The effect of solid objects on the electrostatic potential

J Cork, T Hendrick-Beattie, D Lafferty, M Pereira, V Nordgren and N Warrack

School of Physics and Astronomy, University of Glasgow, Glasgow, UK

Abstract.

Introduction

Electromagnetism is one of four fundamental forces of nature. It describes how electrically charged particles interact and how they generate electromagnetic fields [1]. These fields permeate the space around them, influencing the behaviour of other charges by exerting forces on them. Electromagnetic forces determine the atomic and macroscopic properties of matter that dominate most of the physical phenomena encountered in daily life such as sound, light and biological processes. Understanding these principles in depth can facilitate significant advances in science and technology. For instance, successful modelling of a simple system of conductors makes it possible to investigate arbitrarily complex configurations of charges which can aid the design of detectors for particle physics.

Electrostatics is the study of stationary or slow-moving electric charges [2]. They exert electrostatic forces on each other, which in turn are governed by Coulomb's force law and most conveniently described by electric field equations. The relationship between these fields and the distribution of electric charge can be expressed as [2]

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \ , \tag{1}$$

known as the differential form of Guass's law, where $\nabla \cdot \mathbf{E}$ is the divergence of the electric field, ϵ_0 is the permittivity of free space and ρ is the charge density. Additionally, for any static charge distribution the electric field is irrotational, $\nabla \times \mathbf{E} = 0$, therefore the line integral of the electric field is independent of the path taken [2]. This means that $\mathbf{E}$ can be written as the gradient of a scalar potential

$$\mathbf{E} = -\nabla V \ , \tag{2}$$

where V is called the electric potential, defined as the potential energy per unit charge [1]. In regions where there is no charge, such that $\rho = 0$, the divergence of the electric field is zero. Hence, using Eq.(1) and Eq.(2) the electric potential can be described by

$$\nabla^2 V = 0 \ . \tag{3}$$

This is Laplace's equation [3].

In this work, the electrostatic potential and the electric field are evaluated at all points in two different systems. The systems consist of different geometric configurations of conductors held at constant potentials. This study shows how V is modified in the presence of these conductors.

Methods

In general, there are two methods to analyse the electric potential in all space: analytical and numerical techniques. Often there is no analytical solution, even for a simple electrostatic system, and thus numerical methods are useful tools in obtaining approximate solutions.

Analytical Approach

For systems with certain symmetries Laplace's equation can be expressed in cylindrical coordinates [3], namely

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0\ . \tag{4}$$

For an axisymmetrical system the electrostatic potential will not depend on z [2]. This implies that ${}^{\partial^2 V}/_{\partial z^2}=0$. Multiplying through by s^2 the equation is simplified to

$$s\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{\partial^2 V}{\partial \phi^2}=0\ . \tag{5}$$

Separation of variables is used to solve the equation above. Let $V(s,\phi)=S(s)\Phi(\phi)$, where the factors S and Φ are functions of s and ϕ respectively. Substituting for and dividing through by V, Eq.(5) can be written as

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=0\ . \tag{6}$$

This form shows that the first term depends only on s, and the second only on ϕ. It follows that each term is equal to a constant defined, for convenience, to be k^2:

$$\frac{d^2\Phi}{d\phi^2}=-k^2\Phi\ . \tag{7a}$$

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right)=k^2 S, \tag{7b}$$

The partial differential equation, Eq.(5), has been converted into two independent ordinary differential equations. The general solution for the angular part, Eq.(7a), is

$$\Phi(\phi)=A_k cos(k\phi)+B_k sin(k\phi)\ , \tag{8}$$

where A_k and B_k are arbitrary constants [3]. The condition $\Phi(\phi)=\Phi(\phi+2\pi)$ is required so that the potential is single valued, this implies that k must be an integer. When $k=0$ Eq.(7a) has solution $\Phi(\phi)=B_0\phi+A_0$, where $B_0=0$ in order to satisfy this condition and A_0 is an arbitrary constant.

Eq.(7b) has solutions s^k and s^{-k} by inspection [2]. The general solution has the form

$$S(s) = C_k s^k + D_k s^{-k} \tag{9}$$

where C_k and D_k are arbitrary constants. For $k = 0$, Eq.(7b) takes the form

$$S(s) = C_0 \ln s + D_0 \ , \tag{10}$$

where C_0 and D_0 are, again, arbitrary constants. This formula is known as the potential of a long line of charge.

