A

Answers to Exercises

EVERY EXERCISE is answered here (at least briefly), and some of these answers go beyond what was asked. Readers will learn best if they make a serious attempt to find their own answers BEFORE PEEKING at this appendix.

The authors will be interested to learn of any solutions (or partial solutions) to the research problems, or of any simpler (or more correct) ways to solve the non-research ones.

(The first finder of every error in this book will receive a reward of \$2.56.)

Does that mean I have to find every error?

(We meant to say "any error.")

Does that mean only one person gets a reward?

(Hmmm. Try it and see.)

1.1 The proof is fine except when $n = 2$. If all sets of two horses have horses of the same color, the statement is true for any number of horses.

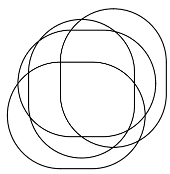
1.2 If X_n is the number of moves, we have $X_0 = 0$ and $X_n = X_{n-1} + 1 + X_{n-1} + 1 + X_{n-1}$ when $n > 0$. It follows (for example by adding 1 to both sides) that $X_n = 3^n - 1$. (After $\frac{1}{2}X_n$ moves, it turns out that the entire tower will be on the middle peg, halfway home!)

1.3 There are 3^n possible arrangements, since each disk can be on any of the pegs. We must hit them all, since the shortest solution takes $3^n - 1$ moves. (This construction is equivalent to a "ternary Gray code," which runs through all numbers from $(0\ldots0)_3$ to $(2\ldots2)_3$, changing only one digit at a time.)

1.4 No. If the largest disk doesn't have to move, $2^{n-1} - 1$ moves will suffice (by induction); otherwise $(2^{n-1} - 1) + 1 + (2^{n-1} - 1)$ will suffice (again by induction).

The number of intersection points turns out to give the whole story; convexity was a red herring.

1.5 No; different circles can intersect in at most two points, so the fourth circle can increase the number of regions to at most 14. However, it is possible to do the job with ovals:



Venn [359] claimed that there is no way to do the five-set case with ellipses, but a five-set construction with ellipses was found by Grünbaum [167].

1.6 If the nth line intersects the previous lines in $k > 0$ distinct points, we get $k-1$ new bounded regions (assuming that none of the previous lines were mutually parallel) and two new infinite regions. Hence the maximum number of bounded regions is $(n-2)+(n-3)+\cdots = S_{n-2} = (n-1)(n-2)/2 = L_n - 2n$.

This answer assumes that $n > 0$.

1.7 The basis is unproved; and in fact, $H(1) \neq 2$.

1.8 $Q_2 = (1+\beta)/\alpha$; $Q_3 = (1+\alpha+\beta)/\alpha\beta$; $Q_4 = (1+\alpha)/\beta$; $Q_5 = \alpha$; $Q_6 = \beta$. So the sequence is periodic!

1.9 (a) We get $P(n-1)$ from the inequality

$$x_1 \ldots x_{n-1}\left(\frac{x_1 + \cdots + x_{n-1}}{n-1}\right) \leqslant \left(\frac{x_1 + \cdots + x_{n-1}}{n-1}\right)^n .$$

(b) $x_1 \ldots x_n x_{n+1} \ldots x_{2n} \leqslant \bigl(((x_1 + \cdots + x_n)/n)((x_{n+1} + \cdots + x_{2n})/n)\bigr)^n$ by $P(n)$; the product inside is $\leqslant \bigl((x_1 + \cdots + x_{2n})/2n\bigr)^2$ by $P(2)$. (c) For example, $P(5)$ follows from $P(6)$ from $P(3)$ from $P(4)$ from $P(2)$.

1.10 First show that $R_n = R_{n-1} + 1 + Q_{n-1} + 1 + R_{n-1}$, when $n > 0$. Incidentally, the methods of Chapter 7 will tell us that $Q_n = \bigl((1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}\bigr)/(2\sqrt{3}) - 1$.

1.11 (a) We cannot do better than to move a double $(n-1)$-tower, then move (and invert the order of) the two largest disks, then move the double $(n-1)$-tower again; hence $A_n = 2A_{n-1} + 2$ and $A_n = 2T_n = 2^{n+1} - 2$. This solution interchanges the two largest disks but returns the other $2n-2$ to their original order.

(b) Let B_n be the minimum number of moves. Then $B_1 = 3$, and it can be shown that no strategy does better than $B_n = A_{n-1} + 2 + A_{n-1} + 2 + B_{n-1}$ when $n > 1$. Hence $B_n = 2^{n+2} - 5$, for all $n > 0$. Curiously this is just $2A_n - 1$, and we also have $B_n = A_{n-1} + 1 + A_{n-1} + 1 + A_{n-1} + 1 + A_{n-1}$.

1.12 If all $m_k > 0$, then $A(m_1, \ldots, m_n) = 2A(m_1, \ldots, m_{n-1}) + m_n$. This is an equation of the "generalized Josephus" type, with solution $(m_1 \ldots m_n)_2 = 2^{n-1}m_1 + \cdots + 2m_{n-1} + m_n$.

Incidentally, the corresponding generalization of exercise 11b appears to satisfy the recurrence

$$B(m_1, \ldots, m_n) = \begin{cases} A(m_1, \ldots, m_n), & \text{if } m_n = 1; \\ 2m_n - 1, & \text{if } n = 1; \\ 2A(m_1, \ldots, m_{n-1}) + 2m_n & \\ \quad + B(m_1, \ldots, m_{n-1}), & \text{if } n > 1 \text{ and } m_n > 1. \end{cases}$$

1.13 Given n straight lines that define L_n regions, we can replace them by extremely narrow zig-zags with segments sufficiently long that there are nine intersections between each pair of zig-zags. This shows that $ZZ_n = ZZ_{n-1}+9n-8$, for all $n>0$; consequently $ZZ_n = 9S_n-8n+1 = \frac{9}{2}n^2-\frac{7}{2}n+1$.

1.14 The number of new 3-dimensional regions defined by each new cut is the number of 2-dimensional regions defined in the new plane by its intersections with the previous planes. Hence $P_n = P_{n-1} + L_{n-1}$, and it turns out that $P_5 = 26$. (Six cuts in a cubical piece of cheese can make 27 cubelets, or up to $P_6 = 42$ cuts of weirder shapes.)

Incidentally, the solution to this recurrence fits into a nice pattern if we express it in terms of binomial coefficients (see Chapter 5):

$$\begin{aligned} X_n &= \binom{n}{0}+\binom{n}{1}; \\ L_n &= \binom{n}{0}+\binom{n}{1}+\binom{n}{2}; \\ P_n &= \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}. \end{aligned}$$

I bet I know what happens in four dimensions!

Here X_n is the maximum number of 1-dimensional regions definable by n points on a line.

1.15 The function I satisfies the same recurrence as J when $n>1$, but $I(1)$ is undefined. Since $I(2)=2$ and $I(3)=1$, there's no value of $I(1)=\alpha$ that will allow us to use our general method; the "end game" of unfolding depends on the two leading bits in n's binary representation.

If $n = 2^m+2^{m-1}+k$, where $0 \leqslant k < 2^{m+1}+2^m-(2^m+2^{m-1}) = 2^m+2^{m-1}$, the solution is $I(n)=2k+1$ for all $n>2$. Another way to express this, in terms of the representation $n = 2^m + l$, is to say that

$$I(n) = \begin{cases} J(n)+2^{m-1}, & \text{if } 0 \leqslant l < 2^{m-1}; \\ J(n)-2^m, & \text{if } 2^{m-1} \leqslant l < 2^m. \end{cases}$$

1.16 Let $g(n) = a(n)\alpha + b(n)\beta_0 + c(n)\beta_1 + d(n)\gamma$. We know from (1.18) that $a(n)\alpha + b(n)\beta_0 + c(n)\beta_1 = (\alpha\,\beta_{b_{m-1}}\,\beta_{b_{m-2}}\ldots\beta_{b_1}\,\beta_{b_0})_3$ when $n = (1\,b_{m-1}\ldots b_1\,b_0)_2$; this defines $a(n)$, $b(n)$, and $c(n)$. Setting $g(n)=n$ in the recurrence implies that $a(n)+c(n)-d(n)=n$; hence we know everything. [Setting $g(n)=1$ gives the additional identity $a(n)-2b(n)-2c(n)=1$, which can be used to define $b(n)$ in terms of the simpler functions $a(n)$ and $a(n)+c(n)$.]

1.17 In general we have $W_m \leqslant 2W_{m-k}+T_k$, for $0 \leqslant k \leqslant m$. (This relation corresponds to transferring the top $m-k$, then using only three pegs to move the bottom k, then finishing with the top $m-k$.) The stated relation turns out to be based on the unique value of k that minimizes the right-hand side of this general inequality, when $m = n(n+1)/2$. (However, we cannot conclude that equality holds; many other strategies for transferring the tower are conceivable.) If we set $Y_n = (W_{n(n+1)/2} - 1)/2^n$, we find that $Y_n \leqslant Y_{n-1} + 1$; hence $W_{n(n+1)/2} \leqslant 2^n(n-1) + 1$.

1.18 It suffices to show that both of the lines from $(n^{2j}, 0)$ intersect both of the lines from $(n^{2k}, 0)$, and that all these intersection points are distinct.

A line from $(x_j, 0)$ through $(x_j - a_j, 1)$ intersects a line from $(x_k, 0)$ through $(x_k - a_k, 1)$ at the point $(x_j - ta_j, t)$ where $t = (x_k - x_j)/(a_k - a_j)$. Let $x_j = n^{2j}$ and $a_j = n^j + (0 \text{ or } n^{-n})$. Then the ratio $t = (n^{2k} - n^{2j})/\big(n^k - n^j + (-n^{-n} \text{ or } 0 \text{ or } n^{-n})\big)$ lies strictly between $n^j + n^k - 1$ and $n^j + n^k + 1$; hence the y coordinate of the intersection point uniquely identifies j and k. Also the four intersections that have the same j and k are distinct.

1.19 Not when $n > 11$. A bent line whose half-lines run at angles θ and $\theta + 30°$ from its apex can intersect four times with another whose half-lines run at angles ϕ and $\phi + 30°$ only if $|\theta - \phi| > 30°$. We can't choose more than 11 angles this far apart from each other. (Is it possible to choose 11?)

1.20 Let $h(n) = a(n)\alpha + b(n)\beta_0 + c(n)\beta_1 + d(n)\gamma_0 + e(n)\gamma_1$. We know from (1.18) that $a(n)\alpha + b(n)\beta_0 + c(n)\beta_1 = (\alpha\, \beta_{b_{m-1}}\, \beta_{b_{m-2}} \ldots \beta_{b_1}\, \beta_{b_0})_4$ when $n = (1\, b_{m-1} \ldots b_1\, b_0)_2$; this defines $a(n)$, $b(n)$, and $c(n)$. Setting $h(n) = n$ in the recurrence implies that $a(n) + c(n) - 2d(n) - 2e(n) = n$; setting $h(n) = n^2$ implies that $a(n) + c(n) + 4e(n) = n^2$. Hence $d(n) = \big(3a(n) + 3c(n) - n^2 - 2n\big)/4$; $e(n) = \big(n^2 - a(n) - c(n)\big)/4$.

1.21 We can let m be the least (or any) common multiple of $2n$, $2n-1$, $\ldots$, $n+1$. [A non-rigorous argument suggests that a "random" value of m will succeed with probability

$$\frac{n}{2n}\frac{n-1}{2n-1}\cdots\frac{1}{n+1} = 1\bigg/\binom{2n}{n} \sim \frac{\sqrt{\pi n}}{4^n},$$

so we might expect to find such an m less than 4^n.]

1.22 Take a regular polygon with 2^n sides and label the sides with the elements of a "de Bruijn cycle" of length 2^n. (This is a cyclic sequence of 0's and 1's in which all n-tuples of adjacent elements are different; see [207, exercise 2.3.4.2–23] and [208, exercise 3.2.2–17].) Attach a very thin convex extension to each side that's labeled 1. The n sets are copies of the resulting polygon, rotated by the length of k sides for $k = 0, 1, \ldots, n-1$.

I once rode a de Bruijn cycle (when visiting at his home in Nuenen, The Netherlands).

1.23 Yes. (We need principles of elementary number theory from Chapter 4.) Let $L(n) = \operatorname{lcm}(1, 2, \ldots, n)$. We can assume that $n > 2$; hence by Bertrand's postulate there is a prime p between $n/2$ and n. We can also assume that $j > n/2$, since $q' = L(n) + 1 - q$ leaves $j' = n + 1 - j$ if and only if q leaves j. Choose q so that $q \equiv 1 \pmod{L(n)/p}$ and $q \equiv j + 1 - n \pmod{p}$. The people are now removed in order $1, 2, \ldots, n - p, j + 1, j + 2, \ldots, n, n - p + 1, \ldots, j - 1$.

1.24 The only known examples are: $X_n = 1/X_{n-1}$, which has period 2; Gauss's recurrence of period 5 in exercise 8; H. Todd's even more remarkable recurrence $X_n = (1+X_{n-1}+X_{n-2})/X_{n-3}$, which has period 8 (see [261]); and recurrences derived from these when we replace X_n by a constant times X_{mn}. We can assume that the first nonzero coefficient in the denominator is unity, and that the first nonzero coefficient in the numerator (if any) has nonnegative real part. Computer algebra shows easily that there are no further solutions of period $\leqslant 5$ when $k = 2$. A partial theory has been developed by Lyness [261, 262] and by Kurshan and Gopinath [231].

An interesting example of another type, with period 9 when the starting values are real, is the recurrence $X_n = |X_{n-1}| - X_{n-2}$ discovered by Morton Brown [43]. Nonlinear recurrences having any desired period $\geqslant 5$ can be based on continuants [65].

1.25 If $T^{(k)}(n)$ denotes the minimum number of moves needed to transfer n disks with k auxiliary pegs (hence $T^{(1)}(n) = T_n$ and $T^{(2)}(n) = W_n$), we have $T^{(k)}(\binom{n+1}{k}) \leqslant 2T^{(k)}(\binom{n}{k}) + T^{(k-1)}(\binom{n}{k-1})$. No examples (n, k) are known where this inequality fails to be an equality. When k is small compared with n, the formula $2^{n+1-k}\binom{n-1}{k-1}$ gives a convenient (but non-optimum) upper bound on $T^{(k)}(\binom{n}{k})$.

1.26 The execution-order permutation can be computed in $O(n \log n)$ steps for all m and n [209, exercises 5.1.1–2 and 5.1.1–5]. Bjorn Poonen has proved that non-Josephus sets with exactly four "bad guys" exist whenever $n \equiv 0 \pmod 3$ and $n \geqslant 9$; in fact, the number of such sets is at least $\epsilon\binom{n}{4}$ for some $\epsilon > 0$. He also found by extensive computations that the only other $n < 24$ with non-Josephus sets is $n = 20$, which has 236 such sets with $k = 14$ and two with $k = 13$. (One of the latter is $\{1, 2, 3, 4, 5, 6, 7, 8, 11, 14, 15, 16, 17\}$; the other is its reflection with respect to 21.) There is a unique non-Josephus set with $n = 15$ and $k = 9$, namely $\{3, 4, 5, 6, 8, 10, 11, 12, 13\}$.

2.1 There's no agreement about this; three answers are defensible: (1) We can say that $\sum_{k=m}^{n} q_k$ is always equivalent to $\sum_{m \leqslant k \leqslant n} q_k$; then the stated sum is zero. (2) A person might say that the given sum is $q_4 + q_3 + q_2 + q_1 + q_0$, by summing over decreasing values of k. But this conflicts with the generally accepted convention that $\sum_{k=1}^{n} q_k = 0$ when $n = 0$. (3) We can say that $\sum_{k=m}^{n} q_k = \sum_{k \leqslant n} q_k - \sum_{k < m} q_k$; then the stated sum is equal to $-q_1 - q_2 - q_3$. This convention may appear strange, but it obeys the useful law $\sum_{k=a}^{b} + \sum_{k=b+1}^{c} = \sum_{k=a}^{c}$ for all a, b, c.

It's best to use the notation $\sum_{k=m}^{n}$ only when $n-m \geqslant -1$; then both conventions (1) and (3) agree.

2.2 This is $|x|$. Incidentally, the quantity $\bigl([x>0]-[x<0]\bigr)$ is often called $\operatorname{sign}(x)$ or $\operatorname{signum}(x)$; it is $+1$ when $x>0$, 0 when $x=0$, and -1 when $x<0$.

2.3 The first sum is, of course, $a_0+a_1+a_2+a_3+a_4+a_5$; the second is $a_4+a_1+a_0+a_1+a_4$, because the sum is over the values $k \in \{-2,-1,0,+1,+2\}$. The commutative law doesn't hold here because the function $p(k)=k^2$ is not a permutation. Some values of n (e.g., $n=3$) have no k such that $p(k)=n$; others (e.g., $n=4$) have two such k.

2.4 (a) $\sum_{i=1}^{4}\sum_{j=i+1}^{4}\sum_{k=j+1}^{4} a_{ijk} = \sum_{i=1}^{2}\sum_{j=i+1}^{3}\sum_{k=j+1}^{4} a_{ijk} = \bigl((a_{123}+a_{124})+a_{134}\bigr)+a_{234}$.

(b) $\sum_{k=1}^{4}\sum_{j=1}^{k-1}\sum_{i=1}^{j-1} a_{ijk} = \sum_{k=3}^{4}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1} a_{ijk} = a_{123}+\bigl(a_{124}+(a_{134}+a_{234})\bigr)$.

2.5 The same index 'k' is being used for two different index variables, although k is bound in the inner sum. This is a famous mistake in mathematics (and computer programming). The result turns out to be correct if $a_j=a_k$ for all j and k, $1\leqslant j,k\leqslant n$.

2.6 It's $[1\leqslant j\leqslant n](n-j+1)$. The first factor is necessary here because we should get zero when $j<1$ or $j>n$.

2.7 $mx^{\overline{m-1}}$. A version of finite calculus based on ∇ instead of Δ would therefore give special prominence to *rising* factorial powers.

2.8 0, if $m\geqslant 1$; $1/|m|!$, if $m\leqslant 0$.

2.9 $x^{\overline{m+n}} = x^{\overline{m}}\,(x+m)^{\overline{n}}$, for integers m and n. Setting $m=-n$ tells us that $x^{\overline{-n}} = 1/(x-n)^{\overline{n}} = 1/(x-1)^{\underline{n}}$.

2.10 Another possible right-hand side is $\mathrm{E}u\,\Delta v + v\,\Delta u$.

2.11 Break the left-hand side into two sums, and change k to $k+1$ in the second of these.

2.12 If $p(k)=n$ then $n+c=k+\bigl((-1)^k+1\bigr)c$ and $\bigl((-1)^k+1\bigr)$ is even; hence $(-1)^{n+c}=(-1)^k$ and $k=n-(-1)^{n+c}c$. Conversely, this value of k yields $p(k)=n$.

2.13 Let $R_0=\alpha$, and $R_n=R_{n-1}+(-1)^n(\beta+n\gamma+n^2\delta)$ for $n>0$. Then $R(n)=A(n)\alpha+B(n)\beta+C(n)\gamma+D(n)\delta$. Setting $R_n=1$ yields $A(n)=1$. Setting $R_n=(-1)^n$ yields $A(n)+2B(n)=(-1)^n$. Setting $R_n=(-1)^n n$ yields $-B(n)+2C(n)=(-1)^n n$. Setting $R_n=(-1)^n n^2$ yields $B(n)-2C(n)+2D(n)=(-1)^n n^2$. Therefore $2D(n)=(-1)^n(n^2+n)$; the stated sum is $D(n)$.

2.14 The suggested rewrite is legitimate since we have $k = \sum_{1\leqslant j\leqslant k} 1$ when $1 \leqslant k \leqslant n$. Sum first on k; the multiple sum reduces to

$$\sum_{1\leqslant j\leqslant n} (2^{n+1} - 2^j) \;=\; n2^{n+1} - (2^{n+1} - 2)\,.$$

2.15 The first step replaces $k(k+1)$ by $2\sum_{1\leqslant j\leqslant k} j$. The second step gives $❒_n + \square_n = \bigl(\sum_{k=1}^{n} k\bigr)^2 + \square_n$.

2.16 $x^{\underline{m}}(x-m)^{\underline{n}} = x^{\underline{m+n}} = x^{\underline{n}}(x-n)^{\underline{m}}$, by (2.52).

2.17 Use induction for the first two $=$'s, and (2.52) for the third. The second line follows from the first.

2.18 Use the facts that $(\Re z)^+ \leqslant |z|$, $(\Re z)^- \leqslant |z|$, $(\Im z)^+ \leqslant |z|$, $(\Im z)^- \leqslant |z|$, and $|z| \leqslant (\Re z)^+ + (\Re z)^- + (\Im z)^+ + (\Im z)^-$.

2.19 Multiply both sides by $2^{n-1}/n!$ and let $S_n = 2^nT_n/n! = S_{n-1} + 3\cdot 2^{n-1} = 3(2^n - 1) + S_0$. The solution is $T_n = 3\cdot n! + n!/2^{n-1}$. (We'll see in Chapter 4 that T_n is an integer only when n is 0 or a power of 2.)

"It is a profoundly erroneous truism, repeated by all copybooks and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in a battle — they are strictly limited in number, they require fresh horses, and must only be made at decisive moments."
— A. N. Whitehead [370]

2.20 The perturbation method gives

$$S_n + (n+1)H_{n+1} \;=\; S_n + \Bigl(\sum_{0\leqslant k\leqslant n} H_k\Bigr) + n + 1\,.$$

2.21 Extracting the final term of S_{n+1} gives $S_{n+1} = 1 - S_n$; extracting the first term gives

$$\begin{aligned} S_{n+1} \;&=\; (-1)^{n+1} + \sum_{1\leqslant k\leqslant n+1} (-1)^{n+1-k} \;=\; (-1)^{n+1} + \sum_{0\leqslant k\leqslant n} (-1)^{n-k} \\ &=\; (-1)^{n+1} + S_n\,. \end{aligned}$$

Hence $2S_n = 1 + (-1)^n$ and we have $S_n = [n \text{ is even}]$. Similarly, we find

$$T_{n+1} \;=\; n + 1 - T_n \;=\; \sum_{k=0}^{n} (-1)^{n-k}(k+1) \;=\; T_n + S_n\,,$$

hence $2T_n = n + 1 - S_n$ and we have $T_n = \frac{1}{2}\bigl(n + [n \text{ is odd}]\bigr)$. Finally, the same approach yields

$$\begin{aligned} U_{n+1} \;=\; (n+1)^2 - U_n \;&=\; U_n + 2T_n + S_n \\ &=\; U_n + n + [n \text{ is odd}] + [n \text{ is even}] \\ &=\; U_n + n + 1\,. \end{aligned}$$

Hence U_n is the triangular number $\frac{1}{2}(n+1)n$.

2.22 Twice the general sum gives a "vanilla" sum over $1 \leqslant j, k \leqslant n$, which splits and yields twice $\bigl(\sum_k a_k A_k\bigr)\bigl(\sum_k b_k B_k\bigr) - \bigl(\sum_k a_k B_k\bigr)\bigl(\sum_k b_k A_k\bigr)$.

2.23 (a) This approach gives four sums that evaluate to $2n + H_n - 2n + (H_n + \frac{1}{n+1} - 1)$. (It would have been easier to replace the summand by $1/k + 1/(k+1)$.) (b) Let $u(x) = 2x + 1$ and $\Delta v(x) = 1/x(x+1) = (x-1)^{\underline{-2}}$; then $\Delta u(x) = 2$ and $v(x) = -(x-1)^{\underline{-1}} = -1/x$. The answer is $2H_n - \frac{n}{n+1}$.

2.24 Summing by parts, $\sum x^{\underline{m}} H_x \,\delta x = x^{\underline{m+1}} H_x/(m+1) - x^{\underline{m+1}}/(m+1)^2 + C$; hence $\sum_{0 \leqslant k < n} k^{\underline{m}} H_k = n^{\underline{m+1}}\bigl(H_n - 1/(m+1)\bigr)/(m+1) + 0^{\underline{m+1}}/(m+1)^2$. In our case $m = -2$, so the sum comes to $1 - (H_n + 1)/(n+1)$.

2.25 Here are some of the basic analogies:

$$\begin{aligned}
\sum_{k\in K} c a_k &= c\sum_{k\in K} a_k &&\longleftrightarrow& \prod_{k\in K} a_k^c &= \Bigl(\prod_{k\in K} a_k\Bigr)^c\\
\sum_{k\in K} (a_k+b_k) &= \sum_{k\in K} a_k + \sum_{k\in K} b_k &&\longleftrightarrow& \prod_{k\in K} a_k b_k &= \Bigl(\prod_{k\in K} a_k\Bigr)\Bigl(\prod_{k\in K} b_k\Bigr)\\
\sum_{k\in K} a_k &= \sum_{p(k)\in K} a_{p(k)} &&\longleftrightarrow& \prod_{k\in K} a_k &= \prod_{p(k)\in K} a_{p(k)}\\
\sum_{\substack{j\in J\\ k\in K}} a_{j,k} &= \sum_{j\in J}\sum_{k\in K} a_{j,k} &&\longleftrightarrow& \prod_{\substack{j\in J\\ k\in K}} a_{j,k} &= \prod_{j\in J}\prod_{k\in K} a_{j,k}\\
\sum_{k\in K} a_k &= \sum_k a_k[k\in K] &&\longleftrightarrow& \prod_{k\in K} a_k &= \prod_k a_k^{[k\in K]}\\
\sum_{k\in K} 1 &= \#K &&\longleftrightarrow& \prod_{k\in K} c &= c^{\#K}
\end{aligned}$$

2.26 $P^2 = \bigl(\prod_{1\leqslant j,k\leqslant n} a_j a_k\bigr)\bigl(\prod_{1\leqslant j=k\leqslant n} a_j a_k\bigr)$. The first factor is equal to $\bigl(\prod_{k=1}^n a_k^n\bigr)^2$; the second factor is $\prod_{k=1}^n a_k^2$. Hence $P = \bigl(\prod_{k=1}^n a_k\bigr)^{n+1}$.

2.27 $\Delta(c^{\underline{x}}) = c^{\underline{x}}(c - x - 1) = c^{\underline{x+2}}/(c - x)$. Setting $c = -2$ and decreasing x by 2 yields $\Delta(-(-2)^{\underline{x-2}}) = (-2)^{\underline{x}}/x$, hence the stated sum is $(-2)^{\underline{-1}} - (-2)^{\underline{n-1}} = (-1)^n n! - 1$.

2.28 The interchange of summation between the second and third lines is not justifiable; the terms of this sum do not converge absolutely. Everything else is perfectly correct, except that the result of $\sum_{k\geqslant 1}[k = j-1]k/j$ should perhaps have been written $[j - 1 \geqslant 1](j-1)/j$ and simplified explicitly.

As opposed to imperfectly correct.

2.29 Use partial fractions to get

$$\frac{k}{4k^2-1} = \frac{1}{4}\left(\frac{1}{2k+1} + \frac{1}{2k-1}\right).$$

The $(-1)^k$ factor now makes the two halves of each term cancel with their neighbors. Hence the answer is $-1/4+(-1)^n/(8n+4)$.

2.30 $\sum_a^b x\,\delta x = \frac12(b^{\underline{2}}-a^{\underline{2}}) = \frac12(b-a)(b+a-1)$. So we have

$$(b-a)(b+a-1) \;=\; 2100 \;=\; 2^2\cdot 3\cdot 5^2\cdot 7\,.$$

There is one solution for each way to write $2100 = x\cdot y$ where x is even and y is odd; we let $a=\frac12|x-y|+\frac12$ and $b=\frac12(x+y)+\frac12$. So the number of solutions is the number of divisors of $3\cdot 5^2\cdot 7$, namely 12. In general, there are $\prod_{p>2}(n_p+1)$ ways to represent $\prod_p p^{n_p}$, where the products range over primes.

2.31 $\sum_{j,k\geqslant 2} j^{-k} = \sum_{j\geqslant 2} 1/j^2(1-1/j) = \sum_{j\geqslant 2} 1/j(j-1)$. The second sum is, similarly, 3/4.

2.32 If $2n \leqslant x < 2n+1$, the sums are $0+\cdots+n+(x-n-1)+\cdots+(x-2n) = n(x-n) = (x-1)+(x-3)+\cdots+(x-2n+1)$. If $2n-1 \leqslant x < 2n$ they are, similarly, both equal to $n(x-n)$. (Looking ahead to Chapter 3, the formula $\lfloor\frac12(x+1)\rfloor\bigl(x-\lfloor\frac12(x+1)\rfloor\bigr)$ covers both cases.)

2.33 If K is empty, $\bigwedge_{k\in K} a_k = \infty$. The basic analogies are:

$$\begin{aligned}
\sum_{k\in K} ca_k &= c\sum_{k\in K} a_k & &\longleftrightarrow & \bigwedge_{k\in K}(c+a_k) &= c+\bigwedge_{k\in K} a_k\\
\sum_{k\in K}(a_k+b_k) &= \sum_{k\in K} a_k+\sum_{k\in K} b_k & &\longleftrightarrow & \bigwedge_{k\in K}\min(a_k,b_k) &\\
& & & & &= \min\Bigl(\bigwedge_{k\in K} a_k,\bigwedge_{k\in K} b_k\Bigr)\\
\sum_{k\in K} a_k &= \sum_{p(k)\in K} a_{p(k)} & &\longleftrightarrow & \bigwedge_{k\in K} a_k &= \bigwedge_{p(k)\in K} a_{p(k)}\\
\sum_{\substack{j\in J\\ k\in K}} a_{j,k} &= \sum_{j\in J}\sum_{k\in K} a_{j,k} & &\longleftrightarrow & \bigwedge_{\substack{j\in J\\ k\in K}} a_{j,k} &= \bigwedge_{j\in J}\bigwedge_{k\in K} a_{j,k}\\
\sum_{k\in K} a_k &= \sum_k a_k[k\in K] & &\longleftrightarrow & \bigwedge_{k\in K} a_k &= \bigwedge_k a_k\cdot\infty^{[k\notin K]}
\end{aligned}$$

A permutation that consumes terms of one sign faster than those of the other can steer the sum toward any value that it likes.

2.34 Let $K^+ = \{k \mid a_k \geqslant 0\}$ and $K^- = \{k \mid a_k < 0\}$. Then if, for example, n is odd, we choose F_n to be $F_{n-1}\cup E_n$, where $E_n \subseteq K^-$ is sufficiently large that $\sum_{k\in(F_{n-1}\cap K^+)} a_k - \sum_{k\in E_n}(-a_k) < A^-$.

2.35 Goldbach's sum can be shown to equal

$$\sum_{m,n\geqslant 2} m^{-n} \;=\; \sum_{m\geqslant 2}\frac{1}{m(m-1)} \;=\; 1$$

as follows: By unsumming a geometric series, it equals $\sum_{k\in P,\, l\geqslant 1} k^{-l}$; therefore the proof will be complete if we can find a one-to-one correspondence between ordered pairs (m,n) with $m,n \geqslant 2$ and ordered pairs (k,l) with $k \in P$ and $l \geqslant 1$, where $m^n = k^l$ when the pairs correspond. If $m \notin P$ we let $(m,n) \longleftrightarrow (m^n,1)$; but if $m = a^b \in P$, we let $(m,n) \longleftrightarrow (a^n,b)$.

2.36 (a) By definition, $g(n) - g(n-1) = f(n)$. (b) By part (a), $g\bigl(g(n)\bigr) - g\bigl(g(n-1)\bigr) = \sum_k f(k)\bigl[g(n-1)<k\leqslant g(n)\bigr] = n\bigl(g(n)-g(n-1)\bigr) = nf(n)$. (c) By part (a) again, $g\bigl(g(g(n))\bigr) - g\bigl(g(g(n-1))\bigr)$ is

With this self-description, Golomb's sequence wouldn't do too well on the Dating Game.

$$\begin{aligned}
&\sum_k f(k)\bigl[g(g(n-1))<k\leqslant g(g(n))\bigr]\\
&\qquad= \sum_{j,k} j\,\bigl[j=f(k)\bigr]\bigl[g(g(n-1))<k\leqslant g(g(n))\bigr]\\
&\qquad= \sum_{j,k} j\,\bigl[j=f(k)\bigr]\bigl[g(n-1)<j\leqslant g(n)\bigr]\\
&\qquad= \sum_{j} j\,\bigl(g(j)-g(j-1)\bigr)\bigl[g(n-1)<j\leqslant g(n)\bigr]\\
&\qquad= \sum_{j} jf(j)\,\bigl[g(n{-}1)<j\leqslant g(n)\bigr] \;=\; n\sum_j j\,\bigl[g(n{-}1)<j\leqslant g(n)\bigr]\,.
\end{aligned}$$

Colin Mallows observes that the sequence can also be defined by the recurrence

$$f(1) \;=\; 1\,; \qquad f(n+1) \;=\; 1 + f\bigl(n+1-f(f(n))\bigr)\,, \quad \text{for } n \geqslant 0.$$

2.37 (RLG thinks they probably won't fit; DEK thinks they probably will; OP is not committing himself.)

3.1 $m = \lfloor \lg n \rfloor$; $l = n - 2^m = n - 2^{\lfloor \lg n\rfloor}$.

3.2 (a) $\lfloor x+.5\rfloor$. (b) $\lceil x-.5\rceil$.

3.3 This is $\lfloor mn - \{m\alpha\}n/\alpha\rfloor = mn-1$, since $0<\{m\alpha\}<1$.

3.4 Something where no proof is required, only a lucky guess (I guess).

3.5 We have $\lfloor nx\rfloor = \bigl\lfloor n\lfloor x\rfloor + n\{x\}\bigr\rfloor = n\lfloor x\rfloor + \bigl\lfloor n\{x\}\bigr\rfloor$ by (3.8) and (3.6). Therefore $\lfloor nx\rfloor = n\lfloor x\rfloor \iff \bigl\lfloor n\{x\}\bigr\rfloor = 0 \iff 0\leqslant n\{x\}<1 \iff \{x\}<1/n$, assuming that n is a positive integer. (Notice that $n\lfloor x\rfloor \leqslant \lfloor nx\rfloor$ for all x in this case.)

3.6 $\lfloor f(x)\rfloor = \lfloor f(\lceil x\rceil)\rfloor$.

3.7 $\lfloor n/m\rfloor + n \bmod m$.

3.8 If all boxes contain $<\lceil n/m\rceil$ objects, then $n \leqslant \bigl(\lceil n/m\rceil - 1\bigr)m$, so $n/m+1\leqslant\lceil n/m\rceil$, contradicting (3.5). The other proof is similar.

