# Soft computing and datamining

# Fuzzy modelling

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# Outline

- Fuzzy logic, fuzzy modelling
- Fuzzification
- Inference
- Defuzzification
- If - Then rules - Inference system

# Fuzzy theory 1

- A. Einstein: “If mathematics describes the reality, it is not precise. If it is precise, it does not describe the reality.”

- 1965 - Lofti Zadeh (4.2.1921- 6.9.2017)
  - father of fuzzy modelling

- fuzzy - vague, blurred, uncertain…
- fuzzy theory - it tries to cover the reality in its inaccuracy, imprecision, and uncertainty

# Fuzzy theory 2

- in the ordinary world, we are used to using words such as low, high
- the computer world usually needs precise boundaries from to

- and what about the problem of the type
  - middle speed 40 - 65 km / hour
  - high speed 66 km - 100 km / hour
    - 65km / hour middle speed, why it is not middle speed also 66 km/hour?

- Paradox from Ancient Greek
  - Have a small pile of stones. If we add one stone we obtain again the small pile of the stones. Thus, each pile of stones is small.

# Fuzzy sets 1

- classical sets (crisp): either belong (full membership) or not belong (no membership in the set)

- fuzzy admit also a partial membership (non crisp)
  - an element belongs to the set with a degree of truth (membership degree)
  - membership function - function which assigns the membership degree to the element

# Fuzzy sets 2 - example a) 1

- task
- How is the person high? A linguistic variable corresponding to a category - height.

| Person | Height [m] | Membership degree |
|---|---|---|
| Petr | 1,45 | 0 |
| Martin | 1,65 | 0,333 |
| Michal | 1,73 | 0,622 |
| Ondřej | 1,87 | 0,933 |
| Jakub | 2,05 | 1 |

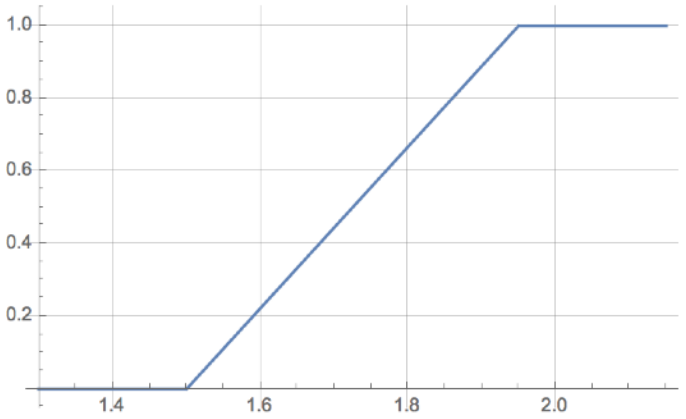
not high = 0, $x < 1.5$ m

is high = 1, $x > 1.95$ m

function $(x - 1.5)/0.45$ for $1.5 \text{ m} \leq x \leq 1.95$ m

- membership function - function, which assigns to an element a membership degree

and its graph



# Fuzzy sets 3 - example a) 2

- Michal is high - degree membership 0,462
- popularly it is possible to say that he is high from 46.2%
- or Michal belongs to the group of high persons from 46.2%
- but IT IS NOT ABOUT THE probability or likelihood of some event or condition
- however, it might be suitable to add also the dependency to the age - then we would have fuzzy relation (two-dimensional membership functions) - height and dependency to the age)

# Crisp sets (classical sets)

- classical sets (crisp set)
  - listing of elements
  - rule, which elements has to satisfy
  - characteristic function
- Strict description of the reality - two-element set - {0,1}
  - sharp sets - sharp decision about the membership

$$M = \{x_1, x_2, x_3, x_4\} \qquad m_M(x) = \begin{cases} 1 & if \quad x \in M \\ 0 & if \quad x \notin M \end{cases}$$

# Fuzzy sets 4

- if we state the degree of the element which belong to the set with then we label these sets as vague - fuzzy sets
- it is extended domain of the definition from $\{0,1\}$ to $\langle 0,1\rangle$
  - an element carries the dominant attributes of the set A but also it can have attributes of the set B in minor extent

# Fuzzy sets 5

- fuzzy set is defined as an assignment which assigns to each element u from universe U the value of the membership function into the fuzzy set F directly

$$\mu_F(u), \text{ tj. } F = \{(\mu_F(u))/u \in U\}$$

- discrete universe with n elements, fuzzy set is then listing of pairs - membership degree / element

$$F = \{\mu_F(u_1)/u_1, \mu_F(u_2)/u_2, \ldots, \mu_F(u_n)/u_n\}.$$

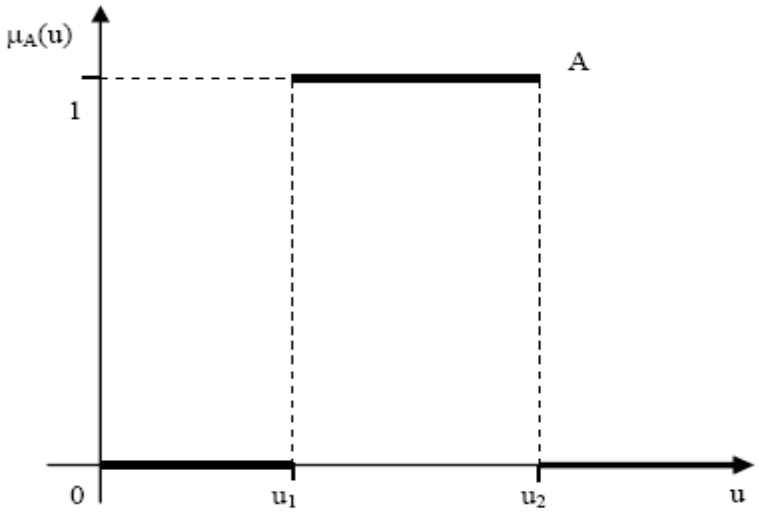
$$F = \mu_F(u_1)/u_1 + \mu_F(u_2)/u_2 + \ldots + \mu_F(u_n)/u_n.$$

$$F = \sum_{u \in U} \mu_F(u)/u\,.$$

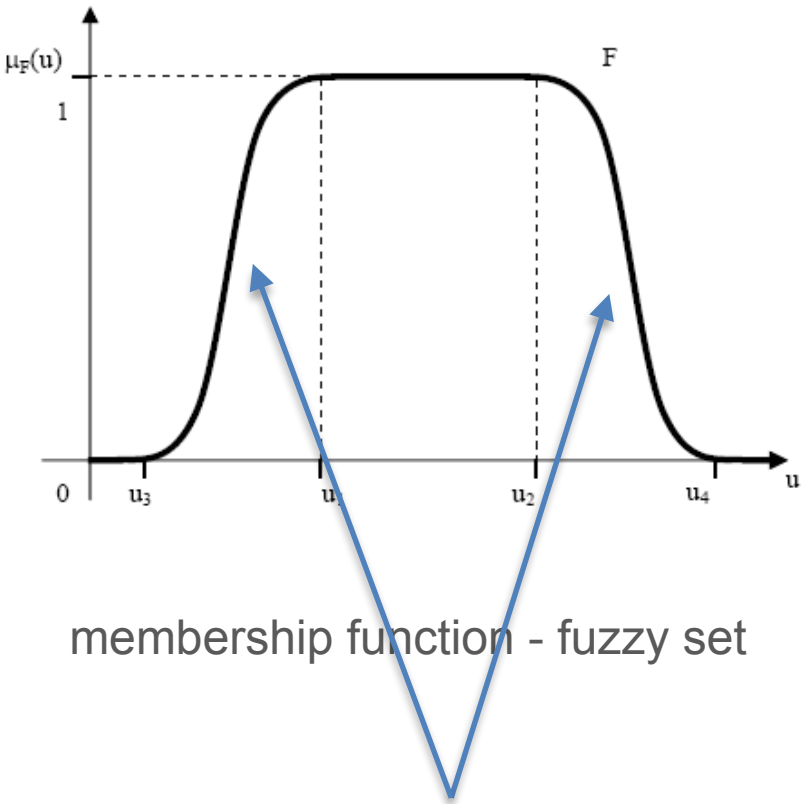
- for continuous or countless the integral can be used formally

$$\int_U \mu_F(u)/u$$

# Fuzzy sets 6




membership function - classical set




membership function - fuzzy set

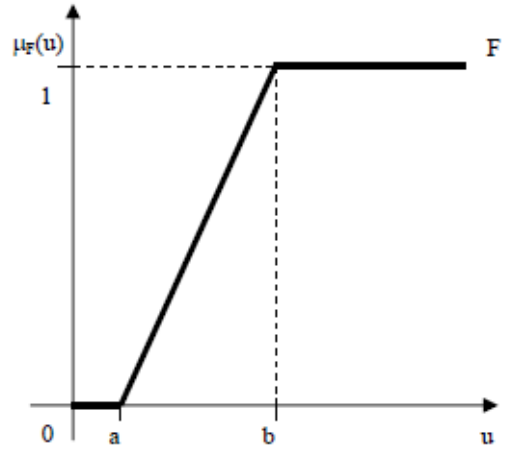
partial assignment

# Fuzzy sets 7

- practical use of fuzzy sets require analytical formulation of the membership function
- in practise - the most often - we use the approximation via angled lines
- also continuous function might be used
- membership function are parametrized via 4 points of break, i.e. values a, b, c, d

