NEWMAN-JANIS ALGORITHM REVISITED

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The purpose of the present article is to show that the Newman-Janis and Newman et al algorithm used to derive the Kerr and Kerr-Newman metrics respectively, automatically leads to the extension of the initial non negative polar radial coordinate r to a cartesian coordinate r' running from $-\infty$ to $+\infty$, thus introducing in a natural way the region $-\infty < r' < 0$ in the above spacetimes. Using Boyer-Lindquist and ellipsoidal coordinates, we discuss some geometrical aspects of the positive and negative regions of r', like horizons, ergosurfaces, and foliation structures.

1. Introduction

The uncharged Kerr (K) (Kerr, 1963) and the charged Kerr-Newman (KN) (Newman et al, 1965) axially symmetric stationary spacetimes have, in contradistinction to the charged static spherical Reissner-Nördstrom (RN) solution (Reissner, 1916; Nördstrom, 1918), and the chargeless static spherical Schwarzschild solution (Schwarzschild, 1916), an asymptotically flat region which involves, both in the Eddington-Finkelstein (EF) (Eddington, 1924; Finkelstein, 1958) and Boyer-Lindquist (BL) (Boyer and Lindquist, 1967) coordinates, a radial cordinate taking not only positive and zero values, but also negative ones, form 0 to $-\infty$. This strange situation is usually explained by the fact that, since the curvature singularity of both the K and KN solutions is in the circular boundary of a non singular open disk in the equatorial plane, the spacetimes can be continued through it, to regions in which the radial coordinate becomes negative (regions III and III' in the Penrose-Carter diagram (Penrose, 1963; Carter, 1966), section 8). As it stands, the argument, though correct, does not imply the necessity of this continuation, but only it allows for its possibility.

In this note we show that the complexification procedure involved in the Newman-Janis algorithm (NJA) (Newman and Janis, 1965) used to derive the K and KN metrics, automatically leads to the extension to negative values of the originally non negative polar radial coordinate, leaving from the outset with coordinates $(u', r', \theta, \varphi)$ (or (t', r', θ, ϕ)) taking values in $\mathbb{R}^2 \times S^2$ i.e. two cartesian (u', r' or t', r') and two compact $(\theta, \varphi \text{ or } \theta, \phi)$ coordinates (u') is the retarded EF time and t the BL time). Though at the epoque of its inception and for many years, the Newman-Janis and Newman et al derivations respectively of the K and KN metrics were considered as flukes, recent work by Drake and Szekeres (Drake and Szekeres, 2000) has put the algorithm on a more solid ground by proving uniqueness theorems for the kind of solutions which can be derived using the algorithm.

In sections 2 to 5 we review the NJA derivation of the K and KN metrics emphasizing, in section 4, how the complexification of the radial coordinate automatically implies the range

 $(-\infty, +\infty)$ for its real part r'. Using BL (section **6**) and ellipsoidal coordinates, in section **7** we exhibit the horizons, ergosurfaces, and foliation structures of the KN and K solutions (respectively for the cases $M^2 > a^2 + Q^2$ and $M^2 > a^2$) in both regions r' > 0 and r' < 0, and give a brief discussion of the spatial topology of these constructions. For completeness, the basic cell of the Penrose-Carter diagram of the K and KN spacetimes is exhibited in section **8**.

2. Reissner-Nördstrom spacetime

Our starting point is the Reissner-Nördstrom (RN) spacetime written in terms of the retarded Eddington -Finkelstein coordinates (u, r, θ, φ) with $u \in \mathbb{R} = (-\infty, +\infty)$, $r \in \mathbb{R}_{\geq} = [0, +\infty)$, and $\theta, \varphi \in S^2$ i.e. $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$:

$$ds_{RN}^2 = f du^2 + 2 du dr - r^2 d^2 \Omega, \quad d^2 \Omega = d\theta^2 + \sin^2 \theta d\varphi^2$$
 (1)

with

$$f = \frac{r^2 - 2Mr + Q^2}{r^2} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad [f] = [L]^0, \tag{2}$$

where M is the gravitating (positive) mass and $Q^2 = q^2 + p^2$ where q is the electric charge and p is the hypotetical abelian (Dirac) magnetic charge. The metric corresponding to (1) is

$$g_{\mu\nu_{RN}} = \begin{pmatrix} f & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & -r^2 & 0\\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}, \quad det g_{\mu\nu_{RN}} = -r^4 \sin^2\theta, \tag{3}$$

with inverse

$$g^{\mu\nu}{}_{RN} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -f & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & -r^{-2}sin^{-2}\theta \end{pmatrix}. \tag{4}$$