Since Laplace's equation is a linear PDE, the general solution is a superposition of all solutions corresponding to different allowed values of k [2]. The linear combination can be written as

$$V(s,\phi) = A_0(C_0 \ln s + D_0) + \sum_{k=1}^{\infty} \left[(A_k \cos(k\phi) + B_k \sin(k\phi))(C_k s^k + D_k s^{-k})\right] \tag{11}$$

and also expressed as

$$V(s,\phi) = c_0 \ln s + d_0 + \sum_{k=1}^{\infty} \left[s^k(a_k \cos(k\phi) + b_k \sin(k\phi)) + s^{-k}(c_k \cos(k\phi) + d_k \sin(k\phi))\right] \tag{12}$$

where c_0, d_0, a_k, b_k, c_k and d_k are arbitrary constants that will be determined by boundary conditions.

Numerical Techniques

To find a particular solution for a differential equation, boundary conditions must be imposed that describe all or part of the region in which the solution is desired. The nature of these boundary conditions determines the numerical method required to obtain an approximate solution [4].

Laplace's equation is an elliptic PDE due to its mathematical characteristics and these are fundamental to its physical significance [3]. Problems that are governed by elliptic PDEs and subjected to boundary conditions are called Boundary value problems. These problems can be solved by Finite-difference and Relaxation methods.

The first step of the numerical technique is to discretize the problem by defining a mesh; a grid of spatial points that cover the domain of interest. The points that lie on the mesh can be written in cartesian coordinates as

$$x_i = ih \tag{13a}$$

$$y_j = jh \tag{13b}$$

where h is the distance between the grid points and i, j is a pair of indices describing each grid point. The differential operator ∇^2 is then approximated to a discrete form through the use of finite differences. The standard approximation for the second derivatives is [4]

$$f''(x) \approx \frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)]. \tag{14}$$

Since there are two variables, x and y, the discrete approximation to the the Laplacian can be written as

$$\nabla^2 V \approx \frac{V(x+h,y)+V(x-h,y)-2V(x,y)}{h^2}+\frac{V(x,y+h)+V(x,y-h)-2V(x,y)}{h^2}\ , \quad (15)$$

or in grid notation,

$$(\nabla^2 V)_{ij} \approx \frac{1}{h^2}[V_{i+1,j}+V_{i-1,j}+V_{i,j+1}+V_{i,j-1}-4V_{i,j}]\ . \quad (16)$$

This equation is called the Five-point formula [4]. The name comes from the fact that the value of V at some point (x, y) is related to the four nearest grid points. Substituting $(\nabla^2 V)_{ij} = 0$ into Eq.(16) gives

$$V_{i,j} = \frac{1}{4}(V_{i+1,j}+V_{i-1,j}+V_{i,j+1}+V_{i,j-1})\ , \quad (17)$$

which shows that a solution to Laplace's equation has the property that at any point it is equal to the average of the values at neighbouring points. The inherent truncation error in the Five-point formula is of order $\mathcal{O}(h^2)$ [4].

An alternative method is the Nine-point formula where the eight nearest neighbours are used: horizontal, vertical and diagonal. It can be written as

$$V_{i,j} = \frac{1}{20}(4V_{i+1,j}+4V_{i-1,j}+4V_{i,j+1}+4V_{i,j-1}+V_{i+1,j+1}V_{i-1,j+1}+V_{i+1,j-1}+V_{i-1,j-1})\ . \quad (18)$$

This formula provides an approximation with an error of $\mathcal{O}(h^4)$ when applied to Laplace's equation [5].

The boundary conditions are then specified in the mesh and the interior points of the region desired are assigned to an arbitrary values. The chosen values will not influence the final solution but they may affect the rate of convergence of the scheme.

The iterative method to solve Laplace's equation re-evaluates each $V_{i,j}$ to the average of the four or eight nearest points. The process is repeated until the change between successive iterations is too small according to a specified error tolerance, and an approximate solution is obtained. This procedure is a relaxation technique called the Jacobi method. This method is generally slow but Gauss-Seidel method can be implemented in order to improve it.