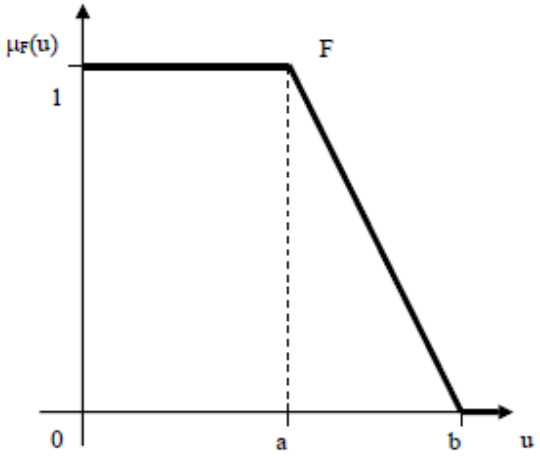
# Fuzzy sets 8

Γ - gamma function




$$\Gamma(u,a,b)=\begin{cases}0 & u<a\\ (u-a)/(b-a) & a\le u\le b\\ 1 & u>b\end{cases}$$

L - L function

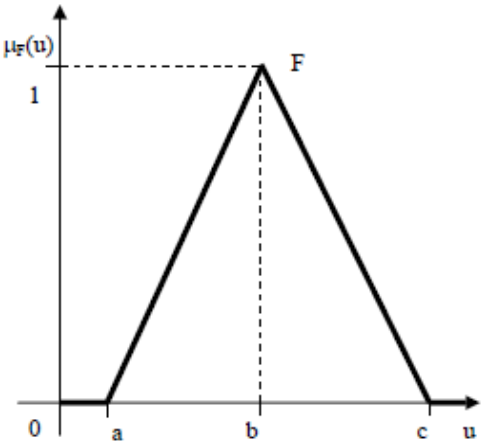



$$L(u,a,b)=\begin{cases}1 & u<a\\ (b-u)/(b-a) & a\le u\le b\\ 0 & u>b\end{cases}$$

# Fuzzy sets 9

triangle function

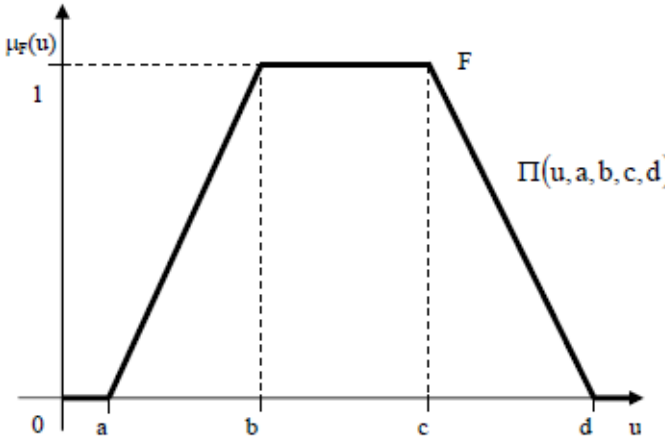
Λ - lambda function




$$\Lambda(u,a,b,c)=\begin{cases}0 & u<a\\ (u-a)/(b-a) & a\le u\le b\\ (c-u)/(c-b) & b\le u\le c\\ 0 & u>c\end{cases}$$

trapezoid function

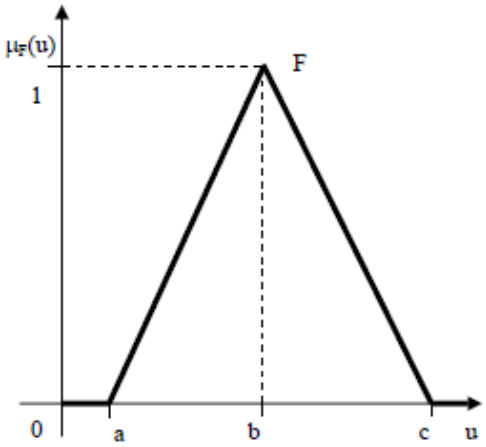
Π - pi function




$$\Pi(u,a,b,c,d)=\begin{cases}0 & u<a\\ (u-a)/(b-a) & a\le u\le b\\ 1 & b\le u\le c\\ (d-u)/(c-d) & c\le u\le d\\ 0 & u>d\end{cases}$$

# Fuzzy sets 10

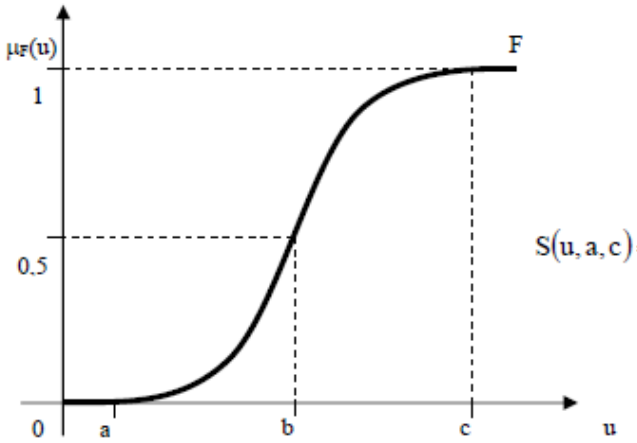
- fuzzy number - approximately “b”
- singleton - fuzzy set with one element - sharp value



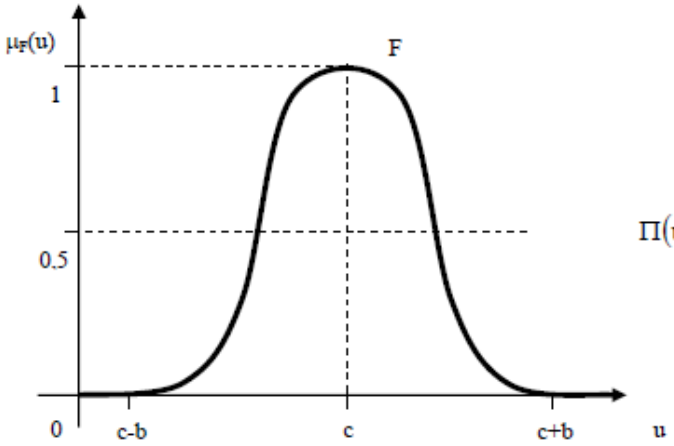

$$\Lambda(u,a,b,c)=\begin{cases}0 & u<a\\ (u-a)/(b-a) & a\le u\le b\\ (c-u)/(c-b) & b\le u\le c\\ 0 & u>c\end{cases}$$

# Fuzzy sets 11

there exist also
continuous versions
of membership
functions

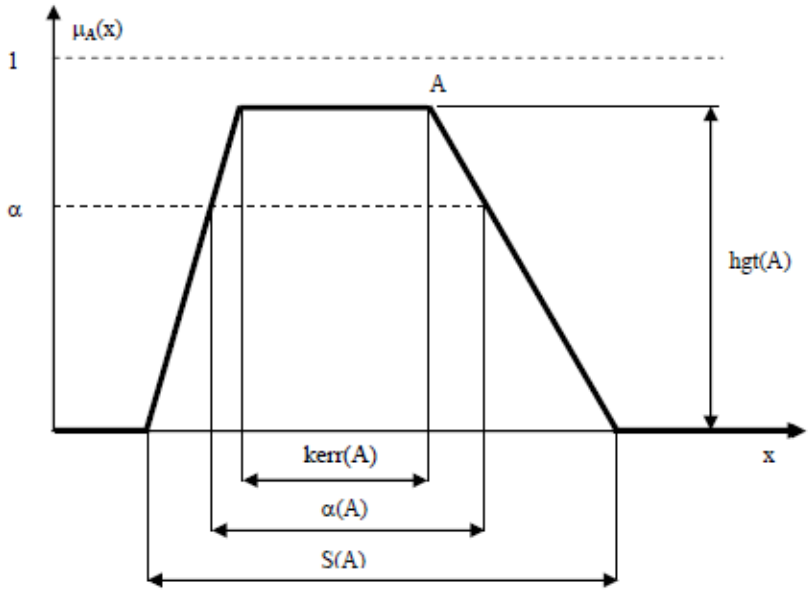



$$S(u,a,c)=\begin{cases}0 & u\le a\\ 2(u-a)^2/(c-a)^2 & a<u\le b\\ 1-2(u-c)^2/(c-a)^2 & b<u\le c\\ 1 & u>c\end{cases}$$




$$\Pi(u,b,c)=\begin{cases}S(u,c-b,c) & u\le c\\ 1-S(u,c,c+b) & u\ge b\end{cases}$$

