

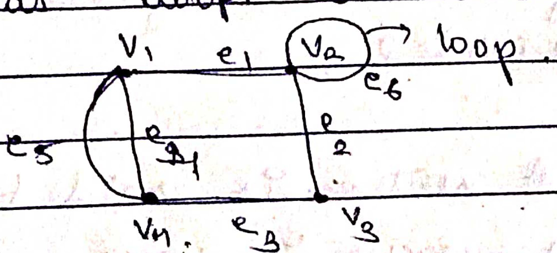



★

∴ G is a simple graph.

★ **Multigraphs**: A graph which contains multiple edges as well as loops is called a

→ Ex.

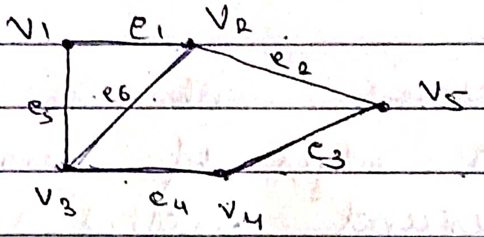


* Pseudograph : A graph in which loops and multiple edges are allowed is a.

* Walk : A walk is a sequence of vertices and edges that begins at $v_i$ and travel along edges to $v_j$. (~~so that no edge appears more than once.~~)

→ Let $v_i$ and $v_j$ (not necessarily distinct) be any two vertices in an undirected $G(V,E)$. Then a walk $v_i - v_j$ (Read as $v_i$ to $v_j$) in $G(V,E)$ is a loop free finite alternating sequence $v_i = v_0, e_1, v_1, e_2, v_2, e_3 \ldots v_3 = v_{n-1}$, $e_n, v_n = v_j$ of vertices and edges starting from $v_i$ and end at $v_j$ involving $n$ edges $e_i = (v_{i-1}, v_i), 1 \le i \le n$

→ Ex : G :

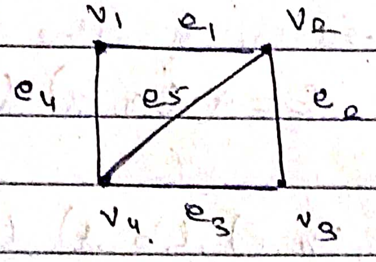



walk : ① $v_1 e_1 v_2 e_2 v_5 e_3, v_4$
~~② $v_1 e_1, v_2, e_6, v_3, e_4, v_4$~~ ② $v_2\ e_2, v_5, e_3, v_4, e_4, v_3$
~~③ $v_1, e_5, v_3, e_4, v_4$~~ ③ $v_1, e_1, v_2, e_6, v_3, e_5, v_1$

* Closed walk and open walk :

A " is said to be closed walk if it is possible that begin and end at the same vertex. otherwise walk is an open walk.

→ Ex : G :

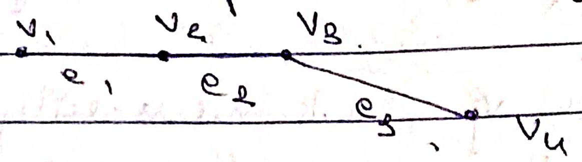



Open walk : ① $v_1, e_1, v_2, e_2, v_3, e_3, v_4$
② $v_2, e_2, v_3, e_3, v_4$

Closed walk : ① $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1$
② $v_1, e_1, v_2, e_5, v_4, e_4, v_1$
③ $v_2, e_2, v_3, e_3, v_4, e_5, v_2$

★ Path: A path is a walk through a sequence $v_0, v_1, v_2 \ldots v_n$ of vertices, each adjacent to the next without repetition of vertices.

For eg: G:




∴ Path is $v_1, e_1, v_2, e_2, v_3, e_3, v_4$.

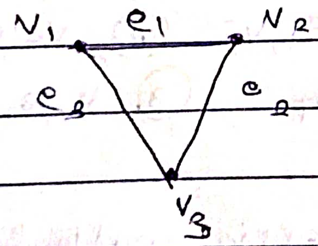
★ Length of a path: The no. of edges in a path is the length of the path.

→ Eg: In above ex. " " " is 3.

★ Cycle: A closed walk is said to cycle if vertices and edges of walk should not be repeated except initial vertices.