Gauss-Seidel method generally has a faster convergence and requires less memory [4]. It uses the updated values immediately to compute the next ones. In this way most points will be calculated using the values from previous and current iterations at the same time. For the Five-point formula this can be expressed as

$$V^{n+1}_{i,j} = \frac{1}{4}(V^{n}_{i+1,j}+V^{n+1}_{i-1,j}+V^{n}_{i,j+1}+V^{n+1}_{i,j-1})\ . \quad (19)$$

where n is the iteration number.

The accuracy of the numerical approximation improves as the number of points in the mesh increases for both numerical formulas, Five-point and Nine-point. This can be done by implementing Adaptive meshing [6]. The Relaxation method is applied to an uniform mesh and then the solution is examined to determine where more grid points should be added. A finer subgrid is superimposed in the regions requiring more resolution. The procedure is then repeated with the improved mesh until the local error has dropped below a desired level. This refinement decreases the uncertainty in the numerical solution.

Solving a physical system

There are simple electrostatic systems for which an analytical solution can be derived. These systems can be extremely helpful when developing a numerical solution. They are used to ensure the correctness and the accuracy of the numerical approximation. In this work, a system consisting of two infinitely long plates and a conducting cylinder was used to serve this purpose. Due to symmetry this 3D system can be approximated to a 2D

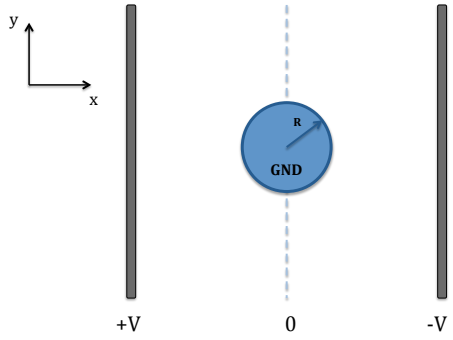

Figure 1: Illustration of physical system A. A perfectly uniform field is defined between plates $+V$ and $-V$. An uncharged conducting cylinder of radius R is placed in the middle of the system (cross-section view).

When an uncharged cylindrical conductor is placed between two parallel charged plates, the electric field is altered. The field pushes free positive charges to the right and the negative ones to the left. The charges accumulate in the edges of the conductor distorting the field in the surroundings of the cylinder. The potential outside the conductor can be described mathematically using principles of electrostatics, namely Laplace's Equation Eq.(3). A particular solution defining System A was obtained using Eq.(12) and the required boundary conditions.

The potential at the surface of a grounded conductor is zero [2]. Also, far from the cylinder the field is perpendicular to the plates, so at infinity $\mathbf{E} = E_0\hat{\mathbf{x}}$, hence $V = -E_0x + C$, where C is an arbitrary constant and E_0 is the uniform electric field. Due to the symmetry of the system, V is known to be zero at the same distance from the plates. In cylindrical coordinates, this becomes $V(s,\phi) = -E_0 s\cos\phi$. Therefore, the boundary conditions can be expressed as

$$V = 0 \qquad s = R\ , \tag{20a}$$

$$V \to -E_0 s\cos\phi \qquad s \gg R\ , \tag{20b}$$

where R is the radius of the sphere as shown in Fig.1. For these conditions to hold $C_0 = D_0 = b_k = d_k = 0$, and $a_k = b_k = 0$ except for $k = 1$. Therefore,

$$V(s,\phi) = \left(a_1 s + \frac{c_1}{s}\right)\cos\phi\ . \tag{21}$$

The first boundary condition implies that Eq.(21) is zero, hence

$$\left(a_1 R + \frac{c_1}{R}\right)\cos\phi = 0\ , \tag{22}$$

or

$$c_1 = -a_1 R^2 \ . \tag{23}$$

The second condition requires Eq.(21) to be equal to $-E_0 s\cos\phi$. By rearranging the expression and substituting in c_1 this takes the form

$$a_1 s = -E_0 s + \frac{a_1 R^2}{s} \ . \tag{24}$$

For $s \gg R$ the term $({}^{a_1 R^2}/_s)$ goes to zero, so $a_1 = -E_0$. Using Eq.(21), a_1 and c_1 the potential $V(s,\phi)$ can be written as