# Parameters of fuzzy sets 1




normal fuzzy set
height = 1

support of fuzzy set A: $S(A) = \{x / \mu_A(x) > 0\}$

kernel of fuzzy set A: $\text{kerr}(A) = \{x \in X / \mu_A(x) = 1\}$

height of fuzzy set A: $\text{hgt}(A) = \sup_{x \in X}(\mu_A(x))$

$\alpha$- cut of fuzzy set A: $\alpha(A) = \{x \in X / \mu_A(x) \geq \alpha\}$

# Parameters of fuzzy sets 2

- Height of fuzzy set is the lowest high border of the fuzzy set
- Support of fuzzy set is the sharp set for which elements it is valid that their values of membership function are different from zero
- Kernel of fuzzy set is the sharp set for which elements it is valid that their values of membership function are equal to one
- α-cut is the sharp set for which elements it is valid that their values of membership function are greater or equal to value α
- α-level is the sharp set for which elements it is valid that their values of membership function are equal to value α

# Linguistic variables 1

- from the reason of empiric knowledge of the experts,  expertů, crew, staff
- linguistic variable - values are the expression of some language
- values of linguistic variables can be interpreted as fuzzy sets
- set of linguistic variables = set of terms

# Linguistic variables 2

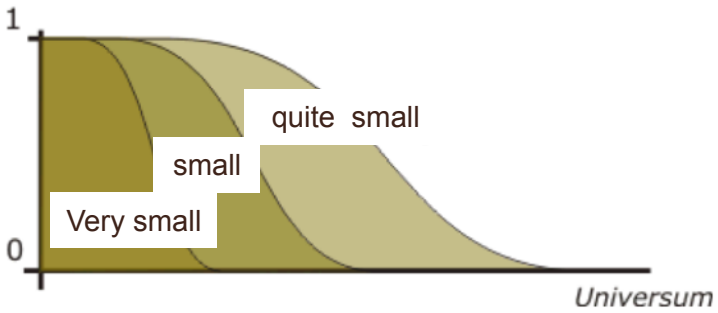
- not - the same as complement (see further)
- very
- quite

„not“: $\mu_{notA}(x) = 1 - \mu_A(x)$

„very“: $\mu_{velmi\ A}(x) = [\mu_A(x)]^2$

„quite“: $\mu_{docela\ A}(x) = [\mu_A(x)]^{\frac{1}{2}}$

Influence of the linguistic operators



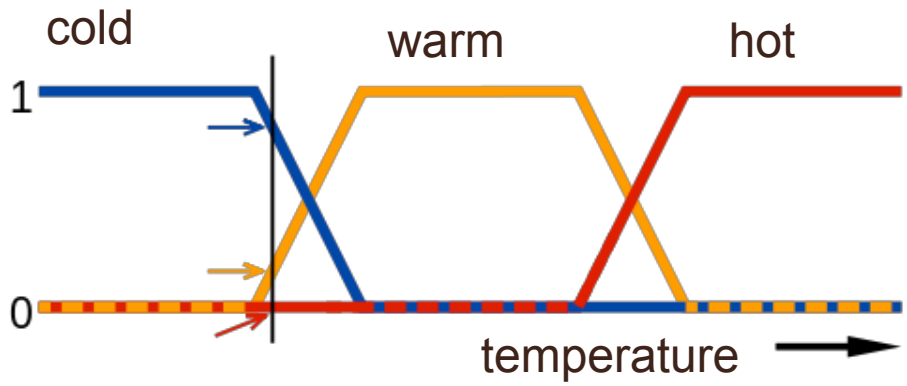
# Terms - fuzzy values, abbreviations

- for fuzzy modelling, different approaches are considered
- on of those, the table of terms is used and its typical values

| Meaning | Czech | English |
|---|---|---|
| Value large negative | ZV | NL |
| Value middle negative | ZS | NM |
| Value small negative | ZM | NS |
| Value negative close to zero | ZN | NZ |
| Value zero | NU | Z |
| Value positive close to zero | KN | PZ |
| Value small positive | KM | PS |
| Value middle positive | KS | PM |
| Value large positive | KV | PL |

# Example b) 1

- temperature of the bath (quantity, universum)
- it is measured in degrees of Celsius
- it is possible to create several sets (blue, orange, red)
- via composition, it is possible to create the final action for the decision, etc.
- for quantitative expression, there are used informally common words - ice, cold, lukewarm, warm, hot
  - terms - non crisp sets, for which it is necessary to introduce the membership degree



# Processing of the task with fuzzy logic 1

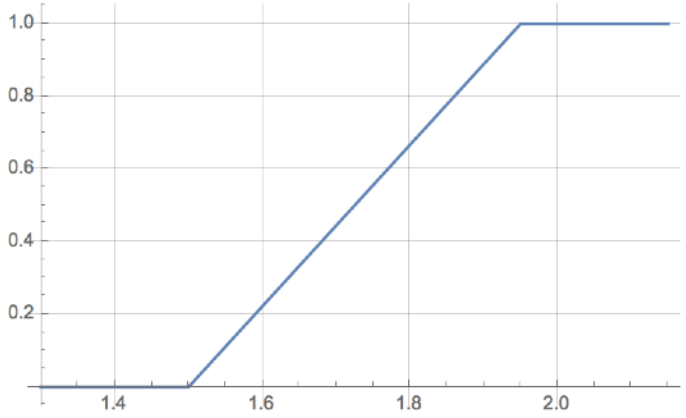
- To find out if fuzzy system is the right choice for the problem. If the knowledge about the system behavior is described in the approximate form or via heuristic rules - then **fuzzy logic is suitable**. Fuzzy logic can be useful in understanding and simplification of processing if the system behavior requires a complex mathematical model.
- To identify inputs and outputs and their extent. The extent of sensors corresponds to extent of input variables typically and the extent of control action gives the scale of output variables.
- To define the membership functions for each input and output parameter. Number of required membership functions is for consideration and depends on the system behavior.
- To create rules. It is necessary to determine how many rules are necessary.
- To verify if rules for inputs gives the output in the given extent and further verify if this output is correct and if it correct also according to rules for given set of inputs.

# Processing of the task with fuzzy logic 2

- fuzzyfication ⇨ inference ⇨ defuzzyfication

# Fuzzyfication

- fuzzyfication is the process of assignment of measured values of input variables into fuzzy sets via membership function
- thus designing of the function ala the example a) with the height of persons




not high = 0, $x < 1.5$ m

is high = 1, $x > 1.95$ m

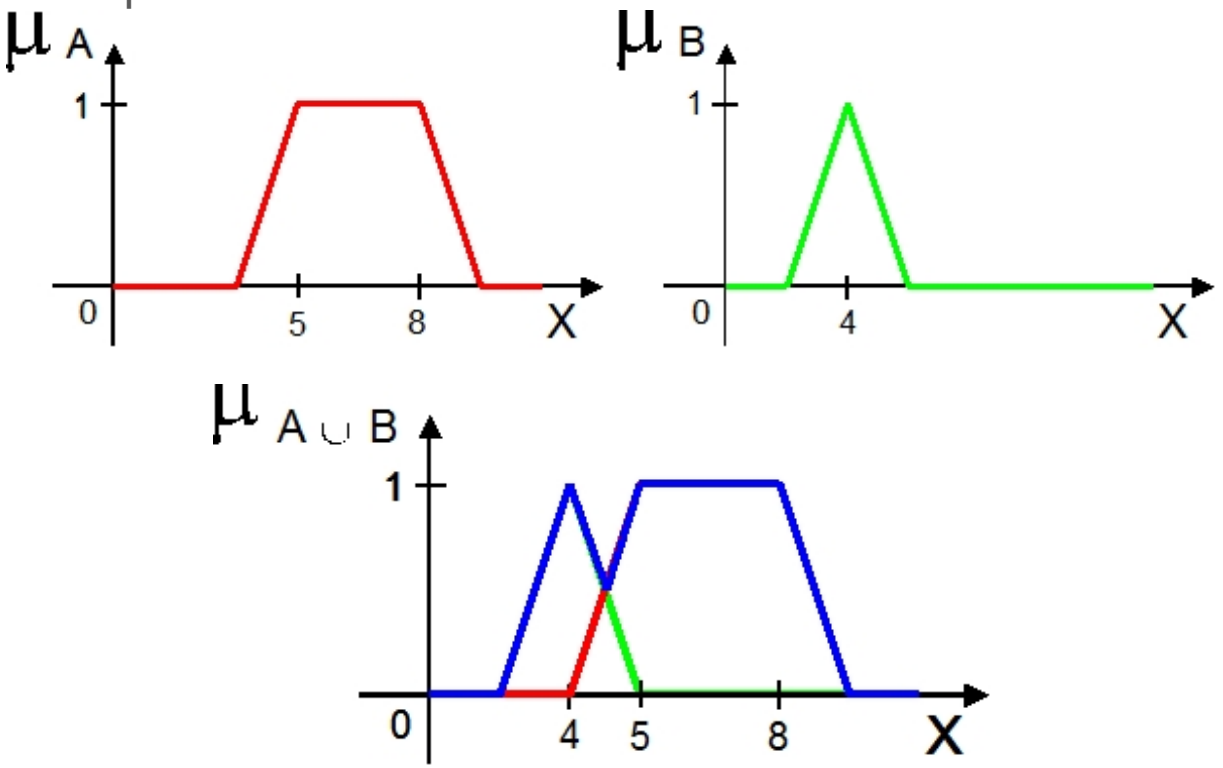
function $(x - 1.5)/0.45$ for $1.5 \text{ m} \leq x \leq 1.95$ m