 $(\mu = 0, 1, 2, 3 \text{ respectively correspond to } u, r, \theta, \varphi.)$

3. Null tetrad

At each point of the RN manifold we can choose as a basis of the corresponding tangent space the null tetrad consisting of the following linear independent 4-vectors:

$$l = l^{\mu} \frac{\partial}{\partial x^{\mu}} = \delta_1^{\mu} \frac{\partial}{\partial x^{\mu}} = \frac{\partial}{\partial x^1} = \frac{\partial}{\partial r}, \tag{5a}$$

$$n = n^{\mu} \frac{\partial}{\partial x^{\mu}} = (\delta_0^{\mu} - \frac{f}{2} \delta_1^{\mu}) \frac{\partial}{\partial x^{\mu}} = \frac{\partial}{\partial u} - \frac{f}{2} \frac{\partial}{\partial r}, \tag{5b}$$

$$m = m^{\mu} \frac{\partial}{\partial x^{\mu}} = \frac{1}{\sqrt{2}r} (\delta_{2}^{\mu} + \frac{i}{\sin\theta} \delta_{3}^{\mu}) \frac{\partial}{\partial x^{\mu}} = \frac{1}{\sqrt{2}r} (\frac{\partial}{\partial \theta} + \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi}), \tag{5c}$$

$$\bar{m} = \bar{m}^{\mu} \frac{\partial}{\partial x^{\mu}} = \frac{1}{\sqrt{2}r} (\delta_{2}^{\mu} - \frac{i}{\sin\theta} \delta_{3}^{\mu}) \frac{\partial}{\partial x^{\mu}} = \frac{1}{\sqrt{2}r} (\frac{\partial}{\partial \theta} - \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi}), \tag{5d}$$

i.e.

$$l^{\mu} = (l^{u}, l^{r}, l^{\theta}, l^{\varphi}) = (0, 1, 0, 0), \tag{5a'}$$

$$n^{\mu} = (n^{u}, n^{r}, n^{\theta}, n^{\varphi}) = (1, -\frac{f}{2}, 0, 0), \tag{5b'}$$

$$m^{\mu} = (m^{u}, m^{r}, m^{\theta}, m^{\varphi}) = \frac{1}{\sqrt{2r}}(0, 0, 1, \frac{i}{\sin\theta}),$$
 (5c')

$$\bar{m}^{\mu} = (\bar{m}^{u}, \bar{m}^{r}, \bar{m}^{\theta}, \bar{m}^{\varphi}) = \frac{1}{\sqrt{2}r}(0, 0, 1, -\frac{i}{\sin\theta}),$$
 (5d')

with covariant components $b_{\mu} = g_{\mu\nu}{}_{RN}b^{\nu}$ given by

$$l_{\mu} = (1, 0, 0, 0,), \tag{5a''}$$

$$n_{\mu} = (\frac{f}{2}, 1, 0, 0),$$
 (5b")

$$m_{\mu} = \frac{1}{\sqrt{2}r}(0, 0, -r^2, -ir^2 \sin\theta),$$
 (5c")

$$\bar{m}_{\mu} = \frac{1}{\sqrt{2}r}(0, 0, -r^2, ir^2 sin\theta).$$
 (5d")

The scalar products $a \cdot b = g_{\mu\nu}{}_{RN} a^{\mu} b^{\nu}$ of the members of the tetrad are given in the following table:

Table I

It is an easy exercise to verify that the quantity

$$\tilde{g}^{\mu\nu} = (l^{\mu}n^{\nu} + l^{\nu}n^{\mu}) - (m^{\mu}\bar{m}^{\nu} + m^{\nu}\bar{m}^{\mu}) \tag{6}$$

is nothing but the inverse RN metric:

$$\tilde{g}^{\mu\nu} = g^{\mu\nu}_{RN}.\tag{7}$$

It is interesting to note that if the tetrad is denoted by

$$(e_1, e_2, e_3, e_4) = (l, n, m, \bar{m}),$$
 (8)

then

$$\tilde{g}^{\mu\nu} = e_a{}^{\mu} J^{ab} e_b{}^{\nu} \tag{9}$$

with

$$J = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix},\tag{10}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Pauli matrix. J is related to the $\alpha^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$, k = 1, 2, 3, matrices of the standard representation of the Dirac equation, with $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, through the similarity transformations

$$S_k^{-1}JS_k = \alpha^k, \ S_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ S_2 = \begin{pmatrix} 1 & \sigma_1\sigma_2 \\ 1 & -\sigma_1\sigma_2 \end{pmatrix}, \ S_3 = \begin{pmatrix} 1 & \sigma_1\sigma_3 \\ 1 & -\sigma_1\sigma_3 \end{pmatrix}, \tag{11}$$

with

$$S_1^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ S_2^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i\sigma_3 & i\sigma_3 \end{pmatrix}, \ S_3^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i\sigma_2 & -i\sigma_2 \end{pmatrix}. \tag{12}$$

It is curious that a relation with the Pauli and Dirac matrices appears here, since the RN solution is static without rotation. The next complexification and appearance of the KN metric does not change the situation, since the J matrix tacitly involved in eq. (22) is the same as that in eq. (9). We leave this apparent "accident" opened to further research.

4. Complexification

The following step is the crucial element of the NJA: the coordinates r and u are complexified and a new real positive parameter a (later identified with the angular momentum per unit mass) is introduced:

$$r \in \mathbb{R}_{>} \longrightarrow r \in \mathbb{C}, \quad r = r' - iacos\theta,$$
 (13)

$$u \in \mathbb{R} \longrightarrow u \in \mathbb{C}, \quad u = u' + iacos\theta,$$
 (14)

with $\bar{r} = r' + iacos\theta$ and $\bar{u} = u' - iacos\theta$. But now

$$r' \in (-\infty, +\infty),\tag{15}$$

i.e. r' has become a cartesian coordinate. In particular, this will imply that when dealing with the Boyer-Lindquist coordinates in section **6**, no appeal for an analytic continuation to the asymptotically flat (A.F.) region r' < 0 through the open disk $y^2 + z^2 < a^2$ where no singularity occurs will be required: implicitly, this analytic continuation was already done through the complexification (13). The domains of definition of r and u are shown in Figure 1. Clearly, $[a] = [L]^1$.

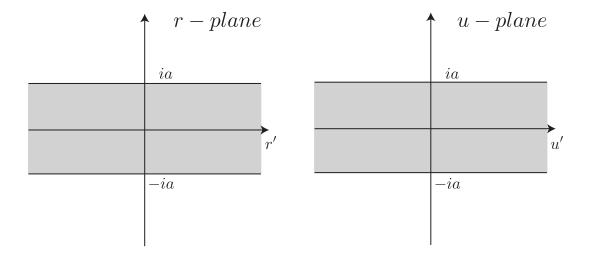


Figure 1: Domains of definition of the coordinates r and u after the complexification (13).

The new set of real coordinates is $(u', r', \theta', \varphi')$ with $u' \in (-\infty, +\infty)$, $\theta' = \theta$, and $\varphi' = \varphi$. The total domain of these coordinates is $\mathbb{R}^2 \times S^2$.

Since in the end one needs a real spacetime, f must remain real and so its change is given by

$$f(r) \longrightarrow f(r,\bar{r}) = 1 - M(\frac{1}{r} + \frac{1}{\bar{r}}) + \frac{Q^2}{r\bar{r}} = 1 - \frac{2Mr' - Q^2}{\Sigma}$$
 (16)

where

$$\Sigma = r'^2 + a^2 \cos^2 \theta. \tag{17}$$

 $([\Sigma]=[L]^2.)$ The transformation of the tetrad is

$$e_a{}^{\mu} \longrightarrow e_a'{}^{\mu} = \frac{\partial x'{}^{\mu}}{\partial x^{\nu}} e_a{}^{\nu} \equiv (l'{}^{\mu}, n'{}^{\mu}, m'{}^{\mu}, \bar{m'}{}^{\mu})$$
 (18)

