STOCHASTIC PROCESSES IN FINANCE

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Abstract

A project investigating the properties of financial stochastic processes. Specifically, we will consider the implementation of both the Black-Scholes model for determining the price of European call options on stock and the Vasicek model for calculating the future price of bonds. We discuss the background, benefits and drawbacks of both models before conducting our investigation and discussing our findings.

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1 Introduction

The Black-Scholes model is a mathematical model developed by the economists Fischer Black, Myron Scholes and Robert Merton which is widely used in the pricing of options contacts. The model assumes that the price of heavily traded assets follows a geometric Brownian motion with constant drift and volatility. When applied to a stock option, the model incorporates the constant price variation of the stock, the time value of money, the option's strike price, and the time to the option's expiry.

The Vasicek model is a mathematical model used in financial economics to estimate potential pathways for future interest rate changes. The model states that the movement of interest rates is affected only by random (stochastic) market movements and models interest rate movements as a factor composed of market risk, time, and equilibrium value - where the rate tends to revert towards the mean of those factors over time.

We will first investigate how the price of a European call option - as given by the Black-Scholes model - varies with changes in time, interest rates, strike price and volatility. We will then simulate an Ornstein-Uhlenbeck process and use this to simulate the price of a bond - using the Vasicek model - over a given time period before evaluating the distribution of the simulated prices.

2 The Black Scholes Model

The price at time $t_0 = 0$ of a European call option (ECO) on a stock with strike price c, expiry time t_0, initial stock price S_0, interest rate ρ and volatility σ, is given by the Black-Scholes formula as:

$$P_{t_0} = S_0 \Phi\left(\frac{\log(S_0/c) + (\rho + \sigma^2/2)t_0}{\sigma\sqrt{t_0}}\right) - c\exp(-\rho t_0)\Phi\left(\frac{\log(S_0/c) + (\rho - \sigma^2/2)t_0}{\sigma\sqrt{t_0}}\right). \quad (2.1)$$

We plot the price of the ECO P_t for $0 \le t \le 10$ with $S_0 = 1$, $\sigma^2 = 0.02$, $\rho = 0.03$ and $c = 1$, shown in Figure 1.

We see that P_t is increasing over time, which makes sense as the value of the ECO should be dependent on the time the underlying stock has to increase in value. From equation (2.1) we can see that infact $P_t \to S_0$ as $t \to \infty$.

We plot the price P_{10} at $t = 10$ as we vary each of σ, ρ and c in turn - shown in Figure 2, Figure 3 and Figure 4 respectively.

As both σ and ρ increase we see that $P_{10} \to S_0$. This is because high underlying stock volatility and interest rates both increase the potential option return which increases the options value up to the initial price of

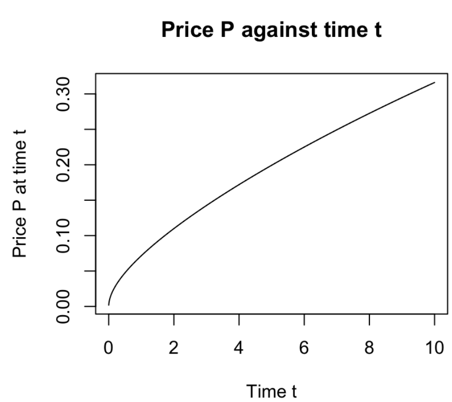
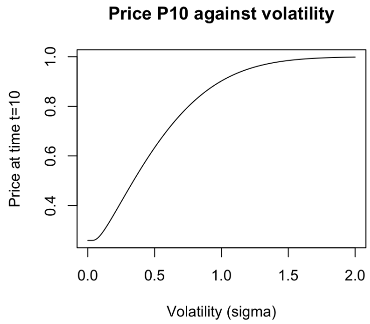
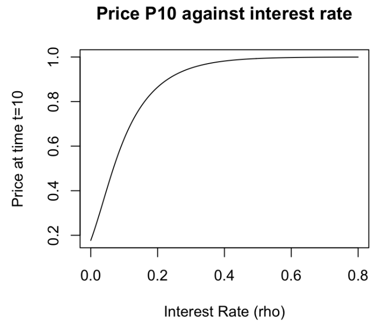


Figure 1: ECO price P_t over time t.