# Fuzzy Inference system - FIS 1

- logical operation
  - AND (intersection, logical multiplication)
  - OR (union, logical total)
  - NOT (complement)
- fuzzy rules IF - THEN - fuzzy implication

# Union of fuzzy sets - OR

- Union or logical sum works with the membership function and on a given interval of the universe takes over the value of a higher degree of membership

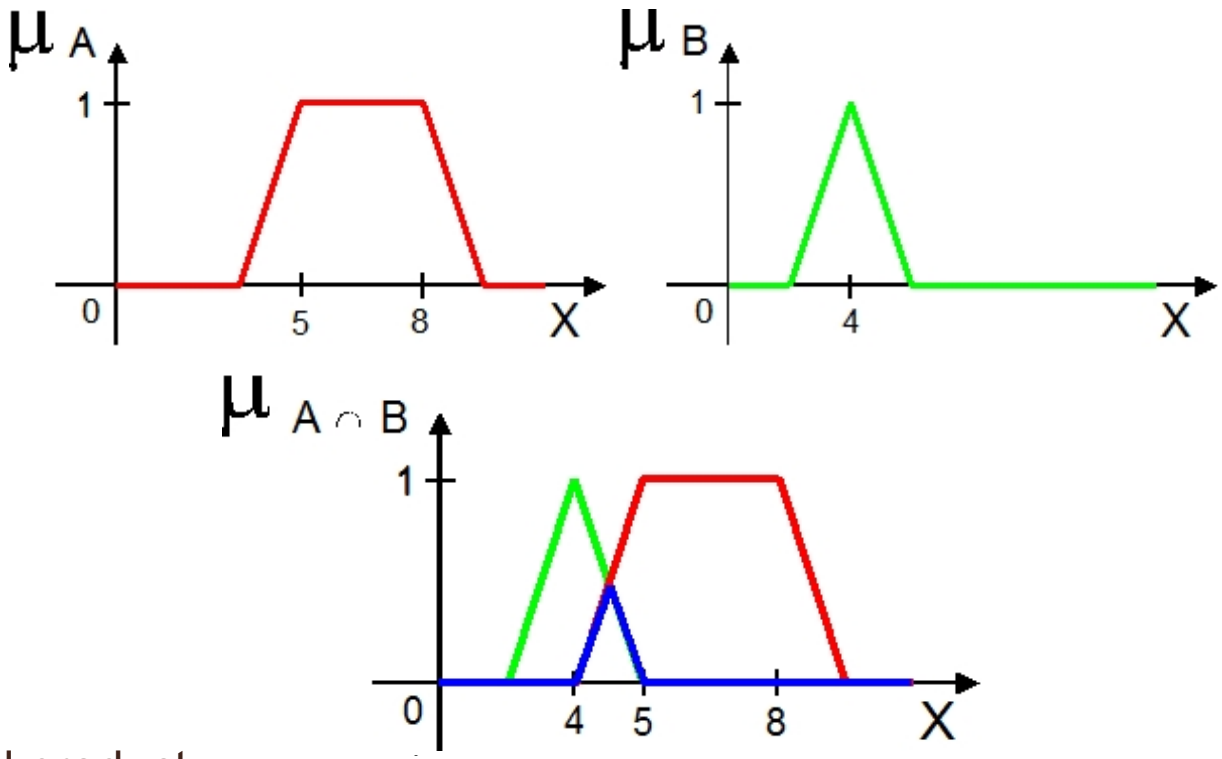



Logical sum

union of A and B $A \cup B : \mu_{A \cup B} = max\{\mu_A(x); \mu_B(x)\}\ pro\ \forall x \in X$

# Intersection of fuzzy set - AND

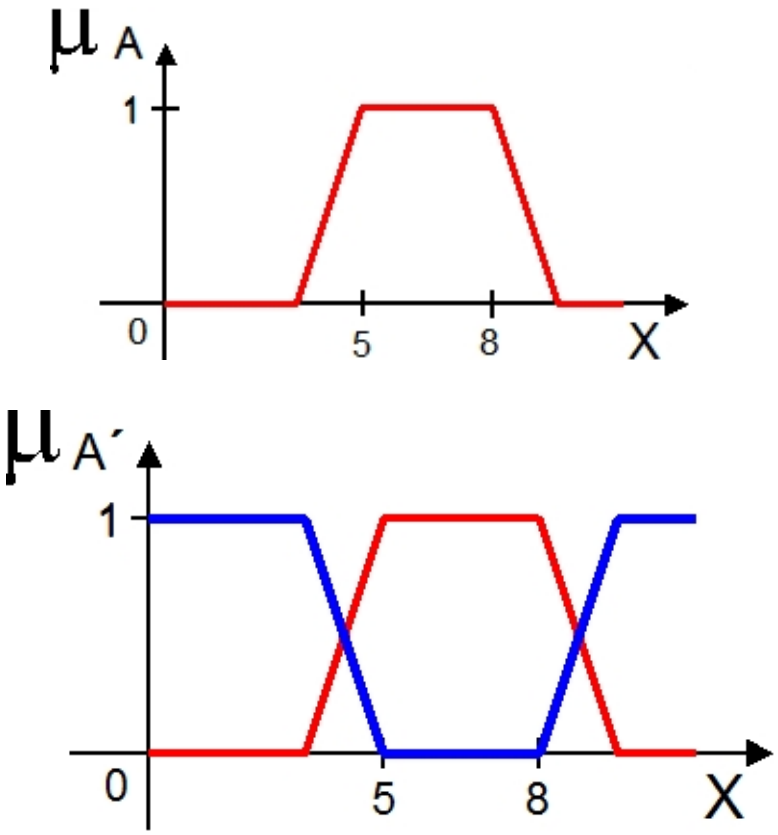
Intersection or logical multiplication works with the membership function and on a given interval of the universe takes over the value of a lower degree of membership




Logical product

Intersection of A and B $A \cap B : \mu_{A \cap B} = min\{\mu_A(x); \mu_B(x)\}\ pro\ \forall\ x \in X$

# Complement of fuzzy set - NOT




Complement of A

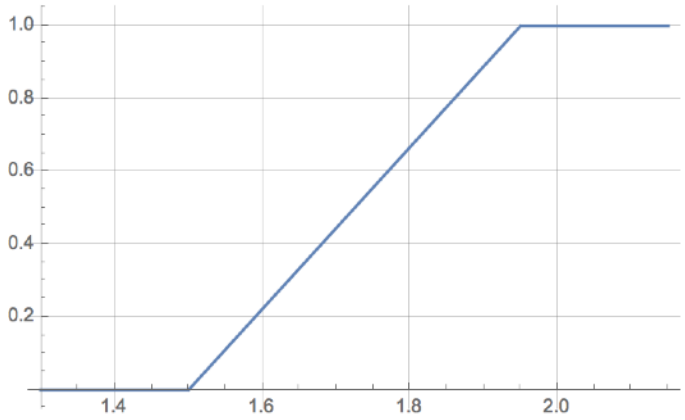
$$A' : \mu_{A'} = 1 - \mu_A(x) \; pro \; \forall x \in X$$

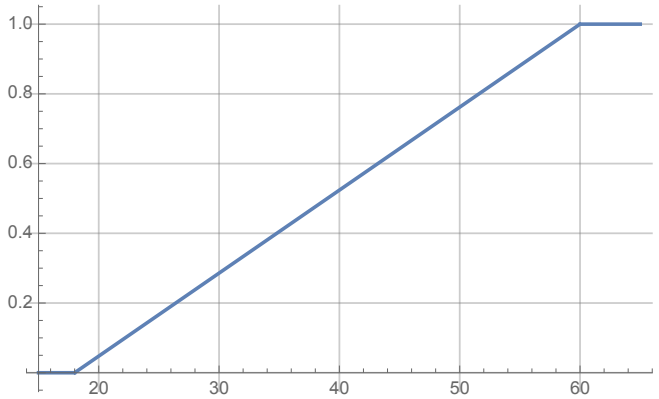
# Example c) 1

example for following logic fuzzy operation and function
"**height**" and "**old**"

- height = 0, height(x) < 1.5 m
- is high = 1, height(x) > 1.95 m
- function (height(x) - 1.5)/0.45 for
  1.5 m ≤ height(x) ≤ 1.95 m

- old = 0, age(x) < 18 years
- is old = 1, age(x) > 60 years
- function (age(x) - 18) / 42 for
  18 years ≤ age(x) ≤ 60 years



# Example c) 2

logic operation with function “**height**” and “**old**”

total table of height (from example a) and age and fuzzy operations

a = x is **high AND** x is **old**

b = x is **high OR** x is **old**

c = **NOT** (x is **high**)

| Person | Height | Age | x is **high** | x is **old** | a | b | c |
|---|---|---|---|---|---|---|---|
| Petr | 1,45 | 14 | 0 | 0 | 0 | 0 | 1 |
| Martin | 1,65 | 23 | 0,333 | 0,119 | 0,119 | 0,333 | 0,667 |
| Michal | 1,73 | 54 | 0,622 | 0,857 | 0,622 | 0,857 | 0,378 |
| Ondřej | 1,87 | 37 | 0,933 | 0,452 | 0,452 | 0,933 | 0,067 |
| Jakub | 2,05 | 65 | 1 | 1 | 1 | 1 | 0 |