with

$$\begin{pmatrix}
\frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial r} & \frac{\partial u'}{\partial \theta} & \frac{\partial u'}{\partial \varphi} \\
\frac{\partial r'}{\partial u} & \frac{\partial r'}{\partial r} & \frac{\partial r'}{\partial \theta} & \frac{\partial r'}{\partial \varphi} \\
\frac{\partial \theta'}{\partial u} & \frac{\partial \theta'}{\partial r} & \frac{\partial \theta'}{\partial \theta} & \frac{\partial \theta'}{\partial \varphi} \\
\frac{\partial \theta'}{\partial u} & \frac{\partial \theta'}{\partial r} & \frac{\partial \theta'}{\partial \theta} & \frac{\partial \theta'}{\partial \varphi} \\
\frac{\partial \varphi'}{\partial u} & \frac{\partial \varphi'}{\partial r} & \frac{\partial \varphi'}{\partial \theta} & \frac{\partial \varphi'}{\partial \varphi} \\
\frac{\partial \varphi'}{\partial u} & \frac{\partial \varphi'}{\partial r} & \frac{\partial \varphi'}{\partial \theta} & \frac{\partial \varphi'}{\partial \theta}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & iasin\theta & 0 \\
0 & 1 & -iasin\theta & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$
(19)

and

$$l^{\nu} = \delta_{1}^{\nu}, \ n^{\nu} = \delta_{0}^{\nu} - \frac{1}{2}f(r,\bar{r})\delta_{1}^{\nu}, \ m^{\nu} = \frac{1}{\sqrt{2}\bar{r}}(\delta_{2}^{\nu} + \frac{i}{\sin\theta}\delta_{3}^{\nu}), \ \bar{m^{\nu}} = \frac{1}{\sqrt{2}r}(\delta_{2}^{\nu} - \frac{i}{\sin\theta}\delta_{3}^{\nu}). \eqno(20)$$

One obtains:

$$l'^{\mu} = \delta_1^{\mu}, \tag{21a}$$

$$n'^{\mu} = \delta_0^{\mu} - \frac{1}{2} \left(1 - \frac{2Mr' - Q^2}{\Sigma}\right) \delta_1^{\mu},\tag{21b}$$

$$m'^{\mu} = \frac{1}{\sqrt{2}(r' + iacos\theta)} ((\delta_0^{\mu} - \delta_1^{\mu})iasin\theta + \delta_2^{\mu} + \delta_3^{\mu} \frac{i}{sin\theta}), \tag{21c}$$

$$\bar{m'}^{\mu} = \frac{1}{\sqrt{2}(r' - iacos\theta)} \left(-(\delta_0^{\mu} - \delta_1^{\mu})iasin\theta + \delta_2^{\mu} - \delta_3^{\mu} \frac{i}{sin\theta} \right). \tag{21d}$$

5. Kerr-Newman metric

The "miracle", though in some way justified by Drake and Szekeres through the proof of important uniqueness theorems from the NJA for vacuum and Einstein-Maxwell solutions, is that the quantity

$$g^{\prime \mu \nu} = (l^{\prime \mu} n^{\prime \nu} + l^{\prime \nu} n^{\prime \mu}) - (m^{\prime \mu} \bar{m}^{\prime \nu} + m^{\prime \nu} \bar{m}^{\prime \mu})$$
(22)

is the inverse Kerr-Newman (KN) metric in Eddington-Finkelstein retarded coordinates $(u', r', \theta, \varphi)$:

$$g^{\prime \mu \nu} = g^{\mu \nu}{}_{KN}. \tag{23}$$

In fact, a straightforward calculation leads to

$$g'^{\mu\nu} = \begin{pmatrix} g^{u'u'} & g^{u'r'} & g^{u'\theta} & g^{u'\varphi} \\ \vdots & g^{r'r'} & g^{r'\theta} & g^{r'\varphi} \\ \vdots & \vdots & g^{\theta\theta} & g^{\theta\varphi} \\ \vdots & \vdots & \vdots & g^{\varphi\varphi} \end{pmatrix} =$$