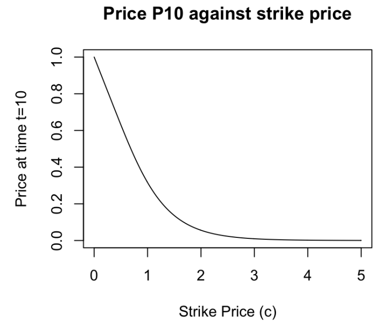


Figure 2: Price P_{10} against σ. Figure 3: Price P_{10} against ρ. Figure 4: Price P_{10} against c

the stock. The option value is limited to this because if the option price were to rise above the stock price then there would be no reason for investors to purchase the option rather than the stock.

As the strike price c increases we see that $P_{10} \to 0$. This is because if the option has a high strike price it is less likely that the underlying stock will reach this price during the period of the option which makes the option less valuable.

3 Ornstein-Uhlenbeck Processes

The spot-rate $\{R_s : s > 0\}$ is an Ornstein-Uhlenbeck (OU) Process - with initial spot rate R_0, long-term mean μ, reversion speed $\theta > 0$ - given by:

$$R_s = e^{-\theta s} R_0 + (1 - e^{-\theta s})\mu + X_s$$

where X_s is an OU process with volatility $\sigma > 0$ and reversion parameter $\theta > 0$. That is $E[X_s] = 0$ & $\mathrm{Cov}(X_s, X_t) = \frac{\sigma^2}{2\theta} e^{-\theta(s+t)} e^{2\theta \min(s,t) - 1}$.

We simulate the OU process $\{R_s : 0 \leq s \leq 10\}$ for $R_0 = 0.1$, $\theta = 0.5$, $\mu = 0.05$ and $\theta = 0.02$ and plotted out results which are shown in Figure 5.

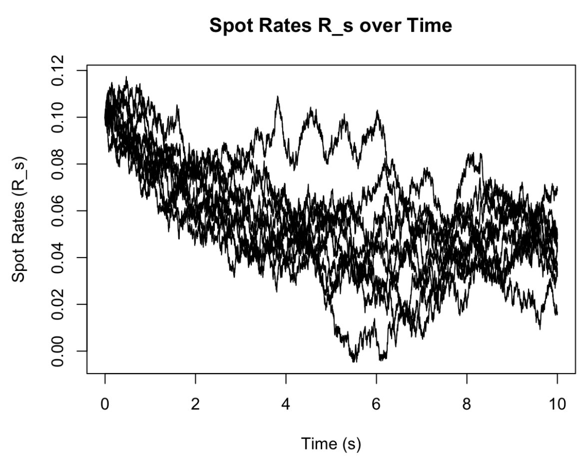


Figure 5: R_s against σ.

We can obtain the following expressions for the mean and variance of this process:

$$E[R_s] = e^{-\theta s} R_0 + (1 + e^{-\theta s})\mu$$
$$\mathrm{Var}(R_s) = \frac{\sigma^2}{2\theta}(1 - \exp(-2\theta s))$$

We plot $E[R_s]$ and $\mathrm{Var}(R_s)$ against time which are shown in Figure 6 and Figure 7 respectively.

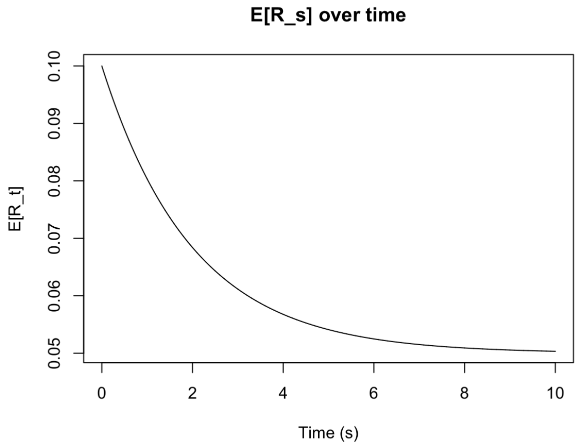


Figure 6: $E[R_s]$ over time.