# Fuzzy rules 1

- If …. Then rules (Jestliže …. Pak) (suitable for instance into expert systems)
- If (fuzzy declaration)…. Then (fuzzy declaration)
  - condition = production rule
  - first part - usually with logic connectors - antecedent
  - second part - consequent

IF x is A

THEN y is B

- where x, y are linguistic variables and A, B linguistic values determined by fuzzy sets

# Fuzzy rules 2 - example d) 1

IF costs are greater than 1000 (Boolean logic)
THEN production process is ineffective

IF costs are less than 100
THEN production process is effective

---

IF costs are big (fuzzy logic)
THEN production process is ineffective

IF costs are small
THEN production process is effective

- costs (Bolean e.g. 0-5000) - fuzzy sets - big, small, middle
- production process (e.g. 0,1 or 0-1) - **effective**, **ineffective**, acceptable, suitable

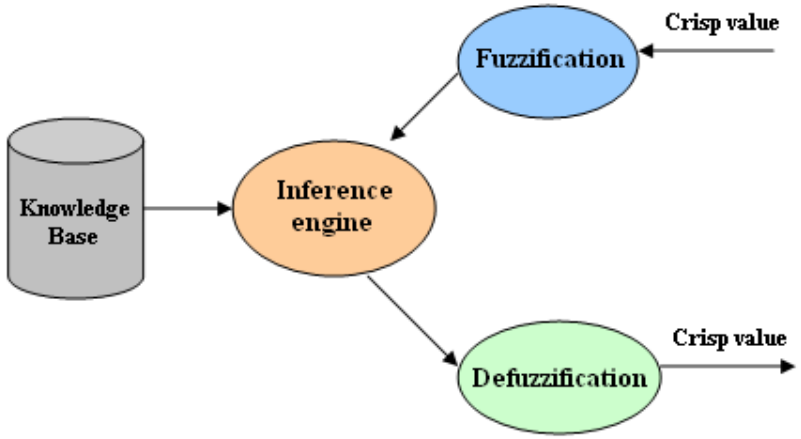
# Fuzzy rules 3 - example e)

IF duration of the project is long
AND number of employees working on the project is big
AND funding of the project is insufficient
THEN project is risk

IF service is excellent
OR food is delicious
THEN tip is generous

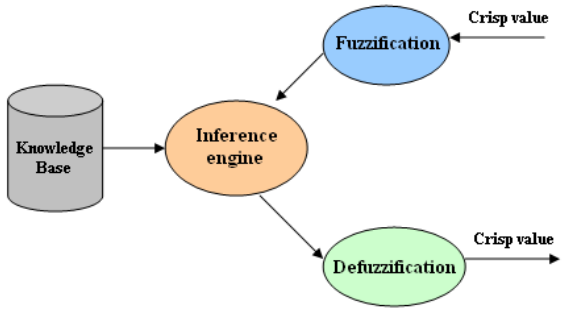
# Fuzzy Inference System - FIS 2

- basically, it belongs among the systems which imitate the human reasoning
- usage at every day problems
- it is used in decision processes when more rules exist and it is necessary to "create" one fuzzy value on the output



# Fuzzy Inference System - FIS 2

- fuzzyfication - from crisp values to fuzzy sets via membership function




- Knowledge base - base of IF THEN rules determined by experts
- Inference engine - executing of fuzzy inference (consequence) on inputs and IF THEN rules
- defuzzyfication - from non crisp values to make crisp after the inference

# Fuzzy Inference System - FIS 3

- methods of fuzzy reasoning
  - direct methods
    - **Mamdani method**
    - **Larsen method**
    - Takagi and Sugeno method
    - Simplified method
  - indirect methods

# Reasoning - Applying of If-Then

- Function of the membership of the consequent is defined as

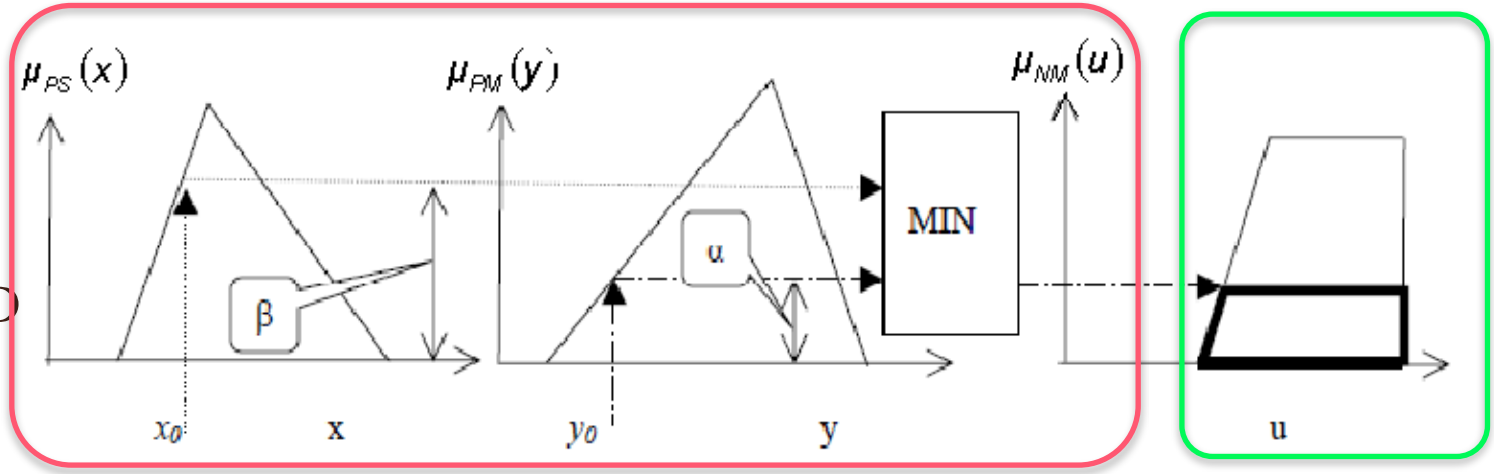
$$\mu_{MI}(x_1, x_2) = \min\{\mu_A(x_1), \mu_B(x_2)\}$$

- minimization expresses the reality that the consequent can be maximally of the membership degree as antecedent (condition)

# Mamdani method

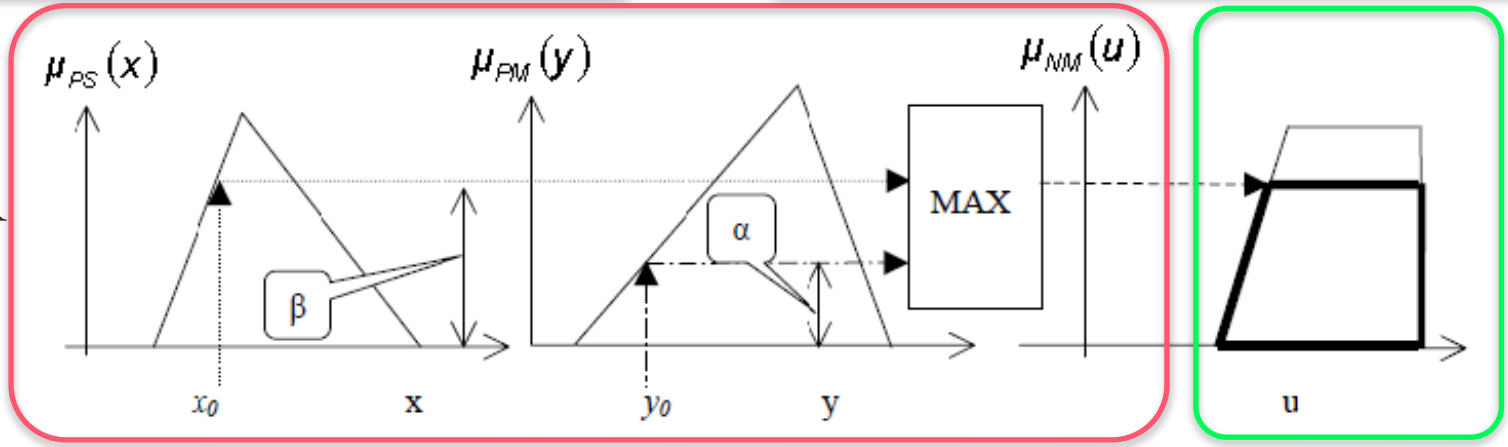
- Mamdani method “cuts” the consequent (after the antecedent processing)

