Expectation and Variance / Examples and Computation of Mean Time to Failure

ECE 313
Probability with Engineering Applications
Lecture 28 - December 6, 1999
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Mean Time to Failure

- Let X denote the lifetime of a component so that its reliability *R(t) = P(X>t)* and *R'(t) = -f(t).*
- The **expected life** or the **mean time to failure** (MTTF) of the component is given by:

$$E[X] = \int_0^\infty t f(t)dt = -\int_0^\infty tR'(t)dt$$

- Integrating by parts:

$$E[X] = -tR(t)\Big|_0^\infty + \int_0^\infty R9t)dt$$

- Since *R(t)* approaches zero faster than *t* approaches ∞:

$$E[X] = \int_0^\infty R(t)dt$$

Mean Time to Failure (cont.)

- This latter expression for MTTF is in more common use in reliability theory. Generally:

$$\begin{aligned} E[X^k] &= \int_0^\infty t^k f(t)dt \\ &= -\int_0^\infty t^k R'(t)dt \\ &= -t^k R(t)\Big|_0^\infty + \int_0^\infty kt^{k-1}R(t)dt \end{aligned}$$

- Thus:

$$E[X^k] = \int_0^\infty kt^{k-1}R(t)dt$$

Mean Time to Failure (cont.)

- In particular:

$$Var[X] = \int_0^\infty 2tR(t)dt - \left[\int_0^\infty R(t)dt\right]^2$$

- If the component lifetime is exponentially distributed, then $R(t)=e^{-\lambda t}$ and:

$$\begin{aligned} E[X] &= \int_0^\infty e^{-\lambda t}dt = \frac{1}{\lambda}, \\ Var[X] &= \int_0^\infty 2te^{-\lambda t}dt - \frac{1}{\lambda^2} \\ &= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \end{aligned}$$

as derived earlier.

Series System

- Assume that the lifetime of the *i*th component is exponentially distributed with parameter λ_i. System reliability is given by:

$$R(t) = \prod_{i=1}^{n} R_i(t) = \prod_{i=1}^{n} e^{-\lambda_i t} = \exp\left[-(\sum_{i=1}^{n} \lambda_i)t\right]$$

- Thus, the lifetime of the system is also exponentially distributed with parameter $\lambda = \sum_{i=1}^{n} \lambda_i$.

- The system series MTTF is:

$$\frac{1}{\sum_{i=1}^{n} \lambda_i}$$

- The MTTF of a series system is much smaller the the MTTF of its components.

Series System (cont.)

- If X_i denotes the lifetime of component *i*, and *X* denotes the series system lifetime, then we can show that:

$$0 \le E[X] \le \min_i\{R_{X_i}(t)\}$$

- To prove inequality:

$$R_X(t) = \prod_{i=1}^{n} R_{X_i}(t) \le \min_i\{\int_0^\infty R_{X_i}(t)\} \quad \text{since } 0 \le R_{X_i}(t) \le 1$$

- Then:

$$E[X] = \int_0^\infty R_X(t)dt \le \min_i\{\int_0^\infty R_{X_i}(t)dt\}$$
$$= \min_i\{E[X_i]\}$$

Parallel System

- Consider a parallel system of n independent components, with X_i denoting the lifetime of component *i* and *X* denoting the lifetime of the system:

- Then $X = \max\{X_1, X_2, ..., X_n\}$, and

$$R_X(t) = 1 - \prod_{i=1}^{n}[1 - R_{X_1}(t)] \geq 1 - [1 - R_{X_i}(t) \quad \text{for all i}$$

- Which implies that the reliability of a parallel redundant system is larger than that of any of its components.

Parallel System (cont.)

- Therefore: $E[X] = \int_0^\infty R_X(t)dt \geq \max_i\{\int_0^\infty R_{X_i}(t)dt\}$

$$= \max_i\{E[X_i]\}$$

- Assume that Xi is exponentially distributed with parameter λ (all components have the same parameter). Then:

$$R_X(t) = 1 - (1 - e^{-\lambda t})^n$$

and

$$E[X] = \int_0^\infty [1 - (1 - e^{-\lambda t})^n]dt$$

Parallel System (cont.)