→ Ex:

★ G:




∴ Cycle is $v_1 e_1, v_2, e_2, v_3, e_3, v_1$

★ Examples:

① Draw a graph for each of the following

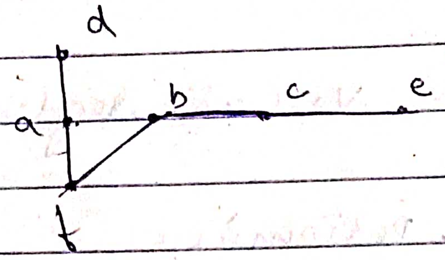
(i) $V = \{a, b, c, d, e, f\}$, $E = \{(a,d), (a,f), (b,c), (b,f), (c,e)\}$

(ii) $V = \{a,b,c,d,e\}$, $E = \{(a,a), (b,c), (b,d), (c,d), (e,a), (e,d)\}$

iii) $V = \{a, b, c, d, e\}$, $E = \{(a,d), (b,c), (b,d), (b,e), (d,e)$
$(c,e)\}$.

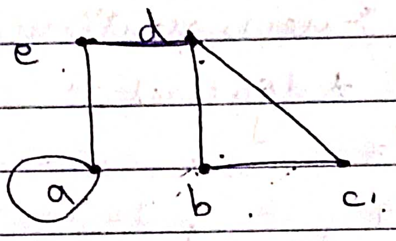
soln → i) $V = \{a,b,c,d,e,f\}$ $E = \{(a,d), (a,f), (b,c), (b,f), (c,e)\}$.

$G = G(V,E)$:

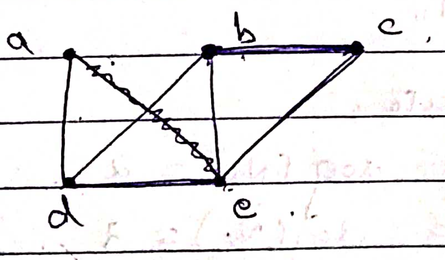



ii) $V = \{a,b,c,d,e\}$ $E = \{(a,a), (b,c), (b,d), (c,d), (e,a), (e,d)\}$

$G = G(V,E)$:

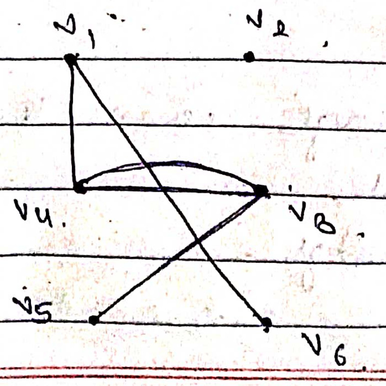



iii)




② Consider a graph as shown below. Find the degree of each vertices, pendant vertices and isolated vertices:

G:



sol^n → degree of each vertices

$deg(v_1) = 2$ $deg(v_2) = 0$ $deg(v_3) = 3$

$deg(v_4) = 2$ $deg(v_5) = 1$ $deg(v_6) = 1$

→ Pendent vertices are $v_5$ and $v_6$. (as $deg(v_5) = deg(v_6) = 1$.

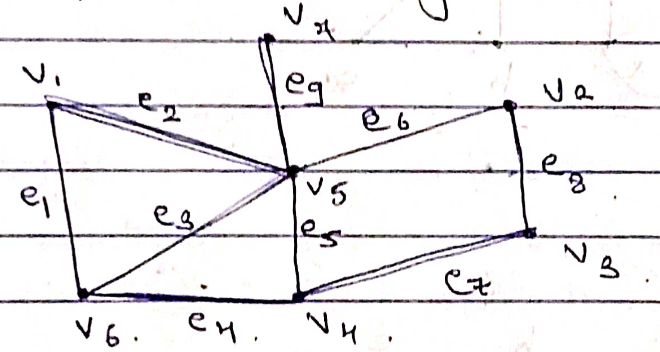
→ Isolated vertex is $v_2$ (as $deg(v_2) = 0$.)