$$\begin{pmatrix}
\frac{-a^{2}sin^{2}\theta}{\Sigma} & \frac{r'^{2}+a^{2}}{\Sigma} & 0 & -\frac{a}{\Sigma} \\
\vdots & -\frac{\Sigma-2Mr'+Q^{2}+a^{2}sin^{2}\theta}{\Sigma} & 0 & \frac{a}{\Sigma} \\
\vdots & \vdots & \vdots & \ddots & -\frac{1}{\Sigma} & 0 \\
\vdots & \vdots & \vdots & \ddots & -\frac{1}{\Sigma} & \frac{1}{\Sigma}
\end{pmatrix}$$
(24)

with inverse

$$g_{\mu\nu}{}_{KN} = \begin{pmatrix} 1 - \frac{2Mr' - Q^2}{\Sigma} & 1 & 0 & asin^2\theta \frac{2Mr' - Q^2}{\Sigma} \\ \cdot & 0 & 0 & -asin^2\theta \\ \cdot & \cdot & -\Sigma & 0 \\ \cdot & \cdot & \cdot & -sin^2\theta \frac{A}{\Sigma} \end{pmatrix}$$
(25)

with

$$A = \Sigma(r'^2 + a^2) + a^2 \sin^2\theta \frac{2Mr' - Q^2}{\Sigma} = (r'^2 + a^2)^2 - a^2 \sin^2\theta \Delta, \tag{26}$$

and

$$\Delta = r'^2 + a^2 - 2Mr' + Q^2. \tag{27}$$

The square of the spacetime element is

$$ds_{KN}^2 =$$

$$(1-\frac{2Mr'-Q^2}{\Sigma})d{u'}^2+2du'dr'+2asin^2\theta\frac{2Mr'-Q^2}{\Sigma}du'd\varphi-2asin^2\theta dr'd\varphi-\Sigma d\theta^2-sin^2\theta\frac{A}{\Sigma}d\varphi^2. \eqno(28)$$

 ds_{KN}^2 reduces to ds_{RN}^2 for a=0 and $r'\geq 0$.

6. Boyer-Lindquist coordinates

The change of coordinates

$$dt = du' - \frac{{r'}^2 + a^2}{\Lambda} dr', \quad d\phi = d\varphi - \frac{a}{\Lambda} dr'$$
 (29)

leads to the Boyer-Lindquist form of the Kerr-Newman spacetime:

$$ds_{KN}^{2}|_{BL} = \frac{\Delta - a^{2}sin^{2}\theta}{\Sigma}dt^{2} - \frac{\Sigma}{\Delta}dr'^{2} - \Sigma d\theta^{2} - sin^{2}\theta\frac{A}{\Sigma}d\phi^{2} + \frac{2asin^{2}\theta}{\Sigma}(r'^{2} + a^{2} - \Delta)dtd\phi \quad (30)$$

i.e.

$$g_{\mu\nu_{KN}}|_{BL} = \begin{pmatrix} 1 - \frac{2Mr' - Q^2}{\Sigma} & 0 & 0 & \frac{a\sin^2\theta}{\Sigma} (r'^2 + a^2 - \Delta) \\ \vdots & -\frac{\Sigma}{\Delta} & 0 & 0 \\ \vdots & \vdots & -\Sigma & 0 \\ \vdots & \vdots & \ddots & -\sin^2\theta \frac{A}{\Sigma} \end{pmatrix}.$$
(31)

Horizons H_+ and H_- are defined by the zeros of Δ ; it is clear that only for r' > 0 and $M^2 \ge a^2 + Q^2$ horizons exist, with

$$r'_{+} = M \pm \sqrt{M^2 - (a^2 + Q^2)}. (32)$$

It can be easily seen that for $M^2 > a^2 + Q^2$, $r'_- < \sqrt{a^2 + Q^2}$, in particular $r'_- < a$ for the Kerr case $Q^2 = 0$; for the extreme cases $M^2 = a^2 + Q^2$, $r'_- = r'_+ = M = \sqrt{a^2 + Q^2}$.