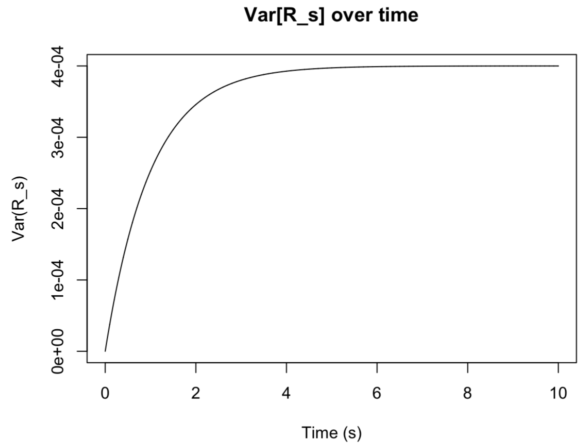


Figure 7: $\mathrm{Var}[R_s]$ over time.

From these we evaluate what the expectation and variance of R_s tend to as $s \to \infty$ for which we attain the following expressions:

$$\lim_{s\to\infty}\{E[R_s]\} = \mu \qquad \& \qquad \lim_{s\to\infty}\{\mathrm{Var}(R_s)\} = \frac{\sigma^2}{2\theta}$$

We can see from Figure 6 that the expectation is tending towards the long-term mean $\mu = 0.05$ as expected.

We can see from Figure 7 that the variance is tending towards 0.04, as predicted by our expression for the limit of $\mathrm{Var}(R_S)$ as $s \to \infty$.

If we change the value of R_0 we will change the starting point of the process. If we change the value of μ we change the value which the expectation of the process will tend towards. If we increase the value of the reversion parameter θ only, we will decrease the value that the variance of the process will tend towards. Similarly, if we increase the value of the volatility σ only, we will increase the value that the variance of the process will tend towards.

4 The Vasicek Model

The Vasicek model defines the price Q_t at time 0 of a bond paying one unit at time t as:

$$Q_t = \exp\left(-\int_0^t R_s \, ds\right)$$

where R_s is the OU process defined in Section 3.

We plot 10 simulations of the bond price Q_t at time 0 of a bond paying one unit at time t for $R_0 = 0.1$, $\theta = 0.5$, $\mu = 0.05$ and $\theta = 0.02$, shown in Figure 8.

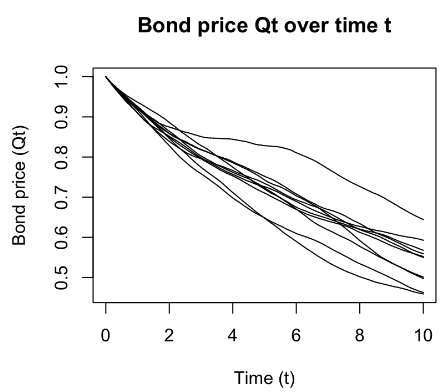


Figure 8: Bond price Q_t against time.

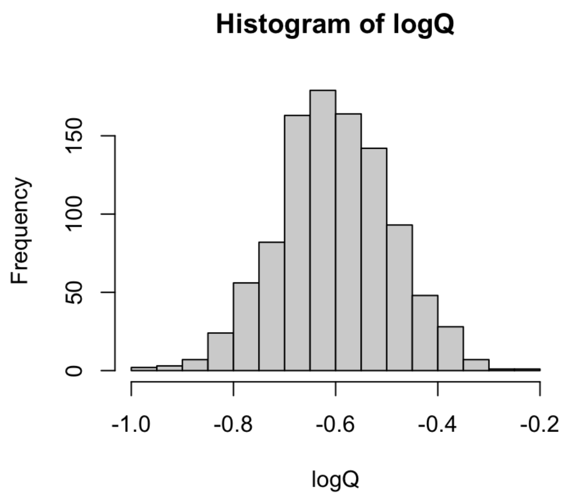


Figure 9: Histogram of 1000 Q_{10} realisations.