AND



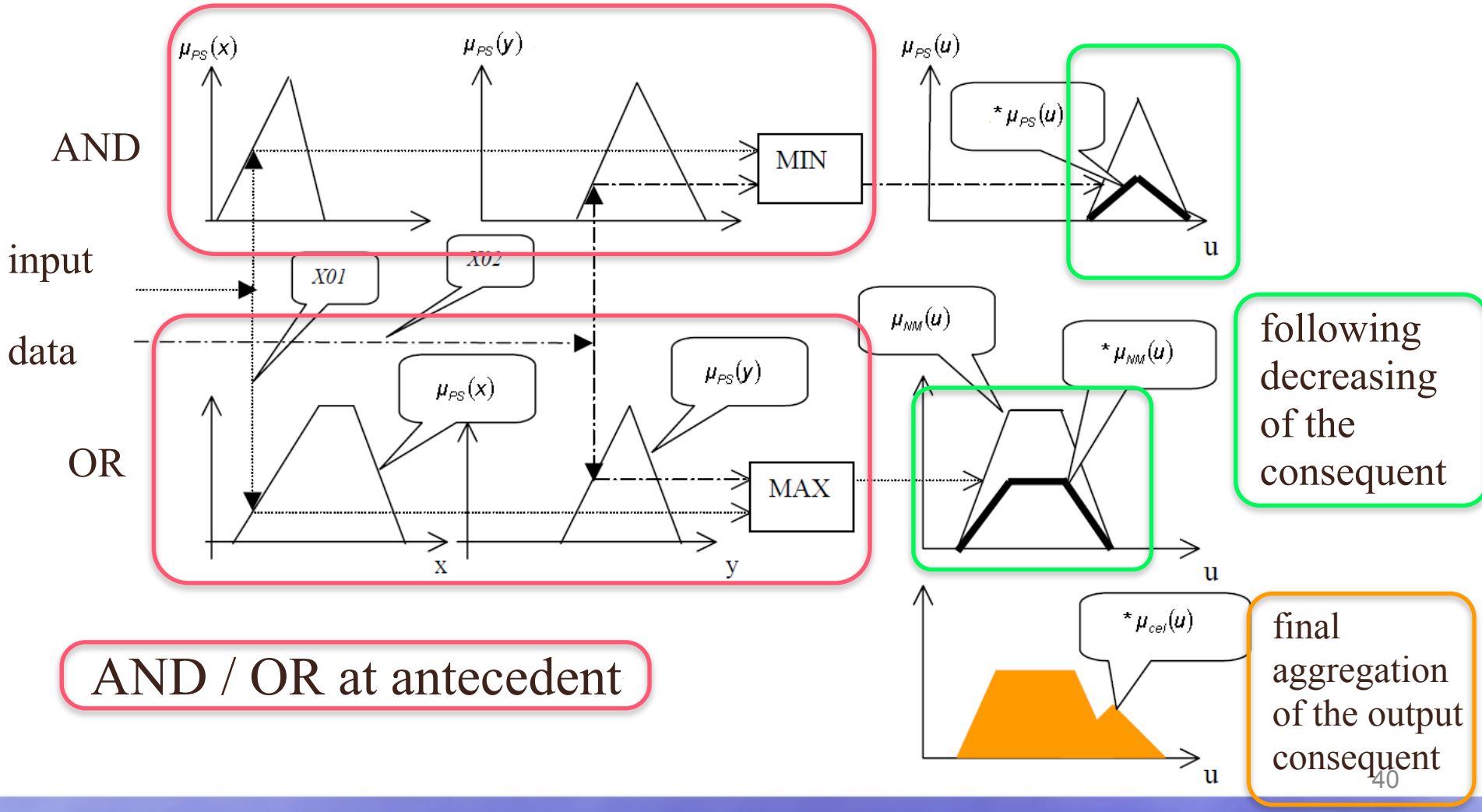

AND / OR at antecedent and following cut of consequent

OR



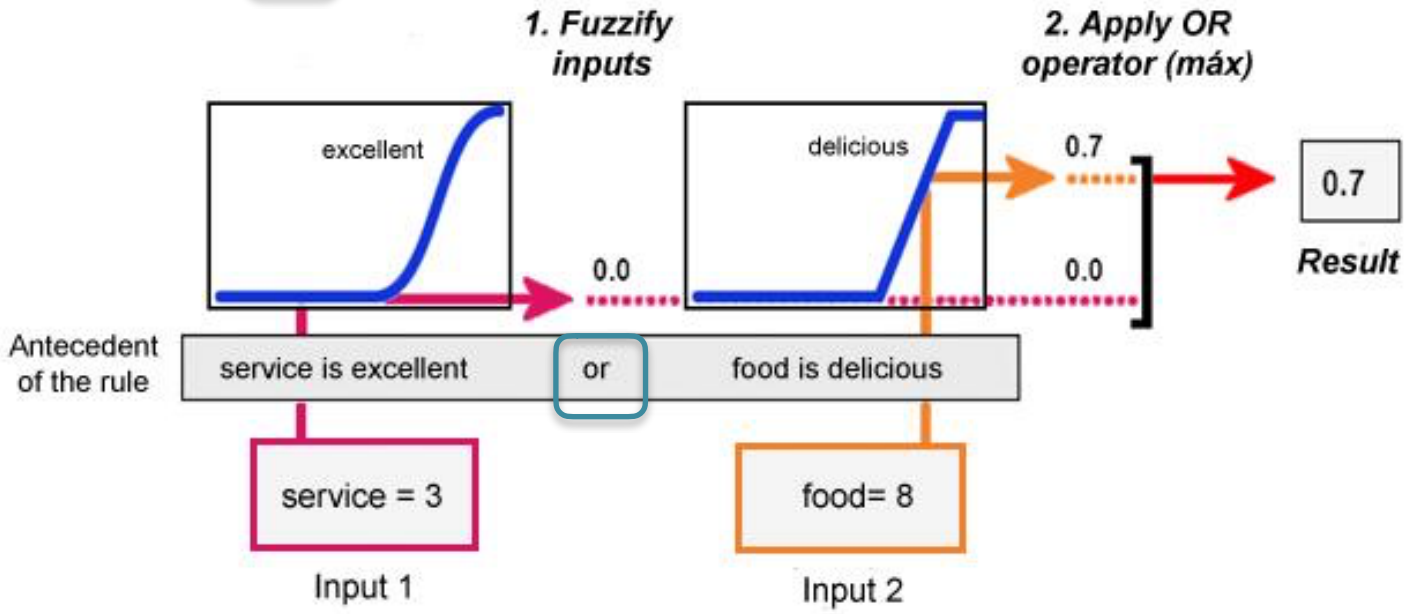
# Larsen method - multiplication (decrease)

- Larsen method “edits the shape” of the consequent (after the antecedent processing)



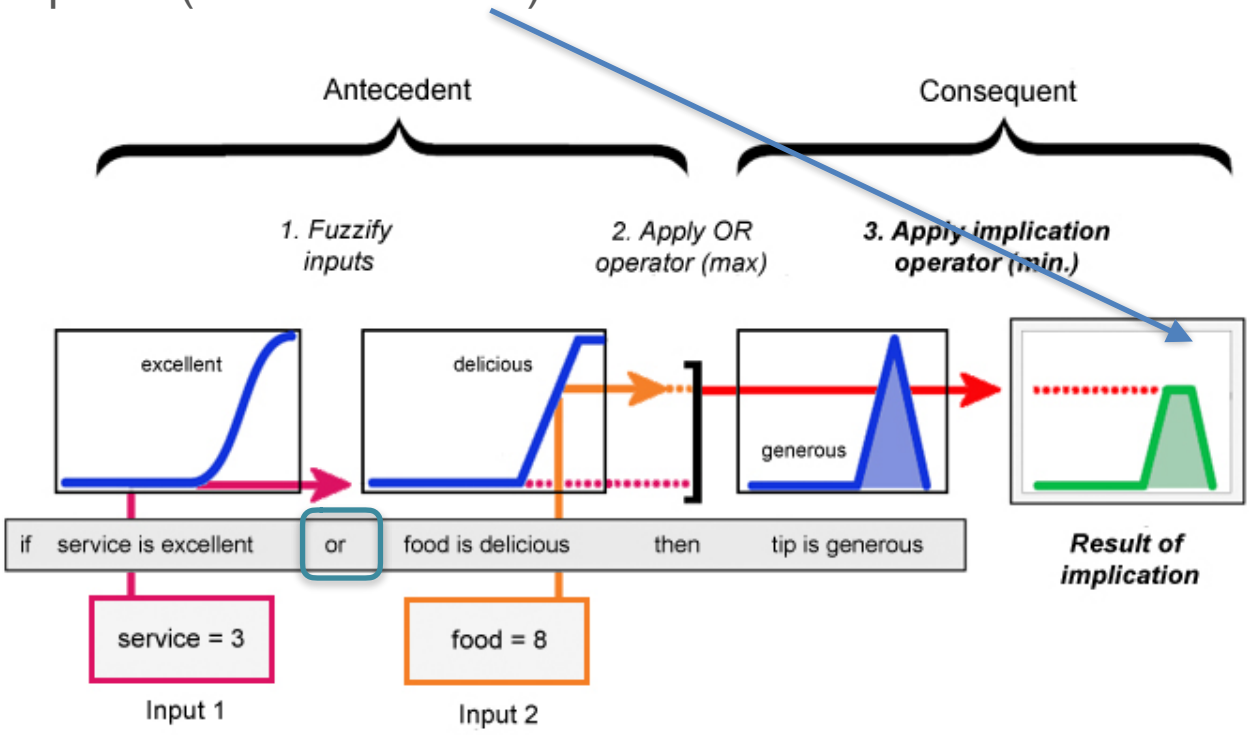
# Mamdani method - example OR f) 1

- An example of use in the evaluation of food or service in a restaurant on the final decision of the tip amount
- Objective scale 1-10, the customer selects a value for the service and for the quality of the food
- The expert set the expert rules in the knowledge base, which are gradually processed
- 1. step - fuzzification of inputs and possible interaction of inputs in the antecedent OR