3.9 We have $m/n-1/q = (n \operatorname{mumble} m)/qn$. The process must terminate, because $0 \leqslant n \operatorname{mumble} m < m$. The denominators of the representation are strictly increasing, hence distinct, because $qn/(n \operatorname{mumble} m) > q$.

3.10 $\lceil x+\frac{1}{2}\rceil - \bigl[(2x+1)/4 \text{ is not an integer}\bigr]$ is the nearest integer to x, if $\{x\} \neq \frac{1}{2}$; otherwise it's the nearest even integer. (See exercise 2.) Thus the formula gives an "unbiased" way to round.

3.11 If n is an integer, $\alpha < n < \beta \iff \lfloor\alpha\rfloor < n < \lceil\beta\rceil$. The number of integers satisfying $a < n < b$ when a and b are integers is $(b-a-1)[b>a]$. We would therefore get the wrong answer if $\alpha = \beta = \text{integer}$.

3.12 Subtract $\lfloor n/m \rfloor$ from both sides, by (3.6), getting $\lceil (n \bmod m)/m \rceil = \lfloor (n \bmod m + m - 1)/m \rfloor$. Both sides are now equal to $[n \bmod m > 0]$, since $0 \leqslant n \bmod m < m$.

A shorter but less direct proof simply observes that the first term in (3.24) must equal the last term in (3.25).

3.13 If they form a partition, the text's formula for $N(\alpha, n)$ implies that $1/\alpha + 1/\beta = 1$, because the coefficients of n in the equation $N(\alpha,n) + N(\beta,n) = n$ must agree if the equation is to hold for large n. Hence α and β are both rational or both irrational. If both are irrational, we do get a partition, as shown in the text. If both can be written with numerator m, the value $m-1$ occurs in neither spectrum, and m occurs in both. (However, Golomb [151] has observed that the sets $\{\lfloor n\alpha \rfloor \mid n \geqslant 1\}$ and $\{\lceil n\beta \rceil - 1 \mid n \geqslant 1\}$ always do form a partition, when $1/\alpha + 1/\beta = 1$.)

3.14 It's obvious by (3.22) if $ny = 0$, otherwise true by (3.21) and (3.6).

3.15 Plug in $\lceil mx \rceil$ for n in (3.24): $\lceil mx \rceil = \lceil x \rceil + \lceil x - \frac{1}{m} \rceil + \cdots + \lceil x - \frac{m-1}{m} \rceil$.

3.16 The formula $n \bmod 3 = 1 + \frac{1}{3}\bigl((\omega-1)\omega^n - (\omega+2)\omega^{2n}\bigr)$ can be verified by checking it when $0 \leqslant n < 3$.

A general formula for $n \bmod m$, when m is any positive integer, appears in exercise 7.25.

3.17 $\sum_{j,k}[0 \leqslant k < m][1 \leqslant j \leqslant x + k/m] = \sum_{j,k}[0 \leqslant k < m][1 \leqslant j \leqslant \lceil x \rceil] \times \bigl[k \geqslant m(j-x)\bigr] = \sum_{1 \leqslant j \leqslant \lceil x \rceil} \sum_k [0 \leqslant k < m] - \sum_{j=\lceil x \rceil} \sum_k \bigl[0 \leqslant k < m(j-x)\bigr] = m\lceil x \rceil - \bigl\lceil m(\lceil x \rceil - x) \bigr\rceil = -\lceil -mx \rceil = \lfloor mx \rfloor$.

3.18 We have

$$S = \sum_{0 \leqslant j < \lceil n\alpha \rceil} \sum_{k \geqslant n} \bigl[j\alpha^{-1} \leqslant k < (j+v)\alpha^{-1}\bigr].$$

If $j \leqslant n\alpha - 1 \leqslant n\alpha - v$, there is no contribution, because $(j+v)\alpha^{-1} \leqslant n$. Hence $j = \lfloor n\alpha \rfloor$ is the only case that matters, and the value in that case equals $\bigl\lceil (\lfloor n\alpha \rfloor + v)\alpha^{-1} \bigr\rceil - n \leqslant \lceil v\alpha^{-1} \rceil$.

3.19 If and only if b is an integer. (If b is an integer, $\log_b x$ is a continuous, increasing function that takes integer values only at integer points. If b is not an integer, the condition fails when $x = b$.)

3.20 We have $\sum_k kx[\alpha \leqslant kx \leqslant \beta] = x \sum_k k\bigl[\lceil \alpha/x \rceil \leqslant k \leqslant \lfloor \beta/x \rfloor\bigr]$, which sums to $\frac{1}{2}x\bigl(\lfloor \beta/x \rfloor \lfloor \beta/x + 1 \rfloor - \lceil \alpha/x \rceil \lceil \alpha/x - 1 \rceil\bigr)$.

3.21 If $10^n \leqslant 2^M < 10^{n+1}$, there are exactly $n+1$ such powers of 2, because there's exactly one such k-digit power of 2 for each k. Therefore the answer is $1 + \lfloor M \log 2 \rfloor$.

Note: The number of powers of 2 with leading digit l is more difficult, when $l > 1$; it's $\sum_{0 \leqslant n \leqslant M} \bigl(\lfloor n \log 2 - \log l \rfloor - \lfloor n \log 2 - \log(l+1) \rfloor\bigr)$.

3.22 All terms are the same for n and $n-1$ except the kth, where $n = 2^{k-1}q$ and q is odd; we have $S_n = S_{n-1} + 1$ and $T_n = T_{n-1} + 2^k q$. Hence $S_n = n$ and $T_n = n(n+1)$.

3.23 $X_n = m \iff \frac{1}{2}m(m-1) < n \leqslant \frac{1}{2}m(m+1) \iff m^2 - m + \frac{1}{4} < 2n < m^2 + m + \frac{1}{4} \iff m - \frac{1}{2} < \sqrt{2n} < m + \frac{1}{2}$.

3.24 Let $\beta = \alpha/(\alpha+1)$. Then the number of times the nonnegative integer m occurs in $\mathrm{Spec}(\beta)$ is exactly one more than the number of times it occurs in $\mathrm{Spec}(\alpha)$. Why? Because $N(\beta, n) = N(\alpha, n) + n + 1$.

3.25 Continuing the development in the text, if we could find a value of m such that $K_m \leqslant m$, we could violate the stated inequality at $n+1$ when $n = 2m+1$. (Also when $n = 3m+1$ and $n = 3m+2$.) But the existence of such an $m = n' + 1$ requires that $2K_{\lfloor n'/2 \rfloor} \leqslant n'$ or $3K_{\lfloor n'/3 \rfloor} \leqslant n'$, i.e., that

$$K_{\lfloor n'/2 \rfloor} \leqslant \lfloor n'/2 \rfloor \qquad \text{or} \qquad K_{\lfloor n'/3 \rfloor} \leqslant \lfloor n'/3 \rfloor .$$

Aha. This goes down further and further, implying that $K_0 \leqslant 0$; but $K_0 = 1$.

What we really want to prove is that K_n is strictly *greater* than n, for all $n > 0$. In fact, it's easy to prove this by induction, although it's a stronger result than the one we couldn't prove!

(This exercise teaches an important lesson. It's more an exercise about the nature of induction than about properties of the floor function.)

"In trying to devise a proof by mathematical induction, you may fail for two opposite reasons. You may fail because you try to prove too much: Your $P(n)$ is too heavy a burden. Yet you may also fail because you try to prove too little: Your $P(n)$ is too weak a support. In general, you have to balance the statement of your theorem so that the support is just enough for the burden."
—G. Pólya [297]

3.26 Induction, using the stronger hypothesis

$$D_n^{(q)} \leqslant (q-1)\left(\left(\frac{q}{q-1}\right)^{n+1} - 1\right), \qquad \text{for } n \geqslant 0.$$

3.27 If $D_n^{(3)} = 2^m b - a$, where a is 0 or 1, then $D_{n+m}^{(3)} = 3^m b - a$.

3.28 The key observation is that $a_n = m^2$ implies $a_{n+2k+1} = (m+k)^2 + m-k$ and $a_{n+2k+2} = (m+k)^2+2m$, for $0 \leqslant k \leqslant m$; hence $a_{n+2m+1} = (2m)^2$. The solution can be written in a nice form discovered by Carl Witty:

$$a_{n-1} \;=\; 2^l + \left\lfloor \left(\frac{n-l}{2}\right)^2 \right\rfloor, \qquad \text{when } 2^l + l \leqslant n < 2^{l+1} + l + 1.$$

3.29 $D\bigl(\alpha', \lfloor \alpha n \rfloor\bigr)$ is at most the maximum of the right-hand side of

$$s\bigl(\alpha', \lfloor n\alpha \rfloor, \nu'\bigr) \;=\; -s(\alpha, n, \nu) + S - \epsilon - \{0 \text{ or } 1\} - \nu' + \{0 \text{ or } 1\}.$$

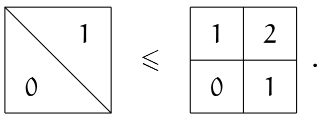
3.30 $X_n = \alpha^{2^n} + \alpha^{-2^n}$, by induction; and X_n is an integer.

This logic is seriously floored.

3.31 Here's an "elegant," "impressive" proof that gives no clue about how it was discovered:

$$\begin{aligned} \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor \;&=\; \bigl\lfloor x + \lfloor y \rfloor \bigr\rfloor + \lfloor x+y \rfloor \\ &\leqslant\; \bigl\lfloor x + \tfrac12 \lfloor 2y \rfloor \bigr\rfloor + \bigl\lfloor x + \tfrac12 \lfloor 2y \rfloor + \tfrac12 \bigr\rfloor \\ &=\; \bigl\lfloor 2x + \lfloor 2y \rfloor \bigr\rfloor \;=\; \lfloor 2x \rfloor + \lfloor 2y \rfloor . \end{aligned}$$

But there's also a simple, graphical proof based on the observation that we need to consider only the case $0 \leqslant x, y < 1$. Then the functions look like this in the plane:



A slightly stronger result is possible, namely

$$\lceil x \rceil + \lfloor y \rfloor + \lfloor x+y \rfloor \;\leqslant\; \lceil 2x \rceil + \lfloor 2y \rfloor ;$$

but this is stronger only when $\{x\} = \frac12$. If we replace (x, y) by $(-x, x+y)$ in this identity and apply the reflective law (3.4), we get

$$\lfloor y \rfloor + \lfloor x+y \rfloor + \lfloor 2x \rfloor \;\leqslant\; \lfloor x \rfloor + \lfloor 2x+2y \rfloor .$$

3.32 Let $f(x)$ be the sum in question. Since $f(x) = f(-x)$, we may assume that $x \geqslant 0$. The terms are bounded by 2^k as $k \to -\infty$ and by $x^2/2^k$ as $k \to +\infty$, so the sum exists for all real x.

We have $f(2x) = 2\sum_k 2^{k-1} \|x/2^{k-1}\|^2 = 2f(x)$. Let $f(x) = l(x) + r(x)$ where $l(x)$ is the sum for $k \leqslant 0$ and $r(x)$ is the sum for $k > 0$. Then $l(x+1) = l(x)$, and $l(x) \leqslant 1/2$ for all x. When $0 \leqslant x < 1$, we have $r(x) = x^2/2 + x^2/4 + \cdots = x^2$, and $r(x+1) = (x-1)^2/2 + (x+1)^2/4 + (x+1)^2/8 + \cdots = x^2 + 1$. Hence $f(x+1) = f(x) + 1$, when $0 \leqslant x < 1$.

We can now prove by induction that $f(x+n) = f(x)+n$ for all integers $n \geqslant 0$, when $0 \leqslant x < 1$. In particular, $f(n) = n$. Therefore in general, $f(x) = 2^{-m}f(2^m x) = 2^{-m}\lfloor 2^m x\rfloor + 2^{-m}f\bigl(\{2^m x\}\bigr)$. But $f\bigl(\{2^m x\}\bigr) = l\bigl(\{2^m x\}\bigr) + r\bigl(\{2^m x\}\bigr) \leqslant \frac{1}{2} + 1$; so $\bigl|f(x) - x\bigr| \leqslant \bigl|2^{-m}\lfloor 2^m x\rfloor - x\bigr| + 2^{-m}\cdot\frac{3}{2} \leqslant 2^{-m}\cdot\frac{5}{2}$ for all integers m.

The inescapable conclusion is that $f(x) = |x|$ for all real x.

3.33 Let $r = n - \frac{1}{2}$ be the radius of the circle. (a) There are $2n-1$ horizontal lines and $2n-1$ vertical lines between cells of the board, and the circle crosses each of these lines twice. Since r^2 is not an integer, the Pythagorean theorem tells us that the circle doesn't pass through the corner of any cell. Hence the circle passes through as many cells as there are crossing points, namely $8n - 4 = 8r$. (The same formula gives the number of cells at the edge of the board.) (b) $f(n,k) = 4\lfloor\sqrt{r^2 - k^2}\rfloor$.

It follows from (a) and (b) that

$$\tfrac{1}{4}\pi r^2 - 2r \;\leqslant\; \sum_{0<k<r} \lfloor\sqrt{r^2 - k^2}\rfloor \;\leqslant\; \tfrac{1}{4}\pi r^2\,, \qquad r = n - \tfrac{1}{2}.$$

The task of obtaining more precise estimates of this sum is a famous problem in number theory, investigated by Gauss and many others; see Dickson [78, volume 2, chapter 6].

3.34 (a) Let $m = \lceil \lg n\rceil$. We can add $2^m - n$ terms to simplify the calculations at the boundary:

$$\begin{aligned} f(n) + (2^m - n)m \;=\; \sum_{k=1}^{2^m} \lceil \lg k\rceil \;&=\; \sum_{j,k} j[j = \lceil \lg k\rceil][1 \leqslant k \leqslant 2^m] \\ &=\; \sum_{j,k} j[2^{j-1} < k \leqslant 2^j][1 \leqslant j \leqslant m] \\ &=\; \sum_{j=1}^{m} j\,2^{j-1} \;=\; 2^m(m-1) + 1\,. \end{aligned}$$

Consequently $f(n) = nm - 2^m + 1$.

(b) We have $\lceil n/2\rceil = \lfloor (n+1)/2\rfloor$, and it follows that the solution to the general recurrence $g(n) = a(n) + g\bigl(\lceil n/2\rceil\bigr) + g\bigl(\lfloor n/2\rfloor\bigr)$ must satisfy $\Delta g(n) = \Delta a(n) + \Delta g\bigl(\lfloor n/2\rfloor\bigr)$. In particular, when $a(n) = n-1$, $\Delta f(n) = 1 + \Delta f\bigl(\lfloor n/2\rfloor\bigr)$ is satisfied by the number of bits in the binary representation of n, namely $\lceil \lg(n+1)\rceil$. Now convert from Δ to Σ.

A more direct solution can be based on the identities $\lceil \lg 2j\rceil = \lceil \lg j\rceil + 1$ and $\lceil \lg(2j-1)\rceil = \lceil \lg j\rceil + [j > 1]$, for $j \geqslant 1$.

3.35 $(n+1)^2 n!\,e = A_n + (n+1)^2 + (n+1) + B_n$, where

$$A_n = \frac{(n+1)^2 n!}{0!} + \frac{(n+1)^2 n!}{1!} + \cdots + \frac{(n+1)^2 n!}{(n-1)!}$$

is a multiple of n and

$$\begin{aligned} B_n &= \frac{(n+1)^2 n!}{(n+2)!} + \frac{(n+1)^2 n!}{(n+3)!} + \cdots \\ &= \frac{n+1}{n+2}\left(1 + \frac{1}{n+3} + \frac{1}{(n+3)(n+4)} + \cdots\right) \\ &< \frac{n+1}{n+2}\left(1 + \frac{1}{n+3} + \frac{1}{(n+3)(n+3)} + \cdots\right) \\ &= \frac{(n+1)(n+3)}{(n+2)^2} \end{aligned}$$

is less than 1. Hence the answer is $2 \bmod n$.

3.36 The sum is

$$\begin{aligned} &\sum_{k,l,m} 2^{-l} 4^{-m} \big[m = \lfloor \lg l \rfloor\big] \big[l = \lfloor \lg k \rfloor\big] [1 < k < 2^{2^n}] \\ &\quad = \sum_{k,l,m} 2^{-l} 4^{-m} [2^m \leqslant l < 2^{m+1}][2^l \leqslant k < 2^{l+1}][0 \leqslant m < n] \\ &\quad = \sum_{l,m} 4^{-m} [2^m \leqslant l < 2^{m+1}][0 \leqslant m < n] \\ &\quad = \sum_{m} 2^{-m} [0 \leqslant m < n] \;=\; 2(1 - 2^{-n}) \,. \end{aligned}$$

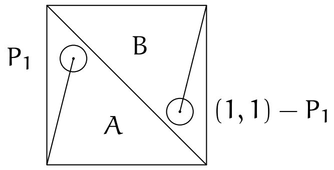
3.37 First consider the case $m < n$, which breaks into subcases based on whether $m < \frac{1}{2}n$; then show that both sides change in the same way when m is increased by n.

This is really only a level 4 problem, in spite of the way it's stated.

3.38 At most one x_k can be noninteger. Discard all integer x_k, and suppose that n are left. When $\{x\} \neq 0$, the average of $\{mx\}$ as $m \to \infty$ lies between $\frac{1}{4}$ and $\frac{1}{2}$; hence $\{mx_1\} + \cdots + \{mx_n\} - \{mx_1 + \cdots + mx_n\}$ cannot have average value zero when $n > 1$.

But the argument just given relies on a difficult theorem about uniform distribution. An elementary proof is possible, sketched here for $n = 2$: Let P_m be the point $\big(\{mx\}, \{my\}\big)$. Divide the unit square $0 \leqslant x, y < 1$ into triangular regions A and B according as $x + y < 1$ or $x + y \geqslant 1$. We want to show that $P_m \in B$ for some m, if $\{x\}$ and $\{y\}$ are nonzero. If $P_1 \in B$, we're done. Otherwise there is a disk D of radius $\epsilon > 0$ centered at P_1 such that

$D \subseteq A$. By Dirichlet's box principle, the sequence $P_1, \ldots, P_N$ must contain two points with $|P_k - P_j| < \epsilon$ and $k > j$, if N is large enough.



It follows that P_{k-j-1} is within ϵ of $(1,1) - P_1$; hence $P_{k-j-1} \in B$.

3.39 Replace j by $b - j$ and add the term $j = 0$ to the sum, so that exercise 15 can be used for the sum on j. The result,

$$\lceil x/b^k \rceil - \lceil x/b^{k+1} \rceil + b - 1\,,$$

telescopes when summed on k.

3.40 Let $\lfloor 2\sqrt{n} \rfloor = 4k + r$ where $-2 \leqslant r < 2$, and let $m = \lfloor \sqrt{n} \rfloor$. Then the following relationships can be proved by induction:

segment	r	m	x	y	if and only if
W_k	-2	$2k-1$	$m(m+1) - n - k$	k	$(2k-1)(2k-1) \leqslant n \leqslant (2k-1)(2k)$
S_k	-1	$2k-1$	$-k$	$m(m+1) - n + k$	$(2k-1)(2k) < n < (2k)(2k)$
E_k	0	$2k$	$n - m(m+1) + k$	$-k$	$(2k)(2k) \leqslant n \leqslant (2k)(2k+1)$
N_k	1	$2k$	k	$n - m(m+1) - k$	$(2k)(2k+1) < n < (2k+1)(2k+1)$

Thus, when $k \geqslant 1$, W_k is a segment of length $2k$ where the path travels west and $y(n) = k$; S_k is a segment of length $2k - 2$ where the path travels south and $x(n) = -k$; etc. (a) The desired formula is therefore

$$y(n) \;=\; (-1)^m \Big(\big(n - m(m+1)\big) \cdot \big[\lfloor 2\sqrt{n} \rfloor \text{ is odd} \big] - \lceil \tfrac{1}{2} m \rceil \Big)\,.$$

(b) On all segments, $k = \max\big(|x(n)|, |y(n)|\big)$. On segments W_k and S_k we have $x < y$ and $n + x + y = m(m+1) = (2k)^2 - 2k$; on segments E_k and N_k we have $x \geqslant y$ and $n - x - y = m(m+1) = (2k)^2 + 2k$. Hence the sign is $(-1)^{[x(n) < y(n)]}$.

3.41 Since $1/\phi + 1/\phi^2 = 1$, the stated sequences do partition the positive integers. Since the condition $g(n) = f\big(f(n)\big) + 1$ determines f and g uniquely, we need only show that $\big\lfloor \lfloor n\phi \rfloor \phi \big\rfloor + 1 = \lfloor n\phi^2 \rfloor$ for all $n > 0$. This follows from exercise 3, with $\alpha = \phi$ and $n = 1$.

3.42 No; an argument like the analysis of the two-spectrum case in the text and in exercise 13 shows that a tripartition occurs if and only if $1/\alpha + 1/\beta + 1/\gamma = 1$ and

$$\left\{\frac{n+1}{\alpha}\right\} + \left\{\frac{n+1}{\beta}\right\} + \left\{\frac{n+1}{\gamma}\right\} \;=\; 1\,,$$

for all $n > 0$. But the average value of $\{(n+1)/\alpha\}$ is $1/2$ if α is irrational, by the theorem on uniform distribution. The parameters can't all be rational, and if $\gamma = m/n$ the average is $3/2 - 1/(2n)$. Hence γ must be an integer, but this doesn't work either. (There's also a proof of impossibility that uses only simple principles, without the theorem on uniform distribution; see [155].)

3.43 One step of unfolding the recurrence for K_n gives the minimum of the four numbers $1 + a + a\cdot b\cdot K_{\lfloor (n-1-a)/(a\cdot b)\rfloor}$, where a and b are each 2 or 3. (This simplification involves an application of (3.11) to remove floors within floors, together with the identity $x + \min(y,z) = \min(x+y, x+z)$. We must omit terms with negative subscripts; i.e., with $n - 1 - a < 0$.)

Continuing along such lines now leads to the following interpretation: K_n is the least number $> n$ in the multiset S of all numbers of the form

$$1 + a_1 + a_1a_2 + a_1a_2a_3 + \cdots + a_1a_2a_3\ldots a_m\,,$$

where $m \geqslant 0$ and each a_k is 2 or 3. Thus,

$$S \;=\; \{1, 3, 4, 7, 9, 10, 13, 15, 19, 21, 22, 27, 28, 31, 31, \ldots\};$$

the number 31 is in S "twice" because it has two representations $1+2+4+8+16 = 1+3+9+18$. (Incidentally, Michael Fredman [134] has shown that $\lim_{n\to\infty} K_n/n = 1$, i.e., that S has no enormous gaps.)

3.44 Let $d_n^{(q)} = D_{n-1}^{(q)} \operatorname{mumble}(q-1)$, so that $D_n^{(q)} = (qD_{n-1}^{(q)} + d_n^{(q)})/(q-1)$ and $a_n^{(q)} = \lceil D_{n-1}^{(q)}/(q-1)\rceil$. Now $D_{k-1}^{(q)} \leqslant (q-1)n \iff a_k^{(q)} \leqslant n$, and the results follow. (This is the solution found by Euler [116], who determined the a's and d's sequentially without realizing that a single sequence $D_n^{(q)}$ would suffice.)

Too easy.

3.45 Let $\alpha > 1$ satisfy $\alpha + 1/\alpha = 2m$. Then we find $2Y_n = \alpha^{2^n} + \alpha^{-2^n}$, and it follows that $Y_n = \lceil \alpha^{2^n}/2\rceil$.

3.46 The hint follows from (3.9), since $2n(n+1) = \lfloor 2(n+\frac12)^2\rfloor$. Let $n+\theta = \bigl(\sqrt2^{\,l} + \sqrt2^{\,l-1}\bigr)m$ and $n' + \theta' = \bigl(\sqrt2^{\,l+1} + \sqrt2^{\,l}\bigr)m$, where $0 \leqslant \theta, \theta' < 1$. Then $\theta' = 2\theta \bmod 1 = 2\theta - d$, where d is 0 or 1. We want to prove that $n' = \lfloor \sqrt2(n+\frac12)\rfloor$; this equality holds if and only if

$$0 \;\leqslant\; \theta'(2-\sqrt2\,) + \sqrt2(1-d) \;<\; 2\,.$$

To solve the recurrence, note that $\mathrm{Spec}(1+1/\sqrt{2}\,)$ and $\mathrm{Spec}(1+\sqrt{2}\,)$ partition the positive integers; hence any positive integer a can be written uniquely in the form $a = \lfloor(\sqrt{2}^{\,l} + \sqrt{2}^{\,l-1})m\rfloor$, where l and m are integers with m odd and $l \geqslant 0$. It follows that $L_n = \lfloor(\sqrt{2}^{\,l+n} + \sqrt{2}^{\,l+n-1})m\rfloor$.

3.47 (a) $c = -\frac{1}{2}$. (b) c is an integer. (c) $c = 0$. (d) c is arbitrary. See the answer to exercise 1.2.4–40 in [207] for more general results.

3.48 Let $x^{:0} = 1$ and $x^{:(k+1)} = x\lfloor x^{:k}\rfloor$; also let $a_k = \{x^{:k}\}$ and $b_k = \lfloor x^{:k}\rfloor$, so that the stated identity reads $x^3 = 3x^{:3} + 3a_1a_2 + a_1^3 - 3b_1b_2 + b_1^3$. Since $a_k + b_k = x^{:k} = xb_{k-1}$ for $k \geqslant 0$, we have $(1-xz)(1+b_1z+b_2z^2+\cdots) = 1 - a_1z - a_2z^2 - \cdots$; thus

$$\frac{1}{1-xz} = \frac{1+b_1z+b_2z^2+\cdots}{1-a_1z-a_2z^2-\cdots}.$$

Take the logarithm of both sides, to separate the a's from the b's. Then differentiate with respect to z, obtaining

$$\frac{x}{1-xz} = \frac{a_1+2a_2z+3a_3z^2+\cdots}{1-a_1z-a_2z^2-\cdots} + \frac{b_1+2b_2z+3b_3z^2+\cdots}{1+b_1z+b_2z^2+\cdots}.$$

The coefficient of z^{n-1} on the left is x^n; on the right it is a formula that matches the given identity when $n = 3$.

Similar identities for the more general product $x_0x_1\ldots x_{n-1}$ can also be derived [170].

3.49 (Solution by Heinrich Rolletschek.) We can replace (α,β) by $(\{\beta\}, \alpha+\lfloor\beta\rfloor)$ without changing $\lfloor n\alpha\rfloor + \lfloor n\beta\rfloor$. Hence the condition $\alpha = \{\beta\}$ is necessary. It is also sufficient: Let $m = \lfloor\beta\rfloor$ be the least element of the given multiset, and let S be the multiset obtained from the given one by subtracting mn from the nth smallest element, for all n. If $\alpha = \{\beta\}$, consecutive elements of S differ by either 0 or 2, hence the multiset $\frac{1}{2}S = \mathrm{Spec}(\alpha)$ determines α.

A more interesting (still unsolved) problem: Restrict both α and β to be < 1, and ask when the given multiset determines the unordered pair $\{\alpha,\beta\}$.

3.50 According to unpublished notes of William A. Veech, it is sufficient to have $\alpha\beta$, β, and 1 linearly independent over the rationals.

3.51 H. S. Wilf observes that the functional equation $f(x^2-1) = f(x)^2$ would determine $f(x)$ for all $x \geqslant \phi$ if we knew $f(x)$ on any interval $(\phi\,..\,\phi+\epsilon)$.

3.52 There are infinitely many ways to partition the positive integers into three or more generalized spectra with *irrational* α_k; for example,

$$\mathrm{Spec}(2\alpha;0) \cup \mathrm{Spec}(4\alpha;-\alpha) \cup \mathrm{Spec}(4\alpha;-3\alpha) \cup \mathrm{Spec}(\beta;0)$$

works. But there's a precise sense in which all such partitions arise by "expanding" a basic one, $\mathrm{Spec}(\alpha) \cup \mathrm{Spec}(\beta)$; see [158]. The only known rational examples, e.g.,

$$\mathrm{Spec}(7;-3) \cup \mathrm{Spec}(\tfrac{7}{2};-1) \cup \mathrm{Spec}(\tfrac{7}{4};0)\,,$$

are based on parameters like those in the stated conjecture, which is due to A. S. Fraenkel [128].

3.53 Partial results are discussed in [95, pages 30–31]. The greedy algorithm probably does not terminate.

4.1 1, 2, 4, 6, 16, 12.

4.2 Note that $m_p + n_p = \min(m_p, n_p) + \max(m_p, n_p)$. The recurrence $\mathrm{lcm}(m,n) = \bigl(n/(n \bmod m)\bigr)\,\mathrm{lcm}(n \bmod m, m)$ is valid but not really advisable for computing lcm's; the best way known to compute $\mathrm{lcm}(m,n)$ is to compute $\gcd(m,n)$ first and then to divide mn by the gcd.

4.3 This holds if x is an integer, but $\pi(x)$ is defined for all real x. The correct formula,

$$\pi(x) - \pi(x-1) \;=\; \bigl[\lfloor x \rfloor \text{ is prime}\bigr]\,,$$

is easy to verify.

4.4 Between $\frac{1}{0}$ and $\frac{0}{-1}$ we'd have a left-right reflected Stern–Brocot tree with all denominators negated, etc. So the result is *all* fractions m/n with $m \perp n$. The condition $m'n - mn' = 1$ still holds throughout the construction. (This is called the *Stern–Brocot wreath*, because we can conveniently regard the final $\frac{0}{1}$ as identical to the first $\frac{0}{1}$, thereby joining the trees in a cycle at the top. The Stern–Brocot wreath has interesting applications to computer graphics because it represents all rational directions in the plane.)

4.5 $L^k = \binom{1\;k}{0\;1}$ and $R^k = \binom{1\;0}{k\;1}$; this holds even when $k < 0$. (We will find a general formula for any product of L's and R's in Chapter 6.)

After all, 'mod y' sort of means "pretend y is zero." So if it already is, there's nothing to pretend.

4.6 $a = b$. (Chapter 3 defined $x \bmod 0 = x$, primarily so that this would be true.)

4.7 We need $m \bmod 10 = 0$, $m \bmod 9 = k$, and $m \bmod 8 = 1$. But m can't be both even and odd.

4.8 We want $10x + 6y \equiv 10x + y \pmod{15}$; hence $5y \equiv 0 \pmod{15}$; hence $y \equiv 0 \pmod 3$. We must have $y = 0$ or 3, and $x = 0$ or 1.

4.9 $3^{2k+1} \bmod 4 = 3$, so $(3^{2k+1} - 1)/2$ is odd. The stated number is divisible by $(3^7 - 1)/2$ and $(3^{11} - 1)/2$ (and by other numbers).

4.10 $999(1 - \frac{1}{3})(1 - \frac{1}{37}) = 648$.

4.11 $\sigma(0) = 1$; $\sigma(1) = -1$; $\sigma(n) = 0$ for $n > 1$. (Generalized Möbius functions defined on arbitrary partially ordered structures have interesting and important properties, first explored by Weisner [366] and developed by many other people, notably Gian-Carlo Rota [313].)

4.12 $\sum_{d\backslash m}\sum_{k\backslash d}\mu(d/k)\,g(k) = \sum_{k\backslash m}\sum_{d\backslash(m/k)}\mu(d)\,g(k) = \sum_{k\backslash m} g(k)\times [m/k=1] = g(m)$, by (4.7) and (4.9).

4.13 (a) $n_p \leqslant 1$ for all p; (b) $\mu(n) \neq 0$.

4.14 True when $k > 0$. Use (4.12), (4.14), and (4.15).

4.15 No. For example, $e_n \bmod 5 = [2 \text{ or } 3]$; $e_n \bmod 11 = [2, 3, 7, \text{or } 10]$.

4.16 $1/e_1 + 1/e_2 + \cdots + 1/e_n = 1 - 1/\bigl(e_n(e_n - 1)\bigr) = 1 - 1/(e_{n+1} - 1)$.

4.17 We have $f_n \bmod f_m = 2$; hence $\gcd(f_n, f_m) = \gcd(2, f_m) = 1$. (Incidentally, the relation $f_n = f_0 f_1 \ldots f_{n-1} + 2$ is very similar to the recurrence that defines the Euclid numbers e_n.)

4.18 If $n = qm$ and q is odd, $2^n+1 = (2^m+1)(2^{n-m}-2^{n-2m}+\cdots-2^m+1)$.

4.19 The first sum is $\pi(n)$, since the summand is $[k+1 \text{ is prime}]$. The inner sum in the second is $\sum_{1\leqslant k<m}[k\backslash m]$, so it is greater than 1 if and only if m is composite; again we get $\pi(n)$. Finally $\bigl\lceil\{m/n\}\bigr\rceil = [n\nmid m]$, so the third sum is an application of Wilson's theorem. To evaluate $\pi(n)$ by any of these formulas is, of course, sheer lunacy.

4.20 Let $p_1 = 2$ and let p_n be the smallest prime greater than $2^{p_{n-1}}$. Then $2^{p_{n-1}} < p_n < 2^{p_{n-1}+1}$, and it follows that we can take $b = \lim_{n\to\infty} \lg^{(n)} p_n$ where $\lg^{(n)}$ is the function lg iterated n times. The stated numerical value comes from $p_2 = 5$, $p_3 = 37$. It turns out that $p_4 = 2^{37} + 9$, and this gives the more precise value

$$b \approx 1.2516475977905$$

(but no clue about p_5).

4.21 By Bertrand's postulate, $P_n < 10^n$. Let

$$K = \sum_{k\geqslant 1} 10^{-k^2} P_k = .200300005\ldots\,.$$

Then $10^{n^2} K \equiv P_n + \text{fraction} \pmod{10^{2n-1}}$.

4.22 $(b^{mn} - 1)/(b - 1) = \bigl((b^m - 1)/(b - 1)\bigr)(b^{mn-m} + \cdots + 1)$. [The only prime numbers of the form $(10^p - 1)/9$ for $p < 20000$ occur when $p = 2$, 19, 23, 317, 1031.] Numbers of this form are called "repunits."

4.23 $\rho(2k+1) = 0$; $\rho(2k) = \rho(k) + 1$, for $k \geqslant 1$. By induction we can show that $\rho(n) = \rho(n - 2^m)$, if $n > 2^m$ and $m > \rho(n)$. The kth Hanoi move is disk $\rho(k)$, if we number the disks $0, 1, \ldots, n-1$. This is clear if k is a power of 2. And if $2^m < k < 2^{m+1}$, we have $\rho(k) < m$; moves k and $k - 2^m$ correspond in the sequence that transfers $m+1$ disks in $T_m + 1 + T_m$ steps.

4.24 The digit that contributes dp^m to n contributes $dp^{m-1} + \cdots + d = d(p^m - 1)/(p-1)$ to $\epsilon_p(n!)$, hence $\epsilon_p(n!) = \bigl(n - \nu_p(n)\bigr)/(p-1)$.

