LA2 Komplexe Zahlen

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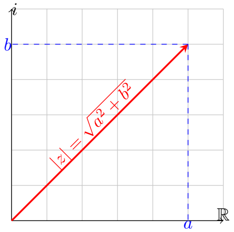
LaTeX

Allgemein

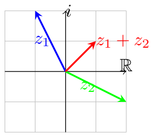


$z = a + bi$

a: Realteil
b: Imaginärteil
i: Imaginäre Einheit

$i = \sqrt{-1} \to i^2 = -1$

Addition / Subtraktion:



$z_1 = a_1 + b_1 i \quad z_2 = a_2 + b_2 i$

$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$
$z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$

Multiplikation / Division:

$z_1 = a_1 + b_1 i \qquad z_2 = a_2 + b_2 i$

$z_1 \cdot z_2 = (a_1 \cdot a_2 - b_1 \cdot b_2) + (a_1 \cdot b_2 + a_2 \cdot b_1)i$

Division $\to \frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = \frac{1}{|z_2|^2} \cdot z_1 \cdot \overline{z_2}$

Neuer Vektor ist um Summe der Winkel der beiden Vektoren rotiert

Betrag:
$|z| = \sqrt{a^2 + b^2}$

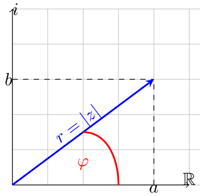
Inverse:
$z = a + bi \qquad z^{-1} = \frac{1}{|z_1|^2} \cdot (a - bi)$

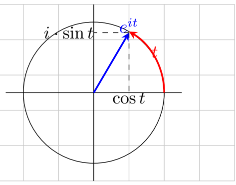
Konjugiert #:
$z = a + bi \qquad \overline{z} = a - bi$

$z \cdot \overline{z} = |z|^2 \qquad \to w = \frac{z}{v} \cdot \frac{\overline{v}}{\overline{v}} = \frac{z \cdot \overline{v}}{|v|^2}$

Spiegelt Vektor an x-Achse

Polar Darstellung





$e^{it} = \cos(t) + i \cdot \sin(t) \qquad \to t$ Winkel (Bogenmass)

$z = |z| \cdot e^{it} = |z| \cdot (\cos(t) + i \cdot \sin(t))$

Addition / Subtraktion:
$z_1 = r_1 \cdot (\cos(\varphi_1) + i \cdot \sin(\varphi_1)) \; z_2 = r_2 \cdot (\cos(\varphi_2) + i \cdot \sin(\varphi_2))$

$$
\begin{aligned}
z_1 \pm z_2 &= r_1(\cos(\varphi_1) + i\,\sin(\varphi_1)) \pm r_2(\cos(\varphi_2) + i\,\sin(\varphi_2)) \\
&= r_1 \cos(\varphi_1) + i\, r_1 \sin(\varphi_1) \pm r_2 \cos(\varphi_2) \pm i\, r_2 \sin(\varphi_2) \\
&= r_1 \cos(\varphi_1) \pm r_2 \cos(\varphi_2) + i\,(r_1 \sin(\varphi_1) \pm r_2 \sin(\varphi_2))
\end{aligned}
$$

Multiplikation / Division:
$z_1 = r_1 \cdot (\cos(\varphi_1) + i \cdot \sin(\varphi_1)) \; z_2 = r_2 \cdot (\cos(\varphi_2) + i \cdot \sin(\varphi_2))$

$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot (\cos(\varphi_1 + \varphi_2) + i \cdot \sin(\varphi_1 + \varphi_2))$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot (\cos(\varphi_1 - \varphi_2) + i \cdot \sin(\varphi_1 - \varphi_2))$

Eulerische Darstellung

$$z = r(\cos(\varphi) + i \cdot \sin(\varphi)) = r \cdot e^{i\,\varphi}$$

Multiplikation/Division:

$z_1 \cdot z_2 = r_1 r_2 \cdot e^{i\,(\varphi_1 + \varphi_2)} \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot e^{i\,(\varphi_1 - \varphi_2)}$

$\cos(\varphi) = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \qquad \sin(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$

Umrechnung Kartesisch, Polar, Euler

Kartesisch $\to$ Polar:
Umrechnung (ohne $z = 0 \to z = 0/\{\}/\emptyset$)

$r = |z| = \sqrt{a^2+b^2}$

Polar: $z = r(\cos(\varphi) + i\sin(\varphi)) = \underbrace{r\cos(\varphi)}_{a} + \underbrace{i\, r\sin(\varphi)}_{b}$

Kartesisch: $z = a + bi$

$\cos(\varphi) = \frac{a}{\sqrt{a^2+b^2}} \qquad \to \qquad \varphi = \arccos(\frac{a}{\sqrt{a^2+b^2}}) \in [0, \pi]$

$\sin(\varphi) = \frac{b}{\sqrt{a^2+b^2}} \qquad \to \qquad \varphi = \arcsin(\frac{b}{\sqrt{a^2+b^2}}) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\tan(\varphi) = \frac{b}{a} \qquad \to \qquad \varphi = \arctan(\frac{b}{a}) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Polar $\to$ Euler:

$z = r \cdot (\cos(\varphi) + i \cdot \sin(\varphi)) \qquad z = r \cdot e^{i\,\varphi}$

Vorzeichen von i beachten!