$$V(s,\phi) = -E_0\left(s - \frac{R^2}{s}\right)\cos\phi \ . \tag{25}$$

The particular solution describing system A is then

$$V(s,\phi) = \begin{cases} -E_0\big(s - \frac{R^2}{s}\big)\cos\phi & \text{for } s > R \\ 0 & \text{for } s \leq R \end{cases} \tag{26}$$

or in cartesian coordinates,

$$V(x,y) = \begin{cases} -E_0 x\big(1 - \frac{R^2}{x^2+y^2}\big) & \text{for } \sqrt{x^2+y^2} > R \\ 0 & \text{for } \sqrt{x^2+y^2} \leq R \ . \end{cases} \tag{27}$$

Values of the analytical solution of system A were calculated using Eq.(27) for discrete values of x and y to allow for comparison with the numerical. The uniform electric field, E_0, was defined as

$$E_0 = \frac{\Delta V}{d}, \tag{28}$$

where d is the plate separation and ΔV is the potential difference between the plates. The potentials at the plates were defined as +1V and -1V.

The Gauss-Seidel method described was implemented in C++. Initial values pertaining to system A were imposed, and with a starting matrix containing only the boundary values, the algorithm was run until no point on the matrix changed by more than a user-defined percentage between two iterations. A method of adaptive meshing was then applied, the level of granulation depending on the gradient of the potential and the maximum granulation being user-defined. This meshed system was refined further by applying the Gauss-Seidel method again for several more iterations, and finally plotted as a heatmap showing potential. (See Fig.(2) for a more detailed programme flow.)