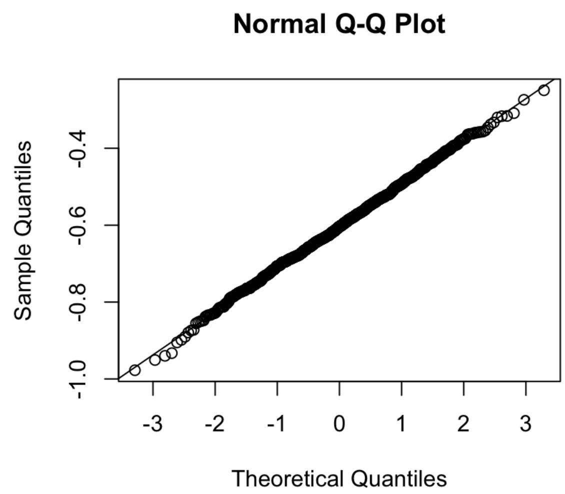


Figure 10: Normal QQ plot of Q_t.

We expect the distribution of Q_t to be Log-Normal as the integral $\int_0^t R_s \, ds$ is Normal - as it is a linear combination of Normal random variables.

We can check the distribution of Q_t for the fixed value of $t = 10$ by simulating 1000 realisations of $\log(Q_{10})$ and plotting them on a histogram, shown in Figure 9. From this histogram we can see that the distribution of $\log(Q_{10})$ appears to be Normal, as predicted.

We can illustrate this by simulation by using a QQ-plot, shown in Figure 10. We can see that the points seem to (roughly) lie on a straight line which suggests they follow a Normal distribution as we expected.

5 Conclusion

Both models discussed in this project come with several limitations. Firstly, the Black-Scholes model assumes that an option can only be exercised at expiration which limits its use to European options (as US options can be exercised before expiration). The model also makes assumptions that do not tend to hold in real world applications such as that no dividends are paid out during the life of the option, that markets are efficient, that there are no transaction costs in buying the option, that the risk-free rate and volatility of the underlying assets are known and constant and that the returns on the underlying assets are known and constant. .

The main disadvantage of the Vasicek model that has come to light since the global financial crisis is that the model does not allow for interest rates to dip below zero and become negative. This issue has been fixed in several models that have been developed since the Vasicek model such as the exponential Vasicek model and the Cox-Ingersoll-Ross model for estimating interest rate changes and further investigation into these models would be a useful topic of further research.

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A R-Code

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##MATH580 Project

###--------------------------------------------------
##Black-Scholes--------------------------------------
#Define our values
S0=1
sigma=sqrt(0.02)
rho=0.03
c=1
t=seq(0, 10, by=0.001)

#Define our formula for the price
P=S0*pnorm((log(S0/c)+(rho+(sigma^2)/2)*t)/(sigma*sqrt(t)))-(c*exp(-rho*t))*pnorm((log(S0/c)+(rho-(sigma^2)/2)*t)/(sigma*
      sqrt(t)))
plot(t, P,  type="line",xlab="Time t", ylab="Price P at time t", main="Price P against time t")

##Investigation--------------------------------

##Changing Volatility (sigma)-----------------

#Set t=10
t10=10

sigma1=seq(0,2, by=0.01)

P1=S0*pnorm((log(S0/c)+(rho+(sigma1^2)/2)*t10)/(sigma1*sqrt(t10)))-(c*exp(-rho*t10))*pnorm((log(S0/c)+(rho-(sigma1^2)/2)*
      t10)/(sigma1*sqrt(t10)))
plot(sigma1, P1, xlim=c(0, max(sigma1)),type="l", xlab="Volatility (sigma)", ylab="Price at time t=10", main="Price P10
      against volatility")

##Changing Interest Rate (rho)-------------------

rho2=seq(0, 0.8, by=0.005) #Create a vector of different rho values

P2=S0*pnorm((log(S0/c)+(rho2+(sigma^2)/2)*t10)/(sigma*sqrt(t10)))-(c*exp(-rho2*t10))*pnorm((log(S0/c)+(rho2-(sigma^2)/2)*
      t10)/(sigma*sqrt(t10)))
plot(rho2, P2, type="line", xlab="Interest Rate (rho)", ylab="Price at time t=10", main="Price P10 against interest rate")

##Changing price (c)--------------------

c3=seq(0,5, by=0.01)

P3=S0*pnorm((log(S0/c3)+(rho+(sigma^2)/2)*t10)/(sigma*sqrt(t10)))-(c3*exp(-rho*t10))*pnorm((log(S0/c3)+(rho-(sigma^2)/2)*
      t10)/(sigma*sqrt(t10)))
plot(c3, P3, type="line", xlab="Strike Price (c)", ylab="Price at time t=10", main="Price P10 against strike price")

#--------------------------------------------------
#OU Process----------------------------------------

##CLEAR ENVIRONMENT!!!