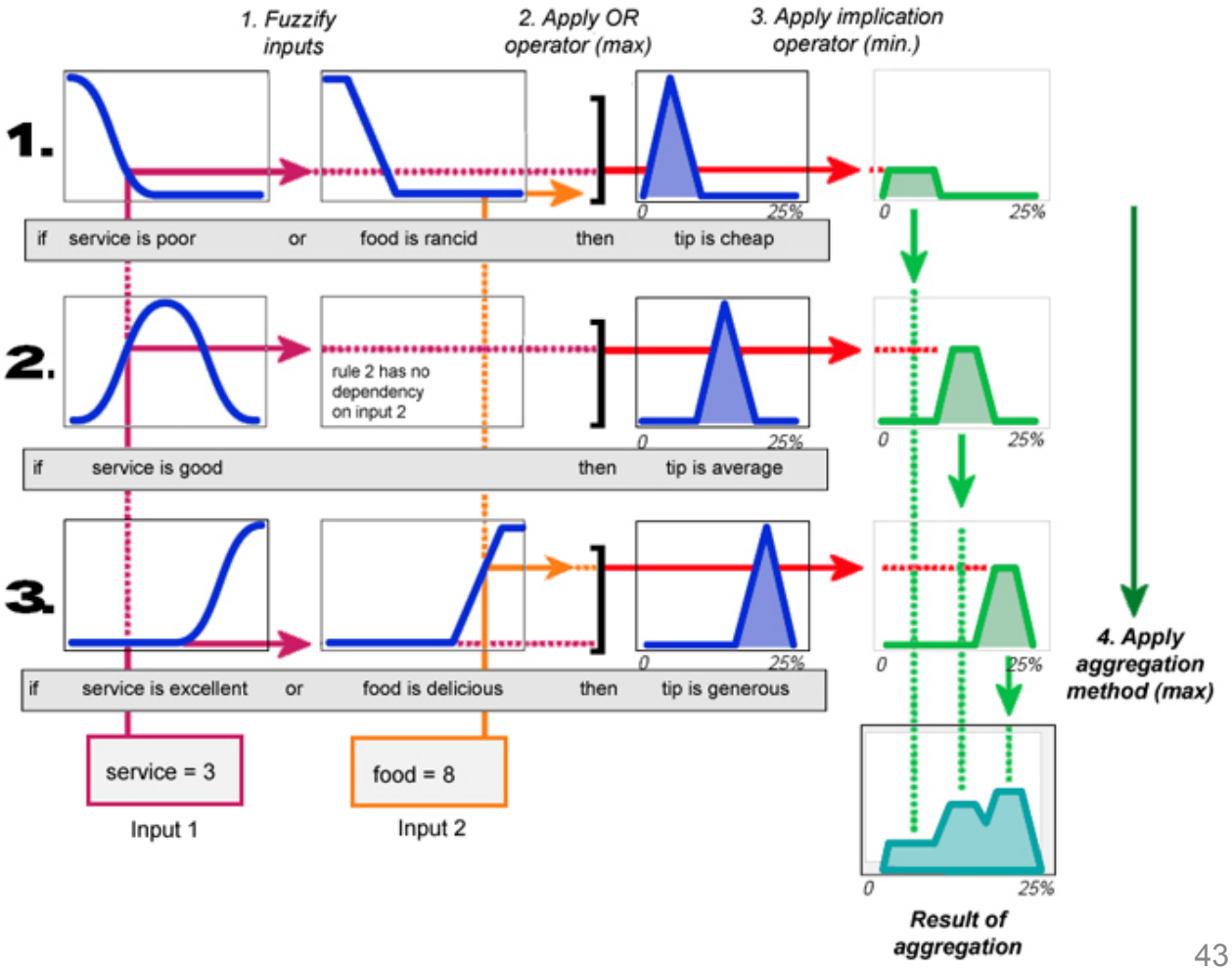
# Mamdani metoda - example OR f) 2

- After processing the antecendent, it can be applied to the function in the consequent
- the rule must be observed - the maximum degree of membership from the antecendent will be reflected in the maximum degree of membership in the consequent (Mamdani “cuts”)



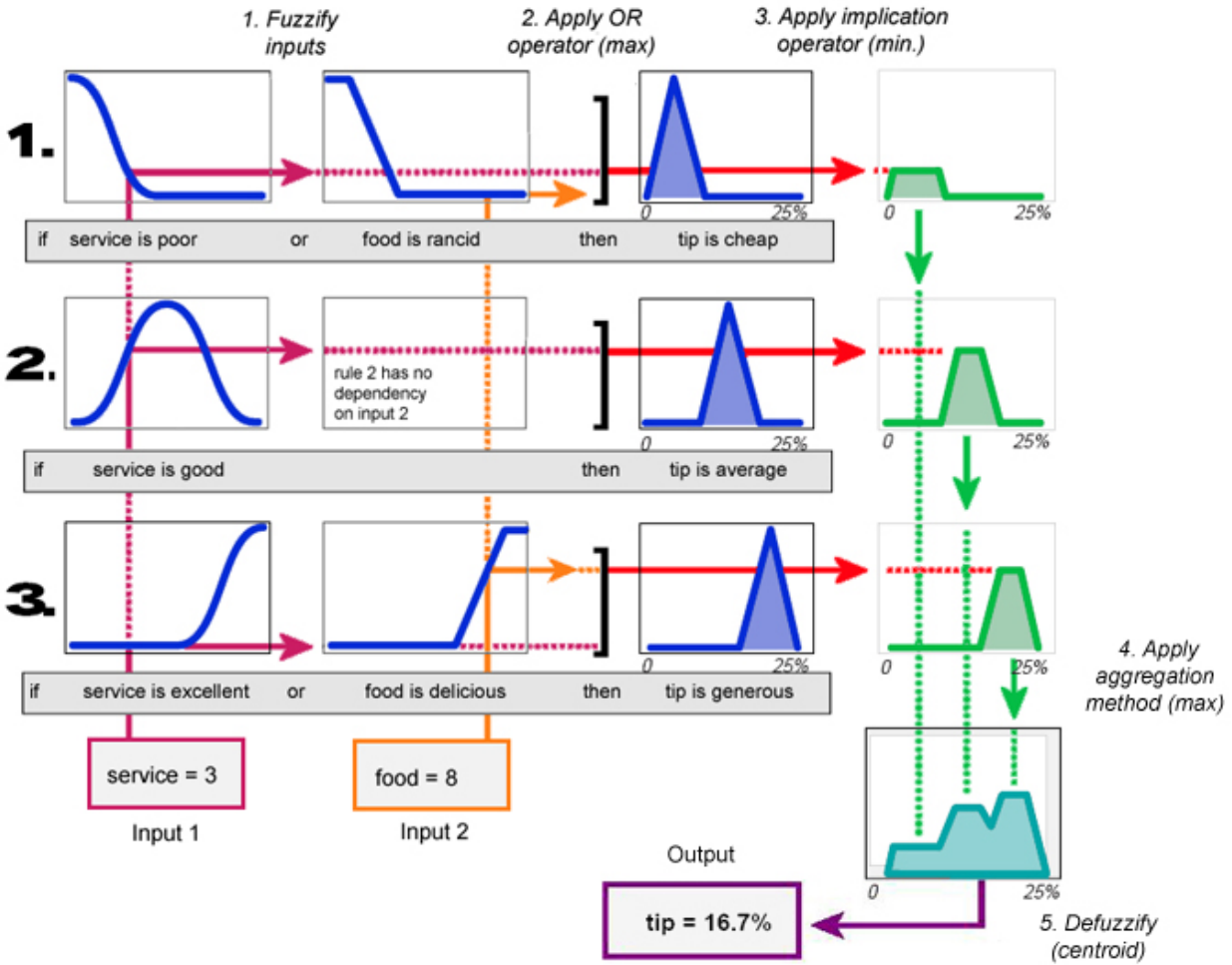
# Mamdani method - example OR f) 3

- 3 rules gradually performed, the result of individual consequents eventually aggregated



# Mamdani method - example OR f) 4

- and from the resulting shape of the aggregated area, the sharp result is determined by the defuzzification



# Defuzzyfication 1

- for practise it is necessary to assign a crisp value of action quantity in the permissible extent to output linguistic values
- defuzzyfication - a process of approximation of non crisp terms by a crisp value
  - methods coming out from empirical verification to heuristic approaches
- the names of methods can differ in literature
- output values differ based on the method, usage according to the application