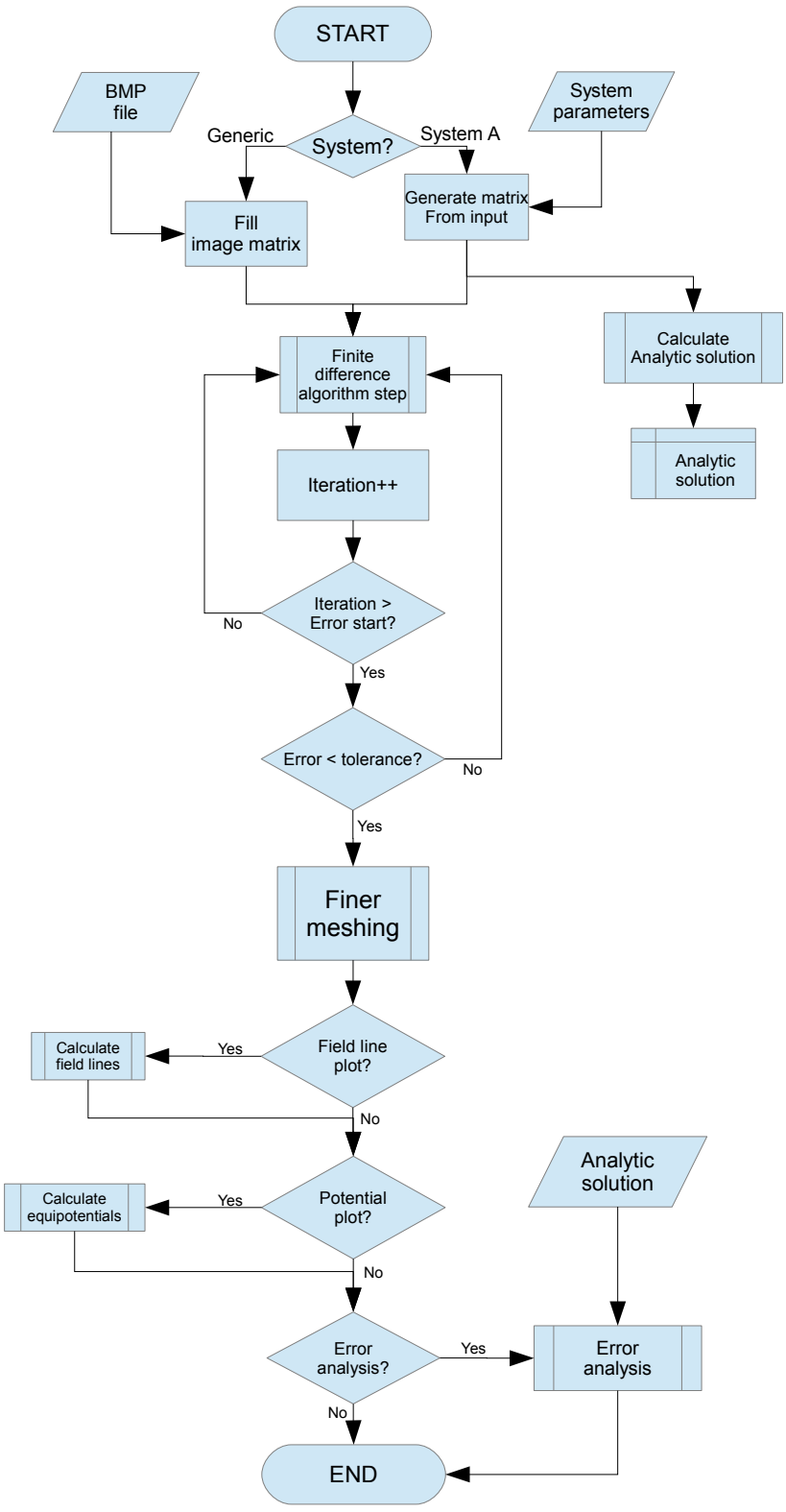

Figure 2: Program Flow

Results and Discussion

Error Analysis

Complex systems

Most electrostatic systems can only be solved numerically due to their complex geometry. These systems are extremely important as they have many applications in physics. The code developed was modular and generic enabling different geometric configurations of conductors. This was achieved using a pixelated drawing describing the components of the system desired, known as bitmap. When this was passed to the program each pixel was mapped to a matrix holding values between -1 and 1.

Silicon detectors are commonly used in high energy physics because of their outstanding performance. They are essential to the disentanglement of decay events and for the discovery of new particles due to their precise tracking. The detectors consist of silicon wafers with segmented doped implants on one side and a uniform implant on the other [7]. This design is known as planar strip detectors and is used in many particle physics experiments.

An alternative set up are 3D detectors which are able to operate under extreme radiation environments [7]. Columnar implants penetrate perpendicularly the silicon wafer and extend into the depth of the sensor instead of being limited by the surface as planar detectors. Ionising radiation produces electron-hole pairs and under the influence of an electric field these pairs will drift inducing a signal on the implants [8]. A top view of this geometry can be seen in Fig.3. Using the generic computer program designed a numerical approximation was obtained for the electrostatic potential.

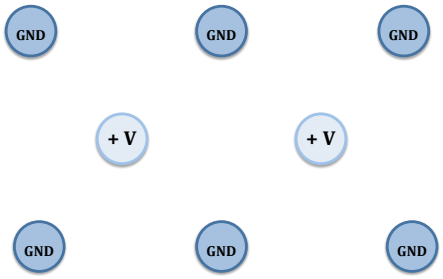

Figure 3: Illustration of physical system E. Shows a grid of cylindrical implants where in the middle they are held at potential $+V$ and in the outside at ground potential.

Conclusion

References

[1] Sears and Zemansky, *University Physics*. Pearson, 13th ed., 2012.

[2] D. J. Griffiths, *Introduction to Electrodynamics*. Prentice Hall, 3rd ed., 1999.

[3] K. F. Riley, M. P. Hobson, and S. J. Bence, *Mathematical Methods for Physics and Engineering*. Cambridge University Press, 3rd ed., 2006.

[4] W. Cheney and D. Kincaid, *Numerical Mathematics and Computing*. Thomson Brooks/Cole, 6th ed., 2008.

[5] A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*. Cambridge University Press, 2nd ed., 2009.

[6] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P.Flannery, *Numerical Recipes in C : The Art of Scientific Computing*. Cambridge University Press, Numerical Recipes Software, 2nd ed., 1992.

[7] M. Kohler, *Double-Sided 3D Silicon Detectors for the High-Luminosity LHC*. PhD thesis, University of Freiburg, 2011.

[8] G. F. Knoll, *Radiation Detection and Measurement*. John Wiley and Sons, 3rd ed., 2000.