4.25 $m \backslash\backslash n \iff m_p = 0$ or $m_p = n_p$, for all p. It follows that (a) is true. But (b) fails, in our favorite example $m = 12$, $n = 18$. (This is a common fallacy.)

4.26 Yes, since $\mathcal{G}_N$ defines a subtree of the Stern–Brocot tree.

4.27 Extend the shorter string with M's (since M lies alphabetically between L and R) until both strings are the same length, then use dictionary order. For example, the topmost levels of the tree are $LL < LM < LR < MM < RL < RM < RR$. (Another solution is to append the infinite string RL^∞ to both inputs, and to keep comparing until finding $L < R$.)

4.28 We need to use only the first part of the representation:

$$\begin{array}{cccccccccccccccccc} & R & R & R & L & L & L & L & L & L & L & R & R & R & R & R & R & \\ \frac{1}{1}, & \frac{2}{1}, & \frac{3}{1}, & \frac{4}{1}, & \frac{7}{2}, & \frac{10}{3}, & \frac{13}{4}, & \frac{16}{5}, & \frac{19}{6}, & \frac{22}{7}, & \frac{25}{8}, & \frac{47}{15}, & \frac{69}{22}, & \frac{91}{29}, & \frac{113}{36}, & \frac{135}{43}, & \ldots . \end{array}$$

The fraction $\frac{4}{1}$ appears because it's a better upper bound than $\frac{1}{0}$, not because it's closer than $\frac{3}{1}$. Similarly, $\frac{25}{8}$ is a better lower bound than $\frac{3}{1}$. The simplest upper bounds and the simplest lower bounds all appear, but the next really good approximation doesn't occur until just before the string of R's switches back to L.

4.29 $1/\alpha$. To get $1-x$ from x in binary notation, we interchange 0 and 1; to get $1/\alpha$ from α in Stern–Brocot notation, we interchange L and R. (The finite cases must also be considered, but they must work since the correspondence is order preserving.)

4.30 The m integers $x \in [A \mathinner{..} A+m)$ are different mod m; so their residues $(x \bmod m_1, \ldots, x \bmod m_r)$ run through all $m_1 \ldots m_r = m$ possible values, one of which must be equal to $(a_1 \bmod m_1, \ldots, a_r \bmod m_r)$ by the pigeonhole principle.

4.31 A number in radix b notation is divisible by d if and only if the sum of its digits is divisible by d, whenever $b \equiv 1 \pmod d$. This follows because $(a_m \ldots a_0)_b = a_m b^m + \cdots + a_0 b^0 \equiv a_m + \cdots + a_0$.

4.32 The $\varphi(m)$ numbers $\{\, kn \bmod m \mid k \perp m \text{ and } 0 \leqslant k < m \,\}$ are the numbers $\{\, k \mid k \perp m \text{ and } 0 \leqslant k < m \,\}$ in some order. Multiply them together and divide by $\prod_{0 \leqslant k < m,\, k \perp m} k$.

4.33 Obviously $h(1) = 1$. If $m \perp n$ then $h(mn) = \sum_{d\backslash mn} f(d)\, g(mn/d) = \sum_{c\backslash m,\, d\backslash n} f(cd)\, g\big((m/c)(n/d)\big) = \sum_{c\backslash m} \sum_{d\backslash n} f(c)\, g(m/c)\, f(d)\, g(n/d)$; this is $h(m)\, h(n)$, since $c \perp d$ for every term in the sum.

4.34 $g(m) = \sum_{d\backslash m} f(d) = \sum_{d\backslash m} f(m/d) = \sum_{d\geqslant 1} f(m/d)$ if $f(x)$ is zero when x is not an integer.

4.35 The base cases are

$$I(0,n) \;=\; 0\,; \qquad I(m,0) \;=\; 1\,.$$

When $m, n > 0$, there are two rules, where the first is trivial if $m > n$ and the second is trivial if $m < n$:

$$\begin{aligned} I(m,n) \;&=\; I(m, n \bmod m) - \lfloor n/m \rfloor I(n \bmod m, m)\,; \\ I(m,n) \;&=\; I(m \bmod n, n)\,. \end{aligned}$$

4.36 A factorization of any of the given quantities into nonunits must have $m^2 - 10n^2 = \pm 2$ or ± 3, but this is impossible mod 10.

4.37 Let $a_n = 2^{-n} \ln(e_n - \frac{1}{2})$ and $b_n = 2^{-n} \ln(e_n + \frac{1}{2})$. Then

$$e_n \;=\; \lfloor E^{2^n} + \tfrac{1}{2} \rfloor \quad\Longleftrightarrow\quad a_n \;\leqslant\; \ln E \;<\; b_n\,.$$

And $a_{n-1} < a_n < b_n < b_{n-1}$, so we can take $E = \lim_{n\to\infty} e^{a_n}$. In fact, it turns out that

$$E^2 \;=\; \frac{3}{2} \prod_{n\geqslant 1} \left(1 + \frac{1}{(2e_n - 1)^2}\right)^{1/2^n},$$

a product that converges rapidly to $(1.26408473530530111\ldots)^2$. But these observations don't tell us what e_n is, unless we can find another expression for E that doesn't depend on Euclid numbers.

4.38 Let $r = n \bmod m$. Then $a^n - b^n = (a^m - b^m)(a^{n-m}b^0 + a^{n-2m}b^m + \cdots + a^r b^{n-m-r}) + b^{m\lfloor n/m\rfloor}(a^r - b^r)$.

4.39 If $a_1 \ldots a_t$ and $b_1 \ldots b_u$ are perfect squares, so is

$$a_1 \ldots a_t b_1 \ldots b_u / c_1^2 \ldots c_v^2\,,$$

where $\{a_1, \ldots, a_t\} \cap \{b_1, \ldots, b_u\} = \{c_1, \ldots, c_v\}$. (It can be shown, in fact, that the sequence $\langle S(1), S(2), S(3), \ldots, \rangle$ contains every nonprime positive integer exactly once.)

4.40 Let $f(n) = \prod_{1\le k\le n,\, p\nmid k} k = n!/p^{\lfloor n/p\rfloor}\lfloor n/p\rfloor!$ and $g(n) = n!/p^{\epsilon_p(n!)}$. Then

$$g(n) \;=\; f(n)f\bigl(\lfloor n/p\rfloor\bigr)f\bigl(\lfloor n/p^2\rfloor\bigr)\ldots \;=\; f(n)g\bigl(\lfloor n/p\rfloor\bigr)\,.$$

Also $f(n) \equiv a_0!(p-1)!^{\lfloor n/p\rfloor} \equiv a_0!(-1)^{\lfloor n/p\rfloor} \pmod p$, and $\epsilon_p(n!) = \lfloor n/p\rfloor + \epsilon_p\bigl(\lfloor n/p\rfloor!\bigr)$. These recurrences make it easy to prove the result by induction. (Several other solutions are possible.)

4.41 (a) If $n^2 \equiv -1 \pmod p$ then $(n^2)^{(p-1)/2} \equiv -1$; but Fermat says it's $+1$. (b) Let $n = \bigl((p-1)/2\bigr)!$; we have $n \equiv (-1)^{(p-1)/2}\prod_{1\le k<p/2}(p-k) = (p-1)!/n$, hence $n^2 \equiv (p-1)!$.

4.42 First we observe that $k \perp l \iff k \perp l + ak$ for any integer a, since $\gcd(k,l) = \gcd(k, l+ak)$ by Euclid's algorithm. Now

$$\begin{aligned} m \perp n \quad\text{and}\quad n' \perp n \quad &\iff\quad mn' \perp n \\ &\iff\quad mn' + nm' \perp n\,. \end{aligned}$$

Similarly

$$m' \perp n' \quad\text{and}\quad n \perp n' \quad\iff\quad mn' + nm' \perp n'\,.$$

Hence

$$m \perp n \quad\text{and}\quad m' \perp n' \quad\text{and}\quad n \perp n' \quad\iff\quad mn' + nm' \perp nn'\,.$$

4.43 We want to multiply by $L^{-1}R$, then by $R^{-1}L^{-1}RL$, then $L^{-1}R$, then $R^{-2}L^{-1}RL^2$, etc.; the nth multiplier is $R^{-\rho(n)}L^{-1}RL^{\rho(n)}$, since we must cancel $\rho(n)$ R's. And $R^{-m}L^{-1}RL^m = \binom{0\ \ -1}{1\ \ 2m+1}$.

4.44 We can find the simplest rational number that lies in

$$[0.3155\,..\,0.3165) \;=\; \bigl[\tfrac{631}{2000}\,..\,\tfrac{633}{2000}\bigr)$$

John .316

—banner displayed during the 1993 World Series, when John Kruk came to bat.

by looking at the Stern–Brocot representations of $\frac{631}{2000}$ and $\frac{633}{2000}$ and stopping just before the former has L where the latter has R:

$(m_1, n_1, m_2, n_2) := (631, 2000, 633, 2000)$;
while $m_1 > n_1$ **or** $m_2 < n_2$ **do**
  if $m_2 < n_2$ **then** $\bigl(\text{output}(L);\ (n_1, n_2) := (n_1, n_2) - (m_1, m_2)\bigr)$
    else $\bigl(\text{output}(R);\ (m_1, m_2) := (m_1, m_2) - (n_1, n_2)\bigr)$.

The output is LLLRRRRR $= \frac{6}{19} \approx .3158$. Incidentally, an average of .334 implies at least 287 at bats.

4.45 $x^2 \equiv x \pmod{10^n} \iff x(x-1) \equiv 0 \pmod{2^n}$ and $x(x-1) \equiv 0 \pmod{5^n} \iff x \bmod 2^n = [0 \text{ or } 1]$ and $x \bmod 5^n = [0 \text{ or } 1]$. (The last step is justified because $x(x-1) \bmod 5 = 0$ implies that either x or $x-1$ is a multiple of 5, in which case the other factor is relatively prime to 5^n and can be divided from the congruence.)

So there are at most four solutions, of which two ($x = 0$ and $x = 1$) don't qualify for the title "n-digit number" unless $n = 1$. The other two solutions have the forms x and $10^n + 1 - x$, and at least one of these numbers is $\geqslant 10^{n-1}$. When $n = 4$ the other solution, $10001 - 9376 = 625$, is not a four-digit number. We expect to get two n-digit solutions for about 90% of all n, but this conjecture has not been proved.

(Such self-reproducing numbers have been called "automorphic.")

4.46 (a) If $j'j - k'k = \gcd(j,k)$, we have $n^{k'k}n^{\gcd(j,k)} = n^{j'j} \equiv 1$ and $n^{k'k} \equiv 1$. (b) Let $n = pq$, where p is the smallest prime divisor of n. If $2^n \equiv 1 \pmod n$ then $2^n \equiv 1 \pmod p$. Also $2^{p-1} \equiv 1 \pmod p$; hence $2^{\gcd(p-1,n)} \equiv 1 \pmod p$. But $\gcd(p-1, n) = 1$ by the definition of p.

4.47 If $n^{m-1} \equiv 1 \pmod m$ we must have $n \perp m$. If $n^k \equiv n^j$ for some $1 \leqslant j < k < m$, then $n^{k-j} \equiv 1$ because we can divide by n^j. Therefore if the numbers $n^1 \bmod m, \ldots, n^{m-1} \bmod m$ are not distinct, there is a $k < m-1$ with $n^k \equiv 1$. The least such k divides $m-1$, by exercise 46(a). But then $kq = (m-1)/p$ for some prime p and some positive integer q; this is impossible, since $n^{kq} \not\equiv 1$. Therefore the numbers $n^1 \bmod m, \ldots, n^{m-1} \bmod m$ are distinct and relatively prime to m. Therefore the numbers $1, \ldots, m-1$ are relatively prime to m, and m must be prime.

4.48 By pairing numbers up with their inverses, we can reduce the product $\pmod m$ to $\prod_{1 \leqslant n < m,\, n^2 \bmod m = 1} n$. Now we can use our knowledge of the solutions to $n^2 \bmod m = 1$. By residue arithmetic we find that the result is $m-1$ if $m = 4$, p^k, or $2p^k$ $(p > 2)$; otherwise it's $+1$.

4.49 (a) Either $m < n$ ($\Phi(N-1)$ cases) or $m = n$ (one case) or $m > n$ ($\Phi(N-1)$ again). Hence $R(N) = 2\Phi(N-1) + 1$. (b) From (4.62) we get

$$2\Phi(N-1) + 1 \;=\; 1 + \sum_{d \geqslant 1} \mu(d) \lfloor N/d \rfloor \lfloor N/d - 1 \rfloor\,;$$

hence the stated result holds if and only if

$$\sum_{d \geqslant 1} \mu(d) \lfloor N/d \rfloor \;=\; 1\,, \qquad \text{for } N \geqslant 1.$$

And this is a special case of (4.61) if we set $f(x) = [x \geqslant 1]$.

4.50 (a) If f is any function,

$$\begin{aligned}\sum_{0\leqslant k<m} f(k) &= \sum_{d\backslash m}\ \sum_{0\leqslant k<m} f(k)\big[d=\gcd(k,m)\big]\\ &= \sum_{d\backslash m}\ \sum_{0\leqslant k<m} f(k)\big[k/d\perp m/d\big]\\ &= \sum_{d\backslash m}\ \sum_{0\leqslant k<m/d} f(kd)\big[k\perp m/d\big]\\ &= \sum_{d\backslash m}\ \sum_{0\leqslant k<d} f(km/d)\big[k\perp d\big]\,;\end{aligned}$$

we saw a special case of this in the derivation of (4.63). An analogous derivation holds for $\prod$ instead of $\sum$. Thus we have

$$z^m-1 \;=\; \prod_{0\leqslant k<m}(z-\omega^k) \;=\; \prod_{d\backslash m}\ \prod_{\substack{0\leqslant k<d\\ k\perp d}}(z-\omega^{km/d}) \;=\; \prod_{d\backslash m}\Psi_d(z)$$

because $\omega^{m/d}=e^{2\pi i/d}$.

Part (b) follows from part (a) by the analog of (4.56) for products instead of sums. Incidentally, this formula shows that $\Psi_m(z)$ has integer coefficients, since $\Psi_m(z)$ is obtained by multiplying and dividing polynomials whose leading coefficient is 1.

4.51 $(x_1+\cdots+x_n)^p = \sum_{k_1+\cdots+k_n=p} p!/(k_1!\ldots k_n!)x_1^{k_1}\ldots x_n^{k_n}$, and the coefficient is divisible by p unless some $k_j=p$. Hence $(x_1+\cdots+x_n)^p \equiv x_1^p+\cdots+x_n^p \pmod p$. Now we can set all the x's to 1, obtaining $n^p\equiv n$.

4.52 If $p>n$ there is nothing to prove. Otherwise $x\perp p$, so $x^{k(p-1)}\equiv 1 \pmod p$; this means that at least $\lfloor (n-1)/(p-1)\rfloor$ of the given numbers are multiples of p. And $(n-1)/(p-1)\geqslant n/p$ since $n\geqslant p$.

"Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk."
— L. Kronecker [365]

4.53 First show that if $m\geqslant 6$ and m is not prime then $(m-2)!\equiv 0 \pmod m$. (If $m=p^2$, the product for $(m-2)!$ includes p and 2p; otherwise it includes d and m/d where $d<m/d$.) Next consider cases:

Case 0, $n<5$. The condition holds for $n=1$ only.

Case 1, $n\geqslant 5$ and n is prime. Then $(n-1)!/(n+1)$ is an integer and it can't be a multiple of n.

Case 2, $n\geqslant 5$, n is composite, and $n+1$ is composite. Then n and $n+1$ divide $(n-1)!$, and $n\perp n+1$; hence $n(n+1)\backslash(n-1)!$.

Case 3, $n\geqslant 5$, n is composite, and $n+1$ is prime. Then $(n-1)!\equiv 1 \pmod{n+1}$ by Wilson's theorem, and

$$\lfloor (n-1)!/(n+1)\rfloor \;=\; \big((n-1)!+n\big)/(n+1)\,;$$

this is divisible by n.

Therefore the answer is: Either $n = 1$ or $n \neq 4$ is composite.

4.54 $\epsilon_2(1000!) > 500$ and $\epsilon_5(1000!) = 249$, hence $1000! = a \cdot 10^{249}$ for some even integer a. Since $1000 = (1300)_5$, exercise 40 tells us that $a \cdot 2^{249} = 1000!/5^{249} \equiv -1 \pmod 5$. Also $2^{249} \equiv 2$, hence $a \equiv 2$, hence $a \bmod 10 = 2$ or 7; hence the answer is $2 \cdot 10^{249}$.

4.55 One way is to prove by induction that $P_{2n}/P_n^4(n+1)$ is an integer; this stronger result helps the induction go through. Another way is based on showing that each prime p divides the numerator at least as often as it divides the denominator. This reduces to proving the inequality

$$\sum_{k=1}^{2n} \lfloor k/m \rfloor \;\geqslant\; 4\sum_{k=1}^{n} \lfloor k/m \rfloor\,,$$

which follows from

$$\bigl\lfloor (2n-1)/m \bigr\rfloor + \bigl\lfloor 2n/m \bigr\rfloor \;\geqslant\; \lfloor n/m \rfloor\,.$$

The latter is true when $0 \leqslant n < m$, and both sides increase by 4 when n is increased by m.

4.56 Let $f(m) = \sum_{k=1}^{2n-1} \min(k, 2n-k)[m\backslash k]$, $g(m) = \sum_{k=1}^{n-1} (2n-2k-1) \times \bigl[m\backslash(2k+1)\bigr]$. The number of times p divides the numerator of the stated product is $f(p) + f(p^2) + f(p^3) + \cdots$, and the number of times p divides the denominator is $g(p) + g(p^2) + g(p^3) + \cdots$. But $f(m) = g(m)$ whenever m is odd, by exercise 2.32. The stated product therefore reduces to $2^{n(n-1)}$, by exercise 3.22.

4.57 The hint suggests a standard interchange of summation, since

$$\sum_{1\leqslant m\leqslant n} [d\backslash m] \;=\; \sum_{0<k\leqslant n/d} [m=dk] \;=\; \lfloor n/d \rfloor\,.$$

Calling the hinted sum $\Sigma(n)$, we have

$$\Sigma(m+n) - \Sigma(m) - \Sigma(n) \;=\; \sum_{d\in S(m,n)} \varphi(d)\,.$$

On the other hand, we know from (4.54) that $\Sigma(n) = \frac{1}{2}n(n+1)$. Hence $\Sigma(m+n) - \Sigma(m) - \Sigma(n) = mn$.

4.58 The function $f(m)$ is multiplicative, and when $m = p^k$ it equals $1 + p + \cdots + p^k$. This is a power of 2 if and only if p is a Mersenne prime and $k = 1$. For k must be odd, and in that case the sum is

$$(1+p)(1+p^2+p^4+\cdots+p^{k-1})$$

and $(k-1)/2$ must be odd, etc. The necessary and sufficient condition is that m be a product of distinct Mersenne primes.

4.59 Proof of the hint: If $n = 1$ we have $x_1 = \alpha = 2$, so there's no problem. If $n > 1$ we can assume that $x_1 \leqslant \cdots \leqslant x_n$. Case 1: $x_1^{-1} + \cdots + x_{n-1}^{-1} + (x_n - 1)^{-1} \geqslant 1$ and $x_n > x_{n-1}$. Then we can find $\beta \geqslant x_n - 1 \geqslant x_{n-1}$ such that $x_1^{-1} + \cdots + x_{n-1}^{-1} + \beta^{-1} = 1$; hence $x_n \leqslant \beta + 1 \leqslant e_n$ and $x_1 \ldots x_n \leqslant x_1 \ldots x_{n-1}(\beta + 1) \leqslant e_1 \ldots e_n$, by induction. There is a positive integer m such that $\alpha = x_1 \ldots x_n/m$; hence $\alpha \leqslant e_1 \ldots e_n = e_{n+1} - 1$, and we have $x_1 \ldots x_n(\alpha + 1) \leqslant e_1 \ldots e_n e_{n+1}$. Case 2: $x_1^{-1} + \cdots + x_{n-1}^{-1} + (x_n - 1)^{-1} \geqslant 1$ and $x_n = x_{n-1}$. Let $a = x_n$ and $a^{-1} + (a-1)^{-1} = (a-2)^{-1} + \zeta^{-1}$. Then we can show that $a \geqslant 4$ and $(a-2)(\zeta + 1) \geqslant a^2$. So there's a $\beta \geqslant \zeta$ such that $x_1^{-1} + \cdots + x_{n-2}^{-1} + (a-2)^{-1} + \beta^{-1} = 1$; it follows by induction that $x_1 \ldots x_n \leqslant x_1 \ldots x_{n-2}(a-2)(\zeta+1) \leqslant x_1 \ldots x_{n-2}(a-2)(\beta+1) \leqslant e_1 \ldots e_n$, and we can finish as before. Case 3: $x_1^{-1} + \cdots + x_{n-1}^{-1} + (x_n - 1)^{-1} < 1$. Let $a = x_n$, and let $a^{-1} + \alpha^{-1} = (a-1)^{-1} + \beta^{-1}$. It can be shown that $(a-1)(\beta+1) > a(\alpha+1)$, because this identity is equivalent to

$$a\alpha^2 - a^2\alpha + a\alpha - a^2 + \alpha + a \; > \; 0\,,$$

which is a consequence of $a\alpha(\alpha - a) + (1+a)\alpha \geqslant (1+a)\alpha > a^2 - a$. Hence we can replace x_n and α by $a - 1$ and β, repeating this transformation until cases 1 or 2 apply.

Another consequence of the hint is that $1/x_1 + \cdots + 1/x_n < 1$ implies $1/x_1 + \cdots + 1/x_n \leqslant 1/e_1 + \cdots + 1/e_n$; see exercise 16.

4.60 The main point is that $\theta < \frac{2}{3}$. Then we can take p_1 sufficiently large (to meet the conditions below) and p_n to be the least prime greater than p_{n-1}^3. With this definition let $a_n = 3^{-n}\ln p_n$ and $b_n = 3^{-n}\ln(p_n + 1)$. If we can show that $a_{n-1} \leqslant a_n < b_n \leqslant b_{n-1}$, we can take $P = \lim_{n\to\infty} e^{a_n}$ as in exercise 37. But this hypothesis is equivalent to $p_{n-1}^3 \leqslant p_n < (p_{n-1}+1)^3$. If there's no prime p_n in this range, there must be a prime $p < p_{n-1}^3$ such that $p + cp^\theta > (p_{n-1} + 1)^3$. But this implies that $cp^\theta > 3p^{2/3}$, which is impossible when p is sufficiently large.

"Man made the integers: All else is Dieudonné." —*R. K. Guy*

We can almost certainly take $p_1 = 2$, since all available evidence indicates that the known bounds on gaps between primes are much weaker than the truth (see exercise 69). Then $p_2 = 11$, $p_3 = 1361$, $p_4 = 2521008887$, and $1.306377883863 < P < 1.306377883869$.

4.61 Let $\hat{m}$ and $\hat{n}$ be the right-hand sides; observe that $\hat{m}n' - m'\hat{n} = 1$, hence $\hat{m} \perp \hat{n}$. Also $\hat{m}/\hat{n} > m'/n'$ and $N = \big((n+N)/n'\big)n' - n \geqslant \hat{n} > \big((n+N)/n' - 1\big)n' - n = N - n' \geqslant 0$. So we have $\hat{m}/\hat{n} \geqslant m''/n''$. If equality doesn't hold, we have $n'' = (\hat{m}n' - m'\hat{n})n'' = n'(\hat{m}n'' - m''\hat{n}) + \hat{n}(m''n' - m'n'') \geqslant n' + \hat{n} > N$, a contradiction.

Incidentally, this exercise implies that $(m+m'')/(n+n'') = m'/n'$, although the former fraction is not always reduced.

4.62 $2^{-1}+2^{-2}+2^{-3}-2^{-6}-2^{-7}+2^{-12}+2^{-13}-2^{-20}-2^{-21}+2^{-30}+2^{-31}-2^{-42}-2^{-43}+\cdots$ can be written

$$\frac{1}{2}+3\sum_{k\geqslant 0}\bigl(2^{-4k^2-6k-3}-2^{-4k^2-10k-7}\bigr)\,.$$

This sum, incidentally, can be expressed in closed form using the "theta function" $\theta(z,\lambda)=\sum_k e^{-\pi\lambda k^2+2izk}$; we have

$$e \quad\leftrightarrow\quad \tfrac{1}{2}+\tfrac{3}{8}\theta(\tfrac{4}{\pi}\ln 2,\, 3i\ln 2)-\tfrac{3}{128}\theta(\tfrac{4}{\pi}\ln 2,\, 5i\ln 2)\,.$$

4.63 Any $n>2$ either has a prime divisor d or is divisible by $d=4$. In either case, a solution with exponent n implies a solution $(a^{n/d})^d+(b^{n/d})^d=(c^{n/d})^d$ with exponent d. Since $d=4$ has no solutions, d must be prime.

I have discovered a wonderful proof of Fermat's Last Theorem, but there's no room for it here.

The hint follows from the binomial theorem, since $(a^p+(x-a)^p)/x\equiv pa^{p-1} \pmod x$ when p is odd. The smallest counterexample, if (4.46) fails, has $a\perp x$. If x is not divisible by p then x is relatively prime to c^p/x; this means that whenever q is prime and $q^e\backslash\backslash x$ and $q^f\backslash\backslash c$, we have $e=fp$. Hence $x=m^p$ for some m. On the other hand if x is divisible by p, then c^p/x is divisible by p but not by p^2, and c^p has no other factors in common with x.

4.64 Equal fractions in $\mathcal{P}_N$ appear in "organ-pipe order":

$$\frac{2m}{2n},\ \frac{4m}{4n},\ \ldots,\ \frac{rm}{rn},\ \ldots,\ \frac{3m}{3n},\ \frac{m}{n}\,.$$

Suppose that $\mathcal{P}_N$ is correct; we want to prove that $\mathcal{P}_{N+1}$ is correct. This means that if kN is odd, we want to show that

$$\frac{k-1}{N+1} \;=\; \mathcal{P}_{N,kN}\,;$$

if kN is even, we want to show that

$$\mathcal{P}_{N,kN-1}\;\;\mathcal{P}_{N,kN}\;\;\frac{k-1}{N+1}\;\;\mathcal{P}_{N,kN}\;\;\mathcal{P}_{N,kN+1}\,.$$

In both cases it will be helpful to know the number of fractions that are strictly less than $(k-1)/(N+1)$ in $\mathcal{P}_N$; this is

$$\begin{aligned}\sum_{n=1}^{N}\sum_m\Bigl[0\leqslant\frac{m}{n}<\frac{k-1}{N+1}\Bigr] \;&=\; \sum_{n=1}^{N}\left\lceil\frac{(k-1)n}{N+1}\right\rceil \;=\; \sum_{n=0}^{N}\left\lfloor\frac{(k-1)n+N}{N+1}\right\rfloor\\ &=\; \frac{(k-2)N}{2}+\frac{d-1}{2}+d\left\lfloor\frac{N}{d}\right\rfloor\end{aligned}$$

by (3.32), where $d = \gcd(k-1, N+1)$. And this reduces to $\frac{1}{2}(kN - d + 1)$, since $N \bmod d = d - 1$.

Furthermore, the number of fractions equal to $(k-1)/(N+1)$ in $\mathcal{P}_N$ that should precede it in $\mathcal{P}_{N+1}$ is $\frac{1}{2}\bigl(d-1-[d \text{ even}]\bigr)$, by the nature of organ-pipe order.

If kN is odd, then d is even and $(k-1)/(N+1)$ is preceded by $\frac{1}{2}(kN-1)$ elements of $\mathcal{P}_N$; this is just the correct number to make things work. If kN is even, then d is odd and $(k-1)/(N+1)$ is preceded by $\frac{1}{2}(kN)$ elements of $\mathcal{P}_N$. If $d = 1$, none of these equals $(k-1)/(N+1)$ and $\mathcal{P}_{N,kN}$ is '$<$'; otherwise $(k-1)/(N+1)$ falls between two equal elements and $\mathcal{P}_{N,kN}$ is '$=$'. (C. S. Peirce [288] independently discovered the Stern–Brocot tree at about the same time as he discovered $\mathcal{P}_N$.)

"No square less than 25×10^{14} divides a Euclid number."
— Ilan Vardi

4.65 The analogous question for the (analogous) Fermat numbers f_n is a famous unsolved problem. This one might be easier or harder.

4.66 It is known that no square less than 36×10^{18} divides a Mersenne number or Fermat number. But there has still been no proof of Schinzel's conjecture that there exist infinitely many squarefree Mersenne numbers. It is not even known if there are infinitely many p such that $p \backslash\backslash (a \pm b)$, where all prime factors of a and b are $\leqslant 31$.

4.67 M. Szegedy has proved this conjecture for all large n; see [348], [95, pp. 78–79], and [55].

4.68 This is a much weaker conjecture than the result in the following exercise.

4.69 Cramér [66] showed that this conjecture is plausible on probabilistic grounds, and computational experience bears this out: Brent [37] has shown that $P_{n+1} - P_n \leqslant 602$ for $P_{n+1} < 2.686 \times 10^{12}$. But the much weaker bounds in exercise 60 are the best that have been published so far [255]. Exercise 68 has a "yes" answer if $P_{n+1} - P_n < 2P_n^{1/2}$ for all sufficiently large n. According to Guy [169, problem A8], Paul Erdős offers \$10,000 for proof that there are infinitely many n such that

$$P_{n+1} - P_n \;>\; \frac{c \ln n \,\ln\ln n\, \ln\ln\ln\ln n}{(\ln\ln\ln n)^2}$$

for all $c > 0$.

4.70 This holds if and only if $\nu_2(n) = \nu_3(n)$, according to exercise 24. The methods of [96] may help to crack this conjecture.

4.71 When $k = 3$ the smallest solution is $n = 4700063497 = 19 \cdot 47 \cdot 5263229$; no other solutions are known in this case.

4.72 This is known to be true for infinitely many values of a, including -1 (of course) and 0 (not so obviously). Lehmer [244] has a famous conjecture that $\varphi(n)\backslash(n-1)$ if and only if n is prime.

4.73 This is known to be equivalent to the Riemann hypothesis (that the complex zeta function $\zeta(z)$ is nonzero when the real part of z is greater than $1/2$).

4.74 Experimental evidence suggests that there are about $p(1-1/e)$ distinct values, just as if the factorials were randomly distributed modulo p.

5.1 $(11)_r^4 = (14641)_r$, in any number system of radix $r \geqslant 7$, because of the binomial theorem.

What's 11^4 *in radix* 11*?*

5.2 The ratio $\binom{n}{k+1}/\binom{n}{k} = (n-k)/(k+1)$ is $\leqslant 1$ when $k \geqslant \lfloor n/2 \rfloor$ and $\geqslant 1$ when $k < \lceil n/2 \rceil$, so the maximum occurs when $k = \lfloor n/2 \rfloor$ and $k = \lceil n/2 \rceil$.

5.3 Expand into factorials. Both products are equal to $f(n)/f(n-k)f(k)$, where $f(n) = (n+1)!\,n!\,(n-1)!$.

5.4 $\binom{-1}{k} = (-1)^k\binom{k+1-1}{k} = (-1)^k\binom{k}{k} = (-1)^k[k \geqslant 0]$.

5.5 If $0 < k < p$, there's a p in the numerator of $\binom{p}{k}$ with nothing to cancel it in the denominator. Since $\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1}$, we must have $\binom{p-1}{k} \equiv (-1)^k \pmod p$, for $0 \leqslant k < p$.

5.6 The crucial step (after second down) should be

$$\begin{aligned}
\frac{1}{n+1}\sum_k \binom{n+k}{k}\binom{n+1}{k+1}(-1)^k \\
&= \frac{1}{n+1}\sum_{k\geqslant 0} \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k \\
&= \frac{1}{n+1}\sum_{k} \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k \\
&\quad - \frac{1}{n+1}\binom{n-1}{n}\binom{n+1}{0}(-1)^{-1}\,.
\end{aligned}$$

The original derivation forgot to include this extra term, which is $[n=0]$.

5.7 Yes, because $r^{\underline{-k}} = (-1)^k/(-r-1)^{\underline{k}}$. We also have

$$r^{\overline{k}}(r+\tfrac{1}{2})^{\overline{k}} \;=\; (2r)^{\overline{2k}}/2^{2k}\,.$$

5.8 $f(k) = (k/n-1)^n$ is a polynomial of degree n whose leading coefficient is n^{-n}. By (5.40), the sum is $n!/n^n$. When n is large, Stirling's approximation says that this is approximately $\sqrt{2\pi n}/e^n$. (This is quite different from $(1-1/e)$, which is what we get if we use the approximation $(1-k/n)^n \sim e^{-k}$, valid for fixed k as $n \to \infty$.)

5.9 $\mathcal{E}_t(z)^t = \sum_{k\geqslant 0} t(tk+t)^{k-1}z^k/k! = \sum_{k\geqslant 0}(k+1)^{k-1}(tz)^k/k! = \mathcal{E}_1(tz)$, by (5.60).

5.10 $\sum_{k\geqslant 0} 2z^k/(k+2) = F(2,1;3;z)$, since $t_{k+1}/t_k = (k+2)z/(k+3)$.

But not Imbesselian.

5.11 The first is Besselian and the second is Gaussian:

$$\begin{aligned} z^{-1}\sin z &= \textstyle\sum_{k\geqslant 0}(-1)^k z^{2k}/(2k+1)! = F(1;1,\frac{3}{2};-z^2/4)\,;\\ z^{-1}\arcsin z &= \textstyle\sum_{k\geqslant 0} z^{2k}(\frac{1}{2})^{\overline{k}}/(2k+1)k! = F(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2)\,. \end{aligned}$$

Each value of a hypergeometric term $t(k)$ can be written $0^{e(k)}v(k)$, where $e(k)$ is an integer and $v(k) \neq 0$. Suppose the term ratio $t(k+1)/t(k)$ is $p(k)/q(k)$, and that p and q have been completely factored over the complex numbers. Then, for each k, $e(k+1)$ is $e(k)$ plus the number of zero factors of $p(k)$ minus the number of zero factors of $q(k)$, and $v(k+1)$ is $v(k)$ times the product of the nonzero factors of $p(k)$ divided by the product of the nonzero factors of $q(k)$.