For
$$r' < 0$$
,

$$\Delta = {r'}^2 + a^2 + 2M|r'| + Q^2 > 0,$$
(33)

which has no real roots. The same occurs for the ergosurfaces S_+ and S_- whose equations are given by the zeros of $g_{ttKN}|_{BL}$; for r' < 0,

$$g_{tt_{KN}}|_{BL} = 1 + \frac{2M|r'| + Q^2}{\Sigma} > 1.$$
 (34)

Also, as is well known, from (17), (26), (27) and (30),

$$ds_{KN}^2|_{BL} \longrightarrow dt^2 - dr'^2 - r'^2(d\theta^2 + \sin^2\theta d\phi^2)$$
(35)

as $r' \to \pm \infty$ i.e. the metric is A.F. in both the r' > 0 and r' < 0 regions.

7. Generalized ellipsoidal coordinates

As is well known, the use of Kerr-Schild (KS) coordinates for the KN metric and their restriction to ellipsoidal coordinates (EC) for the Kerr (K) metric $(ds_K^2 = ds_{KN}^2|_{Q^2=0})$, allows to show that both metrics are flat (Minkowskian) for $M^2 = Q^2 = 0$ in the KN case and $M^2 = 0$ in the K case. The general form of the spatial part of these coordinates allowing for both positive and negative values of r' is

$$x_{\pm} = \sqrt{r'^2 + a^2} \sin\theta \cos(\phi + F(r')), \tag{36a}$$

$$y_{\pm} = \sqrt{r'^2 + a^2} \sin\theta \sin(\phi + F(r')), \tag{36b}$$

$$z_{\pm} = r' cos\theta \tag{36c}$$

with the + (-) sign corresponding to r' > 0 (r' < 0) and

$$F(r') = \left\{ \begin{array}{c} -arc \ tan\left(\frac{a}{r'}\right), \ KS \\ 0, \ EC \end{array} \right. \tag{37}$$

 x_{\pm} , y_{\pm} , and z_{\pm} are cartesian coordinates defining two \mathbb{R}^3 spaces with opposite orientations: right handed for r' > 0 and left-handed for r' < 0 (Reall, 2008). In terms of these coordinates,

$$ds_{KN}^2|_{BL}|_{M=0, Q^2=0} = ds_K^2|_{BL}|_{M=0} = dt^2 - (dx_{\pm}^2 + dy_{\pm}^2 + dz_{\pm}^2).$$
(38)

In particular, from (30),

$$ds_K^2|_{BL}|_{M=0} = dt^2 - (\frac{\Sigma}{\Delta} dr'^2 + \Sigma d\theta^2 + \Delta sin^2 \theta d\phi^2).$$
 (39)

In terms of (x', y', z'), r' is given by

$$r' = \pm \frac{1}{\sqrt{2}} \sqrt{(x_{\pm}^2 + y_{\pm}^2 + z_{\pm}^2 - a^2) + \sqrt{(x_{\pm}^2 + y_{\pm}^2 + z_{\pm}^2 - a^2) + 4a^2 z_{\pm}^2}}.$$
 (40)

Both the KS and the EC coordinate systems admit, at each t, the same foliations of the two \mathbb{R}^3 spaces:

i) Confocal ellipsoids of revolution r' = const., foci at $x_+^2 + y_+^2 = a^2$, $z_+ = 0$:

$$\frac{x_{\pm}^2 + y_{\pm}^2}{r'^2 + a^2} + \frac{z_{\pm}^2}{r'^2} = 1,\tag{41}$$

with larger semi-axis = $\sqrt{r'^2 + a^2}$ and smaller semi-axis=|r'|. For r' > 0, the ellipsoids corresponding to r'_{\pm} are the horizons H_{\pm} , embedded in the (x_+, y_+, z_+) space.

ii) Confocal 1-sheet hyperboloids of revolution $\theta = const.$, foci at $x_+^2 + y_+^2 = a^2$, $z_+ = 0$:

$$\frac{x_{\pm}^2 + y_{\pm}^2}{a^2 \sin^2 \theta} - \frac{z_{\pm}^2}{a^2 \cos^2 \theta} = 1. \tag{42}$$

The surfaces $\phi = const.$ for the KS coordinates are discussed e.g. in Krasiński and Plebański (2006), but for the EC coordinates they are simply given by:

iii) Planes through the z_{\pm} axis.