* ③ Consider the graph. Determine

i) Pendent vertex ii) Pendent edge

iii) odd vertices. iv) Even vertices

v) Incident edge. vi) Adjacent "




sol^n → Degree of each vertex

$deg(v_1) = 2$. $deg(v_2) = 2$ $deg(v_3) = 2$

$deg(v_4) = 3$ $deg(v_5) = 5$ $deg(v_6) = 3$

$deg(v_7) = 1$

i) Pendent vertex: $v_7$ is a pendent vertex.

[∵ $deg(v_7) = 1$)

ii) Pendent edge: An edge connected to a " vertex is called,

∴ $e_9$ is a pendent edge.

$e_9 = (v_5, v_7)$

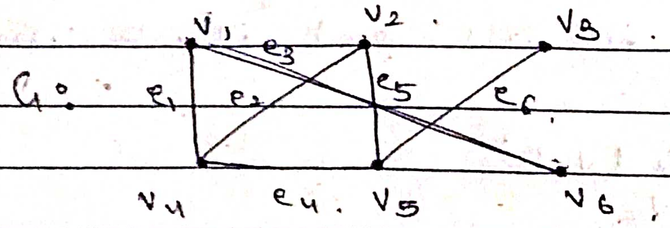
iii) Odd vertices : A vertex is whose degree is odd is called an odd vertex.
∴ Odd vertices are $V_4, V_5, V_6, V_7$.

iv) Even vertices : ... is even is called an even vertex.
∴ ... vertices are $V_1, V_2, V_3$.

v) Incident Edge : An edge e is called an Incident with $V_i$ and $V_j$ if it connects on its end points.
( $V_i$ —— $V_j$ ✓    $V_i$ —— X )
∴ All edges are incident edges coz each of them is connected with pair of vertices.

vi) Adjacent vertex :
1) $V_1$ is adjacent to $V_5$ and $V_6$.
2) $V_2$ is " to $V_3$ and $V_5$.
3) $V_3$ is " to $V_4$.
4) $V_4$ is " to $V_5$ and $V_6$.
5) $V_5$ is " to $V_6$ and $V_7$.

④ Let G be a graph as shown below. Find all simple paths from vertex $V_1$ to $V_3$ and find the length of its shortest path.




Soln → Simple paths from $V_1$ to $V_3$ are,
i) $V_1, e_1, V_4, e_4, V_5, e_6, V_3$ or $(V_1, V_4, V_5, V_3)$
ii) $V_1, e_1, V_4, e_2, V_2, e_5, V_5, e_6, V_3$ or $(V_1, V_4, V_2, V_5, V_3)$

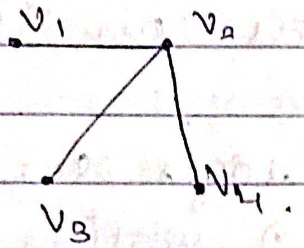
∴ length of shortest path is 3.

* **Connected graph:**

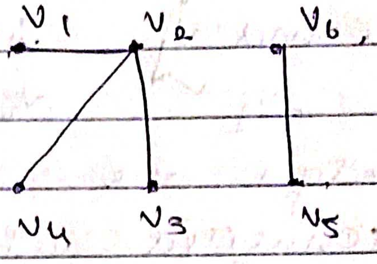
A " is called connected, if there is a path from any vertex to any other vertex in a graph otherwise the graph is disconnected.

→ Ex:

G:




G':




Connected graph. Disconnected graph

IA2

* **Sub-graphs:**

If G(V,E) is a graph (directed or undirected) then ~~edge~~ H(V',E') is called subgraph of G if non-empty ~~subsets~~ $V' \subseteq V$ & $E' \subset E$, where each edge in E' is incident with vertices in V'.

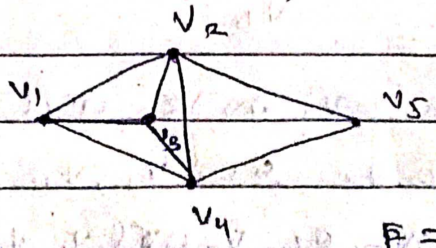
OR

sub

If G(V,E) is ... then the graph H(V',E') obtained by deleting few vertices and edges from G, provided the set of vertices V' in ~~edge~~ H contains all its end points in vertex set V'

→ Ex. consider G(V,E)

G(V,E):




where

$V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{(v_1, v_2), (v_1, v_3), (v_1, v_4)$