5.12 (a) Yes, if $n \neq 0$, since the term ratio is n. (b) Yes, when n is an integer; the term ratio is $(k+1)^n/k^n$. Notice that we get this term from (5.115) by setting $m = n+1$, $a_1 = \cdots = a_m = 1$, $b_1 = \cdots = b_n = 0$, $z = 1$, and multiplying by 0^n. (c) Yes, the term ratio is $(k+1)(k+3)/(k+2)$. (d) No, the term ratio is $1 + 1/(k+1)H_k$; and $H_k \sim \ln k$ isn't a rational function. (e) Yes, the reciprocal of any hypergeometric term is a hypergeometric term. The fact that $t(k) = \infty$ when $k < 0$ or $k > n$ does not exclude $t(k)$ from hypergeometric termhood. (f) Of course. (g) Not when, say, $t(k) = 2^k$ and $T(k) = 1$. (h) Yes; the term ratio $t(n-1-k)/t(n-1-(k+1))$ is a rational function (the reciprocal of the term ratio for t, with k replaced by $n-1-k$), for arbitrary n. (i) Yes; the term ratio can be written

$$\frac{a\,t(k{+}1)/t(k) + b\,t(k{+}2)/t(k) + c\,t(k{+}3)/t(k)}{a + b\,t(k{+}1)/t(k) + c\,t(k{+}2)/t(k)}\,,$$

and $t(k+m)/t(k) = \bigl(t(k+m)/t(k+m-1)\bigr)\ldots\bigl(t(k+1)/t(k)\bigr)$ is a rational function of k. (j) No. Whenever two rational functions $p_1(k)/q_1(k)$ and $p_2(k)/q_2(k)$ are equal for infinitely many k, they are equal for all k, because $p_1(k)q_2(k) = q_1(k)p_2(k)$ is a polynomial identity. Therefore the term ratio $\lceil (k{+}1)/2\rceil/\lceil k/2\rceil$ would have to equal 1 if it were a rational function. (k) No. The term ratio would have to be $(k+1)/k$, since it is $(k+1)/k$ for all $k > 0$; but then $t(-1)$ can be zero only if $t(0)$ is a multiple of 0^2, while $t(1)$ can be 1 only if $t(0) = 0^1$.

5.13 $R_n = n!^{n+1}/P_n^2 = Q_n/P_n = Q_n^2/n!^{n+1}$.

5.14 The first factor in (5.25) is $\binom{l-k}{l-k-m}$ when $k \leqslant l$, so it's $(-1)^{l-k-m} \times \binom{-m-1}{l-k-m}$. The sum for $k \leqslant l$ is the sum over all k, since $m \geqslant 0$. (The condition $n \geqslant 0$ isn't really needed, although k must assume negative values if $n < 0$.)

To go from (5.25) to (5.26), first replace s by $-1-n-q$.

5.15 If n is odd, the sum is zero, since we can replace k by $n-k$. If $n = 2m$, the sum is $(-1)^m(3m)!/m!^3$, by (5.29) with $a = b = c = m$.

5.16 This is just $(2a)!\,(2b)!\,(2c)!/(a+b)!\,(b+c)!\,(c+a)!$ times (5.29), if we write the summands in terms of factorials.

5.17 The formulas $\binom{2n-1/2}{n} = \binom{4n}{2n}/2^{2n}$ and $\binom{2n-1/2}{2n} = \binom{4n}{2n}/2^{4n}$ yield $\binom{2n-1/2}{n} = 2^{2n}\binom{2n-1/2}{2n}$.

5.18 $\binom{3r}{3k}\binom{3k}{k,k,k}/3^{3k}$.

5.19 $\mathcal{B}_{1-t}(-z)^{-1} = \sum_{k\geqslant 0}\binom{k-tk-1}{k}\bigl(-1/(k-tk-1)\bigr)(-z)^k$, by (5.60), and this is $\sum_{k\geqslant 0}\binom{tk}{k}\bigl(1/(tk-k+1)\bigr)z^k = \mathcal{B}_t(z)$.

5.20 It equals $F(-a_1,\ldots,-a_m;-b_1,\ldots,-b_n;(-1)^{m+n}z)$; see exercise 2.17.

5.21 $\lim_{n\to\infty}(n+m)^{\underline{m}}/n^m = 1$.

5.22 Multiplying and dividing instances of (5.83) gives

$$\begin{aligned}\frac{(-1/2)!}{x!\,(x-1/2)!} &= \lim_{n\to\infty}\binom{n+x}{n}\binom{n+x-1/2}{n}n^{-2x}\Big/\binom{n-1/2}{n}\\ &= \lim_{n\to\infty}\binom{2n+2x}{2n}n^{-2x},\end{aligned}$$

by (5.34) and (5.36). Also

$$1/(2x)! = \lim_{n\to\infty}\binom{2n+2x}{2n}(2n)^{-2x}.$$

Hence, etc. The Gamma function equivalent, incidentally, is

$$\Gamma(x)\,\Gamma(x+\tfrac{1}{2}) = \Gamma(2x)\,\Gamma(\tfrac{1}{2})/2^{2x-1}.$$

5.23 $(-1)^n n¡$, see (5.50).

5.24 This sum is $\binom{n}{m}F\bigl({m-n,-m \atop 1/2}\big|1\bigr) = \binom{2n}{2m}$, by (5.35) and (5.93).

5.25 This is equivalent to the easily proved identity

$$(a-b)\frac{a^{\overline{k}}}{(b+1)^{\overline{k}}} = a\,\frac{(a+1)^{\overline{k}}}{(b+1)^{\overline{k}}} - b\,\frac{a^{\overline{k}}}{b^{\overline{k}}}$$

as well as to the operator formula $a-b = (\vartheta+a)-(\vartheta+b)$.

Similarly, we have

$$\begin{aligned}&(a_1-a_2)\,F\left({a_1,a_2,a_3,\ldots,a_m \atop b_1,\ldots,b_n}\middle|\,z\right)\\ &\quad= a_1\,F\left({a_1{+}1,a_2,a_3,\ldots,a_m \atop b_1,\ldots,b_n}\middle|\,z\right) - a_2\,F\left({a_1,a_2{+}1,a_3,\ldots,a_m \atop b_1,\ldots,b_n}\middle|\,z\right),\end{aligned}$$

because $a_1-a_2=(a_1+k)-(a_2+k)$. If a_1-b_1 is a nonnegative integer d, this second identity allows us to express $F(a_1,\ldots,a_m;b_1,\ldots,b_n;z)$ as a linear combination of $F(a_2+j,a_3,\ldots,a_m;b_2,\ldots,b_n;z)$ for $0\leqslant j\leqslant d$, thereby eliminating an upper parameter and a lower parameter. Thus, for example, we get closed forms for $F(a,b;a-1;z)$, $F(a,b;a-2;z)$, etc.

Gauss [143, §7] derived analogous relations between $F(a,b;c;z)$ and any two "contiguous" hypergeometrics in which a parameter has been changed by ± 1. Rainville [301] generalized this to cases with more parameters.

5.26 If the term ratio in the original hypergeometric series is $t_{k+1}/t_k=r(k)$, the term ratio in the new one is $t_{k+2}/t_{k+1}=r(k+1)$. Hence

$$F\left({a_1,\,\ldots,\,a_m \atop b_1,\,\ldots,\,b_n}\middle|\,z\right) = 1+\frac{a_1\ldots a_m\,z}{b_1\ldots b_n}\,F\left({a_1{+}1,\ldots,a_m{+}1,1 \atop b_1{+}1,\ldots,b_n{+}1,2}\middle|\,z\right).$$

5.27 This is the sum of the even terms of $F(2a_1,\ldots,2a_m;2b_1,\ldots,2b_m;z)$. We have $(2a)^{\overline{2k+2}}/(2a)^{\overline{2k}}=4(k+a)(k+a+\frac12)$, etc.

Equating coefficients of z^n gives the Pfaff-Saalschütz formula (5.97).

5.28 $F\left({a,\,b \atop c}\middle|z\right) = (1-z)^{-a}F\left({a,\,c-b \atop c}\middle|\frac{-z}{1-z}\right) = (1-z)^{-a}F\left({c-b,\,a \atop c}\middle|\frac{-z}{1-z}\right) = (1-z)^{c-a-b}F\left({c-a,\,c-b \atop c}\middle|z\right)$. (Euler proved the identity by showing that both sides satisfy the same differential equation. The reflection law is often attributed to Euler, but it does not seem to appear in his published papers.)

5.29 The coefficients of z^n are equal, by Vandermonde's convolution. (Kummer's original proof was different: He considered $\lim_{m\to\infty}F(m,b-a;b;z/m)$ in the reflection law (5.101).)

5.30 Differentiate again to get $z(1-z)F''(z)+(2-3z)F'(z)-F(z)=0$. Therefore $F(z)=F(1,1;2;z)$ by (5.108).

5.31 The condition $f(k)=T(k+1)-T(k)$ implies that $f(k+1)/f(k) = \bigl(T(k+2)/T(k+1)-1\bigr)/\bigl(1-T(k)/T(k+1)\bigr)$ is a rational function of k.

5.32 When summing a polynomial in k, Gosper's method reduces to the "method of undetermined coefficients." We have $q(k)=r(k)=1$, and we try to solve $p(k)=s(k+1)-s(k)$. The method suggests letting $s(k)$ be a polynomial whose degree is $d=\deg(p)+1$.

5.33 The solution to $k=(k-1)s(k+1)-(k+1)s(k)$ is $s(k)=-k+\frac12$; hence the answer is $(1-2k)/2k(k-1)+C$.

5.34 The limiting relation holds because all terms for $k>c$ vanish, and $\epsilon-c$ cancels with $-c$ in the limit of the other terms. Therefore the second partial sum is $\lim_{\epsilon\to0}F(-m,-n;\epsilon-m;1)=\lim_{\epsilon\to0}(\epsilon+n-m)^{\overline{m}}/(\epsilon-m)^{\overline{m}} = (-1)^m\binom{n-1}{m}$.

5.35 (a) $2^{-n}3^n[n\geqslant0]$. (b) $(1-\frac12)^{-k-1}[k\geqslant0]=2^{k+1}[k\geqslant0]$.

5.36 The sum of the digits of $m+n$ is the sum of the digits of m plus the sum of the digits of n, minus $p-1$ times the number of carries, because each carry decreases the digit sum by $p-1$. [See [226] for extensions of this result to generalized binomial coefficients.]

5.37 Dividing the first identity by $n!$ yields $\binom{x+y}{n} = \sum_k \binom{x}{k}\binom{y}{n-k}$, Vandermonde's convolution. The second identity follows, for example, from the formula $x^{\overline{k}} = (-1)^k(-x)^{\underline{k}}$ if we negate both x and y.

5.38 Choose c as large as possible such that $\binom{c}{3} \leqslant n$. Then $0 \leqslant n - \binom{c}{3} < \binom{c+1}{3} - \binom{c}{3} = \binom{c}{2}$; replace n by $n - \binom{c}{3}$ and continue in the same fashion. Conversely, any such representation is obtained in this way. (We can do the same thing with

$$n = \binom{a_1}{1} + \binom{a_2}{2} + \cdots + \binom{a_m}{m}, \qquad 0 \leqslant a_1 < a_2 < \cdots < a_m$$

for any fixed m.)

5.39 $x^m y^n = \sum_{k=1}^{m} \binom{m+n-1-k}{n-1} a^n b^{m-k} x^k + \sum_{k=1}^{n} \binom{m+n-1-k}{m-1} a^{n-k} b^m y^k$ for all $mn > 0$, by induction on $m+n$.

5.40 $(-1)^{m+1} \sum_{k=1}^{n} \sum_{j=1}^{m} \binom{r}{j}\binom{m-rk-s-1}{m-j} = (-1)^m \sum_{k=1}^{n} \left(\binom{m-rk-s-1}{m} - \binom{m-r(k-1)-s-1}{m}\right) = (-1)^m\left(\binom{m-rn-s-1}{m} - \binom{m-s-1}{m}\right) = \binom{rn+s}{m} - \binom{s}{m}$.

5.41 $\sum_{k\geqslant 0} n!/(n-k)!\,(n+k+1)! = \bigl(n!/(2n+1)!\bigr) \sum_{k>n} \binom{2n+1}{k}$, which is $2^{2n} n!/(2n+1)!$.

5.42 We treat n as an indeterminate real variable. Gosper's method with $q(k) = k+1$ and $r(k) = k-1-n$ has the solution $s(k) = 1/(n+2)$; hence the desired indefinite sum is $(-1)^{x-1}\frac{n+1}{n+2}\big/\binom{n+1}{x}$. And

$$\sum_{k=0}^{n} (-1)^k \Big/ \binom{n}{k} = (-1)^{x-1} \frac{n+1}{n+2} \Big/ \binom{n+1}{x} \Bigg|_0^{n+1} = 2\,\frac{n+1}{n+2}\,[n \text{ even}]\,.$$

This exercise, incidentally, implies the formula

$$\frac{1}{n\dbinom{n-1}{k}} = \frac{1}{(n+1)\dbinom{n}{k+1}} + \frac{1}{(n+1)\dbinom{n}{k}},$$

a "dual" to the basic recurrence (5.8).

5.43 After the hinted first step we can apply (5.21) and sum on k. Then (5.21) applies again and Vandermonde's convolution finishes the job. (A combinatorial proof of this identity has been given by Andrews [10]. There's a quick way to go from this identity to a proof of (5.29), explained in [207, exercise 1.2.6–62].)

5.44 Cancellation of factorials shows that

$$\binom{m}{j}\binom{n}{k}\binom{m+n}{m} \;=\; \binom{m+n-j-k}{m-j}\binom{j+k}{j}\binom{m+n}{j+k},$$

so the second sum is $1/\binom{m+n}{m}$ times the first. And the first is just the special case $l=0$, $n=b$, $r=a$, $s=m+n-b$ of (5.32), so it is $\binom{a+b}{a}\binom{m+n-a-b}{n-a}$.

5.45 According to (5.9), $\sum_{k\leqslant n}\binom{k-1/2}{k} = \binom{n+1/2}{n}$. If this form of the answer isn't "closed" enough, we can apply (5.35) and get $(2n+1)\binom{2n}{n}4^{-n}$.

5.46 By (5.69), this convolution is the negative of the coefficient of z^{2n} in $\mathcal{B}_{-1}(z)\mathcal{B}_{-1}(-z)$. Now $(2\mathcal{B}_{-1}(z)-1)(2\mathcal{B}_{-1}(-z)-1) = \sqrt{1-16z^2}$; hence $\mathcal{B}_{-1}(z)\mathcal{B}_{-1}(-z) = \frac14\sqrt{1-16z^2}+\frac12\mathcal{B}_{-1}(z)+\frac12\mathcal{B}_{-1}(-z)-\frac14$. By the binomial theorem,

$$(1-16z^2)^{1/2} \;=\; \sum_n\binom{1/2}{n}(-16)^n z^{2n} \;=\; -\sum_n\binom{2n}{n}\frac{4^n z^{2n}}{2n-1},$$

so the answer is $\binom{2n}{n}4^{n-1}/(2n-1)+\binom{4n-1}{2n}/(4n-1)$.

5.47 It's the coefficient of z^n in $\bigl(\mathcal{B}_r(z)^s/\mathcal{Q}_r(z)\bigr)\bigl(\mathcal{B}_r(z)^{-s}/\mathcal{Q}_r(z)\bigr) = \mathcal{Q}_r(z)^{-2}$, where $\mathcal{Q}_r(z) = 1-r+r\mathcal{B}_r(z)^{-1}$, by (5.61).

The boxed sentence on the other side of this page is true.

5.48 $F(2n+2,1;n+2;\frac12) = 2^{2n+1}/\binom{2n+1}{n+1}$, a special case of (5.111).

5.49 Saalschütz's identity (5.97) yields

$$\binom{x+n}{n}\frac{y}{y+n}\,F\left({-x,\,-n,\,-n-y \atop -x-n,\,1-n-y}\,\middle|\,1\right) \;=\; \frac{(y-x)^{\overline{n}}}{(y+1)^{\overline{n}}}\,.$$

5.50 The left-hand side is

$$\begin{aligned}\sum_{k\geqslant 0}\frac{a^{\overline{k}}\,b^{\overline{k}}}{c^{\overline{k}}}\frac{(-z)^k}{k!}&\sum_{m\geqslant 0}\binom{k+a+m-1}{m}z^m\\ &=\sum_{n\geqslant 0}z^n\sum_{k\geqslant 0}\frac{a^{\overline{k}}\,b^{\overline{k}}}{c^{\overline{k}}\,k!}(-1)^k\binom{n+a-1}{n-k}\end{aligned}$$

and the coefficient of z^n is

$$\binom{n+a-1}{n}F\left({a,b,-n \atop c,a}\,\middle|\,1\right)\frac{a^{\overline{n}}}{n!} \;=\; \frac{(c-b)^{\overline{n}}}{c^{\overline{n}}}$$

by Vandermonde's convolution (5.92).

5.51 (a) Reflection gives $F(a,-n;2a;2) = (-1)^nF(a,-n;2a;2)$. (Incidentally, this formula implies the remarkable identity $\Delta^{2m+1}f(0) = 0$, when $f(n) = 2^n x^{\underline{n}}/(2x)^{\underline{n}}$.)

(b) The term-by-term limit is $\sum_{0\leqslant k\leqslant m}\binom{m}{k}\frac{2m+1}{2m+1-k}(-2)^k$ plus an additional term for $k=2m-1$. The additional term is

$$\frac{(-m)\ldots(-1)\,(1)\ldots(m)\,(-2m+1)\ldots(-1)\,2^{2m+1}}{(-2m)\ldots(-1)\,(2m-1)!}$$
$$= (-1)^{m+1}\frac{m!\,m!\,2^{2m+1}}{(2m)!} = \frac{-2}{\binom{-1/2}{m}};$$

hence, by (5.104), this limit is $-1/\binom{-1/2}{m}$, the negative of what we had.

5.52 The terms of both series are zero for $k>N$. This identity corresponds to replacing k by $N-k$. Notice that

$$\begin{aligned} a^{\overline{N}} &= a^{\overline{N-k}}\,(a+N-k)^{\overline{k}} \\ &= a^{\overline{N-k}}\,(a+N-1)^{\underline{k}} = a^{\overline{N-k}}\,(1-a-N)^{\overline{k}}(-1)^k\,. \end{aligned}$$

5.53 When $b=-\frac{1}{2}$, the left side of (5.110) is $1-2z$ and the right side is $(1-4z+4z^2)^{1/2}$, independent of a. The right side is the formal power series

$$1+\binom{1/2}{1}4z(z-1)+\binom{1/2}{2}16z^2(z-1)^2+\cdots,$$

which can be expanded and rearranged to give $1-2z+0z^2+0z^3+\cdots$; but the rearrangement involves divergent series in its intermediate steps when $z=1$, so it is not legitimate.

The boxed sentence on the other side of this page is false.

5.54 If $m+n$ is odd, say $2N-1$, we want to show that

$$\lim_{\epsilon\to 0} F\left({N-m-\frac{1}{2},\,-N+\epsilon \atop -m+\epsilon}\middle|\,1\right) = 0\,.$$

Equation (5.92) applies, since $-m+\epsilon > -m-\frac{1}{2}+\epsilon$, and the denominator factor $\Gamma(c-b)=\Gamma(N-m)$ is infinite since $N\leqslant m$; the other factors are finite. Otherwise $m+n$ is even; setting $n=m-2N$ we have

$$\lim_{\epsilon\to 0} F\left({-N,\,N-m-\frac{1}{2}+\epsilon \atop -m+\epsilon}\middle|\,1\right) = \frac{(N-1/2)^{\underline{N}}}{m^{\underline{N}}}$$

by (5.93). The remaining job is to show that

$$\binom{m}{m-2N}\frac{(N-1/2)!}{(-1/2)!}\frac{(m-N)!}{m!} = \binom{m-N}{m-2N}2^{-2N},$$

and this is the case $x=N$ of exercise 22.

5.55 Let $Q(k) = (k+A_1)\ldots(k+A_M)Z$ and $R(k) = (k+B_1)\ldots(k+B_N)$. Then $t(k+1)/t(k) = P(k)Q(k-1)/P(k-1)R(k)$, where $P(k) = Q(k) - R(k)$ is a nonzero polynomial.

5.56 The solution to $-(k+1)(k+2) = s(k+1)+s(k)$ is $s(k) = -\frac{1}{2}k^2 - k - \frac{1}{4}$; hence $\sum \binom{-3}{k}\,\delta k = \frac{1}{8}(-1)^{k-1}(2k^2+4k+1) + C$. Also

$$\begin{aligned}
&(-1)^{k-1}\left\lfloor\frac{k+1}{2}\right\rfloor\left\lfloor\frac{k+2}{2}\right\rfloor\\
&\quad= \frac{(-1)^{k-1}}{4}\left(k+1-\frac{1+(-1)^k}{2}\right)\left(k+2-\frac{1-(-1)^k}{2}\right)\\
&\quad= \frac{(-1)^{k-1}}{8}(2k^2+4k+1)+\frac{1}{8}\,.
\end{aligned}$$

5.57 We have $t(k+1)/t(k) = (k-n)(k+1+\theta)(-z)/(k+1)(k+\theta)$. Therefore we let $p(k) = k+\theta$, $q(k) = (k-n)(-z)$, $r(k) = k$. The secret function $s(k)$ must be a constant α_0, and we have

$$k+\theta \;=\; \bigl(-z(k-n)-k\bigr)\,\alpha_0\,;$$

hence $\alpha_0 = -1/(1+z)$ and $\theta = -nz/(1+z)$. The sum is

$$\sum\binom{n}{k}z^k\left(k-\frac{nz}{1+z}\right)\delta k \;=\; -\frac{n}{1+z}\binom{n-1}{k-1}z^k + C\,.$$

(The special case $z = 1$ was mentioned in (5.18).)

5.58 If $m > 0$ we can replace $\binom{k}{m}$ by $\frac{k}{m}\binom{k-1}{m-1}$ and derive the formula $T_{m,n} = \frac{n}{m}T_{m-1,n-1} - \frac{1}{m}\binom{n-1}{m}$. The summation factor $\binom{n}{m}^{-1}$ is therefore appropriate:

$$\frac{T_{m,n}}{\binom{n}{m}} \;=\; \frac{T_{m-1,n-1}}{\binom{n-1}{m-1}} - \frac{1}{m} + \frac{1}{n}\,.$$

We can unfold this to get

$$\frac{T_{m,n}}{\binom{n}{m}} \;=\; T_{0,n-m} - H_m + H_n - H_{n-m}\,.$$

Finally $T_{0,n-m} = H_{n-m}$, so $T_{m,n} = \binom{n}{m}(H_n - H_m)$. (It's also possible to derive this result by using generating functions; see Example 2 in Section 7.5.)

5.59 $\sum_{j\geqslant 0,\,k\geqslant 1}\binom{n}{j}\bigl[j=\lfloor\log_m k\rfloor\bigr] = \sum_{j\geqslant 0,\,k\geqslant 1}\binom{n}{j}[m^j\leqslant k<m^{j+1}]$, which is $\sum_{j\geqslant 0}\binom{n}{j}(m^{j+1}-m^j) = (m-1)\sum_{j\geqslant 0}\binom{n}{j}m^j = (m-1)(m+1)^n$.

5.60 $\binom{2n}{n} \approx 4^n/\sqrt{\pi n}$ is the case $m = n$ of

$$\binom{m+n}{n} \;\approx\; \sqrt{\frac{1}{2\pi}\Bigl(\frac{1}{m}+\frac{1}{n}\Bigr)}\Bigl(1+\frac{m}{n}\Bigr)^n\Bigl(1+\frac{n}{m}\Bigr)^m .$$

5.61 Let $\lfloor n/p \rfloor = q$ and $n \bmod p = r$. The polynomial identity $(x+1)^p \equiv x^p + 1 \pmod p$ implies that

$$(x+1)^{pq+r} \;\equiv\; (x+1)^r (x^p+1)^q \pmod p .$$

The coefficient of x^m on the left is $\binom{n}{m}$. On the right it's $\sum_k \binom{r}{m-pk}\binom{q}{k}$, which is just $\binom{r}{m \bmod p}\binom{q}{\lfloor m/p \rfloor}$ because $0 \leqslant r < p$.

5.62 $\binom{np}{mp} = \sum_{k_1+\cdots+k_n=mp}\binom{p}{k_1}\ldots\binom{p}{k_n} \equiv \binom{n}{m} \pmod{p^2}$, because all terms of the sum are multiples of p^2 except for the $\binom{n}{m}$ terms in which exactly m of the k's are equal to p. (Stanley [335, exercise 1.6(d)] shows that the congruence actually holds modulo p^3 when $p > 3$.)

5.63 This is $S_n = \sum_{k=0}^{n}(-4)^k\binom{n+k}{n-k} = \sum_{k=0}^{n}(-4)^{n-k}\binom{2n-k}{k}$. The denominator of (5.74) is zero when $z = -1/4$, so we can't simply plug into that formula. The recurrence $S_n = -2S_{n-1} - S_{n-2}$ leads to the solution $S_n = (-1)^n(2n+1)$.

5.64 $\sum_{k\geqslant 0}\bigl(\binom{n}{2k} + \binom{n}{2k+1}\bigr)/(k+1) = \sum_{k\geqslant 0}\binom{n+1}{2k+1}/(k+1)$, which is

$$\frac{2}{n+2}\sum_{k\geqslant 0}\binom{n+2}{2k+2} \;=\; \frac{2^{n+2}-2}{n+2} .$$

5.65 Multiply both sides by n^{n-1} and replace k by $n-1-k$ to get

$$\begin{aligned}\sum_k \binom{n-1}{k} n^k (n-k)! \;&=\; (n-1)!\sum_{k=0}^{n-1}\bigl(n^{k+1}/k! - n^k/(k-1)!\bigr)\\ &=\; (n-1)!\,n^n/(n-1)! .\end{aligned}$$

(The partial sums can, in fact, be found by Gosper's algorithm.) Alternatively, $\binom{n}{k}kn^{n-1-k}k!$ can be interpreted as the number of mappings of $\{1,\ldots,n\}$ into itself with $f(1), \ldots, f(k)$ distinct but $f(k+1) \in \{f(1),\ldots,f(k)\}$; summing on k must give n^n.

5.66 This is a walk-the-garden-path problem where there's only one "obvious" way to proceed at every step. First replace $k-j$ by l, then replace $\lfloor\sqrt{l}\,\rfloor$ by k, getting

$$\sum_{j,k\geqslant 0}\binom{-1}{j-k}\binom{j}{m}\frac{2k+1}{2^j} .$$

The infinite series converges because the terms for fixed j are dominated by a polynomial in j divided by 2^j. Now sum over k, getting

$$\sum_{j\geqslant 0}\binom{j}{m}\frac{j+1}{2^j}\,.$$

Absorb the $j+1$ and apply (5.57) to get the answer, $4(m+1)$.

5.67 $3\binom{2n+2}{n+5}$ by (5.26), because

$$\binom{\binom{k}{2}}{2} \;=\; 3\binom{k+1}{4}\,.$$

5.68 Using the fact that

$$\sum_{k\leqslant n/2}\binom{n}{k} \;=\; 2^{n-1}+\frac{1}{2}\binom{n}{n/2}[n \text{ is even}]\,,$$

we get $n\bigl(2^{n-1}-\binom{n-1}{\lfloor n/2\rfloor}\bigr)$.

The boxed sentence on the other side of this page is not a sentence.

5.69 Since $\binom{k+1}{2}+\binom{l-1}{2}\leqslant\binom{k}{2}+\binom{l}{2}\iff k<l$, the minimum occurs when the k's are as equal as possible. Hence, by the equipartition formula of Chapter 3, the minimum is

$$\begin{aligned}(n \bmod m)&\binom{\lceil n/m\rceil}{2}+\bigl(n-(n \bmod m)\bigr)\binom{\lfloor n/m\rfloor}{2}\\ &= n\binom{\lfloor n/m\rfloor}{2}+(n \bmod m)\left\lfloor\frac{n}{m}\right\rfloor\,.\end{aligned}$$

A similar result holds for any lower index in place of 2.

5.70 This is $F(-n,\frac12;1;2)$; but it's also $(-2)^{-n}\binom{2n}{n}F(-n,-n;\frac12-n;\frac12)$ if we replace k by $n-k$. Now $F(-n,-n;\frac12-n;\frac12)=F(-\frac{n}{2},-\frac{n}{2};\frac12-n;1)$ by Gauss's identity (5.111). (Alternatively, $F(-n,-n;\frac12-n;\frac12)=2^{-n}F(-n,\frac12;\frac12-n;-1)$ by the reflection law (5.101), and Kummer's formula (5.94) relates this to (5.55).) The answer is 0 when n is odd, $2^{-n}\binom{n}{n/2}$ when n is even. (See [164, §1.2] for another derivation. This sum arises in the study of a simple search algorithm [195].)

5.71 (a) Observe that

$$S(z) \;=\; \sum_{k\geqslant 0}a_k\frac{z^{m+k}}{(1-z)^{m+2k+1}} \;=\; \frac{z^m}{(1-z)^{m+1}}A\bigl(z/(1-z)^2\bigr)\,.$$

(b) Here $A(z)=\sum_{k\geqslant 0}\binom{2k}{k}(-z)^k/(k+1)=\bigl(\sqrt{1+4z}-1\bigr)/2z$, so we have $A\bigl(z/(1-z)^2\bigr)=1-z$. Thus $S_n=[z^n]\,\bigl(z/(1-z)\bigr)^m=\binom{n-1}{n-m}$.

5.72 The stated quantity is $m(m-n)\ldots\bigl(m-(k-1)n\bigr)n^{k-\nu(k)}/k!$. Any prime divisor p of n divides the numerator at least $k-\nu(k)$ times and divides the denominator at most $k-\nu(k)$ times, since this is the number of times 2 divides $k!$. A prime p that does not divide n must divide the product $m(m-n)\ldots\bigl(m-(k-1)n\bigr)$ at least as often as it divides $k!$, because $m(m-n)\ldots\bigl(m-(p^r-1)n\bigr)$ is a multiple of p^r for all $r \geqslant 1$ and all m.

5.73 Plugging in $X_n = n!$ yields $\alpha = \beta = 1$; plugging in $X_n = n¡$ yields $\alpha = 1$, $\beta = 0$. Therefore the general solution is $X_n = \alpha n¡ + \beta(n! - n¡)$.

5.74 $\binom{n+1}{k} - \binom{n-1}{k-1}$, for $1 \leqslant k \leqslant n$.

5.75 The recurrence $S_k(n+1) = S_k(n) + S_{(k-1) \bmod 3}(n)$ makes it possible to verify inductively that two of the S's are equal and that $S_{(-n) \bmod 3}(n)$ differs from them by $(-1)^n$. These three values split their sum $S_0(n)+S_1(n)+S_2(n) = 2^n$ as equally as possible, so there must be $2^n \bmod 3$ occurrences of $\lceil 2^n/3 \rceil$ and $3-(2^n \bmod 3)$ occurrences of $\lfloor 2^n/3 \rfloor$.

5.76 $Q_{n,k} = (n+1)\binom{n}{k} - \binom{n}{k+1}$.

5.77 The terms are zero unless $k_1 \leqslant \cdots \leqslant k_m$, when the product is the multinomial coefficient

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$$\binom{k_m}{k_1,\, k_2-k_1,\, \ldots,\, k_m-k_{m-1}}.$$

Therefore the sum over $k_1, \ldots, k_{m-1}$ is m^{k_m}, and the final sum over k_m yields $(m^{n+1}-1)/(m-1)$.

5.78 Extend the sum to $k = 2m^2+m-1$; the new terms are $\binom{1}{4} + \binom{2}{6} + \cdots + \binom{m-1}{2m} = 0$. Since $m \perp (2m+1)$, the pairs $\bigl(k \bmod m, k \bmod (2m+1)\bigr)$ are distinct. Furthermore, the numbers $(2j+1) \bmod (2m+1)$ as j varies from 0 to $2m$ are the numbers $0, 1, \ldots, 2m$ in some order. Hence the sum is

$$\sum_{\substack{0\leqslant k<m\\0\leqslant j<2m+1}} \binom{k}{j} \;=\; \sum_{0\leqslant k<m} 2^k \;=\; 2^m - 1\,.$$

5.79 (a) The sum is 2^{2n-1}, so the gcd must be a power of 2. If $n = 2^k q$ where q is odd, $\binom{2n}{1}$ is divisible by 2^{k+1} and not by 2^{k+2}. Each $\binom{2n}{2j+1}$ is divisible by 2^{k+1} (see exercise 36), so this must be the gcd. (b) If $p^r \leqslant n+1 < p^{r+1}$, we get the most radix p carries by adding k to $n-k$ when $k = p^r-1$. The number of carries in this case is $r - \epsilon_p(n+1)$, and $r = \epsilon_p\bigl(L(n+1)\bigr)$.

5.80 First prove by induction that $k! \geqslant (k/e)^k$.

5.81 Let $f_{l,m,n}(x)$ be the left-hand side. It is sufficient to show that we have $f_{l,m,n}(1) > 0$ and that $f'_{l,m,n}(x) < 0$ for $0 \leqslant x \leqslant 1$. The value of $f_{l,m,n}(1)$ is $(-1)^{n-m-1}\binom{l+m+\theta}{l+n}$ by (5.23), and this is positive because the binomial coefficient has exactly $n-m-1$ negative factors. The inequality is true when $l = 0$, for the same reason. If $l > 0$, we have $f'_{l,m,n}(x) = -l\,f_{l-1,m,n+1}(x)$, which is negative by induction.

5.82 Let $\epsilon_p(a)$ be the exponent by which the prime p divides a, and let $m = n-k$. The identity to be proved reduces to

$$\begin{aligned}&\min\bigl(\epsilon_p(m){-}\epsilon_p(m{+}k),\, \epsilon_p(m{+}k{+}1){-}\epsilon_p(k{+}1),\, \epsilon_p(k){-}\epsilon_p(m{+}1)\bigr)\\ &\quad= \min\bigl(\epsilon_p(k){-}\epsilon_p(m{+}k),\, \epsilon_p(m){-}\epsilon_p(k{+}1),\, \epsilon_p(m{+}k{+}1){-}\epsilon_p(m{+}1)\bigr)\,.\end{aligned}$$

For brevity let's write this as $\min(x_1, y_1, z_1) = \min(x_2, y_2, z_2)$. Notice that $x_1 + y_1 + z_1 = x_2 + y_2 + z_2$. The general relation

$$\epsilon_p(a) \;<\; \epsilon_p(b) \qquad\Longrightarrow\qquad \epsilon_p(a) \;=\; \epsilon_p\bigl(|a \pm b|\bigr)$$

allows us to conclude that $x_1 \neq x_2 \Longrightarrow \min(x_1, x_2) = 0$; the same holds also for (y_1, y_2) and (z_1, z_2). It's now a simple matter to complete the proof.