- **method of the mean of maximum** - methods searching for the acceptable solutions
- **method of heights (centre of singletons)**
- **method of centre** - computation as the best compromise
- method bisector - position of the axis of the area under the curve which divides the given area on the two areas of the same content

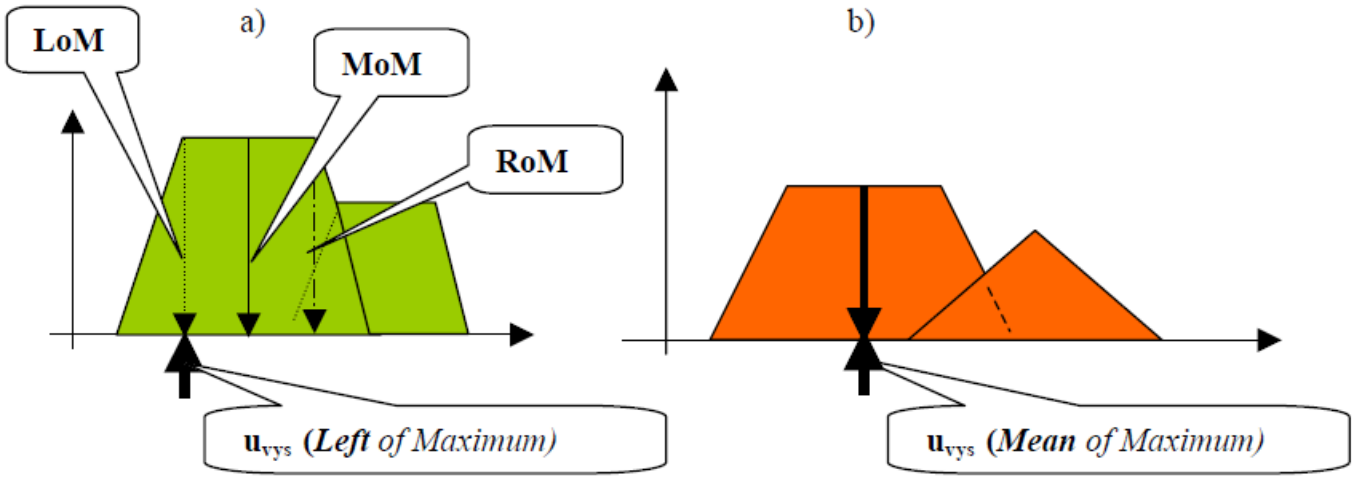
# Method of mean of maximum 1

- method of mean of maximum
  - median of the universum among elements with maximal degree of membership
- acceptable solution, which fits to conditions in decision rules
- from all terms - the term with the highest value of the membership function
    - maximal value gives the output crisp quantity based on the position in the selected method

- Left of Maximum (LoM), respectively First of Maxima (FoM) - result is the most left positioned value from the highest values of the membership function
- **Mean of Maximum** (MoM) - in the middle positioned value, the most often used approach
- Right of Maximum (RoM), respectively Last of Maxima (LoM) - the most right positioned value

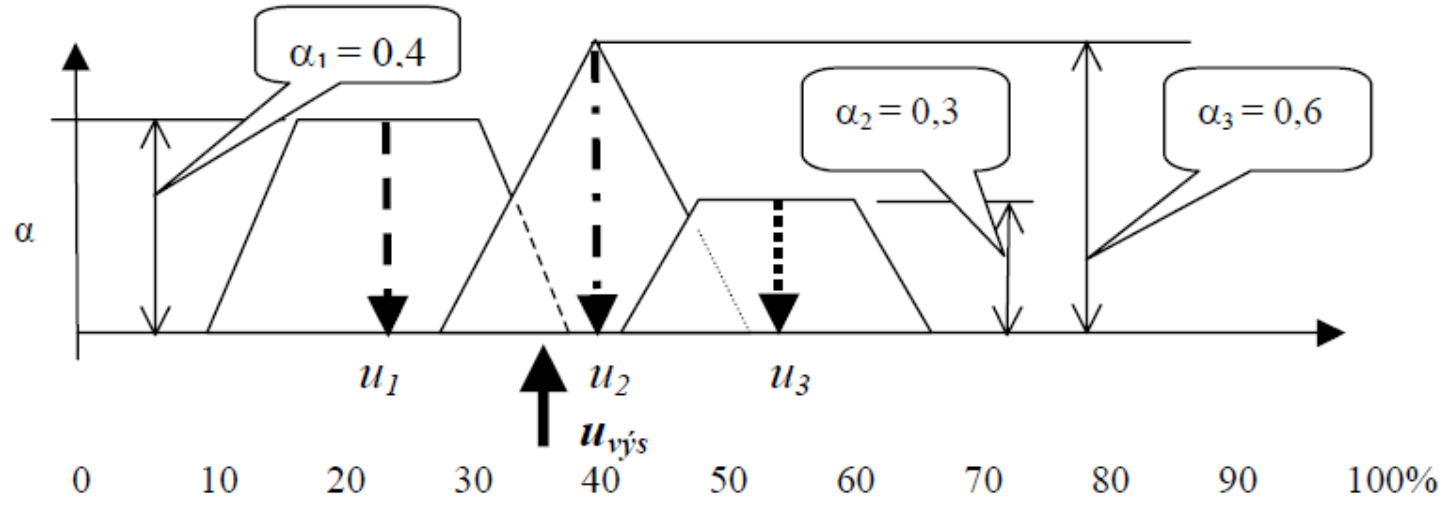
# Mean of maximum 3

- advantage - high speed, because only maxims are searched
- disadvantage - output action quantity can be changed discontinuously



# Method of heights 1

- advantage - high speed because it avoids the complex computation of the union via consideration of only heights of particular fuzzy sets (if flat maximum then the middle of this maximum).
- To each value, a membership function corresponds and the result of the defuzzified value is calculated as a middle value of these heights.



# Metoda heights 2 (sometimes also centre of singletons)

- Method of heights, also known as centre of singletons (Center of Maximum)
  - replaces the functional dependency of each output term by its typical value and the crisp output quantity $u_{out}$ is determined as its centre

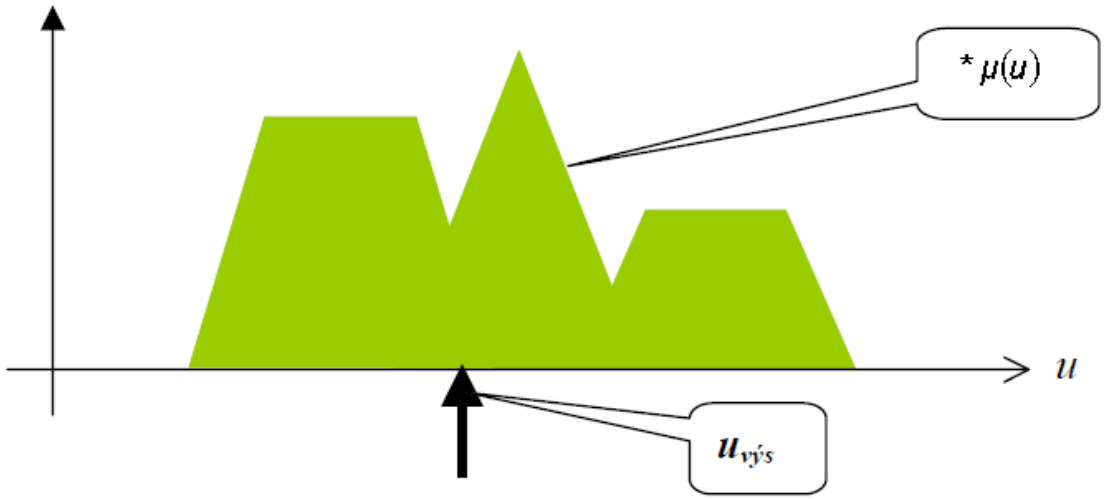
$$u_{out} = \frac{\sum_{k=1}^{r} a_k \cdot u_k}{\sum_{k=1}^{r} a_k}$$

  - where $u_{out}$ is the resulting value of the output quantity
  - $u_k$ is the coordinates of output quantity of k-th term
  - $a_k$ is the value of membership of k-th term

# Method of centre of gravity

- Method of Center of Gravity
  - the centre of gravity of the membership function of the output quantity (e.g. aggregated after the inference rules application).
  - The resulting value of the action quantity is determined as the coordinates of the centre of the gravity of the area created as the union of particular areas which were determined by edging of output term function with non zero values of membership functions

$$u_{vys} = \frac{\int * \mu(u) \cdot u \mathrm{d}u}{\int * \mu(u) \cdot \mathrm{d}u}$$



# Applications

- different models for decision - making systems (fuzzy VIKOR..)
- different modelling of systems
- control
- searching of suitable partner / mate
- the most suitable mortage, searching of a suitable real property
- temperature watching - air-condition
- automatic transmission in the car
- dishwasher, toaster, vacuum cleaner, camera (hand shaking)
- lift control - decreasing of waiting time
- process control
- pattern recognition - hand writing characters (fuzzy neural nets…)
- estimation of the selling price of the flat
- …..