Kartesisch $\to$ Euler:

Gleich wie Kartesisch $\to$ Polar, dann Polar $\to$ Euler

Korrektur Winkel (Umrechnung in Polar/Euler):

$z \in Q_1: \quad \varphi = 0° \quad + |(\arctan / \arccos / \arcsin)|$ $\to$ Betrag
$z \in Q_2: \quad \varphi = 180° - |(\arctan / \arccos / \arcsin)|$ $\to$ Betrag
$z \in Q_3: \quad \varphi = 180° + |(\arctan / \arccos / \arcsin)|$ $\to$ Betrag
$z \in Q_4: \quad \varphi = 360° - |(\arctan / \arccos / \arcsin)|$ $\to$ Betrag

Bogenmass: $rad = \deg \cdot \frac{\pi}{180°}$ Pi Anteil: $\frac{bogen}{\pi} = \frac{1}{x} \cdot \pi$

Vektor & Kartesische Darstellung

$\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

Vektor
z_1: Realteil
z_2: Imaginärteil

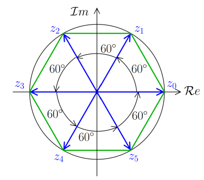
$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 + z_2 \cdot i$

Kartesisch
z_1: Realteil
z_2: Imaginärteil

Satz von Moivre

Eulerische Form: $z^n = r^n \cdot e^{i\,n\varphi}$
Polar Form: $z^n = r^n \cdot (\cos(n\varphi) + i \cdot \sin(n\varphi))$
Kartesische Form: $z^n = (a + i \cdot b)^n$

Wurzeln



n-Lösungen: $z^{\frac{1}{n}} = \sqrt[n]{x}$

mit Polar/Euler Form rechnen!

n = 2: $az^2 + bz + c = 0$

$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$D = b^2 - 4ac$

$z^n = a \cdot e^{i\,\varphi} \qquad a_0 = a \quad \alpha = \varphi \qquad \to \qquad r = \sqrt[n]{a_0}$

Lösungen: $\varphi_k = \frac{\alpha + k \cdot 360°}{n} \qquad k = 0, 1, 2, \ldots, n-1$

$k = 0: \quad z_0 = r \cdot e^{i\,\varphi_k} = r(\cos(\varphi_k) + i \cdot \sin(\varphi_k))$
$k = 1: \quad z_1 = r \cdot e^{i\,\varphi_k} = r(\cos(\varphi_k) + i \cdot \sin(\varphi_k))$
$\vdots$

Sinus/Cosinus Cheatsheet

φ_{Deg}	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
φ_{Rad}	0π	$\frac{1}{12}\pi$	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{5}{12}\pi$	$\frac{1}{2}\pi$	$\frac{7}{12}\pi$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	$\frac{11}{12}\pi$	π
sin	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	0
cos	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	0	$-\frac{\sqrt{6}-\sqrt{2}}{4}$	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{6}+\sqrt{2}}{4}$	-1
tan	0	$2-\sqrt{3}$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\sqrt{3}+2$	–	$-(\sqrt{3}+2)$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	$\sqrt{3}-2$	0

φ_{Deg}	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°
φ_{Rad}	$\frac{13}{12}\pi$	$\frac{7}{6}\pi$	$\frac{5}{4}\pi$	$\frac{4}{3}\pi$	$\frac{17}{12}\pi$	$\frac{3}{2}\pi$	$\frac{19}{12}\pi$	$\frac{5}{3}\pi$	$\frac{7}{4}\pi$	$\frac{11}{6}\pi$	$\frac{23}{12}\pi$	2π
sin	$-\frac{\sqrt{6}-\sqrt{2}}{4}$	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{6}+\sqrt{2}}{4}$	-1	$-\frac{\sqrt{6}+\sqrt{2}}{4}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{6}-\sqrt{2}}{4}$	0
cos	$-\frac{\sqrt{6}+\sqrt{2}}{4}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{6}-\sqrt{2}}{4}$	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	1
tan	$-(\sqrt{3}-2)$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\sqrt{3}+2$	–	$-(\sqrt{3}+2)$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	$\sqrt{3}-2$	0