5.83 (Solution by P. Paule.) Let r be a nonnegative integer. The given sum is the coefficient of $x^l y^m$ in

$$\begin{aligned}&\sum_{j,k}(-1)^{j+k}\frac{(1+x)^{j+k}}{x^k}\binom{r}{j}\binom{n}{k}(1+y)^{s+n-j-k}y^j\\ &\qquad= \left(1-\frac{(1+x)y}{1+y}\right)^{r}\left(1-\frac{1+x}{(1+y)x}\right)^{n}(1+y)^{s+n}\\ &\qquad= (-1)^n(1-xy)^{n+r}(1+y)^{s-r}/x^n\,,\end{aligned}$$

so it is clearly $(-1)^l\binom{n+r}{n+l}\binom{s-r}{m-n-l}$. (See also exercise 106.)

5.84 Following the hint, we get

$$z\mathcal{B}_t(z)^{r-1}\mathcal{B}'_t(z) \;=\; \sum_{k\geqslant 0}\binom{tk+r}{k}\frac{kz^k}{tk+r}\,,$$

and a similar formula for $\mathcal{E}_t(z)$. Thus the formulas $\bigl(zt\mathcal{B}_t^{-1}(z)\mathcal{B}'_t(z){+}1\bigr)\mathcal{B}_t(z)^r$ and $\bigl(zt\mathcal{E}_t^{-1}(z)\mathcal{E}'_t(z)+1\bigr)\mathcal{E}_t(z)^r$ give the respective right-hand sides of (5.61). We must therefore prove that

$$\begin{aligned}\bigl(zt\mathcal{B}_t^{-1}(z)\mathcal{B}'_t(z)+1\bigr)\mathcal{B}_t(z)^r \;&=\; \frac{1}{1-t+t\mathcal{B}_t(z)^{-1}}\,,\\ \bigl(zt\mathcal{E}_t^{-1}(z)\mathcal{E}'_t(z)+1\bigr)\mathcal{E}_t(z)^r \;&=\; \frac{1}{1-zt\mathcal{E}(z)^t}\,,\end{aligned}$$

and these follow from (5.59).

5.85 If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ is any polynomial of degree $\leqslant n$, we can prove inductively that

$$\sum_{0\leqslant \epsilon_1,\ldots,\epsilon_n\leqslant 1} (-1)^{\epsilon_1+\cdots+\epsilon_n} f(\epsilon_1 x_1+\cdots+\epsilon_n x_n) \;=\; (-1)^n n!\, a_n x_1 \ldots x_n\,.$$

The stated identity is the special case where $a_n = 1/n!$ and $x_k = k^3$.

5.86 (a) First expand with $n(n-1)$ index variables l_{ij} for all $i \neq j$. Setting $k_{ij} = l_{ij} - l_{ji}$ for $1 \leqslant i < j < n$ and using the constraints $\sum_{i\neq j}(l_{ij} - l_{ji}) = 0$ for all $i < n$ allows us to carry out the sums on l_{jn} for $1 \leqslant j < n$ and then on l_{ji} for $1 \leqslant i < j < n$ by Vandermonde's convolution. (b) $f(z) - 1$ is a polynomial of degree $< n$ that has n roots, so it must be zero. (c) Consider the constant terms in

$$\prod_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \left(1-\frac{z_i}{z_j}\right)^{a_i} = \sum_{k=1}^{n} \prod_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} \left(1-\frac{z_i}{z_j}\right)^{a_i-[i=k]}.$$

5.87 The first term is $\sum_k \binom{n-k}{k} z^{mk}$, by (5.61). The summands in the second term are

$$\begin{aligned}\frac{1}{m}\sum_{k\geqslant 0} &\binom{(n+1)/m + (1{+}1/m)k}{k} (\zeta z)^{k+n+1}\\ &= \frac{1}{m}\sum_{k>n}\binom{(1{+}1/m)k - n - 1}{k-n-1}(\zeta z)^k.\end{aligned}$$

Since $\sum_{0\leqslant j<m}(\zeta^{2j+1})^k = m(-1)^l[k{=}ml]$, these terms sum to

$$\begin{aligned}\sum_{k>n/m} &\binom{(1{+}1/m)mk - n - 1}{mk-n-1}(-z^m)^k\\ &= \sum_{k>n/m}\binom{(m{+}1)k - n - 1}{k}(-z^m)^k = \sum_{k>n/m}\binom{n-mk}{k} z^{mk}.\end{aligned}$$

Incidentally, the functions $\mathcal{B}_m(z^m)$ and $\zeta^{2j+1} z\,\mathcal{B}_{1+1/m}(\zeta^{2j+1}z)^{1/m}$ are the $m+1$ complex roots of the equation $w^{m+1} - w^m = z^m$.

5.88 Use the facts that $\int_0^\infty (e^{-t} - e^{-nt})\,dt/t = \ln n$ and $(1-e^{-t})/t \leqslant 1$. (We have $\binom{x}{k} = O(k^{-x-1})$ as $k \to \infty$, by (5.83); so this bound implies that Stirling's series $\sum_k s_k \binom{x}{k}$ converges when $x > -1$. Hermite [186] showed that the sum is $\ln \Gamma(1+x)$.)

5.89 Adding this to (5.19) gives $y^{-r}(x+y)^{m+r}$ on both sides, by the binomial theorem. Differentiation gives

$$\sum_{k>m}\binom{m+r}{k}\binom{m-k}{n}x^k y^{m-k-n} = \sum_{k>m}\binom{-r}{k}\binom{m-k}{n}(-x)^k(x+y)^{m-k-n}\,,$$

and we can replace k by $k+m+1$ and apply (5.15) to get

$$\sum_{k\geqslant 0}\binom{m+r}{m+1+k}\binom{-n-1}{k}(-x)^{m+1+k}y^{-1-k-n} = \sum_{k\geqslant 0}\binom{-r}{m+1+k}\binom{-n-1}{k}x^{m+1+k}(x+y)^{-1-k-n}\,.$$

In hypergeometric form, this reduces to

$$F\left({1-r,\ n+1 \atop m+2}\,\middle|\,\frac{-x}{y}\right) = \left(1+\frac{x}{y}\right)^{-n-1} F\left({m+1+r,\ n+1 \atop m+2}\,\middle|\,\frac{x}{x+y}\right),$$

which is the special case $(a,b,c,z) = (n+1, m+1+r, m+2, -x/y)$ of the reflection law (5.101). (Thus (5.105) is related to reflection and to the formula in exercise 52.)

The boxed sentence on the other side of this page is self-referential.

5.90 If r is a nonnegative integer, the sum is finite, and the derivation in the text is valid as long as none of the terms of the sum for $0 \leqslant k \leqslant r$ has zero in the denominator. Otherwise the sum is infinite, and the kth term $\binom{k-r-1}{k}/\binom{k-s-1}{k}$ is approximately $k^{s-r}(-s-1)!/(-r-1)!$ by (5.83). So we need $r > s+1$ if the infinite series is going to converge. (If r and s are complex, the condition is $\Re r > \Re s + 1$, because $|k^z| = k^{\Re z}$.) The sum is

$$F\left({-r,\ 1 \atop -s}\,\middle|\,1\right) = \frac{\Gamma(r-s-1)\Gamma(-s)}{\Gamma(r-s)\Gamma(-s-1)} = \frac{s+1}{s+1-r}$$

by (5.92); this is the same formula we found when r and s were integers.

5.91 (It's best to have computer help for this.) Incidentally, when $c = (a+1)/2$, this reduces to an identity that's equivalent to Gauss's identity (5.110), in view of Pfaff's reflection law. For if $w = -z/(1-z)$ we have $4w(1-w) = -4z/(1-z)^2$, and

$$F\left({\frac12 a,\ \frac12 a+\frac12-b \atop 1+a-b}\,\middle|\,4w(1-w)\right) = F\left({a,\ a+1-2b \atop 1+a-b}\,\middle|\,\frac{-z}{1-z}\right) = (1-z)^a\,F\left({a,\ b \atop 1+a-b}\,\middle|\,z\right).$$

5.92 The identities can be proved, as Clausen proved them more than 150 years ago, by showing that both sides satisfy the same differential equation. One way to write the resulting equations between coefficients of z^n is in terms of binomial coefficients:

$$\sum_k \frac{\binom{r}{k}\binom{s}{k}\binom{r}{n-k}\binom{s}{n-k}}{\binom{r+s-1/2}{k}\binom{r+s-1/2}{n-k}} = \frac{\binom{2r}{n}\binom{r+s}{n}\binom{2s}{n}}{\binom{2r+2s}{n}\binom{r+s-1/2}{n}};$$

$$\sum_k \frac{\binom{-1/4+r}{k}\binom{-1/4+s}{k}\binom{-1/4-r}{n-k}\binom{-1/4-s}{n-k}}{\binom{-1+r+s}{k}\binom{-1-r-s}{n-k}} = \frac{\binom{-1/2}{n}\binom{-1/2+r-s}{n}\binom{-1/2-r+s}{n}}{\binom{-1+r+s}{n}\binom{-1-r-s}{n}}.$$

Another way is in terms of hypergeometrics:

$$F\left({a,b,\frac12-a-b-n,-n \atop \frac12+a+b,1-a-n,1-b-n}\,\Big|\,1\right) = \frac{(2a)^{\overline{n}}\,(a+b)^{\overline{n}}\,(2b)^{\overline{n}}}{(2a+2b)^{\overline{n}}\,a^{\overline{n}}\,b^{\overline{n}}};$$

$$F\left({\frac14+a,\frac14+b,a+b-n,-n \atop 1+a+b,\frac34+a-n,\frac34+b-n}\,\Big|\,1\right) = \frac{(1/2)^{\overline{n}}\,(1/2+a-b)^{\overline{n}}\,(1/2-a+b)^{\overline{n}}}{(1+a+b)^{\overline{n}}\,(1/4-a)^{\overline{n}}\,(1/4-b)^{\overline{n}}}.$$

The boxed sentence on the other side of this page is not self-referential.

5.93 $\alpha^{-1}\prod_{j=1}^{k}\bigl(f(j)+\alpha\bigr)/f(j)$.

5.94 Gosper's algorithm finds the answer $-\binom{a-1}{k-1}\binom{-a-1}{n-k}a/n+C$. Consequently, when $m\geqslant 0$ is an integer, we have

$$\sum\binom{a}{k}\binom{m-a}{n-k}\delta k = \sum_j\binom{m}{j}\frac{-a}{n-j}\binom{a-1}{k-1}\binom{-a-1}{n-j-k}+C.$$

5.95 The leading coefficients of p and r should be unity, and p should have no factors in common with q or r. It is easy to fulfill these additional conditions by shuffling factors around.

Now suppose $p(k+1)q(k)/p(k)r(k+1) = P(k+1)Q(k)/P(k)R(k+1)$, where the polynomials (p,q,r) and (P,Q,R) both satisfy the new criteria. Let

$p_0(k) = p(k)/g(k)$ and $P_0(k) = P(k)/g(k)$, where $g(k) = \gcd\bigl(p(k), P(k)\bigr)$ is the product of all common factors of p and P. Then

$$p_0(k+1)q(k)P_0(k)R(k+1) \;=\; p_0(k)r(k+1)P_0(k+1)Q(k)\,.$$

Suppose $p_0(k) \neq 1$. Then there is a complex number α such that $p_0(\alpha) = 0$; this implies $q(\alpha) \neq 0$, $r(\alpha) \neq 0$, and $P_0(\alpha) \neq 0$. Hence we must have $p_0(\alpha+1)R(\alpha+1) = 0$ and $p_0(\alpha-1)Q(\alpha-1) = 0$. Let N be a positive integer such that $p_0(\alpha+N) \neq 0$ and $p_0(\alpha-N) \neq 0$. Repeating the argument N times, we find $R(\alpha+1)\ldots R(\alpha+N) = 0 = Q(\alpha-1)\ldots Q(\alpha-N)$, contradicting (5.118). Therefore $p_0(k) = 1$. Similarly $P_0(k) = 1$, so $p(k) = P(k)$. Now $q(\alpha) = 0$ implies $r(\alpha+1) \neq 0$, by (5.118), hence $q(k)\backslash Q(k)$. Similarly $Q(k)\backslash q(k)$, so $q(k) = Q(k)$ since they have the same leading coefficient. That leaves $r(k) = R(k)$.

5.96 If $r(k)$ is a nonzero rational function and $T(k)$ is a hypergeometric term, then $r(k)T(k)$ is a hypergeometric term, which is called *similar* to $T(k)$. (We allow $r(k)$ to be ∞ and $T(k)$ to be 0, or vice versa, for finitely many values of k.) In particular, $T(k+1)$ is always similar to $T(k)$. If $T_1(k)$ and $T_2(k)$ are similar hypergeometric terms, then $T_1(k) + T_2(k)$ is a hypergeometric term. If $T_1(k)$, …, $T_m(k)$ are mutually dissimilar, and $m > 1$, then $T_1(k) + \cdots + T_m(k)$ cannot be zero for all but finitely many k. For if it could, consider a counterexample for which m is minimum, and let $r_j(k) = T_j(k+1)/T_j(k)$. Since $T_1(k) + \cdots + T_m(k) = 0$, we have $r_m(k)T_1(k) + \cdots + r_m(k)T_m(k) = 0$ and $r_1(k)T_1(k) + \cdots + r_m(k)T_m(k) = T_1(k+1) + \cdots + T_m(k+1) = 0$; hence $\bigl(r_m(k)-r_1(k)\bigr)T_1(k) + \cdots + \bigl(r_m(k)-r_{m-1}(k)\bigr)T_{m-1}(k) = 0$. We cannot have $r_m(k) - r_j(k) = 0$, for any $j < m$, since T_j and T_m are dissimilar. But m was minimum, so this cannot be a counterexample; it follows that $m = 2$. But then $T_1(k)$ and $T_2(k)$ must be similar, since they are both zero for all but finitely many k.

Burma-Shave

Now let $t(k)$ be any hypergeometric term with $t(k+1)/t(k) = r(k)$, and suppose that $t(k) = \bigl(T_1(k+1)+\cdots+T_m(k+1)\bigr) - \bigl(T_1(k)+\cdots+T_m(k)\bigr)$, where m is minimal. Then T_1, …, T_m must be mutually dissimilar. Let $r_j(k)$ be the rational function such that

$$r(k)\bigl(T_j(k+1) - T_j(k)\bigr) - \bigl(T_j(k+2) - T_j(k+1)\bigr) \;=\; r_j(k)T_j(k)\,.$$

Suppose $m > 1$. Since $0 = r(k)t(k) - t(k+1) = r_1(k)T_1(k) + \cdots + r_m(k)T_m(k)$, we must have $r_j(k) = 0$ for all but at most one value of j. If $r_j(k) = 0$, the function $\bar{t}(k) = T_j(k+1) - T_j(k)$ satisfies $\bar{t}(k+1)/\bar{t}(k) = t(k+1)/t(k)$. So Gosper's algorithm will find a solution.

5.97 Suppose first that z is not equal to $-d - 1/d$ for any integer $d > 0$. Then in Gosper's algorithm we have $p(k) = 1$, $q(k) = (k+1)^2$, $r(k) =$

$k^2 + kz + 1$. Since $\deg(Q) < \deg(R)$ and $\deg(p) - \deg(R) + 1 = -1$, the only possibility is $z = d + 2$ where d is a nonnegative integer. Trying $s(k) = \alpha_d k^d + \cdots + \alpha_0$ fails when $d = 0$ but succeeds whenever $d > 0$. (The linear equations obtained by equating coefficients of k^d, k^{d-1}, ..., k^1 in (5.122) express $\alpha_{d-1}, \ldots, \alpha_0$ as positive multiples of α_d, and the remaining equation $1 = \alpha_d + \cdots + \alpha_1$ then defines α_d.) For example, when $z = 3$ the indefinite sum is $(k+2)k!^2 / \prod_{j=1}^{k-1}(j^2 + 3j + 1) + C$.

If $z = -d - 1/d$, on the other hand, the stated terms $t(k)$ are infinite for $k \geqslant d$. There are two reasonable ways to proceed: We can cancel the zero in the denominator by redefining

$$t(k) \;=\; \frac{k!^2}{\prod_{j=d+1}^{k}\bigl(j^2 - j(d+1/d) + 1\bigr)} \;=\; \frac{(d-1/d)!\,k!^2}{(k-1/d)!\,(k-d)!}\,,$$

thereby making $t(k) = 0$ for $0 \leqslant k < d$ and positive for $k \geqslant d$. Then Gosper's algorithm gives $p(k) = k^{\underline{d}}$, $q(k) = k+1$, $r(k) = k - 1/d$, and we can solve (5.122) for $s(k)$ because the coefficient of k^j on the right is $(j + 1 + 1/d)\alpha^j$ plus multiples of $\{\alpha_{j+1}, \ldots, \alpha_d\}$. For example, when $d = 2$ the indefinite sum is $(3/2)!\,k!\,(\frac{2}{7}k^2 - \frac{26}{35}k + \frac{32}{105})/(k-3/2)! + C$.

Alternatively, we can try to sum the original terms, but only in the range $0 \leqslant k < d$. Then we can replace $p(k) = k^{\underline{d}}$ by

Look, any finite sequence is trivially summable, because we can find a polynomial that matches $t(k)$ for $0 \leqslant k < d$.

$$p'(k) \;=\; \sum_{j=1}^{d} (-1)^{d-j} j \begin{bmatrix} d \\ j \end{bmatrix} k^{j-1}\,.$$

This is justified since (5.117) still holds for $0 \leqslant k < d - 1$; we have $p'(k) = \lim_{\epsilon\to 0}\bigl((k+\epsilon)^{\underline{d}} - k^{\underline{d}}\bigr)/\epsilon = \lim_{\epsilon\to 0}(k+\epsilon)^{\underline{d}}/\epsilon$, so this trick essentially cancels a 0 from the numerator and denominator of (5.117) as in L'Hospital's rule. Gosper's method now yields an indefinite sum.

5.98 $nS_{n+1} = 2nS_n$. (Beware: This gives no information about S_1/S_0.)

5.99 Let $p(n,k) = (n+1+k)\beta_0(n) + (n+1+a+b+c+k)\beta_1(n) = \hat{p}(n,k)$, $\bar{t}(n,k) = t(n,k)/(n+1+k)$, $q(n,k) = (n+1+a+b+c+k)(a-k)(b-k)$, $r(n,k) = (n+1+k)(c+k)k$. Then (5.129) is solved by $\beta_0(n) = (n+1+a+b+c)(n+1+a+b)$, $\beta_1(n) = -(n+1+a)(n+1+b)$, $\alpha_0(n) = s(n,k) = -1$. We discover (5.134) by observing that it is true when $n = -a$ and using induction on n.

5.100 The Gosper-Zeilberger algorithm discovers easily that

$$\frac{n+2}{\binom{n}{k}} - \frac{2n+2}{\binom{n+1}{k}} \;=\; \frac{n-k}{\binom{n}{k+1}} - \frac{n+1-k}{\binom{n}{k}}\,, \qquad 0 \leqslant k < n.$$

Summing from $k = 0$ to $n-1$ yields $(n+2)(S_n-1)-(2n+2)(S_{n+1}-1-\frac{1}{n+1}) = -n$. Hence $(2n+2)S_{n+1} = (n+2)S_n + 2n+2$. Applying a summation factor now leads to the expression $S_n = (n+1)2^{-n-1}\sum_{k=1}^{n+1} 2^k/k$.

5.101 (a) If we hold m fixed, the Gosper-Zeilberger algorithm discovers that $(n+2)S_{m,n+2}(z) = (z-1)(n+1)S_{m,n}(z)+(2n+3-z(n-m+1))S_{m,n+1}(z)$. We can also apply the method to the term

$$\beta_0(m,n)t(m,n,k) + \beta_1(m,n)t(m{+}1,n,k) + \beta_2(m,n)t(m,n{+}1,k)\,,$$

in which case we get a simpler recurrence,

$$(m+1)S_{m+1,n}(z) - (n+1)S_{m,n+1}(z) \;=\; (1-z)(m-n)S_{m,n}(z)\,.$$

(b) Now we must work a little harder, with five equations in six unknowns. The algorithm finds

$$\begin{aligned}(n+1)(z-1)^2\binom{n}{k}^2 z^k - (2n+3)(z+1)\binom{n+1}{k}^2 z^k&\\ +\,(n+2)\binom{n+2}{k}^2 z^k &\;=\; T(n,k+1)-T(n,k)\,,\\ T(n,k) \;=\; \binom{n+1}{k-1}^2\frac{s(n,k)}{n+1}\,z^k\,,&\\ s(n,k) \;=\; (z{-}1)k^2 - 2((n{+}2)z{-}2n{-}3)k + (n{+}2)((n{+}2)z{-}4n{-}5)\,.&\end{aligned}$$

Therefore $(n+1)(z-1)^2S_n(z)-(2n+3)(z+1)S_{n+1}(z)+(n+2)S_{n+2}(z) = 0$. Incidentally, this recurrence holds also for negative n, and we have $S_{-n-1}(z) = S_n(z)/(1-z)^{2n+1}$.

The sum $S_n(z)$ can be regarded as a modified form of the Legendre polynomial $P_n(z) = \sum_k \binom{n}{k}^2(z{-}1)^{n-k}(z{+}1)^k/2^n$, since we can write $S_n(z) = (1-z)^nP_n\bigl(\frac{1+z}{1-z}\bigr)$. Similarly, $S_{m,n}(z) = (1-z)^nP_n^{(0,m-n)}\bigl(\frac{1+z}{1-z}\bigr)$ is a modified Jacobi polynomial.

How about $z = 0$?

5.102 The sum is $F(a-\frac{1}{3}n, -n; b-\frac{4}{3}n; -z)$, so we need not consider the case $z = -1$. Let $n = 3m$. We seek solutions to (5.129) when

$$\begin{aligned}p(m,k) &= (3m+3-k)^{\underline{3}}(m+1-k)\beta_0 + (4m+4-b-k)^{\underline{4}}\beta_1\,,\\ q(m,k) &= (3m+3-k)(m+1-a-k)z\,,\\ r(m,k) &= k(4m+1-b-k)\,,\\ s(m,k) &= \alpha_2k^2+\alpha_1k+\alpha_0\,.\end{aligned}$$

The resulting five homogeneous equations have a nonzero solution $(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1)$ if and only if the determinant of coefficients is zero; and this determinant, a polynomial in m, vanishes only in eight cases. One of those cases is, of course, (5.113); but we can now evaluate the sum for all nonnegative integers n, not just $n \not\equiv 2$ (mod 3):

$$\sum_k \binom{n}{k}\binom{\frac{1}{3}n-\frac{1}{6}}{k} 8^k \Big/ \binom{\frac{4}{3}n-\frac{2}{3}}{k} \;=\; [1,1,-\tfrac{1}{2}]\binom{2n}{n}\Big/\binom{\frac{4}{3}n-\frac{2}{3}}{n}.$$

Here the notation $[c_0,c_1,c_2]$ stands for the single value $c_{n \bmod 3}$. Another case, $(a,b,z) = (\frac{1}{2},0,8)$, yields the identity

$$\sum_k \binom{n}{k}\binom{\frac{1}{3}n-\frac{1}{2}}{k} 8^k \Big/ \binom{\frac{4}{3}n}{k} \;=\; [1,0,0]\,16^{n/3}\binom{\frac{2}{3}n}{\frac{1}{3}n}\Big/\binom{\frac{4}{3}n}{n}.$$

(This sum, amazingly, is zero unless n is a multiple of 3; and then the identity can be written

$$\sum_k \binom{3m}{k}\binom{2m}{2k}\binom{2k}{k} 2^k \Big/ \binom{4m}{k}\binom{m}{k} \;=\; 16^m \frac{(3m)!\,(2m)!}{(4m)!\,m!},$$

which might even be useful.) The remaining six cases generate even weirder sums

$$\sum_k \binom{n}{k}\binom{\frac{1}{3}n-a}{k} z^k \Big/ \binom{\frac{4}{3}n-b}{k}$$
$$= [c_0,c_1,c_2]\frac{\dbinom{\frac{1}{3}n-a}{\lfloor n/3\rfloor}\dbinom{\frac{1}{3}n-a'}{\lfloor n/3\rfloor}x^{\lfloor n/3\rfloor}}{\dbinom{\frac{4}{3}n-b}{n}\dbinom{\frac{1}{3}n-b}{\lfloor n/3\rfloor}\dbinom{\frac{1}{3}n-b'}{\lfloor n/3\rfloor}}$$

where the respective values of $(a,b,z,c_0,c_1,c_2,a',b',x)$ are

$$\begin{array}{ll}
(\frac{7}{12},\ \frac{1}{3},\ \ 8,1,-1,\ \ 0,\ \frac{1}{4},\ 0,\ \ 64)\,; & (\ \frac{1}{4},\ 0,\ \ 8,1,2,\ \ 0,\ \frac{7}{12},\frac{1}{3},\ \ 64)\,;\\
(\frac{5}{12},\ \frac{2}{3},\ \ 8,1,\ \ 0,-3,\ \frac{3}{4},\ 0,\ \ 64)\,; & (\frac{1}{12},\ \frac{1}{3},\ \ 8,1,3,\ \ 0,\ \ \frac{3}{4},\ 0,\ \ 64)\,;\\
(\ \frac{1}{2},\ 0,-4,1,\ \ 2,\ \ 0,\ \frac{1}{6},\frac{1}{3},-16)\,; & (\ \frac{1}{6},\ \frac{2}{3},-4,1,0,-3,\ \ \frac{5}{6},\ 0,-16)\,.
\end{array}$$

5.103 We assume that each a_i' and b_i' is nonzero, since the corresponding factors would otherwise have no influence on the degrees in k. Let $\hat{t}(n,k) = \hat{p}(n,k)\bar{t}(n,k)$ where

$$\bar{t}(n,k) \;=\; \frac{\prod_{i=1}^{p}\big(a_i n + a_i' k + a_i l[a_i<0] + a_i''\big)!}{\prod_{i=1}^{q}\big(b_i n + b_i' k + b_i l[b_i>0] + b_i''\big)!}z^k.$$

Then we have $\deg(\hat{p}) = \deg(f) + \max\big(\sum_{i=1}^{q} b_i[b_i>0] - \sum_{i=1}^{p} a_i[a_i<0],\ \sum_{i=1}^{p} a_i[a_i>0] - \sum_{i=1}^{q} b_i[b_i<0]\big) \geqslant \deg(f) + \frac{1}{2}l\big(|a_1| + \cdots + |a_p| + |b_1| +$

$\cdots + |b_q|\big)$, except in unusual cases where cancellation occurs in the leading coefficient. And $\deg(q) = \sum_{i=1}^{p} a_i'[a_i'>0] - \sum_{i=1}^{q} b_i'[b_i'<0]$, $\deg(r) = \sum_{i=1}^{q} b_i'[b_i'>0] - \sum_{i=1}^{p} a_i'[a_i'<0]$, again except in unusual cases.

(These estimates can be used to show directly that, as l increases, the degree of $\hat{p}$ eventually becomes large enough to make a polynomial $s(n,k)$ possible, and the number of unknown α_j and β_j eventually becomes larger than the number of homogeneous linear equations to be solved. So we obtain another proof that the Gosper-Zeilberger algorithm succeeds, if we argue as in the text that there must be a solution with $\beta_0(n), \ldots, \beta_l(n)$ not all zero.)

5.104 Let $t(n,k) = (-1)^k(r-s-k)!\,(r-2k)!/\big((r-s-2k)!\,(r-n-k+1)!\,(n-k)!\,k!\big)$. Then $\beta_0(n)t(n,k)+\beta_1(n)t(n+1,k)$ is not summable in hypergeometric terms, because $\deg(\hat{p}) = 1$, $\deg(q-r) = 3$, $\deg(q+r) = 4$, $\lambda = -8$, $\lambda' = -4$; but $\beta_0(n)t(n,k) + \beta_1(n)t(n+1,k) + \beta_2(n)t(n+2,k)$ is — basically because $\lambda' = 0$ when $q(n,k) = -(r-s-2k)(r-s-2k-1)(n+2-k)(r-n-k+1)$ and $r(k) = (r-s-k+1)(r-2k+2)(r-2k+1)k$. The solution is

$$\begin{aligned}
\beta_0(n) &= (s-n)(r-n+1)(r-2n+1)\,,\\
\beta_1(n) &= (rs-s^2-2rn+2n^2-2r+2n)(r-2n-1)\,,\\
\beta_2(n) &= (s-r+n+1)(n+2)(r-2n-3)\,,\\
\alpha_0(n) &= r-2n-1\,,
\end{aligned}$$

and we may conclude that $\beta_0(n)S_n + \beta_1(n)S_{n+1} + \beta_2(n)S_{n+2} = 0$ when S_n denotes the stated sum. This suffices to prove the identity by induction, after verifying the cases $n = 0$ and $n = 1$.

But S_n also satisfies the simpler recurrence $\bar{\beta}_0(n)S_n + \bar{\beta}_1(n)S_{n+1} = 0$, where $\bar{\beta}_0(n) = (s-n)(r-2n+1)$ and $\bar{\beta}_1(n) = -(n+1)(r-2n-1)$. Why didn't the method discover this? Well, nobody ever said that such a recurrence necessarily forces the terms $\bar{\beta}_0(n)t(n,k) + \bar{\beta}_1(n)t(n+1,k)$ to be indefinitely summable. The surprising thing is that the Gosper-Zeilberger method actually does find the simplest recurrence in so many other cases.

Notice that the second-order recurrence we found can be factored:

$$\begin{aligned}
&\beta_0(n) + \beta_1(n)N + \beta_2(n)N^2\\
&\qquad = \big((r-n+1)N + (r-s-n-1)\big)\ \big(\bar{\beta}_0(n) + \bar{\beta}_1(n)N\big)\,,
\end{aligned}$$

where N is the shift operator in (5.145).

5.105 Set $a = 1$ and compare the coefficients of z^{3n} on both sides of Henrici's "friendly monster" identity,

$$\begin{aligned}
&f(a,z)\,f(a,\omega z)\,f(a,\omega^2 z)\\
&\qquad = F\left({\frac12 a-\frac14,\ \frac12 a+\frac14 \atop \frac13 a,\ \frac13 a+\frac13,\ \frac13 a+\frac23,\ \frac23 a-\frac13,\ \frac23 a,\ \frac23 a+\frac13,\ a} \,\middle|\, \left(\frac{4z}{9}\right)^3\right),
\end{aligned}$$

where $f(a,z) = F(1; a, 1; z)$. The identity can be proved by showing that both sides satisfy the same differential equation.

Peter Paule has found another interesting way to evaluate the sum:

$$\begin{aligned}
\sum_{k,l} \binom{N}{k,l,N-k-l}^2 \omega^{k+2l} &= \sum_{k,l} \binom{N}{k-l,l,N-k}^2 \omega^{k+l} \\
&= \sum_{k,l} \binom{N}{k}^2 \binom{k}{l}^2 \omega^{k+l} \\
&= \sum_{k} \binom{N}{k}^2 \omega^k \,[z^k]\, \bigl((1+z)(\omega+z)\bigr)^k \\
&= [z^0] \sum_{k} \binom{N}{k}^2 \left(\frac{\omega(1+z)(\omega+z)}{z}\right)^k \\
&= [z^0] \sum_{k,j} \binom{N}{k}^2 \binom{k}{j} \left(\frac{\omega(1+z)(\omega+z)}{z} - 1\right)^j \\
&= [z^0] \sum_{k,j} \binom{N}{k}\binom{N-j}{N-k}\binom{N}{j} \left(\frac{(\omega z-1)^2}{\omega z}\right)^j \\
&= \sum_{j} \binom{2N-j}{N}\binom{N}{j} \,[z^j]\, (z-1)^{2j} \\
&= \sum_{j} \binom{2N-j}{N}\binom{N}{j}\binom{2j}{j}(-1)^j\,,
\end{aligned}$$

using the binomial theorem, Vandermonde's convolution, and the fact that $[z^0]g(az) = [z^0]g(z)$. We can now set $N = 3n$ and apply the Gosper-Zeilberger algorithm to this sum S_n, miraculously obtaining the first-order recurrence $(n+1)^2 S_{n+1} = 4(4n+1)(4n+3)S_n$; the result follows by induction.

If $3n$ is replaced by $3n+1$ or $3n+2$, the stated sum is zero. Indeed, $\sum_{k+l+m=N} t(k,l,m)\omega^{l-m}$ is always zero when $N \bmod 3 \neq 0$ and $t(k,l,m) = t(l,m,k)$.

5.106 (Solution by Shalosh B Ekhad.) Let

$$\begin{aligned}
T(r,j,k) &= \frac{((1+n+s)(1+r) - (1+n+r)j + (s-r)k)(j-l)j}{(l-m+n-r+s)(n+r+1)(j-r-1)(j+k)} t(r,j,k)\,; \\
U(r,j,k) &= \frac{(s+n+1)(k+l)k}{(l-m+n-r+s)(n+r+1)(j+k)} t(r,j,k)\,.
\end{aligned}$$

The stated equality is routinely verifiable, and (5.32) follows by summing with respect to j and k. (We sum $T(r,j+1,k) - T(r,j,k)$ first with respect to j, then with respect to k; we sum the other terms $U(r,j,k+1) - U(r,j,k)$ first with respect to k, then with respect to j.)

Well, we also need to verify (5.32) when $r = 0$. In that case it reduces via trinomial revision to $\sum_k (-1)^k \binom{n}{n+l}\binom{n+l}{k+l}\binom{s+n-k}{m} = (-1)^l \binom{n}{n+l}\binom{s}{m-n-l}$. We are assuming that l, m, and n are integers and $n \geqslant 0$. Both sides are clearly zero unless $n + l \geqslant 0$. Otherwise we can replace k by $n - k$ and use (5.24).

Noticee that $1/nk$ is proper, since it's $(n-1)!(k-1)!/n!\,k!$. Also $1/(n^2 - k^2)$ is proper. But $1/(n^2 + k^2)$ isn't.

5.107 If it were proper, there would be a linear difference operator that annihilates it. In other words, we would have a finite summation identity

$$\sum_{i=0}^{I}\sum_{j=0}^{J} \alpha_{i,j}(n)/\bigl((n+i)(k+j)+1\bigr) \;=\; 0\,,$$

where the α's are polynomials in n, not all zero. Choose integers i, j, and n such that $n > 1$ and $\alpha_{i,j}(n) \neq 0$. Then when $k = -1/(n+i) - j$, the (i,j) term in the sum is infinite but the other terms are finite.

5.108 Replace k by $m - k$ in the double sum, then use (5.28) to sum on k, getting

$$A_{m,n} \;=\; \sum_j \binom{m}{j}^2 \binom{m+n-j}{m}^2 ;$$

trinomial revision (5.21) then yields one of the desired formulas.