##Define the OU function
rOU=function(n,N,Delta,theta,sigma){
  times=(0:n)*Delta ##vector of t_0,t_1,..,t_n
  X=matrix(0,nrow=N,ncol=n+1)
  for(i in 1:n){
    x=X[,i]#current value
    m=x*exp(-theta*Delta) #mean of new value
    v=sigma^2*(1-exp(-2*theta*Delta))/(2*theta) ##variance of new value
    X[,i+1]=rnorm(N,m,sqrt(v)) ##simulate new value
  }
  return(list(X=X,times=times))
}

##first simulate X_t
n=10000
N=10
Delta=10/n
theta=0.5
sigma=0.02
OU=rOU(n, N, Delta, theta, sigma)
plot(range(OU$times),range(OU$X),type="n",xlab="Time",ylab="X_t")
for(i in 1:N){
  lines(OU$times,OU$X[i,])
}

##Now transform
R0=0.1
mu=0.05

##Storage for R_t
Rt=matrix(0, nrow=N, ncol=n+1)

##Loop to transform OU to R_t
for(i in 1:N){
  Rt[i,]=exp(-theta*OU$times)*R0+(1-exp(-theta*OU$times))*mu+OU$X[i,]
}

##plot
```

```
plot(range(OU$times), range(Rt), type="n", xlab="Time (s)", ylab="Spot Rates (R_s)", main="Spot Rates R_s over Time")
for(i in 1:N){
  lines(OU$times, Rt[i,])
}

##Plot the Expectation and the Variance-------------------

expRt=exp(-theta*OU$times)*R0+(1-exp(-theta*OU$times))*mu
plot(OU$times, expRt, type="line" , ylab="E[R_s]", xlab="Time (s)", main="E[R_s] over time")

varRt=(sigma^2/(2*theta))*(1-exp(-2*theta*OU$times))
plot(OU$times, varRt, type="line" ,ylab="Var(R_s)", xlab="Time (s)", main="Var[R_s] over time")

##-------------------------------------------------------
##Vasicek Model

##Calculating Q over 0<t<10

#Matrix to store Q
Q=matrix(0,nrow=N,ncol=n+1)

#Loop to generate values of Q
for(i in 1:N){
  Q[i,]=exp(-Delta*cumsum(Rt[i,]))
}

#Plot
plot(range(OU$times),range(Q),type="n",xlab="Time (t)",ylab="Bond price (Qt)", main="Bond price Qt over time t")
for(i in 1:N){
  lines(OU$times,Q[i,])
}

##Distribution of Qt---------------------

#CLEAR ENVIRONMENT!!!

##Define the OU function
rOU=function(n,N,Delta,theta,sigma){
  times=(0:n)*Delta ##vector of t_0,t_1,..,t_n
  X=matrix(0,nrow=N,ncol=n+1)
  for(i in 1:n){
    x=X[,i]#current value
    m=x*exp(-theta*Delta) #mean of new value
    v=sigma^2*(1-exp(-2*theta*Delta))/(2*theta) ##variance of new value
    X[,i+1]=rnorm(N,m,sqrt(v)) ##simulate new value
  }
  return(list(X=X,times=times))
}

#Simulate 1000 realisations of Rt at t=10 (n=1000, delta=10/n so t=n*delta=10)

n=1000
Delta=10/n
N=1000
theta=0.5
sigma=0.02
OU=rOU(n, N, Delta, theta, sigma)

R0=0.1
mu=0.05

Rt=matrix(0, nrow=N, ncol=n+1)

for(i in 1:N){
  Rt[i,]=exp(-theta*OU$times)*R0+(1-exp(-theta*OU$times))*mu+OU$X[i,]
}

Q=rep(0, N)
for(i in 1:N){
  Q[i]=exp(-Delta*sum(Rt[i, 2:(n+1)]))
}

logQ=log(Q)

hist(logQ)
qqnorm(logQ)
qqline(logQ)

mean(logQ)
var(logQ)
```