Since the EC system can accommodate the horizons H_{\pm} and the ergosurfaces S_{\pm} (see below), we shall restrict the discussion to this coordinate system.

The curvature singularity of (30) is given by the condition

$$\Sigma = 0 \tag{43a}$$

which, by (17), implies

$$r' = 0, \ \theta = \frac{\pi}{2}.\tag{43b}$$

By (36), r' = 0 defines the disks D^2

$$0 \le x_+^2 + y_+^2 = a^2 \sin^2 \theta \le a \tag{44}$$

in the equatorial planes $z_{\pm}=0$. It is clear that its interiors \dot{D}^2 must be identified i.e. $(x_++\epsilon,y_++\epsilon)=(x_--\epsilon,y_--\epsilon)$ as $\epsilon\to\pm0$ for those x_{\pm},y_{\pm} satisfying $x_{\pm}^2+y_{\pm}^2< a^2$, and that the boundary

$$x_{\pm}^2 + y_{\pm}^2 = a^2 \tag{45}$$

is the singularity. On one "side" of \dot{D}^2 one has the $\mathbb{R}^2 \times S^2$ region r' > 0 (with horizons and ergosurfaces), on the other "side" one has another $\mathbb{R}^2 \times S^2$ region which corresponds to r' < 0, but without horizons and ergosurfaces. (In the \mathbb{R}^2 factors, one \mathbb{R} comes from r' > 0 and r' < 0, the other from the time coordinate.) It is easily seen that in the $r' \to 0_{\pm}$ limit, the ellipsoids (41) degenerate into the disks (44), while the hyperboloids (42) degenerate into $(z_{\pm} = 0 \text{ planes } \backslash \dot{D}^2)$.

The ergosurfaces S_{\pm} (in the r' > 0 region) are determined by the zeros of g_{tt} . From (31),

$$r'_{S_{+}}(\theta) = M \pm \sqrt{M^2 - (a^2 \cos^2 \theta + Q^2)}.$$
 (46)

In particular

$$r'_{S_{+}}(0) = r'_{S_{+}}(\pi) = r'_{+} \tag{47}$$

i.e. $S_{\pm} \equiv H_{\pm}$ at the "north" and "south" poles. Replacing (46) in (36) with F(r') = 0, we obtain for both S_{+} and S_{-} the surfaces of revolution

$$\frac{x_{+S_{\pm}}(\theta)^2 + y_{+S_{\pm}}(\theta)^2}{(\sqrt{r'_{S_{\pm}}(\theta)^2 + a^2})^2} + \frac{z_{+S_{\pm}}(\theta)^2}{r'_{S_{\pm}}(\theta)^2} = 1$$
(48)

which, together with horizons H_{\pm} , ellipsoids and hyperboloids corresponding to different sets of values of M, a, and Q^2 are plotted in the $y_+ - z_+$ plane in Figures 2, 3 and 4. For $\phi = \frac{\pi}{2} (y_+ - z_+)$ plane or $x_+ = 0$,

$$y_{+S_{\pm}}(\frac{\pi}{2}) = \sqrt{(r'_{S_{\pm}}(\frac{\pi}{2}))^2 + a^2} = \sqrt{(M \pm \sqrt{M^2 - Q^2})^2 + a^2}$$
 (49)

with $r'_{S_+}(\frac{\pi}{2})=M+\sqrt{M^2-Q^2}>r'_+>M$ and $r'_{S_-}(\frac{\pi}{2})=M-\sqrt{M^2-Q^2}< r'_-< M.$ So, for $Q^2=0,$

$$y_{+S_{+}}(\frac{\pi}{2}) = \sqrt{4M^2 + a^2}, \quad y_{+S_{-}}(\frac{\pi}{2}) = a,$$
 (50)

and, for $Q^2 > 0$,

$$y_{+S_{+}}(\frac{\pi}{2}) = \sqrt{(M + \sqrt{M^{2} - Q^{2}})^{2} + a^{2}}, \quad a < y_{+S_{-}}(\frac{\pi}{2}) = \sqrt{(r'_{S_{-}}(\frac{\pi}{2}))^{2} + a^{2}} < \sqrt{r'_{-}^{2} + a^{2}} < \sqrt{M^{2} + a^{2}}.$$
(51)

The same structure of ellipsoids and hyperboloids are in the $y_- - z_-$ plane; however, in this case, no of the ellipsoids corresponds to H_+ or H_- .