It appears to be difficult to find a direct proof that the two symmetrical sums for $A_{m,n}$ are equal. We can, however, prove the equation indirectly with the Gosper-Zeilberger algorithm, by showing that both sums satisfy the recurrence

$$(n+1)^3 A_{m,n} - f(m,n)A_{m,n+1} + (n+2)^3 A_{m,n+2} \;=\; 0\,,$$

where $f(m,n) = (2n+3)(n^2+3n+2m^2+2m+3)$. Setting $t_1(n,k) = \binom{m}{k}\binom{n}{k}\binom{m+k}{k}\binom{n+k}{k}$ and $t_2(n,k) = \binom{m+n-k}{k}^2\binom{m+n-2k}{m-k}^2$, we find

$$\begin{aligned}(n+1)^2 t_j(n,k) - f(m,n)t_j(n+1,k) + (n+2)^2 t_j(n+2,k)&\\ \;=\; T_j(n,k+1) - T_j(n,k)\,,&\end{aligned}$$

where $T_1(n,k) = -2(2n+3)k^4 t_1(n,k)/(n+1-k)(n+2-k)$ and $T_2(n,k) = -\bigl((n+2)(4mn+n+3m^2+8m+2) - 2(3mn+n+m^2+6m+2)k + (2m+1)k^2\bigr)k^2(m+n+1-k)^2 t_2(n,k)/(n+2-k)^2$. This proves the recurrence, so we need only verify equality when $n = 0$ and $n = 1$. (We could also have used the simpler recurrence

$$m^3 A_{m,n-1} - n^3 A_{m-1,n} \;=\; (m-n)(m^2+n^2-mn)A_{m-1,n-1}\,,$$

which can be discovered by the method of exercise 101.)

The fact that the first formula for $A_{m,n}$ equals the third implies a remarkable identity between the generating functions $\sum_{m,n} A_{m,n} w^m z^n$:

$$\sum_k \frac{w^k S_k(z)^2}{(1-z)^{2k+1}} = \sum_k \binom{2k}{k}^2 \frac{w^k}{(1-w)^{2k+1}} \frac{z^k}{(1-z)^{2k+1}},$$

where $S_k(z) = \sum_j \binom{k}{j}^2 z^j$. It turns out, in fact, that

$$\sum_k \frac{w^k S_k(x) S_k(y)}{(1-x)^k (1-y)^k} = \sum_k \binom{2k}{k} \frac{w^k}{(1-w)^{2k+1}} \frac{\sum_j \binom{k}{j}^2 x^j y^{k-j}}{(1-x)^k (1-y)^k};$$

this is a special case of an identity discovered by Bailey [19].

5.109 Let $X_n = \sum_k \binom{n}{k}^{a_0} \binom{n+k}{k}^{a_1} \ldots \binom{n+lk}{k}^{a_l} x^k$ for any positive integers $a_0, a_1, \ldots, a_l$, and any integer x. Then if $0 \leqslant m < p$ we have

$$X_{m+pn} = \sum_{j=0}^{p-1} \sum_k \binom{m+pn}{j+pk}^{a_0} \ldots \binom{m+pn+l(j+pk)}{j+pk}^{a_l} x^{j+pk},$$

$$X_m X_n = \sum_{j=0}^{p-1} \sum_k \binom{m}{j}^{a_0} \binom{n}{k}^{a_0} \ldots \binom{m+lj}{j}^{a_l} \binom{n+lk}{k}^{a_l} x^{j+k}.$$

And corresponding terms are congruent (mod p), because exercise 36 implies that they are multiples of p when $lj + m \geqslant p$, exercise 61 implies that the binomials are congruent when $lj + m < p$, and (4.48) implies that $x^p \equiv x$.

5.110 The congruence surely holds if $2n+1$ is prime. Steven Skiena has also found the example $n = 2953$, when $2n+1 = 3 \cdot 11 \cdot 179$.

Ilan Vardi notes that the condition holds for $2n+1 = p^2$, where p is prime, if and only if $2^{p-1} \bmod p^2 = 1$. This yields two more examples: $n = (1093^2-1)/2$; $n = (3511^2-1)/2$.

5.111 See [96] for partial results. The computer experiments were done by V. A. Vyssotsky.

5.112 If n is not a power of 2, $\binom{2n}{n}$ is a multiple of 4 because of exercise 36. Otherwise the stated phenomenon was verified for $n \leqslant 2^{22000}$ by A. Granville and O. Ramaré, who also sharpened a theorem of Sárközy [317] by showing that $\binom{2n}{n}$ is divisible by the square of a prime for all $n > 2^{22000}$. This established a long-standing conjecture that $\binom{2n}{n}$ is never squarefree when $n > 4$.

The analogous conjectures for cubes are that $\binom{2n}{n}$ is divisible by the cube of a prime for all $n > 1056$, and by either 2^3 or 3^3 for all $n > 2^{29} + 2^{23}$. This has been verified for all $n < 2^{10000}$. Paul Erdős conjectures that, in fact, $\max_p \epsilon_p\bigl(\binom{2n}{n}\bigr)$ tends to infinity as $n \to \infty$; this might be true even if we restrict p to the values 2 and 3.

5.113 The theorem about generating functions in exercise 7.20 may help resolve this conjecture.

5.114 Strehl [344] has shown that $c_n^{(2)} = \sum_k \binom{n}{k}^3 = \sum_k \binom{n}{k}^2\binom{2k}{n}$ is a so-called Franel number [132], and that $c_n^{(3)} = \sum_k \binom{n}{k}^2\binom{2k}{k}^2\binom{2k}{n-k}$. In another direction, H. S. Wilf has shown that $c_n^{(m)}$ is an integer for all m when $n \leqslant 9$.

6.1 2314, 2431, 3241, 1342, 3124, 4132, 4213, 1423, 2143, 3412, 4321.

6.2 ${n \brace k} m^{\underline{k}}$, because every such function partitions its domain into k nonempty subsets, and there are $m^{\underline{k}}$ ways to assign function values for each partition. (Summing over k gives a combinatorial proof of (6.10).)

6.3 Now $d_{k+1} \leqslant (\text{center of gravity}) - \epsilon = 1 - \epsilon + (d_1 + \cdots + d_k)/k$. This recurrence is like (6.55) but with $1-\epsilon$ in place of 1; hence the optimum solution is $d_{k+1} = (1-\epsilon)H_k$. This is unbounded as long as $\epsilon < 1$.

6.4 $H_{2n+1} - \frac{1}{2}H_n$. (Similarly $\sum_{k=1}^{2n}(-1)^{k-1}/k = H_{2n} - H_n$.)

6.5 $U_n(x,y)$ is equal to

$$x\sum_{k\geqslant 1}\binom{n}{k}(-1)^{k-1}k^{-1}(x+ky)^{n-1} + y\sum_{k\geqslant 1}\binom{n}{k}(-1)^{k-1}(x+ky)^{n-1},$$

and the first sum is

$$U_{n-1}(x,y) + \sum_{k\geqslant 1}\binom{n-1}{k-1}(-1)^{k-1}k^{-1}(x+ky)^{n-1}.$$

The remaining k^{-1} can be absorbed, and we have

$$\begin{aligned}\sum_{k\geqslant 1}\binom{n}{k}(-1)^{k-1}(x+ky)^{n-1} &= x^{n-1} + \sum_{k\geqslant 0}\binom{n}{k}(-1)^{k-1}(x+ky)^{n-1}\\ &= x^{n-1}.\end{aligned}$$

This proves (6.75). Let $R_n(x,y) = x^{-n}U_n(x,y)$; then $R_0(x,y) = 0$ and $R_n(x,y) = R_{n-1}(x,y) + 1/n + y/x$, hence $R_n(x,y) = H_n + ny/x$. (Incidentally, the original sum $U_n = U_n(n,-1)$ doesn't lead to a recurrence such as this; therefore the more general sum, which detaches x from its dependence on n, is easier to solve inductively than its special case. This is another instructive example where a strong induction hypothesis makes the difference between success and failure.)

The Fibonacci recurrence is additive, but the rabbits are multiplying.

6.6 Each pair of babies ʙʙ present at the end of a month becomes a pair of adults **aa** at the end of the next month; and each pair **aa** becomes an **aa** and a ʙʙ. Thus each ʙʙ behaves like a drone in the bee tree and each **aa** behaves like a queen, except that the bee tree goes backward in time while the rabbits are going forward. There are F_{n+1} pairs of rabbits after n months; F_n of them are adults and F_{n-1} are babies. (This is the context in which Fibonacci originally introduced his numbers.)

6.7 (a) Set $k = 1 - n$ and apply (6.107). (b) Set $m = 1$ and $k = n - 1$ and apply (6.128).

6.8 $55 + 8 + 2$ becomes $89 + 13 + 3 = 105$; the true value is 104.607361.

That "true value" is the length of 65 international miles, but the international mile is actually only .999998 as big as a U.S. statute mile. There are exactly 6336 kilometers in 3937 U.S. statute miles; the Fibonacci method converts 3937 to 6370.

6.9 21. (We go from F_n to F_{n+2} when the units are squared. The true answer is about 20.72.)

6.10 The partial quotients a_0, a_1, a_2, ... are all equal to 1, because $\phi = 1 + 1/\phi$. (The Stern–Brocot representation is therefore RLRLRLRLRL... .)

6.11 $(-1)^{\overline{n}} = [n=0] - [n=1]$; see (6.11).

6.12 This is a consequence of (6.31) and its dual in Table 264.

6.13 The two formulas are equivalent, by exercise 12. We can use induction. Or we can observe that $z^n D^n$ applied to $f(z) = z^x$ gives $x^{\underline{n}} z^x$ while ϑ^n applied to the same function gives $x^n z^x$; therefore the sequence $\langle \vartheta^0, \vartheta^1, \vartheta^2, \dots \rangle$ must relate to $\langle z^0D^0, z^1D^1, z^2D^2, \dots \rangle$ as $\langle x^0, x^1, x^2, \dots \rangle$ relates to $\langle x^{\underline{0}}, x^{\underline{1}}, x^{\underline{2}}, \dots \rangle$.

6.14 We have

$$x\binom{x+k}{n} = (k+1)\binom{x+k}{n+1} + (n-k)\binom{x+k+1}{n+1},$$

because $(n+1)x = (k+1)(x+k-n) + (n-k)(x+k+1)$. (It suffices to verify the latter identity when $k = 0$, $k = -1$, and $k = n$.)

6.15 Since $\Delta\bigl(\binom{x+k}{n}\bigr) = \binom{x+k}{n-1}$, we have the general formula

$$\sum_k \left\langle {n \atop k} \right\rangle \binom{x+k}{n-m} = \Delta^m(x^n) = \sum_j \binom{m}{j}(-1)^{m-j}(x+j)^n .$$

Set $x = 0$ and appeal to (6.19).

6.16 $A_{n,k} = \sum_{j\geqslant 0} a_j {n-j \brace k}$; this sum is always finite.

6.17 (a) $\left|{n \atop k}\right| = \left[{n+1 \atop n+1-k}\right]$. (b) $\left|{n \atop k}\right| = n^{\underline{n-k}} = n!\,[n\geqslant k]/k!$. (c) $\left|{n \atop k}\right| = k!{n \brace k}$.

6.18 This is equivalent to (6.3) or (6.8).

6.19 Use Table 272.

6.20 $\sum_{1\leqslant j\leqslant k\leqslant n} 1/j^2 = \sum_{1\leqslant j\leqslant n}(n+1-j)/j^2 = (n+1)H_n^{(2)} - H_n$.

6.21 The hinted number is a sum of fractions with odd denominators, so it has the form a/b with a and b odd. (Incidentally, Bertrand's postulate implies that b_n is also divisible by at least one odd prime, whenever $n > 2$.)

6.22 $\bigl|z/k(k+z)\bigr| \le 2|z|/k^2$ when $k > 2|z|$, so the sum is well defined when the denominators are not zero. If $z = n$ we have $\sum_{k=1}^{m}\bigl(1/k - 1/(k+n)\bigr) = H_m - H_{m+n} + H_n$, which approaches H_n as $m \to \infty$. (The quantity $H_{z-1} - \gamma$ is often called the psi function $\psi(z)$.)

6.23 $z/(e^z+1) = z/(e^z-1) - 2z/(e^{2z}-1) = \sum_{n\ge 0}(1-2^n)B_n z^n/n!$.

6.24 When n is odd, $T_n(x)$ is a polynomial in x^2, hence its coefficients are multiplied by even numbers when we form the derivative and compute $T_{n+1}(x)$ by (6.95). (In fact we can prove more: The Bernoulli number B_{2n} always has 2 to the first power in its denominator, by exercise 54; hence $2^{2n-k}\backslash\backslash T_{2n+1} \iff 2^k\backslash\backslash(n+1)$. The odd positive integers $(n+1)T_{2n+1}/2^{2n}$ are called Genocchi numbers $\langle 1, 1, 3, 17, 155, 2073, \ldots\rangle$, after Genocchi [145].)

6.25 $100n - nH_n < 100(n-1) - (n-1)H_{n-1} \iff H_{n-1} > 99$. (The least such n is approximately $e^{99-\gamma}$, while he finishes at $N \approx e^{100-\gamma}$, about e times as long. So he is getting closer during the final 63% of his journey.)

6.26 Let $u(k) = H_{k-1}$ and $\Delta v(k) = 1/k$, so that $u(k) = v(k)$. Then we have $S_n - H_n^{(2)} = \sum_{k=1}^{n} H_{k-1}/k = H_{k-1}^2\big|_1^{n+1} - S_n = H_n^2 - S_n$.

6.27 Observe that when $m > n$ we have $\gcd(F_m, F_n) = \gcd(F_{m-n}, F_n)$ by (6.108). This yields a proof by induction.

6.28 (a) $Q_n = \alpha(L_n - F_n)/2 + \beta F_n$. (The solution can also be written $Q_n = \alpha F_{n-1} + \beta F_n$.) (b) $L_n = \phi^n + \hat\phi^n$.

6.29 When $k = 0$ the identity is (6.133). When $k = 1$ it is, essentially,

$$\begin{aligned} K(x_1,\ldots,x_n)x_m \;=\; & K(x_1,\ldots,x_m)\,K(x_m,\ldots,x_n) \\ & - K(x_1,\ldots,x_{m-2})\,K(x_{m+2},\ldots,x_n)\,; \end{aligned}$$

in Morse code terms, the second product on the right subtracts out the cases where the first product has intersecting dashes. When $k > 1$, an induction on k suffices, using both (6.127) and (6.132). (The identity is also true when one or more of the subscripts on K become -1, if we adopt the convention that $K_{-1} = 0$. When multiplication is not commutative, Euler's identity remains valid if we write it in the form

$$\begin{aligned} & K_{m+n}(x_1,\ldots,x_{m+n})\,K_k(x_{m+k},\ldots,x_{m+1}) \\ &\quad = K_{m+k}(x_1,\ldots,x_{m+k})\,K_n(x_{m+n},\ldots,x_{m+1}) \\ &\qquad + (-1)^k K_{m-1}(x_1,\ldots,x_{m-1})\,K_{n-k-1}(x_{m+n},\ldots,x_{m+k+2})\,. \end{aligned}$$

For example, we obtain the somewhat surprising noncommutative factorizations

$$(abc + a + c)(1 + ba) \;=\; (ab + 1)(cba + a + c)$$

from the case $k = 2$, $m = 0$, $n = 3$.)

6.30 The derivative of $K(x_1, \ldots, x_n)$ with respect to x_m is

$$K(x_1, \ldots, x_{m-1})\, K(x_{m+1}, \ldots, x_n)\,,$$

and the second derivative is zero; hence the answer is

$$K(x_1, \ldots, x_n) + K(x_1, \ldots, x_{m-1})\, K(x_{m+1}, \ldots, x_n)\, y\,.$$

6.31 Since $x^{\overline{n}} = (x+n-1)^{\underline{n}} = \sum_k \binom{n}{k} x^{\underline{k}} (n-1)^{\underline{n-k}}$, we have $\left|{n \atop k}\right| = \binom{n}{k}(n-1)^{\underline{n-k}}$. These coefficients, incidentally, satisfy the recurrence $\left|{n \atop k}\right| = \left|{-k \atop -n}\right|$.

$$\left|{n \atop k}\right| = (n-1+k)\left|{n-1 \atop k}\right| + \left|{n-1 \atop k-1}\right|, \qquad \text{integers } n, k > 0.$$

6.32 $\sum_{k\leqslant m} k{n+k \brace k} = {m+n+1 \brace m}$ and $\sum_{0\leqslant k\leqslant n} {k \brace m}(m+1)^{n-k} = {n+1 \brace m+1}$, both of which appear in Table 265.

6.33 If $n > 0$, we have ${n \brack 3} = \frac{1}{2}(n-1)!\,(H_{n-1}^2 - H_{n-1}^{(2)})$, by (6.71); ${n \brace 3} = \frac{1}{6}(3^n - 3\cdot 2^n + 3)$, by (6.19).

6.34 We have $\left\langle {-1 \atop k} \right\rangle = 1/(k+1)$, $\left\langle {-2 \atop k} \right\rangle = H_{k+1}^{(2)}$, and in general $\left\langle {n \atop k} \right\rangle$ is given by (6.38) for all integers n.

6.35 Let n be the least integer $> 1/\epsilon$ such that $\lfloor H_n \rfloor > \lfloor H_{n-1} \rfloor$.

6.36 Now $d_{k+1} = \bigl(100+(1+d_1)+\cdots+(1+d_k)\bigr)/(100+k)$, and the solution is $d_{k+1} = H_{k+100} - H_{101} + 1$ for $k \geqslant 1$. This exceeds 2 when $k \geqslant 176$.

6.37 The sum (by parts) is $H_{mn} - \bigl(\frac{m}{m} + \frac{m}{2m} + \cdots + \frac{m}{mn}\bigr) = H_{mn} - H_n$. The infinite sum is therefore $\ln m$. (It follows that

$$\sum_{k\geqslant 1} \frac{\nu_m(k)}{k(k+1)} = \frac{m}{m-1} \ln m\,,$$

because $\nu_m(k) = (m-1)\sum_{j\geqslant 1} (k \bmod m^j)/m^j$.)

6.38 $(-1)^k\bigl(\binom{r-1}{k} r^{-1} - \binom{r-1}{k-1} H_k\bigr) + C$. (By parts, using (5.16).)

6.39 Write it as $\sum_{1\leqslant j\leqslant n} j^{-1} \sum_{j\leqslant k\leqslant n} H_k$ and sum first on k via (6.67), to get

$$(n+1)H_n^2 - (2n+1)H_n + 2n\,.$$

6.40 If $6n-1$ is prime, the numerator of

$$\sum_{k=1}^{4n-1} \frac{(-1)^{k-1}}{k} = H_{4n-1} - H_{2n-1}$$

is divisible by $6n-1$, because the sum is

$$\sum_{k=2n}^{4n-1}\frac{1}{k} = \sum_{k=2n}^{3n-1}\left(\frac{1}{k}+\frac{1}{6n-1-k}\right) = \sum_{k=2n}^{3n-1}\frac{6n-1}{k(6n-1-k)}\,.$$

Similarly if $6n+1$ is prime, the numerator of $\sum_{k=1}^{4n}(-1)^{k-1}/k = H_{4n}-H_{2n}$ is a multiple of $6n+1$. For 1987 we sum up to $k=1324$.

6.41 $S_{n+1} = \sum_k \binom{\lfloor (n+1+k)/2\rfloor}{k} = \sum_k \binom{\lfloor (n+k)/2\rfloor}{k-1}$, hence we have $S_{n+1} + S_n = \sum_k \binom{\lfloor (n+k)/2+1\rfloor}{k} = S_{n+2}$. The answer is F_{n+2}.

6.42 F_n.

6.43 Set $z=\frac{1}{10}$ in $\sum_{n\geqslant 0}F_n z^n = z/(1-z-z^2)$ to get $\frac{10}{89}$. The sum is a repeating decimal with period length 44:

$$0.11235\,95505\,61797\,75280\,89887\,64044\,94382\,02247\,19101\,12359\,55+\,.$$

6.44 Replace (m,k) by $(-m,-k)$ or $(k,-m)$ or $(-k,m)$, if necessary, so that $m\geqslant k\geqslant 0$. The result is clear if $m=k$. If $m>k$, we can replace (m,k) by $(m-k,m)$ and use induction.

6.45 $X_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$, where $B(n)=F_n$, $A(n)=F_{n-1}$, $A(n)+B(n)-D(n)=1$, and $B(n)-C(n)+3D(n)=n$.

6.46 $\phi/2$ and $\phi^{-1}/2$. Let $u=\cos 72°$ and $v=\cos 36°$; then $u=2v^2-1$ and $v=1-2\sin^2 18° = 1-2u^2$. Hence $u+v=2(u+v)(v-u)$, and $4v^2-2v-1=0$. We can pursue this investigation to find the five complex fifth roots of unity:

$$1\,,\quad \frac{\phi^{-1}\pm i\sqrt{2+\phi}}{2}\,,\quad \frac{-\phi\pm i\sqrt{3-\phi}}{2}\,.$$

"Let p be any old prime."
(See [171], p. 419.)

6.47 $2^n\sqrt{5}\,F_n = (1+\sqrt{5})^n - (1-\sqrt{5})^n$, and the even powers of $\sqrt{5}$ cancel out. Now let p be an odd prime. Then $\binom{p}{2k+1}\equiv 0$ except when $k=(p-1)/2$, and $\binom{p+1}{2k+1}\equiv 0$ except when $k=0$ or $k=(p-1)/2$; hence $F_p\equiv 5^{(p-1)/2}$ and $2F_{p+1}\equiv 1+5^{(p-1)/2}$ (mod p). It can be shown that $5^{(p-1)/2}\equiv 1$ when p has the form $10k\pm 1$, and $5^{(p-1)/2}\equiv -1$ when p has the form $10k\pm 3$.

6.48 Let $K_{i,j} = K_{j-i+1}(x_i,\ldots,x_j)$. Using (6.133) repeatedly, both sides expand to $(K_{1,m-2}(x_{m-1}+x_{m+1})+K_{1,m-3})K_{m+2,n} + K_{1,m-2}K_{m+3,n}$.

6.49 Set $z=\frac{1}{2}$ in (6.146); the partial quotients are 0, 2^{F_0}, 2^{F_1}, 2^{F_2}, $\ldots$. (Knuth [206] noted that this number is transcendental.)

6.50 (a) $f(n)$ is even $\iff$ $3\backslash n$. (b) If the binary representation of n is $(1^{a_1}0^{a_2}\ldots 1^{a_{m-1}}0^{a_m})_2$, where m is even, we have $f(n)=K(a_1,a_2,\ldots,a_{m-1})$.

6.51 (a) Combinatorial proof: The arrangements of $\{1,2,\ldots,p\}$ into k subsets or cycles are divided into "orbits" of 1 or p arrangements each, if we add 1 to each element modulo p. For example,

$$\begin{aligned}\{1,2,4\}\cup\{3,5\} &\to \{2,3,5\}\cup\{4,1\} \to \{3,4,1\}\cup\{5,2\}\\ &\to \{4,5,2\}\cup\{1,3\} \to \{5,1,3\}\cup\{2,4\} \to \{1,2,4\}\cup\{3,5\}.\end{aligned}$$

We get an orbit of size 1 only when this transformation takes an arrangement into itself; but then $k=1$ or $k=p$. Alternatively, there's an algebraic proof: We have $x^p \equiv x^{\underline{p}} + x^{\underline{1}}$ and $x^{\underline{p}} \equiv x^p - x \pmod p$, since Fermat's theorem tells us that $x^p - x$ is divisible by $(x-0)(x-1)\ldots\bigl(x-(p{-}1)\bigr)$.

(b) This result follows from (a) and Wilson's theorem; or we can use $x^{\overline{p-1}} \equiv x^{\overline{p}}/(x-1) \equiv (x^p-x)/(x-1) = x^{p-1}+x^{p-2}+\cdots+x$.

(c) We have $\left\{{p+1 \atop k}\right\} \equiv \left[{p+1 \atop k}\right] \equiv 0$ for $3 \leqslant k \leqslant p$, then $\left\{{p+2 \atop k}\right\} \equiv \left[{p+2 \atop k}\right] \equiv 0$ for $4 \leqslant k \leqslant p$, etc. (Similarly, we have $\left[{2p-1 \atop p}\right] \equiv -\left\{{2p-1 \atop p}\right\} \equiv 1$.)

(d) $p! = p^{\underline{p}} = \sum_k (-1)^{p-k} p^k \left[{p \atop k}\right] = p^p\left[{p \atop p}\right] - p^{p-1}\left[{p \atop p-1}\right] + \cdots + p^3\left[{p \atop 3}\right] - p^2\left[{p \atop 2}\right] + p\left[{p \atop 1}\right]$. But $p\left[{p \atop 1}\right] = p!$, so

$$\left[{p \atop 2}\right] = p\left[{p \atop 3}\right] - p^2\left[{p \atop 4}\right] + \cdots + p^{p-2}\left[{p \atop p}\right]$$

is a multiple of p^2. (This is called Wolstenholme's theorem.)

6.52 (a) Observe that $H_n = H_n^* + H_{\lfloor n/p\rfloor}/p$, where $H_n^* = \sum_{k=1}^n [k \perp p]/k$. (b) Working mod 5 we have $H_r = \langle 0,1,4,1,0\rangle$ for $0 \leqslant r \leqslant 4$. Thus the first solution is $n=4$. By part (a) we know that $5\backslash a_n \Longrightarrow 5\backslash a_{\lfloor n/5\rfloor}$; so the next possible range is $n = 20+r$, $0 \leqslant r \leqslant 4$, when we have $H_n = H_n^* + \frac15 H_4 = H_{20}^* + \frac15 H_4 + H_r + \sum_{k=1}^r 20/k(20+k)$. The numerator of H_{20}^*, like the numerator of H_4, is divisible by 25. Hence the only solutions in this range are $n=20$ and $n=24$. The next possible range is $n = 100+r$; now $H_n = H_n^* + \frac15 H_{20}$, which is $\frac15 H_{20} + H_r$ plus a fraction whose numerator is a multiple of 5. If $\frac15 H_{20} \equiv m \pmod 5$, where m is an integer, the harmonic number H_{100+r} will have a numerator divisible by 5 if and only if $m + H_r \equiv 0 \pmod 5$; hence m must be $\equiv 0$, 1, or 4. Working modulo 5 we find $\frac15 H_{20} = \frac15 H_{20}^* + \frac1{25} H_4 \equiv \frac1{25}H_4 = \frac1{12} \equiv 3$; hence there are no solutions for $100 \leqslant n \leqslant 104$. Similarly there are none for $120 \leqslant n \leqslant 124$; we have found all three solutions.

(By exercise 6.51(d), we always have $p^2\backslash a_{p-1}$, $p\backslash a_{p^2-p}$, and $p\backslash a_{p^2-1}$, if p is any prime $\geqslant 5$. The argument just given shows that these are the only solutions to $p\backslash a_n$ if and only if there are no solutions to $p^{-2}H_{p-1} + H_r \equiv 0 \pmod p$ for $0 \leqslant r < p$. The latter condition holds not only for $p=5$ but also for $p = 13$, 17, 23, 41, and 67 — perhaps for infinitely many primes. The numerator of H_n is divisible by 3 only when $n = 2$, 7, and 22; it is divisible

(Attention, computer programmers: Here's an interesting condition to test, for as many primes as you can.)

by 7 only when $n = 6$, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, and 102728.)

6.53 Summation by parts yields

$$\frac{n+1}{(n+2)^2}\biggl(\frac{(-1)^m}{\binom{n+1}{m+1}}\bigl((n+2)H_{m+1}-1\bigr)-1\biggr).$$

6.54 (a) If $m \geqslant p$ we have $S_m(p) \equiv S_{m-(p-1)}(p) \pmod p$, since $k^{p-1} \equiv 1$ when $1 \leqslant k < p$. Also $S_{p-1}(p) \equiv p-1 \equiv -1$. If $0 < m < p-1$, we can write

$$S_m(p) \;=\; \sum_{j=0}^{m} {m \brack j}(-1)^{m-j}\sum_{k=0}^{p-1} k^{\underline{j}} \;=\; \sum_{j=0}^{m} {m \brack j}(-1)^{m-j}\frac{p^{\underline{j+1}}}{j+1} \;\equiv\; 0\,.$$

(b) The condition in the hint implies that the denominator of I_{2n} is not divisible by any prime p; hence I_{2n} must be an integer. To prove the hint, we may assume that $n>1$. Then

(The numerators of Bernoulli numbers played an important role in early studies of Fermat's Last Theorem; see Ribenboim [308].)

$$B_{2n} + \frac{\bigl[(p-1)\backslash(2n)\bigr]}{p} + \sum_{k=0}^{2n-2}\binom{2n+1}{k}B_k\frac{p^{2n-k}}{2n+1}$$

is an integer, by (6.78), (6.84), and part (a). So we want to verify that none of the fractions $\binom{2n+1}{k}B_k p^{2n-k}/(2n+1) = \binom{2n}{k}B_k p^{2n-k}/(2n-k+1)$ has a denominator divisible by p. The denominator of $\binom{2n}{k}B_k p$ isn't divisible by p, since B_k has no p^2 in its denominator (by induction); and the denominator of $p^{2n-k-1}/(2n-k+1)$ isn't divisible by p, since $2n-k+1 < p^{2n-k}$ when $k \leqslant 2n-2$; QED. (The numbers I_{2n} are tabulated in [224]. Hermite calculated them through I_{18} in 1875 [184]. It turns out that $I_2 = I_4 = I_6 = I_8 = I_{10} = I_{12} = 1$; hence there *is* actually a "simple" pattern to the Bernoulli numbers displayed in the text, including $\frac{-691}{2730}$(!). But the numbers I_{2n} don't seem to have any memorable features when $2n > 12$. For example, $B_{24} = -86579 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{13}$, and 86579 is prime.)

(c) The numbers $2-1$ and $3-1$ always divide $2n$. If n is prime, the only divisors of $2n$ are 1, 2, n, and $2n$, so the denominator of B_{2n} for prime $n > 2$ will be 6 unless $2n+1$ is also prime. In the latter case we can try $4n+3$, $8n+7$, $\ldots$, until we eventually hit a nonprime (since n divides $2^{n-1}n + 2^{n-1} - 1$). (This proof does not need the more difficult, but true, theorem that there are infinitely many primes of the form $6k+1$.) The denominator of B_{2n} can be 6 also when n has nonprime values, such as 49.

6.55 The stated sum is $\frac{m+1}{x+m+1}\binom{x+n}{n}\binom{n}{m+1}$, by Vandermonde's convolution. To get (6.70), differentiate and set $x = 0$.

6.56 First replace k^{n+1} by $\bigl((k-m)+m\bigr)^{n+1}$ and expand in powers of $k-m$; simplifications occur as in the derivation of (6.72). If $m>n$ or $m<0$, the answer is $(-1)^n n! - m^n/\binom{n-m}{n}$. Otherwise we need to take the limit of (5.41) minus the term for $k=m$, as $x\to -m$; the answer comes to $(-1)^n n! + (-1)^{m+1}\binom{n}{m} m^n(n+1+mH_{n-m}-mH_m)$.

6.57 First prove by induction that the nth row contains at most three distinct values $A_n \geqslant B_n \geqslant C_n$; if n is even they occur in the cyclic order $[C_n, B_n, A_n, B_n, C_n]$, while if n is odd they occur in the cyclic order $[C_n, B_n, A_n, A_n, B_n]$. Also

$$\begin{aligned} A_{2n+1} &= A_{2n}+B_{2n}\,; & A_{2n} &= 2A_{2n-1}\,;\\ B_{2n+1} &= B_{2n}+C_{2n}\,; & B_{2n} &= A_{2n-1}+B_{2n-1}\,;\\ C_{2n+1} &= 2C_{2n}\,; & C_{2n} &= B_{2n-1}+C_{2n-1}\,. \end{aligned}$$

It follows that $Q_n = A_n - C_n = F_{n+1}$. (See exercise 5.75 for wraparound binomial coefficients of order 3.)

6.58 (a) $\sum_{n\geqslant 0} F_n^2 z^n = z(1-z)/(1+z)(1-3z+z^2) = \frac{1}{5}\bigl((2-3z)/(1-3z+z^2) - 2/(1+z)\bigr)$. (Square Binet's formula (6.123) and sum on n, then combine terms so that ϕ and $\hat\phi$ disappear.) (b) Similarly,

$$\sum_{n\geqslant 0} F_n^3 z^n \;=\; \frac{z(1-2z-z^2)}{(1-4z-z^2)(1+z-z^2)} \;=\; \frac{1}{5}\left(\frac{2z}{1-4z-z^2}+\frac{3z}{1+z-z^2}\right).$$

It follows that $F_{n+1}^3 - 4F_n^3 - F_{n-1}^3 = 3(-1)^n F_n$. (The corresponding recurrence for mth powers involves the Fibonomial coefficients of exercise 86; it was discovered by Jarden and Motzkin [194].)

6.59 Let m be fixed. We can prove by induction on n that it is, in fact, possible to find such an x with the additional condition $x \not\equiv 2 \pmod 4$. If x is such a solution, we can move up to a solution modulo 3^{n+1} because

$$F_{8\cdot 3^{n-1}} \;\equiv\; 3^n\,, \qquad F_{8\cdot 3^{n-1}-1} \;\equiv\; 3^n+1 \pmod{3^{n+1}}\,;$$

either x or $x+8\cdot 3^{n-1}$ or $x+16\cdot 3^{n-1}$ will do the job.

6.60 F_1+1, F_2+1, F_3+1, F_4-1, and F_6-1 are the only cases. Otherwise the Lucas numbers of exercise 28 arise in the factorizations

$$\begin{aligned} F_{2m}+(-1)^m &= L_{m+1}F_{m-1}\,; & F_{2m+1}+(-1)^m &= L_m F_{m+1}\,;\\ F_{2m}-(-1)^m &= L_{m-1}F_{m+1}\,; & F_{2m+1}-(-1)^m &= L_{m+1}F_m\,. \end{aligned}$$

(We have $F_{m+n} - (-1)^n F_{m-n} = L_m F_n$ in general.)

6.61 $1/F_{2m} = F_{m-1}/F_m - F_{2m-1}/F_{2m}$ when m is even and positive. The second sum is $5/4 - F_{3\cdot 2^n-1}/F_{3\cdot 2^n}$, for $n \geqslant 1$.