- Let $u = 1 - e^{-\lambda t}$, then $dt = \frac{1}{\lambda}(1-u)du.$
- Thus: $E[X] = \frac{1}{\lambda}\int_0^1 \frac{1-u^n}{1-u}du$
- Since the integrand above is the sum of a finite geometric series:

$$E[X] = \frac{1}{\lambda}\int_0^1 \left(\sum_{i=1}^{n} u^{i-1}\right)du$$
$$= \frac{1}{\lambda}\sum_{i=0}^{n-1}\int_0^1 u^{i-1}du$$

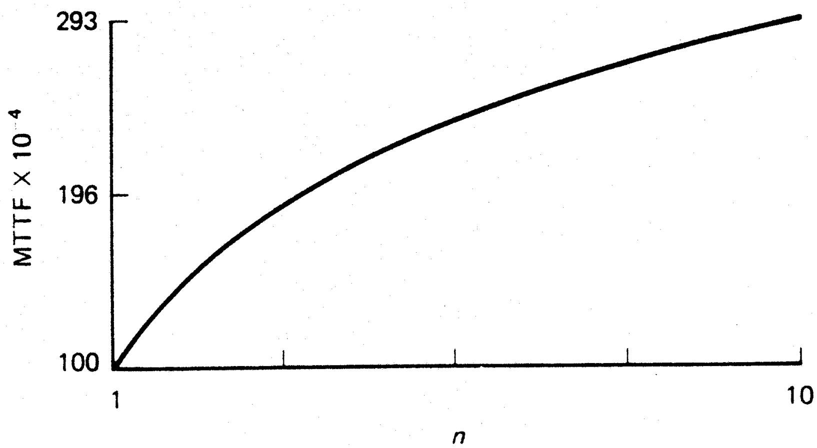
Parallel System (cont.)

- Note that: $\int_0^1 u^{i-1}du = \frac{u^i}{i}\Big|_0^1 = \frac{1}{i}$
- Thus, with the usual exponential assumptions, the MTTF of a parallel redundant system is given by:

$$E[X] = \frac{1}{\lambda}\sum_{i=1}^{n}\frac{1}{i} = \frac{H_n}{\lambda} \approx \frac{1n(n)}{\lambda}$$

- The next figure shows the expected life of a parallel system as a function of *n*. It should be noted that beyond *n* = 2 or 3, the gain in expected life is not very significant. Notes that the rate of increase in the MTTF is 1/(*n*λ).

Parallel System (cont.)



- The variation in the expected life with the degree of (parallel) redundancy (simplex failure rate $\lambda=10^{-6}$)

Parallel System (cont.)

- Alternatively the formula for E[X] can be derived by noting that X is exponentially distributed with parameters $n\lambda, (n-1)\lambda, \cdots, \lambda$.
- In other words, $X = \sum_{i=1}^{n} Y_i$ where Y_i is exponentially distributed with parameter $i\lambda$.
- Then, using the linearity property of expectation:

$$E[X] = \sum_{i=1}^{n} E[Y_i] = \sum_{i=1}^{n} = \frac{H_n}{\lambda}$$

- Also, since the Y_i's are mutually independent:

$$Var[X] = \sum_{i=1}^{n} Var[Y_i] = \sum_{i=1}^{n} \frac{1}{i^2 \lambda^2} = \frac{1}{\lambda^2} H_n^{(2)}$$

- Note that $C_X < 1$. Hence, not only does the parallel configuration increase the MTTF, it also reduces the variability of the system lifetime.

Standby Redundancy

- Assume that the system has one component operating and (n-1) cold (unpowered) spares.
- The failure rate of an operating component is λ, and a cold spare does not fail.
- Furthermore, the switching equipment is failure free.
- Let X_i be the lifetime of the i^{th} component, from the point at which it is put into operation until its failure.
- Then the system lifetime, X, is given by: $X = \sum_{i=1}^{n} X_i$
- Thus X has an n-stage Erlang distribution, and therefore:

$$E[X] = \frac{n}{\lambda} \quad \text{and} \quad Var[X] = \frac{n}{\lambda^2}$$