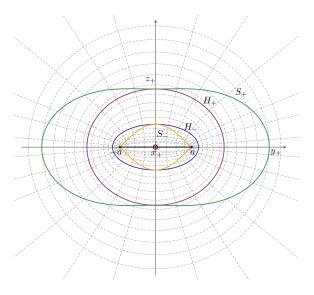


Figure 2: Horizons, ergosurfaces, and ellipsoids and hyperboloids foliations in the r'>0 region for the K case. H_{\pm} and S_{\pm} for a=.9 and M=1.

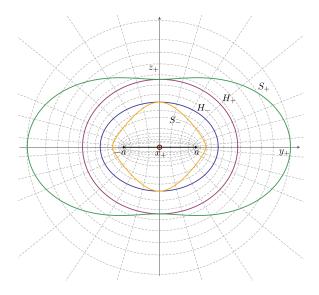


Figure 3: Idem as Fig. 2 for the KN case. H_{\pm} and S_{\pm} for $a=.9,\,M=1.3$ and $Q^2=.81$.

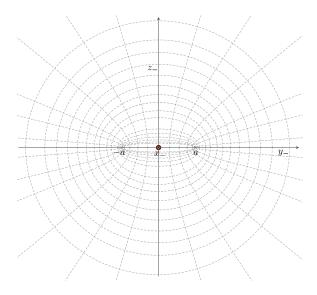


Figure 4: Foliations with ellipsoids and hyperboloids of the region r' < 0 (plane y_-/z_-) for a = .9.

The spatial topology consists of two copies of \mathbb{R}^3 glued by an open disk of radius a with its circular boundary being the singularity; each \mathbb{R}^3 is equivalent to $\{pt.\} \cup \mathbb{R}^+ \times S^2 \cong \{pt.\} \cup \mathbb{R} \times S^2$ where pt. = (0,0,0) is the common origin. Only one of these \mathbb{R}^3 's contains the horizons and ergospheres. Including the time axis, one ends with two copies of $\{pt.\} \cup \mathbb{R}^2 \times S^2$. The spatial topology can be formally "viewed" by reducing the spatial dimension through the elimination of the x_+ and x_- axis: two \mathbb{R}^2 's joined at an open segment with the singularity being its end points (Figure 5).

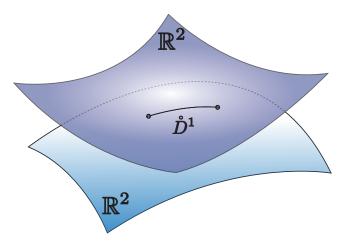


Figure 5: Pictorial view of the spatial topology of the K and/or KN spacetimes with one spatial dimension (x_{\pm} -axis) eliminated.

8. Penrose-Carter diagram

For completeness, we present the Penrose-Carter diagram corresponding to the K and KN cases, consisting in the infinite "vertical" repetition of the elementary cell illustrated in Figure 6. All elements: horizons and singularity are represented in the cell. The infinite tower is necessary to have a geodesically complete spacetime for geodesics that do not end at the singularity ring.

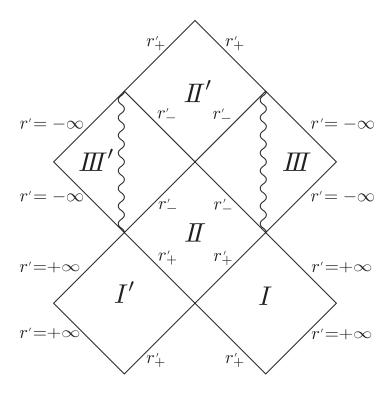


Figure 6: Elementary cell of the Penrose-Carter diagram of the KN spacetime.

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