6.62 (a) $A_n = \sqrt{5}\,A_{n-1} - A_{n-2}$ and $B_n = \sqrt{5}\,B_{n-1} - B_{n-2}$. Incidentally, we also have $\sqrt{5}\,A_n + B_n = 2A_{n+1}$ and $\sqrt{5}\,B_n - A_n = 2B_{n-1}$. (b) A table of small values reveals that

$$A_n = \begin{cases} L_n, & n \text{ even;} \\ \sqrt{5}\,F_n, & n \text{ odd;} \end{cases} \qquad B_n = \begin{cases} \sqrt{5}\,F_n, & n \text{ even;} \\ L_n, & n \text{ odd.} \end{cases}$$

(c) $B_n/A_{n+1} - B_{n-1}/A_n = 1/(F_{2n+1}+1)$ because $B_nA_n - B_{n-1}A_{n+1} = \sqrt{5}$ and $A_nA_{n+1} = \sqrt{5}\,(F_{2n+1}+1)$. Notice that $B_n/A_{n+1} = (F_n/F_{n+1})[n \text{ even}] + (L_n/L_{n+1})[n \text{ odd}]$. (d) Similarly, $\sum_{k=1}^{n} 1/(F_{2k+1}-1) = (A_0/B_1 - A_1/B_2) + \cdots + (A_{n-1}/B_n - A_n/B_{n+1}) = 2 - A_n/B_{n+1}$. This quantity can also be expressed as $(5F_n/L_{n+1})[n \text{ even}] + (L_n/F_{n+1})[n \text{ odd}]$.

6.63 (a) $\left[{n \atop k}\right]$. There are $\left[{n-1 \atop k-1}\right]$ with $\pi_n = n$ and $(n-1)\left[{n-1 \atop k}\right]$ with $\pi_n < n$. (b) $\left\langle{n \atop k}\right\rangle$. Each permutation $\rho_1 \ldots \rho_{n-1}$ of $\{1, \ldots, n-1\}$ leads to n permutations $\pi_1\pi_2 \ldots \pi_n = \rho_1 \ldots \rho_{j-1}\, n\, \rho_{j+1} \ldots \rho_{n-1}\rho_j$. If $\rho_1 \ldots \rho_{n-1}$ has k excedances, there are $k+1$ values of j that yield k excedances in $\pi_1\pi_2 \ldots \pi_n$; the remaining $n-1-k$ values yield $k+1$. Hence the total number of ways to get k excedances in $\pi_1\pi_2 \ldots \pi_n$ is $(k+1)\left\langle{n-1 \atop k}\right\rangle + \bigl((n-1)-(k-1)\bigr)\left\langle{n-1 \atop k-1}\right\rangle = \left\langle{n \atop k}\right\rangle$.

6.64 The denominator of $\binom{1/2}{2n}$ is $2^{4n-\nu_2(n)}$, by the proof in exercise 5.72. The denominator of $\left[{1/2 \atop 1/2-n}\right]$ is the same, by (6.44), because $\left\langle\!\left\langle{n \atop 0}\right\rangle\!\right\rangle = 1$ and $\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$ is even for $k > 0$.

6.65 This is equivalent to saying that $\left\langle{n \atop k}\right\rangle/n!$ is the probability that we have $\lfloor x_1 + \cdots + x_n \rfloor = k$, when $x_1, \ldots, x_n$ are independent random numbers uniformly distributed between 0 and 1. Let $y_j = (x_1 + \cdots + x_j) \bmod 1$. Then $y_1, \ldots, y_n$ are independently and uniformly distributed, and $\lfloor x_1 + \cdots + x_n \rfloor$ is the number of descents in the y's. The permutation of the y's is random, and the probability of k descents is the same as the probability of k ascents.

6.66 $2^{n+1}(2^{n+1}-1)B_{n+1}/(n+1)$, if $n > 0$. (See (7.56) and (6.92); the desired numbers are essentially the coefficients of $1 - \tanh z$.)

6.67 It is $\sum_k \left(\left\{{n \atop k+1}\right\}(k+1)! + \left\{{n \atop k}\right\}k!\right)\binom{n-k}{n-m}(-1)^{m-k} = \sum_k \left\{{n \atop k}\right\}k!(-1)^{m-k} \times \left(\binom{n-k}{n-m} - \binom{n+1-k}{n-m}\right) = \sum_k \left\{{n \atop k}\right\}k!(-1)^{m+1-k}\binom{n-k}{n-m-1} = \left\langle{n \atop n-m-1}\right\rangle$ by (6.3) and (6.40). Now use (6.34). (This identity has a combinatorial interpretation [59].)

6.68 We have the general formula

$$\left\langle\!\!\left\langle{n \atop m}\right\rangle\!\!\right\rangle = \sum_{k=0}^{m} \binom{2n+1}{k} \left\{{n+m+1-k \atop m+1-k}\right\} (-1)^k, \qquad \text{for } n > m \geqslant 0,$$

analogous to (6.38). When $m = 2$ this equals

$$\left\langle\!\!\left\langle {n \atop 2} \right\rangle\!\!\right\rangle = \left\{ {n+3 \atop 3} \right\} - (2n+1)\left\{ {n+2 \atop 2} \right\} + \binom{2n+1}{2}\left\{ {n+1 \atop 1} \right\}$$
$$= \tfrac{1}{2}3^{n+2} - (2n+3)2^{n+1} + \tfrac{1}{2}(4n^2 + 6n + 3)\,.$$

6.69 $\frac{1}{3}n(n+\frac{1}{2})(n+1)(2H_{2n} - H_n) - \frac{1}{36}n(10n^2+9n-1)$. (It would be nice to automate the derivation of formulas such as this.)

6.70 $1/k - 1/(k+z) = z/k^2 - z^2/k^3 + \cdots$, which converges when $|z| < 1$.

6.71 Note that $\prod_{k=1}^{n}(1 + z/k)e^{-z/k} = \binom{n+z}{n} n^{-z} e^{(\ln n - H_n)z}$. If $f(z) = \frac{d}{dz}(z!)$ we find $f(z)/z! + \gamma = H_z$.

6.72 For $\tan z$, we can use $\tan z = \cot z - 2\cot 2z$ (which is equivalent to the identity of exercise 23). Also $z/\sin z = z\cot z + z\tan\frac{1}{2}z$ has the power series $\sum_{n\geqslant 0}(-1)^{n-1}(4^n - 2)B_{2n}z^{2n}/(2n)!$; and

$$\ln\frac{\tan z}{z} = \ln\frac{\sin z}{z} - \ln\cos z$$
$$= \sum_{n\geqslant 1}(-1)^n\frac{4^n B_{2n} z^{2n}}{(2n)(2n)!} - \sum_{n\geqslant 1}(-1)^n\frac{4^n(4^n-1)B_{2n}z^{2n}}{(2n)(2n)!}$$
$$= \sum_{n\geqslant 1}(-1)^{n-1}\frac{4^n(4^n-2)B_{2n}z^{2n}}{(2n)(2n)!}\,,$$

because $\frac{d}{dz}\ln\sin z = \cot z$ and $\frac{d}{dz}\ln\cos z = -\tan z$.

6.73 $\cot(z+\pi) = \cot z$ and $\cot(z + \frac{1}{2}\pi) = -\tan z$; hence the identity is equivalent to

$$\cot z = \frac{1}{2^n}\sum_{k=0}^{2^n-1}\cot\frac{z+k\pi}{2^n}\,,$$

which follows by induction from the case $n = 1$. The stated limit follows since $z\cot z \to 1$ as $z \to 0$. It can be shown that term-by-term passage to the limit is justified, hence (6.88) is valid. (Incidentally, the general formula

$$\cot z = \frac{1}{n}\sum_{k=0}^{n-1}\cot\frac{z+k\pi}{n}$$

is also true. It can be proved from (6.88), or from

$$\frac{1}{e^{nz}-1} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{e^{z+2k\pi i/n}-1},$$

which is equivalent to the partial fraction expansion of $1/(z^n - 1)$.)

6.74 Since $\tan 2z + \sec 2z = (\sin z + \cos z)/(\cos z - \sin z)$, setting $x = 1$ in (6.94) gives $T_n(1) = 2^n T_n$ when n is odd, $T_n(1) = 2^n|E_n|$ when n is even, where $1/\cos z = \sum_{n\geqslant 0} |E_{2n}| z^{2n}/(2n)!$. (The coefficients $|E_n|$ are called *secant numbers*; with alternating signs they are called *Euler numbers*, not to be confused with the Eulerian numbers $\left\langle{n \atop k}\right\rangle$. We have $\langle E_0, E_2, E_4, \ldots\rangle = \langle 1, -1, 5, -61, 1385, -50521, 2702765, \ldots\rangle$.)

6.75 Let $G(w,z) = \sin z/\cos(w+z)$ and $H(w,z) = \cos z/\cos(w+z)$, and let $G(w,z) + H(w,z) = \sum_{m,n} A_{m,n} w^m z^n/m!\,n!$. Then the equations $G(w,0) = 0$ and $\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial w}\right)G(w,z) = H(w,z)$ imply that $A_{m,0} = 0$ when m is odd, $A_{m,n+1} = A_{m+1,n} + A_{m,n}$ when $m+n$ is even; the equations $H(0,z) = 1$ and $\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}\right)H(w,z) = G(w,z)$ imply that $A_{0,n} = [n=0]$ when n is even, $A_{m+1,n} = A_{m,n+1} + A_{m,n}$ when $m+n$ is odd. Consequently the nth row below the apex of the triangle contains the numbers $A_{n,0}, A_{n-1,1}, \ldots, A_{0,n}$. At the left, $A_{n,0}$ is the secant number $|E_n|$; at the right, $A_{0,n} = T_n + [n=0]$.

6.76 Let A_n denote the sum. Looking ahead to equation (7.49), we see that $\sum_n A_n z^n/n! = \sum_{n,k}(-1)^k {n \brace k} 2^{n-k} k!\, z^n/n! = \sum_k (-1)^k 2^{-k}(e^{2z}-1)^k = 2/(e^{2z}+1) = 1 - \tanh z$. Therefore, by exercise 23 or 72,

$$A_n = (2^{n+1} - 4^{n+1})B_{n+1}/(n+1) = (-1)^{(n+1)/2} T_n + [n=0]\,.$$

6.77 This follows by induction on m, using the recurrence in exercise 18. It can also be proved from (6.50), using the fact that

$$\begin{aligned}\frac{(-1)^{m-1}(m-1)!}{(e^z-1)^m} &= (D+1)^{\overline{m-1}}\frac{1}{e^z-1}\\ &= \sum_{k=0}^{m-1}\left[{m \atop m-k}\right]\frac{d^{m-k-1}}{dz^{m-k-1}}\frac{1}{e^z-1}\,, \quad \text{integer } m>0.\end{aligned}$$

The latter equation, incidentally, is equivalent to

$$\frac{d^m}{dz^m}\frac{1}{e^z-1} = (-1)^m \sum_k {m+1 \brace k}\frac{(k-1)!}{(e^z-1)^k}\,, \quad \text{integer } m \geqslant 0.$$

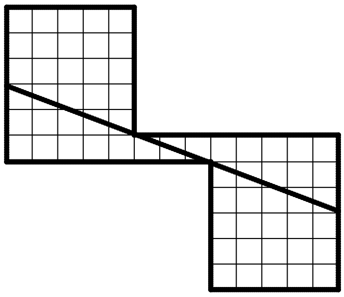
6.78 If $p(x)$ is any polynomial of degree $\leqslant n$, we have

$$p(x) = \sum_k p(-k)\binom{-x}{k}\binom{x+n}{n-k}\,,$$

because this equation holds for $x = 0, -1, \ldots, -n$. The stated identity is the special case where $p(x) = x\sigma_n(x)$ and $x = 1$. Incidentally, we obtain a simpler expression for Bernoulli numbers in terms of Stirling numbers by setting $k = 1$ in (6.99):

$$\sum_{k\geqslant 0} {m \brace k} (-1)^k \frac{k!}{k+1} \;=\; B_m \,.$$

6.79 Sam Loyd [256, pages 288 and 378] gave the construction



and claimed to have invented (but not published) the $64 = 65$ arrangement in 1858. (Similar paradoxes go back at least to the eighteenth century, but Loyd found better ways to present them.)

6.80 We expect $A_m/A_{m-1} \approx \phi$, so we try $A_{m-1} = 618034 + r$ and $A_{m-2} = 381966 - r$. Then $A_{m-3} = 236068 + 2r$, etc., and we find $A_{m-18} = 144 - 2584r$, $A_{m-19} = 154 + 4181r$. Hence $r = 0$, $x = 154$, $y = 144$, $m = 20$.

6.81 If $P(F_{n+1}, F_n) = 0$ for infinitely many *even* values of n, then $P(x, y)$ is divisible by $U(x, y) - 1$, where $U(x, y) = x^2 - xy - y^2$. For if t is the total degree of P, we can write

$$P(x,y) \;=\; \sum_{k=0}^{t} q_k x^k y^{t-k} + \sum_{j+k<t} r_{j,k} x^j y^k \;=\; Q(x,y) + R(x,y)\,.$$

Then

$$\frac{P(F_{n+1}, F_n)}{F_n^t} \;=\; \sum_{k=0}^{t} q_k \left(\frac{F_{n+1}}{F_n}\right)^k + O(1/F_n)$$

and we have $\sum_{k=0}^{t} q_k \phi^k = 0$ by taking the limit as $n \to \infty$. Hence $Q(x, y)$ is a multiple of $U(x, y)$, say $A(x, y)U(x, y)$. But $U(F_{n+1}, F_n) = (-1)^n$ and n is even, so $P_0(x, y) = P(x, y) - \bigl(U(x, y) - 1\bigr)A(x, y)$ is another polynomial such that $P_0(F_{n+1}, F_n) = 0$. The total degree of P_0 is less than t, so P_0 is a multiple of $U - 1$ by induction on t.

Similarly, $P(x, y)$ is divisible by $U(x, y) + 1$ if $P(F_{n+1}, F_n) = 0$ for infinitely many *odd* values of n. A combination of these two facts gives the desired necessary and sufficient condition: $P(x, y)$ is divisible by $U(x, y)^2 - 1$.

6.82 First add the digits without carrying, getting digits 0, 1, and 2. Then use the two carry rules

$$\begin{aligned} 0\,(d{+}1)\,(e{+}1) &\;\rightarrow\; 1\,d\,e\,,\\ 0\,(d{+}2)\,0\,e &\;\rightarrow\; 1\,d\,0\,(e+1)\,, \end{aligned}$$

always applying the leftmost applicable carry. This process terminates because the binary value obtained by reading $(b_m \ldots b_2)_F$ as $(b_m \ldots b_2)_2$ increases whenever a carry is performed. But a carry might propagate to the right of the "Fibonacci point"; for example, $(1)_F{+}(1)_F$ becomes $(10.01)_F$. Such rightward propagation extends at most two positions; and those two digit positions can be zeroed again by using the text's "add 1" algorithm if necessary.

Incidentally, there's a corresponding "multiplication" operation on nonnegative integers: If $m = F_{j_1} + \cdots + F_{j_q}$ and $n = F_{k_1} + \cdots + F_{k_r}$ in the Fibonacci number system, let $m \circ n = \sum_{b=1}^{q} \sum_{c=1}^{r} F_{j_b + k_c}$, by analogy with multiplication of binary numbers. (This definition implies that $m \circ n \approx \sqrt{5}\,mn$ when m and n are large, although $1 \circ n \approx \phi^2 n$.) Fibonacci addition leads to a proof of the associative law $l \circ (m \circ n) = (l \circ m) \circ n$.

Exercise: $m \circ n = mn + \lfloor (m{+}1)/\phi \rfloor n + m\lfloor (n{+}1)/\phi \rfloor$.

6.83 Yes; for example, we can take

$$\begin{aligned} A_0 &= 331635635998274737472200656430763\,;\\ A_1 &= 1510028911088401971189590305498785\,. \end{aligned}$$

The resulting sequence has the property that A_n is divisible by (but unequal to) p_k when $n \bmod m_k = r_k$, where the numbers (p_k, m_k, r_k) have the following 18 respective values:

$(3,4,1)$	$(2,3,2)$	$(5,5,1)$
$(7,8,3)$	$(17,9,4)$	$(11,10,2)$
$(47,16,7)$	$(19,18,10)$	$(61,15,3)$
$(2207,32,15)$	$(53,27,16)$	$(31,30,24)$
$(1087,64,31)$	$(109,27,7)$	$(41,20,10)$
$(4481,64,63)$	$(5779,54,52)$	$(2521,60,60)$

One of these triples applies to every integer n; for example, the six triples in the first column cover every odd value of n, and the middle column covers all even n that are not divisible by 6. The remainder of the proof is based on the fact that $A_{m+n} = A_m F_{n-1} + A_{m+1} F_n$, together with the congruences

$$\begin{aligned} A_0 &\equiv F_{m_k - r_k} \bmod p_k\,,\\ A_1 &\equiv F_{m_k - r_k + 1} \bmod p_k\,, \end{aligned}$$

for each of the triples (p_k, m_k, r_k). (An improved solution, in which A_0 and A_1 are numbers of "only" 17 digits each, is also possible [218].)

6.84 The sequences of exercise 62 satisfy $A_{-m} = A_m$, $B_{-m} = -B_m$, and

$$\begin{aligned} A_m A_n &= A_{m+n} + A_{m-n}\,; \\ A_m B_n &= B_{m+n} - B_{m-n}\,; \\ B_m B_n &= A_{m+n} - A_{m-n}\,. \end{aligned}$$

Let $f_k = B_{mk}/A_{mk+l}$ and $g_k = A_{mk}/B_{mk+l}$, where $l = \frac{1}{2}(n-m)$. Then $f_{k+1} - f_k = A_l B_m/(A_{2mk+n} + A_m)$ and $g_k - g_{k+1} = A_l B_m/(A_{2mk+n} - A_m)$; hence we have

$$\begin{aligned} S^+_{m,n} &= \frac{\sqrt{5}}{A_l B_m} \lim_{k\to\infty} (f_k - f_0) &&= \frac{\sqrt{5}}{\phi^l A_l L_m}\,; \\ S^-_{m,n} &= \frac{\sqrt{5}}{A_l B_m} \lim_{k\to\infty} (g_0 - g_k) &&= \frac{\sqrt{5}}{A_l L_m}\left(\frac{2}{B_l} - \frac{1}{\phi^l}\right) \\ & &&= \frac{2}{F_l L_l L_m} - S^+_{m,n}\,. \end{aligned}$$

6.85 The property holds if and only if N has one of the seven forms 5^k, $2\cdot 5^k$, $4\cdot 5^k$, $3^j\cdot 5^k$, $6\cdot 5^k$, $7\cdot 5^k$, $14\cdot 5^k$.

6.86 For any positive integer m, let $r(m)$ be the smallest index j such that C_j is divisible by m; if no such j exists, let $r(m) = \infty$. Then C_n is divisible by m if and only if $\gcd(C_n, C_{r(m)})$ is divisible by m if and only if $C_{\gcd(n,r(m))}$ is divisible by m if and only if $\gcd(n, r(m)) = r(m)$ if and only if n is divisible by $r(m)$.

(Conversely, the gcd condition is easily seen to be implied by the condition that the sequence C_1, C_2, … has a function $r(m)$, possibly infinite, such that C_n is divisible by m if and only if n is divisible by $r(m)$.)

Now let $\Pi(n) = C_1 C_2 \ldots C_n$, so that

$$\binom{m+n}{m}_{\mathcal{C}} = \frac{\Pi(m+n)}{\Pi(m)\,\Pi(n)}\,.$$

If p is prime, the number of times p divides $\Pi(n)$ is $f_p(n) = \sum_{k\geqslant 1} \lfloor n/r(p^k)\rfloor$, since $\lfloor n/p^k \rfloor$ is the number of elements $\{C_1, \ldots, C_n\}$ that are divisible by p^k. Therefore $f_p(m+n) \geqslant f_p(m) + f_p(n)$ for all p, and $\binom{m+n}{m}_{\mathcal{C}}$ is an integer.

6.87 The matrix product is

$$\begin{pmatrix} K_{n-2}(x_2, \ldots, x_{n-1}) & K_{n-1}(x_2, \ldots, x_{n-1}, x_n) \\ K_{n-1}(x_1, x_2, \ldots, x_{n-1}) & K_n(x_1, x_2, \ldots, x_{n-1}, x_n) \end{pmatrix}.$$

This relates to products of L and R as in (6.137), because we have

$$R^a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L^a .$$

The determinant is $K_n(x_1, \ldots, x_n)$; the more general tridiagonal determinant

$$\det \begin{pmatrix} x_1 & 1 & 0 & \ldots & 0 \\ y_2 & x_2 & 1 & & 0 \\ 0 & y_3 & x_3 & 1 & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & 0 & \ldots & y_n & x_n \end{pmatrix}$$

satisfies the recurrence $D_n = x_n D_{n-1} - y_n D_{n-2}$.

6.88 Let $\alpha^{-1} = a_0 + 1/\bigl(a_1 + 1/(a_2 + \cdots)\bigr)$ be the continued fraction representation of α^{-1}. Then we have

$$\frac{a_0}{z} + \cfrac{1}{A_0(z) + \cfrac{1}{A_1(z) + \cfrac{1}{A_2(z) + \cfrac{1}{\ddots}}}} = \frac{1-z}{z} \sum_{n \geqslant 1} z^{\lfloor n\alpha \rfloor} ,$$

where

$$A_m(z) = \frac{z^{-q_{m+1}} - z^{-q_{m-1}}}{z^{-q_m} - 1} , \qquad q_m = K_m(a_1, \ldots, a_m) .$$

A proof analogous to the text's proof of (6.146) uses a generalization of Zeckendorf's theorem (Fraenkel [129, §4]). If $z = 1/b$, where b is an integer $\geqslant 2$, this gives the continued fraction representation of the transcendental number $(b-1)\sum_{n \geqslant 1} b^{-\lfloor n\alpha \rfloor}$, as in exercise 49.

6.89 Let $p = K(0, a_1, a_2, \ldots, a_m)$, so that p/n is the mth convergent to the continued fraction. Then $\alpha = p/n + (-1)^m/nq$, where $q = K(a_1, \ldots, a_m, \beta)$ and $\beta > 1$. The points $\{k\alpha\}$ for $0 \leqslant k < n$ can therefore be written

$$\frac{0}{n},\ \frac{1}{n} + \frac{(-1)^m \pi_1}{nq},\ \ldots,\ \frac{n-1}{n} + \frac{(-1)^m \pi_{n-1}}{nq} ,$$

where $\pi_1 \ldots \pi_{n-1}$ is a permutation of $\{1, \ldots, n-1\}$. Let $f(\nu)$ be the number of such points $< \nu$; then $f(\nu)$ and νn both increase by 1 when ν increases from k/n to $(k+1)/n$, except when $k = 0$ or $k = n-1$, so they never differ by 2 or more.

6.90 By (6.139) and (6.136), we want to maximize $K(a_1,\ldots,a_m)$ over all sequences of positive integers whose sum is $\leqslant n+1$. The maximum occurs when all the a's are 1, for if $j \geqslant 1$ and $a \geqslant 1$ we have

$$\begin{aligned} &K_{j+k+1}(1,\ldots,1,a+1,b_1,\ldots,b_k)\\ &\quad= K_{j+k+1}(1,\ldots,1,a,b_1,\ldots,b_k)+K_j(1,\ldots,1)\,K_k(b_1,\ldots,b_k)\\ &\quad\leqslant K_{j+k+1}(1,\ldots,1,a,b_1,\ldots,b_k)+K_{j+k}(1,\ldots,1,a,b_1,\ldots,b_k)\\ &\quad= K_{j+k+2}(1,\ldots,1,a,b_1,\ldots,b_k)\,. \end{aligned}$$

(Motzkin and Straus [278] show how to solve more general maximization problems on continuants.)

6.91 A candidate for the case $n \bmod 1 = \frac{1}{2}$ appears in [213, §6], although it may be best to multiply the integers discussed there by some constant involving $\sqrt{\pi}$. Alternatively, Renzo Sprugnoli observes that we can define $\left\{ {n \atop m} \right\} = \sum_k \binom{m}{k} k^n (-1)^{(}m-k)/m!$ for integer $m \geqslant 0$ and arbitrary $n \geqslant 0$; then (6.3) holds for all $n \geqslant 1$.

6.92 (a) If there are only finitely many solutions, it is natural to conjecture that the same holds for all primes. (b) The behavior of b_n is quite strange: We have $b_n = \operatorname{lcm}(1,\ldots,n)$ for $968 \leqslant n \leqslant 1066$; on the other hand, $b_{600} = \operatorname{lcm}(1,\ldots,600)/(3^3\cdot5^2\cdot43)$. Andrew Odlyzko observes that p divides $\operatorname{lcm}(1,\ldots,n)/b_n$ if and only if $kp^m \leqslant n < (k+1)p^m$ for some $m \geqslant 1$ and some $k < p$ such that p divides the numerator of H_k. Therefore infinitely many such n exist if it can be shown, for example, that almost all primes have only one such value of k (namely $k = p-1$).

Another reason to remember 1066?

6.93 (Brent [38] found the surprisingly large partial quotient 1568705 in e^γ, but this seems to be just a coincidence. For example, Gosper has found even larger partial quotients in π: The 453,294th is 12996958 and the 11,504,931st is 878783625.)

6.94 Consider the generating function $\sum_{m,n\geqslant0} \left| {m+n \atop m} \right| w^m z^n$, which has the form $\sum_n \bigl(wF(a,b,c)+zF(a',b',c')\bigr)^n$, where $F(a,b,c)$ is the differential operator $a + b\vartheta_w + c\vartheta_z$.

6.95 Complete success might be difficult or impossible, because Stirling numbers are not "holonomic" in the sense of [382].

7.1 Substitute z^4 for ▯ and z for ▭ in the generating function, getting $1/(1-z^4-z^2)$. This is like the generating function for T, but with z replaced by z^2. Therefore the answer is zero if m is odd, otherwise $F_{m/2+1}$.

7.2 $G(z) = 1/(1-2z)+1/(1-3z)$; $\widehat{G}(z) = e^{2z}+e^{3z}$.

7.3 Set $z = 1/10$ in the generating function, getting $\frac{10}{9}\ln\frac{10}{9}$.

7.4 Divide $P(z)$ by $Q(z)$, getting a quotient $T(z)$ and a remainder $P_0(z)$ whose degree is less than the degree of Q. The coefficients of $T(z)$ must be added to the coefficients $[z^n]\,P_0(z)/Q(z)$ for small n. (This is the polynomial $T(z)$ in (7.28).)

7.5 This is the convolution of $(1+z^2)^r$ with $(1+z)^r$, so

$$S(z) \;=\; (1+z+z^2+z^3)^r\,.$$

Incidentally, no simple form is known for the coefficients of this generating function; hence the stated sum probably has no simple closed form. (We can use generating functions to obtain negative results as well as positive ones.)

7.6 Let the solution to $g_0=\alpha$, $g_1=\beta$, $g_n=g_{n-1}+2g_{n-2}+(-1)^n\gamma$ be $g_n=A(n)\alpha+B(n)\beta+C(n)\gamma$. The function 2^n works when $\alpha=1$, $\beta=2$, $\gamma=0$; the function $(-1)^n$ works when $\alpha=1$, $\beta=-1$, $\gamma=0$; the function $(-1)^n n$ works when $\alpha=0$, $\beta=-1$, $\gamma=3$. Hence $A(n)+2B(n)=2^n$, $A(n)-B(n)=(-1)^n$, and $-B(n)+3C(n)=(-1)^n n$.

7.7 $G(z)=\bigl(z/(1-z)^2\bigr)G(z)+1$, hence

I bet that the controversial "fan of order zero" does have one spanning tree.

$$G(z) \;=\; \frac{1-2z+z^2}{1-3z+z^2} \;=\; 1+\frac{z}{1-3z+z^2}\,;$$

we have $g_n=F_{2n}+[n\!=\!0]$.

7.8 Differentiate $(1-z)^{-x-1}$ twice with respect to x, obtaining

$$\binom{x+n}{n}\Bigl((H_{x+n}-H_x)^2-(H^{(2)}_{x+n}-H^{(2)}_x)\Bigr)\,.$$

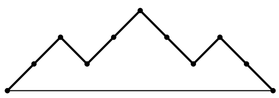
Now set $x=m$.

7.9 $(n+1)(H_n^2-H_n^{(2)})-2n(H_n-1)$.

7.10 The identity $H_{k-1/2}-H_{-1/2}=\frac{2}{2k-1}+\cdots+\frac{2}{1}=2H_{2k}-H_k$ implies that $\sum_k\binom{2k}{k}\binom{2n-2k}{n-k}(2H_{2k}-H_k)=4^nH_n$.

7.11 (a) $C(z)=A(z)B(z^2)/(1-z)$. (b) $zB'(z)=A(2z)e^z$, hence $A(z)=\frac{z}{2}e^{-z/2}B'(\frac{z}{2})$. (c) $A(z)=B(z)/(1-z)^{r+1}$, hence $B(z)=(1-z)^{r+1}A(z)$ and we have $f_k(r)=\binom{r+1}{k}(-1)^k$.

7.12 C_n. The numbers in the upper row correspond to the positions of $+1$'s in a sequence of $+1$'s and -1's that defines a "mountain range"; the numbers in the lower row correspond to the positions of -1's. For example, the given array corresponds to

 .

7.13 Extend the sequence periodically (let $x_{m+k} = x_k$) and define $s_n = x_1 + \cdots + x_n$. We have $s_m = l$, $s_{2m} = 2l$, etc. There must be a largest index k_j such that $s_{k_j} = j$, $s_{k_j+m} = l+j$, etc. These indices $k_1, \ldots, k_l$ (modulo m) specify the cyclic shifts in question.

For example, in the sequence $\langle -2, 1, -1, 0, 1, 1, -1, 1, 1, 1\rangle$ with $m = 10$ and $l = 2$ we have $k_1 = 17$, $k_2 = 24$.

7.14 $\widehat{G}(z) = -2z\widehat{G}(z) + \widehat{G}(z)^2 + z$ (be careful about the final term!) leads via the quadratic formula to

$$\widehat{G}(z) \;=\; \frac{1 + 2z - \sqrt{1+4z^2}}{2}\,.$$

Hence $g_{2n+1} = 0$ and $g_{2n} = (-1)^n (2n)!\, C_{n-1}$, for all $n > 0$.

7.15 There are $\binom{n}{k}\varpi_{n-k}$ partitions with k other objects in the subset containing $n+1$. Hence $\widehat{P}'(z) = e^z\widehat{P}(z)$. The solution to this differential equation is $\widehat{P}(z) = e^{e^z+c}$, and $c = -1$ since $\widehat{P}(0) = 1$. (We can also get this result by summing (7.49) on m, since $\varpi_n = \sum_m {n \brace m}$.)

7.16 One way is to take the logarithm of

$$B(z) \;=\; 1/\bigl((1-z)^{a_1}(1-z^2)^{a_2}(1-z^3)^{a_3}(1-z^4)^{a_4}\ldots\bigr),$$

then use the formula for $\ln\frac{1}{1-z}$ and interchange the order of summation.

7.17 This follows since $\int_0^\infty t^n e^{-t}\,dt = n!$. There's also a formula that goes in the other direction:

$$\widehat{G}(z) \;=\; \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(ze^{-i\theta})\, e^{e^{i\theta}}\, d\theta\,.$$

7.18 (a) $\zeta(z - \frac{1}{2})$; (b) $-\zeta'(z)$; (c) $\zeta(z)/\zeta(2z)$. Every positive integer is uniquely representable as $m^2 q$, where q is squarefree.

7.19 If $n > 0$, the coefficient $[z^n]\, \exp\bigl(x \ln F(z)\bigr)$ is a polynomial of degree n in x that's a multiple of x. The first convolution formula comes from equating coefficients of z^n in $F(z)^x F(z)^y = F(z)^{x+y}$. The second comes from equating coefficients of z^{n-1} in $F'(z)F(z)^{x-1}F(z)^y = F'(z)F(z)^{x+y-1}$, because we have

$$F'(z)F(z)^{x-1} \;=\; x^{-1}\frac{\partial}{\partial z}\bigl(F(z)^x\bigr) \;=\; x^{-1}\sum_{n\geqslant 0} n f_n(x) z^{n-1}\,.$$

(Further convolutions follow by taking $\partial/\partial x$, as in (7.43).)

Still more is true, as shown in [221]: We have

$$\sum_{k=0}^{n} \frac{x f_k(x+tk)}{x+tk}\,\frac{y f_{n-k}(y+t(n-k))}{y+t(n-k)} \;=\; \frac{(x+y) f_n(x+y+tn)}{x+y+tn}\,,$$

for arbitrary x, y, and t. In fact, $xf_n(x+tn)/(x+tn)$ is the sequence of polynomials for the coefficients of $\mathcal{F}_t(z)^x$, where

$$\mathcal{F}_t(z) \;=\; F\bigl(z\mathcal{F}_t(z)^t\bigr)\,.$$

(We saw special cases in (5.59) and (6.52).)

7.20 Let $G(z) = \sum_{n\geqslant 0} g_n z^n$. Then

$$z^l G^{(k)}(z) \;=\; \sum_{n\geqslant 0} n^{\underline{k}} g_n z^{n-k+l} \;=\; \sum_{n\geqslant 0} (n+k-l)^{\underline{k}} g_{n+k-l} z^n$$

for all $k, l \geqslant 0$, if we regard $g_n = 0$ for $n < 0$. Hence if $P_0(z), \ldots, P_m(z)$ are polynomials, not all zero, having maximum degree d, then there are polynomials $p_0(n), \ldots, p_{m+d}(n)$ such that

$$P_0(z)G(z) + \cdots + P_m(z)G^{(m)}(z) \;=\; \sum_{n\geqslant 0}\sum_{j=0}^{m+d} p_j(n)\, g_{n+j-d} z^n\,.$$

Therefore a differentiably finite $G(z)$ implies that

$$\sum_{j=0}^{m+d} p_j(n+d)\, g_{n+j} \;=\; 0\,, \qquad \text{for all } n \geqslant 0.$$

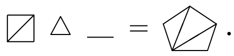
The converse is similar. (One consequence is that $G(z)$ is differentiably finite if and only if the corresponding egf, $\widehat{G}(z)$, is differentiably finite.)

This slow method of finding the answer is just the cashier's way of stalling until the police come.

The USA has two-cent pieces, but they haven't been minted since 1873.

7.21 This is the problem of giving change with denominations 10 and 20, so $G(z) = 1/(1-z^{10})(1-z^{20}) = \check{G}(z^{10})$, where $\check{G}(z) = 1/(1-z)(1-z^2)$. (a) The partial fraction decomposition of $\check{G}(z)$ is $\frac{1}{2}(1-z)^{-2} + \frac{1}{4}(1-z)^{-1} + \frac{1}{4}(1+z)^{-1}$, so $[z^n]\,\check{G}(z) = \frac{1}{4}\bigl(2n+3+(-1)^n\bigr)$. Setting $n = 50$ yields 26 ways to make the payment. (b) $\check{G}(z) = (1+z)/(1-z^2)^2 = (1+z)(1+2z^2+3z^4+\cdots)$, so $[z^n]\,\check{G}(z) = \lfloor n/2 \rfloor + 1$. (Compare this with the value $N_n = \lfloor n/5 \rfloor + 1$ in the text's coin-changing problem. The bank robber's problem is equivalent to the problem of making change with pennies and tuppences.)

7.22 Each polygon has a "base" (the line segment at the bottom). If A and B are triangulated polygons, let $A\triangle B$ be the result of pasting the base of A to the upper left diagonal of $\triangle$, and pasting the base of B to the upper right diagonal. Thus, for example,



(The polygons might need to be warped a bit and/or banged into shape.) Every triangulation arises in this way, because the base line is part of a unique triangle and there are triangulated polygons A and B at its left and right.

Replacing each triangle by z gives a power series in which the coefficient of z^n is the number of triangulations with n triangles, namely the number of ways to decompose an $(n+2)$-gon into triangles. Since $P = 1 + zP^2$, this is the generating function for Catalan numbers $C_0 + C_1 z + C_2 z^2 + \cdots$; the number of ways to triangulate an n-gon is $C_{n-2} = \binom{2n-4}{n-2}/(n-1)$.

7.23 Let a_n be the stated number, and b_n the number of ways with a $2\times1\times1$ notch missing at the top. By considering the possible patterns visible on the top surface, we have

$$\begin{aligned} a_n &= 2a_{n-1} + 4b_{n-1} + a_{n-2} + [n=0]\,; \\ b_n &= a_{n-1} + b_{n-1}\,. \end{aligned}$$

Hence the generating functions satisfy $A = 2zA + 4zB + z^2A + 1$, $B = zA + zB$, and we have

$$A(z) = \frac{1-z}{(1+z)(1-4z+z^2)}\,.$$

This formula relates to the problem of $3 \times n$ domino tilings; we have $a_n = \frac{1}{3}\bigl(U_{2n} + V_{2n+1} + (-1)^n\bigr) = \frac{1}{6}(2+\sqrt{3}\,)^{n+1} + \frac{1}{6}(2-\sqrt{3}\,)^{n+1} + \frac{1}{3}(-1)^n$, which is $(2+\sqrt{3}\,)^{n+1}/6$ rounded to the nearest integer.

7.24 $n\sum_{k_1+\cdots+k_m=n} k_1 \cdot \ldots \cdot k_m/m = F_{2n+1} + F_{2n-1} - 2$. (Consider the coefficient $[z^{n-1}]\,\frac{d}{dz}\ln\bigl(1/(1-G(z))\bigr)$, where $G(z) = z/(1-z)^2$.)

7.25 The generating function is $P(z)/(1-z^m)$, where $P(z) = z + 2z^2 + \cdots + (m-1)z^{m-1} = \bigl((m-1)z^{m+1} - mz^m + z\bigr)/(1-z)^2$. The denominator is $Q(z) = 1 - z^m = (1-\omega^0 z)(1-\omega^1 z)\ldots(1-\omega^{m-1}z)$. By the rational expansion theorem for distinct roots, we obtain

$$n \bmod m = \frac{m-1}{2} + \sum_{k=1}^{m-1} \frac{\omega^{-kn}}{\omega^k - 1}\,.$$

7.26 $(1-z-z^2)\mathfrak{F}(z) = F(z)$ leads to $\mathfrak{F}_n = \bigl(2(n+1)F_n + nF_{n+1}\bigr)/5$ as in equation (7.61).

7.27 Each oriented cycle pattern begins with  or  or a $2 \times k$ cycle (for some $k \geqslant 2$) oriented in one of two ways. Hence

$$Q_n = Q_{n-1} + Q_{n-2} + 2Q_{n-2} + 2Q_{n-3} + \cdots + 2Q_0$$

for $n \geqslant 2$; $Q_0 = Q_1 = 1$. The generating function is therefore

$$\begin{aligned} Q(z) &= zQ(z) + z^2Q(z) + 2z^2Q(z)/(1-z) + 1 \\ &= 1/\bigl(1 - z - z^2 - 2z^2/(1-z)\bigr) \end{aligned}$$

$$= \frac{(1-z)}{(1-2z-2z^2+z^3)}$$
$$= \frac{\phi^2/5}{1-\phi^2 z} + \frac{\phi^{-2}/5}{1-\phi^{-2}z} + \frac{2/5}{1+z}\,,$$

and $Q_n = \bigl(\phi^{2n+2} + \phi^{-2n-2} + 2(-1)^n\bigr)/5 = \bigl((\phi^{n+1} - \hat\phi^{n+1})/\sqrt5\bigr)^2 = F_{n+1}^2$.

7.28 In general if $A(z) = (1 + z + \cdots + z^{m-1})B(z)$, we have $A_r + A_{r+m} + A_{r+2m} + \cdots = B(1)$ for $0 \leqslant r < m$. In this case $m = 10$ and $B(z) = (1 + z + \cdots + z^9)(1 + z^2 + z^4 + z^6 + z^8)(1 + z^5)$.

7.29 $F(z) + F(z)^2 + F(z)^3 + \cdots = z/(1-z-z^2-z) = \bigl(1/(1-(1+\sqrt2\,)z) - (1/(1-(1-\sqrt2\,)z)\bigr)/\sqrt8$, so the answer is $\bigl((1+\sqrt2\,)^n - (1-\sqrt2\,)^n\bigr)/\sqrt8$.

7.30 $\sum_{k=1}^{n} \binom{2n-1-k}{n-1}\bigl(a^n b^{n-k}/(1-\alpha z)^k + a^{n-k} b^n/(1-\beta z)^k\bigr)$, by exercise 5.39.

7.31 The dgf is $\zeta(z)^2/\zeta(z-1)$; hence we find $g(n)$ is the product of $(k+1-kp)$ over all prime powers p^k that exactly divide n.

7.32 We may assume that each $b_k \geqslant 0$. A set of arithmetic progressions forms an exact cover if and only if

$$\frac{1}{1-z} = \frac{z^{b_1}}{1-z^{a_1}} + \cdots + \frac{z^{b_m}}{1-z^{a_m}}\,.$$

Subtract $z^{b_m}/(1-z^{a_m})$ from both sides and set $z = e^{2\pi i/a_m}$. The left side is infinite, and the right side will be finite unless $a_{m-1} = a_m$.

7.33 $(-1)^{n-m+1}[n>m]/(n-m)$.

7.34 We can also write $G_n(z) = \sum_{k_1+(m+1)k_{m+1}=n} \binom{k_1+k_{m+1}}{k_{m+1}}(z^m)^{k_{m+1}}$. In general, if

$$G_n = \sum_{k_1+2k_2+\cdots+rk_r=n} \binom{k_1+k_2+\cdots+k_r}{k_1,k_2,\ldots,k_r} z_1^{k_1} z_2^{k_2} \ldots z_r^{k_r}\,,$$

we have $G_n = z_1 G_{n-1} + z_2 G_{n-2} + \cdots + z_r G_{n-r} + [n=0]$, and the generating function is $1/(1 - z_1 w - z_2 w^2 - \cdots - z_r w^r)$. In the stated special case the answer is $1/(1 - w - z^m w^{m+1})$. (See (5.74) for the case $m = 1$.)

7.35 (a) $\frac1n \sum_{0<k<n}\bigl(1/k + 1/(n-k)\bigr) = \frac2n H_{n-1}$. (b) $[z^n]\bigl(\ln\frac{1}{1-z}\bigr)^2 = \frac{2!}{n!}\left[{n \atop 2}\right] = \frac2n H_{n-1}$ by (7.50) and (6.58). Another way to do part (b) is to use the rule $[z^n]\,F(z) = \frac1n[z^{n-1}]\,F'(z)$ with $F(z) = \bigl(\ln\frac{1}{1-z}\bigr)^2$.

7.36 $\frac{1-z^m}{1-z}A(z^m)$.

7.37 (a) The amazing identity $a_{2n} = a_{2n+1} = b_n$ holds in the table

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	2	4	4	6	6	10	10	14
b_n	1	2	4	6	10	14	20	26	36	46	60

(b) $A(z) = 1/\bigl((1-z)(1-z^2)(1-z^4)(1-z^8)\ldots\bigr)$. (c) $B(z) = A(z)/(1-z)$, and we want to show that $A(z) = (1+z)B(z^2)$. This follows from $A(z) = A(z^2)/(1-z)$.

7.38 $(1-wz)M(w,z) = \sum_{m,n\geqslant 1}\bigl(\min(m,n) - \min(m-1,n-1)\bigr)w^m z^n = \sum_{m,n\geqslant 1} w^m z^n = wz/(1-w)(1-z)$. In general,

$$M(z_1,\ldots,z_m) \;=\; \frac{z_1\ldots z_m}{(1-z_1)\ldots(1-z_m)(1-z_1\ldots z_m)}\,.$$

7.39 The answers to the hint are

$$\sum_{1\leqslant k_1<k_2<\cdots<k_m\leqslant n} a_{k_1}a_{k_2}\ldots a_{k_m} \quad\text{and}\quad \sum_{1\leqslant k_1\leqslant k_2\leqslant\cdots\leqslant k_m\leqslant n} a_{k_1}a_{k_2}\ldots a_{k_m}\,,$$

respectively. Therefore: (a) We want the coefficient of z^m in the product $(1+z)(1+2z)\ldots(1+nz)$. This is the reflection of $(z+1)^{\overline{n}}$, so it is $\left[{n+1 \atop n+1}\right] + \left[{n+1 \atop n}\right]z + \cdots + \left[{n+1 \atop 1}\right]z^n$ and the answer is $\left[{n+1 \atop n+1-m}\right]$. (b) The coefficient of z^m in $1/\bigl((1-z)(1-2z)\ldots(1-nz)\bigr)$ is $\left\{{m+n \atop n}\right\}$ by (7.47).

7.40 The egf for $\langle nF_{n-1} - F_n\rangle$ is $(z-1)\widehat{F}(z)$ where $\widehat{F}(z) = \sum_{n\geqslant 0} F_n z^n/n! = (e^{\phi z} - e^{\hat\phi z})/\sqrt{5}$. The egf for $\langle n¡\rangle$ is $e^{-z}/(1-z)$. The product is

$$5^{-1/2}\bigl(e^{(\hat\phi-1)z} - e^{(\phi-1)z}\bigr) \;=\; 5^{-1/2}(e^{-\phi z} - e^{-\hat\phi z})\,.$$

We have $\widehat{F}(z)e^{-z} = -\widehat{F}(-z)$. So the answer is $(-1)^n F_n$.

7.41 The number of up-down permutations with the largest element n in position $2k$ is $\binom{n-1}{2k-1}A_{2k-1}A_{n-2k}$. Similarly, the number of up-down permutations with the smallest element 1 in position $2k+1$ is $\binom{n-1}{2k}A_{2k}A_{n-2k-1}$, because down-up permutations and up-down permutations are equally numerous. Summing over all possibilities gives

$$2A_n \;=\; \sum_k \binom{n-1}{k} A_k A_{n-1-k} + 2[n=0] + [n=1]\,.$$

The egf $\widehat{A}$ therefore satisfies $2\widehat{A}'(z) = \widehat{A}(z)^2 + 1$ and $\widehat{A}(0) = 1$; the given function solves this differential equation. (Consequently $A_n = |E_n| + T_n$ is a secant number when n is even, a tangent number when n is odd.)

7.42 Let a_n be the number of Martian DNA strings that don't end with c or e; let b_n be the number that do. Then

$$\begin{aligned} a_n &= 3a_{n-1} + 2b_{n-1} + [n=0]\,, & b_n &= 2a_{n-1} + b_{n-1}\,; \\ A(z) &= 3zA(z) + 2zB(z) + 1\,, & B(z) &= 2zA(z) + zB(z)\,; \\ A(z) &= \frac{1-z}{1-4z-z^2}\,, & B(z) &= \frac{2z}{1-4z-z^2}\,; \end{aligned}$$

and the total number is $[z^n](1+z)/(1-4z-z^2) = F_{3n+2}$.

7.43 By (5.45), $g_n = \Delta^n \dot{G}(0)$. The nth difference of a product can be written

$$\Delta^n A(z)B(z) \;=\; \sum_k \binom{n}{k} \bigl(\Delta^k E^{n-k} A(z)\bigr)\bigl(\Delta^{n-k} B(z)\bigr)\,,$$

and $E^{n-k} = (1+\Delta)^{n-k} = \sum_j \binom{n-k}{j}\Delta^j$. Therefore we find

$$h_n \;=\; \sum_{j,k} \binom{n}{k}\binom{n-k}{j} f_{j+k}\, g_{n-k}\,.$$

This is a sum over all trinomial coefficients; it can be put into the more symmetric form

$$h_n \;=\; \sum_{j+k+l=n} \binom{n}{j,k,l} f_{j+k}\, g_{k+l}\,.$$

The empty set is pointless.

7.44 Each partition into k nonempty subsets can be ordered in $k!$ ways, so $b_k = k!$. Thus $\widehat{Q}(z) = \sum_{n,k\geqslant 0} \left\{ {n \atop k} \right\} k!\, z^n/n! = \sum_{k\geqslant 0}(e^z-1)^k = 1/(2-e^z)$. And this is the geometric series $\sum_{k\geqslant 0} e^{kz}/2^{k+1}$, hence $a_k = 1/2^{k+1}$. Finally, $c_k = 2^k$; consider all permutations when the x's are distinct, change each '$>$' between subscripts to '$<$' and allow each '$<$' between subscripts to become either '$<$' or '$=$'. (For example, the permutation $x_1x_3x_2$ produces $x_1 < x_3 < x_2$ and $x_1 = x_3 < x_2$, because $1 < 3 > 2$.)

7.45 This sum is $\sum_{n\geqslant 1} r(n)/n^2$, where $r(n)$ is the number of ways to write n as a product of two relatively prime factors. If n is divisible by t distinct primes, $r(n) = 2^t$. Hence $r(n)/n^2$ is multiplicative and the sum is

$$\begin{aligned} \prod_p \left(1 + \frac{2}{p^2} + \frac{2}{p^4} \cdots\right) &= \prod_p \left(1 + \frac{2}{p^2-1}\right) \\ &= \prod_p \left(\frac{p^2+1}{p^2-1}\right) \;=\; \zeta(2)^2/\zeta(4) \;=\; \frac{5}{2}\,. \end{aligned}$$

7.46 Let $S_n = \sum_{0\leqslant k\leqslant n/2}\binom{n-2k}{k}\alpha^k$. Then $S_n = S_{n-1} + \alpha S_{n-3} + [n=0]$, and the generating function is $1/(1-z-\alpha z^3)$. When $\alpha = -\frac{4}{27}$, the hint tells us that this has a nice factorization $1/(1+\frac{1}{3}z)(1-\frac{2}{3}z)^2$. The general expansion theorem now tells us that $S_n = (\frac{2}{3}n + c)(\frac{2}{3})^n + \frac{1}{9}(-\frac{1}{3})^n$, and the remaining constant c turns out to be $\frac{8}{9}$.

7.47 The Stern–Brocot representation of $\sqrt{3}$ is $R(LR^2)^\infty$, because

$$\sqrt{3}+1 \;=\; 2 + \cfrac{1}{1 + \cfrac{1}{\sqrt{3}+1}}\,.$$

The fractions are $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{7}{4}$, $\frac{12}{7}$, $\frac{19}{11}$, $\frac{26}{15}$, $\ldots$; they eventually have the cyclic pattern

$$\frac{V_{2n-1}+V_{2n+1}}{U_{2n}},\ \frac{U_{2n}+V_{2n+1}}{V_{2n+1}},\ \frac{U_{2n+2}+V_{2n-1}}{U_{2n}+V_{2n+1}},\ \frac{V_{2n+1}+V_{2n+3}}{U_{2n+2}},\ \ldots\,.$$

7.48 We have $g_0 = 0$, and if $g_1 = m$ the generating function satisfies

$$aG(z) + bz^{-1}G(z) + cz^{-2}\bigl(G(z) - mz\bigr) + \frac{d}{1-z} \;=\; 0\,.$$

Hence $G(z) = P(z)/(az^2+bz+c)(1-z)$ for some polynomial $P(z)$. Let ρ_1 and ρ_2 be the roots of cz^2+bz+a, with $|\rho_1| \geqslant |\rho_2|$. If $b^2-4ac \leqslant 0$ then $|\rho_1|^2 = \rho_1\rho_2 = a/c$ is rational, contradicting the fact that $\sqrt[n]{g_n}$ approaches $1+\sqrt{2}$. Hence $\rho_1 = (-b+\sqrt{b^2-4ca})/2c = 1+\sqrt{2}$; and this implies that $a=-c$, $b=-2c$, $\rho_2 = 1-\sqrt{2}$. The generating function now takes the form

$$\begin{aligned} G(z) \;&=\; \frac{z\bigl(m-(r+m)z\bigr)}{(1-2z-z^2)(1-z)} \\ &=\; \frac{-r+(2m+r)z}{2(1-2z-z^2)} + \frac{r}{2(1-z)} \;=\; mz + (2m-r)z^2 + \cdots\,, \end{aligned}$$

where $r = d/c$. Since g_2 is an integer, r is an integer. We also have

$$g_n \;=\; \alpha(1+\sqrt{2}\,)^n + \hat\alpha(1-\sqrt{2}\,)^n + \tfrac{1}{2}r \;=\; \bigl\lfloor \alpha(1+\sqrt{2}\,)^n \bigr\rfloor\,,$$

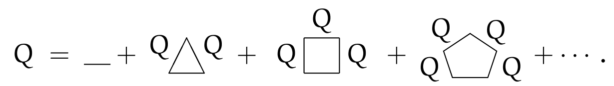
and this can hold only if $r=-1$, because $(1-\sqrt{2}\,)^n$ alternates in sign as it approaches zero. Hence $(a,b,c,d) = \pm(1,2,-1,1)$. Now we find $\alpha = \frac{1}{4}(1+\sqrt{2}\,m)$, which is between 0 and 1 only if $0 \leqslant m \leqslant 2$. Each of these values actually gives a solution; the sequences $\langle g_n\rangle$ are $\langle 0,0,1,3,8,\ldots\rangle$, $\langle 0,1,3,8,20,\ldots\rangle$, and $\langle 0,2,5,13,32,\ldots\rangle$.

7.49 (a) The denominator of $\bigl(1/\bigl(1-(1+\sqrt{2})z\bigr)+1/\bigl(1-(1-\sqrt{2})z\bigr)\bigr)$ is $1-2z-z^2$; hence $a_n = 2a_{n-1}+a_{n-2}$ for $n \geqslant 2$. (b) True because a_n is even and $-1 < 1-\sqrt{2} < 0$. (c) Let

$$b_n \;=\; \left(\frac{p+\sqrt{q}}{2}\right)^n + \left(\frac{p-\sqrt{q}}{2}\right)^n .$$

We would like b_n to be odd for all $n > 0$, and $-1 < (p-\sqrt{q})/2 < 0$. Working as in part (a), we find $b_0 = 2$, $b_1 = p$, and $b_n = pb_{n-1} + \frac{1}{4}(q-p^2)b_{n-2}$ for $n \geqslant 2$. One satisfactory solution has $p = 3$ and $q = 17$.

7.50 Extending the multiplication idea of exercise 22, we have



Replace each n-gon by z^{n-2}. This substitution behaves properly under multiplication, because the pasting operation takes an m-gon and an n-gon into an $(m+n-2)$-gon. Thus the generating function is

$$Q \;=\; 1 + zQ^2 + z^2Q^3 + z^3Q^4 + \cdots \;=\; 1 + \frac{zQ^2}{1-zQ}$$

and the quadratic formula gives $Q = \bigl(1+z-\sqrt{1-6z+z^2}\,\bigr)/4z$. The coefficient of z^{n-2} in this power series is the number of ways to put nonoverlapping diagonals into a convex n-gon. These coefficients apparently have no closed form in terms of other quantities that we have discussed in this book, but their asymptotic behavior is known [207, exercise 2.2.1–12].

Give me Legendre polynomials and I'll give you a closed form.

Incidentally, if each n-gon in Q is replaced by wz^{n-2} we get

$$Q \;=\; \frac{1+z-\sqrt{1-(4w+2)z+z^2}}{2(1+w)z},$$

a formula in which the coefficient of $w^m z^{n-2}$ is the number of ways to divide an n-gon into m polygons by nonintersecting diagonals.

7.51 The key first step is to observe that the square of the number of ways is the number of cycle patterns of a certain kind, generalizing exercise 27. These can be enumerated by evaluating the determinant of a matrix whose eigenvalues are not difficult to determine. When $m = 3$ and $n = 4$, the fact that $\cos 36° = \phi/2$ is helpful (exercise 6.46).

7.52 The first few cases are $p_0(y) = 1$, $p_1(y) = y$, $p_2(y) = y^2+y$, $p_3(y) = y^3+3y^2+3y$. Let $p_n(y) = q_{2n}(x)$ where $y = x(1-x)$; we seek a generating function that defines $q_{2n+1}(x)$ in a convenient way. One such function is $\sum_n q_n(x)z^n/n! = 2e^{ixz}/(e^{iz}+1)$, from which it follows that $q_n(x) = i^n E_n(x)$, where $E_n(x)$ is called an Euler polynomial. We have $\sum(-1)^x x^n\,\delta x = \frac{1}{2}(-1)^{x+1}E_n(x)$, so Euler polynomials are analogous to Bernoulli polynomials, and they have factors analogous to those in (6.98). By exercise 6.23 we have $nE_{n-1}(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}(2-2^{k+1})$; this polynomial has integer coefficients by exercise 6.54. Hence $q_{2n}(x)$, whose coefficients have denominators that are powers of 2, must have integer coefficients. Hence $p_n(y)$ has integer coefficients. Finally, the relation $(4y-1)p_n''(y) + 2p_n'(y) = 2n(2n-1)p_{n-1}(y)$ shows that

$$2m(2m-1)\left|{n \atop m}\right| \;=\; m(m+1)\left|{n \atop m+1}\right| + 2n(2n-1)\left|{n-1 \atop m-1}\right|,$$

and it follows that the $\left|{n \atop m}\right|$'s are positive. (A similar proof shows that the related quantity $(-1)^n(2n+2)E_{2n+1}(x)/(2x-1)$ has positive integer coefficients, when expressed as an nth degree polynomial in y.) It can be shown that $\left|{n \atop 1}\right|$ is the Genocchi number $(-1)^{n-1}(2^{2n+1}-2)B_{2n}$ (see exercise 6.24), and that $\left|{n \atop n-1}\right| = \binom{n}{2}$, $\left|{n \atop n-2}\right| = 2\binom{n+1}{4} + 3\binom{n}{4}$, etc.

7.53 It is $P_{(1+V_{4n+1}+V_{4n+3})/6}$. Thus, for example, $T_{20} = P_{12} = 210$; $T_{285} = P_{165} = 40755$.

7.54 Let E_k be the operation on power series that sets all coefficients to zero except those of z^n where $n \bmod m = k$. The stated construction is equivalent to the operation

$$E_0\,S\,E_0\,S\,(E_0+E_1)\,S\,\ldots\,S\,(E_0+E_1+\cdots+E_{m-1})$$

applied to $1/(1-z)$, where S means "multiply by $1/(1-z)$." There are $m!$ terms

$$E_0\,S\,E_{k_1}\,S\,E_{k_2}\,S\,\ldots\,S\,E_{k_m}$$

where $0 \leqslant k_j < j$, and every such term evaluates to $z^{rm}/(1-z^m)^{m+1}$ if r is the number of places where $k_j < k_{j+1}$. Exactly $\left\langle{m \atop r}\right\rangle$ terms have a given value of r, so the coefficient of z^{mn} is $\sum_{r=0}^{m-1}\left\langle{m \atop r}\right\rangle\binom{n+m-r}{m} = (n+1)^m$ by (6.37). (The fact that operation E_k can be expressed with complex roots of unity seems to be of no help in this problem.)

7.55 Suppose that $P_0(z)F(z) + \cdots + P_m(z)F^{(m)}(z) = Q_0(z)G(z) + \cdots + Q_n(z)G^{(n)}(z) = 0$, where $P_m(z)$ and $Q_n(z)$ are nonzero. (a) Let $H(z) = F(z) + G(z)$. Then there are rational functions $R_{k,l}(z)$ for $0 \leqslant l < m+n$ such that $H^{(k)}(z) = R_{k,0}(z)F^{(0)}(z) + \cdots + R_{k,m-1}(z)F^{(m-1)}(z) + R_{k,m}(z)G^{(0)}(z) + \cdots + R_{k,m+n-1}(z)G^{(n-1)}(z)$. The $m+n+1$ vectors $\bigl(R_{k,0}(z), \ldots, R_{k,m+n-1}(z)\bigr)$ are linearly dependent in the $(m+n)$-dimensional vector space whose components are rational functions; hence there are rational functions $S_l(z)$, not all zero, such that $S_0(z)H^{(0)}(z) + \cdots + S_{m+n}(z)H^{(m+n)}(z) = 0$. (b) Similarly, let $H(z) = F(z)G(z)$. There are rational $R_{k,l}(z)$ for $0 \leqslant l < mn$ with $H^{(k)}(z) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} R_{k,ni+j}(z)F^{(i)}(z)G^{(j)}(z)$, hence $S_0(z)H^{(0)}(z) + \cdots + S_{mn}(z)H^{(mn)}(z) = 0$ for some rational $S_l(z)$, not all zero. (A similar proof shows that if $\langle f_n\rangle$ and $\langle g_n\rangle$ are polynomially recursive, so are $\langle f_n + g_n\rangle$ and $\langle f_n g_n\rangle$. Incidentally, there is no similar result for quotients; for example, $\cos z$ is differentiably finite, but $1/\cos z$ is not.)

7.56 Euler [113] showed that this number is also $[z^n]\,1/\sqrt{1-2z-3z^2}$, and he gave the formula $t_n = \sum_{k\geqslant 0} n^{\underline{2k}}/k!^2 = \sum_k \binom{n}{k}\binom{n-k}{k}$. He also discovered a "memorable failure of induction" while examining these numbers: Although $3t_n - t_{n+1}$ is equal to $F_{n-1}(F_{n-1}+1)$ for $0 \leqslant n \leqslant 8$, this empirical law mysteriously breaks down when n is 9 or more! George Andrews [12] has explained the mystery by showing that the sum $\sum_k [z^{n+10k}]\,(1+z+z^2)^n$ can be expressed as a closed form in terms of Fibonacci numbers.

H. S. Wilf observes that $[z^n]\,(a+bz+cz^2)^n = [z^n]\,1/f(z)$, where $f(z) = \sqrt{1-2bz+(b^2-4ac)z^2}$ (see [373, page 159]), and it follows that the coefficients satisfy

$$(n+1)A_{n+1} - (2n+1)bA_n + n(b^2-4ac)A_{n-1} \;=\; 0\,.$$

The algorithm of Petkovšek [291] can be used to prove that this recurrence has a closed form solution as a finite sum of hypergeometric terms if and only if $abc(b^2-4ac) = 0$. Therefore in particular, the middle trinomial coefficients have no such closed form. The next step is presumably to extend this result to a larger class of closed forms (including harmonic numbers and/or Stirling numbers, for example).

Give me Legendre polynomials and I'll give you a closed form.

7.57 (Paul Erdős currently offers \$500 for a solution.)

8.1 $\frac{1}{24} + \frac{1}{48} + \frac{1}{48} + \frac{1}{48} + \frac{1}{48} + \frac{1}{24} = \frac{1}{6}$. (In fact, we *always* get doubles with probability $\frac{1}{6}$ when at least one of the dice is fair.) Any two faces whose sum is 7 have the same probability in distribution $\Pr_1$, so $S = 7$ has the same probability as doubles.

8.2 There are 12 ways to specify the top and bottom cards and 50! ways to arrange the others; so the probability is $12\cdot 50!/52! = 12/(51\cdot 52) = \frac{1}{17\cdot 13} = \frac{1}{221}$.

8.3 $\frac{1}{10}(3+2+\cdots+9+2) = 4.8$; $\frac{1}{9}(3^2+2^2+\cdots+9^2+2^2-10(4.8)^2) = \frac{388}{45}$, which is approximately 8.6. The true mean and variance with a fair coin are 6 and 22, so Stanford had an unusually heads-up class. The corresponding Princeton figures are 6.4 and $\frac{562}{45} \approx 12.5$. (This distribution has $\kappa_4 = 2974$, which is rather large. Hence the standard deviation of this variance estimate when $n = 10$ is also rather large, $\sqrt{2974/10 + 2(22)^2/9} \approx 20.1$ according to exercise 54. One cannot complain that the students cheated.)

8.4 This follows from (8.38) and (8.39), because $F(z) = G(z)H(z)$. (A similar formula holds for all the cumulants, even though $F(z)$ and $G(z)$ may have negative coefficients.)

8.5 Replace H by p and T by $q = 1-p$. If $S_A = S_B = \frac{1}{2}$ we have $p^2qN = \frac{1}{2}$ and $pq^2N = \frac{1}{2}q + \frac{1}{2}$; the solution is $p = 1/\phi^2$, $q = 1/\phi$.

8.6 In this case $X|y$ has the same distribution as X, for all y, hence $E(X|Y) = EX$ is constant and $V\bigl(E(X|Y)\bigr) = 0$. Also $V(X|Y)$ is constant and equal to its expected value.

8.7 We have $1 = (p_1+p_2+\cdots+p_6)^2 \leqslant 6(p_1^2+p_2^2+\cdots+p_6^2)$ by Chebyshev's monotonic inequality of Chapter 2.

8.8 Let $p = \Pr(\omega \in A \cap B)$, $q = \Pr(\omega \notin A)$, and $r = \Pr(\omega \notin B)$. Then $p + q + r = 1$, and the identity to be proved is $p = (p + r)(p + q) - qr$.

8.9 This is true (subject to the obvious proviso that F and G are defined on the respective ranges of X and Y), because

$$\begin{aligned}\Pr\bigl(F(X)=f \text{ and } G(Y)=g\bigr) &= \sum_{\substack{x\in F^{-1}(f)\\ y\in G^{-1}(g)}} \Pr(X=x \text{ and } Y=y)\\ &= \sum_{\substack{x\in F^{-1}(f)\\ y\in G^{-1}(g)}} \Pr(X=x)\cdot\Pr(Y=y)\\ &= \Pr\bigl(F(X)=f\bigr)\cdot\Pr\bigl(G(y)=g\bigr)\,.\end{aligned}$$

8.10 Two. Let $x_1 < x_2$ be medians; then $1 \leqslant \Pr(X\leqslant x_1) + \Pr(X\geqslant x_2) \leqslant 1$, hence equality holds. (Some discrete distributions have no median elements. For example, let Ω be the set of all fractions of the form $\pm 1/n$, with $\Pr(+1/n) = \Pr(-1/n) = \frac{\pi^2}{12}n^{-2}$.)

8.11 For example, let $K = k$ with probability $4/(k+1)(k+2)(k+3)$, for all integers $k \geqslant 0$. Then $EK = 1$, but $E(K^2) = \infty$. (Similarly we can construct random variables with finite cumulants through κ_m but with $\kappa_{m+1} = \infty$.)

8.12 (a) Let $p_k = \Pr(X=k)$. If $0 < x \leqslant 1$, we have $\Pr(X\leqslant r) = \sum_{k\leqslant r} p_k \leqslant \sum_{k\leqslant r} x^{k-r}p_k \leqslant \sum_k x^{k-r}p_k = x^{-r}P(x)$. The other inequality has a similar proof. (b) Let $x = \alpha/(1-\alpha)$ to minimize the right-hand side. (A more precise estimate for the given sum is obtained in exercise 9.42.)

8.13 (Solution by Boris Pittel.) Let us set $Y = (X_1 + \cdots + X_n)/n$ and $Z = (X_{n+1} + \cdots + X_{2n})/n$. Then

$$\Pr\left(\left|\frac{Y+Z}{2} - \alpha\right| \leqslant |Y-\alpha|\right)$$

$$\begin{aligned}&\geqslant\ \Pr\biggl(\Bigl|\frac{Y-\alpha}{2}\Bigr|+\Bigl|\frac{Z-\alpha}{2}\Bigr|\ \leqslant\ |Y-\alpha|\biggr)\\&=\ \Pr\bigl(|Z-\alpha|\ \leqslant\ |Y-\alpha|\bigr)\ \ \geqslant\ \ \tfrac12\,.\end{aligned}$$

The last inequality is, in fact, '$>$' in any discrete probability distribution, because $\Pr(Y=Z)>0$.

8.14 $\mathrm{Mean}(H)=p\,\mathrm{Mean}(F)+q\,\mathrm{Mean}(G)$; $\mathrm{Var}(H)=p\,\mathrm{Var}(F)+q\,\mathrm{Var}(G)+pq\bigl(\mathrm{Mean}(F)-\mathrm{Mean}(G)\bigr)^2$. (A mixture is actually a special case of conditional probabilities: Let Y be the coin, let $X|\mathtt{H}$ be generated by $F(z)$, and let $X|\mathtt{T}$ be generated by $G(z)$. Then $VX=EV(X|Y)+VE(X|Y)$, where $EV(X|Y)=pV(X|\mathtt{H})+qV(X|\mathtt{T})$ and $VE(X|Y)$ is the variance of $pz^{\mathrm{Mean}(F)}+qz^{\mathrm{Mean}(G)}$.)

8.15 By the chain rule, $H'(z)=G'(z)F'\bigl(G(z)\bigr)$; $H''(z)=G''(z)F'\bigl(G(z)\bigr)+G'(z)^2F''\bigl(G(z)\bigr)$. Hence

$$\begin{aligned}\mathrm{Mean}(H)\ &=\ \mathrm{Mean}(F)\,\mathrm{Mean}(G)\,;\\ \mathrm{Var}(H)\ &=\ \mathrm{Var}(F)\,\mathrm{Mean}(G)^2+\mathrm{Mean}(F)\,\mathrm{Var}(G)\,.\end{aligned}$$

(The random variable corresponding to probability distribution H can be understood as follows: Determine a nonnegative integer n by distribution F; then add the values of n independent random variables that have distribution G. The identity for variance in this exercise is a special case of (8.106), when X has distribution H and Y has distribution F.)

8.16 $e^{w(z-1)}/(1-w)$.

8.17 $\Pr(Y_{n,p}\leqslant m)=\Pr(Y_{n,p}+n\leqslant m+n)=$ probability that we need $\leqslant m+n$ tosses to obtain n heads $=$ probability that $m+n$ tosses yield $\geqslant n$ heads $=\Pr(X_{m+n,p}\geqslant n)$. Thus

$$\begin{aligned}\sum_{k\leqslant m}\binom{n+k-1}{k}p^nq^k\ &=\ \sum_{k\geqslant n}\binom{m+n}{k}p^kq^{m+n-k}\\&=\ \sum_{k\leqslant m}\binom{m+n}{k}p^{m+n-k}q^k\,;\end{aligned}$$

and this is (5.19) with $n=r$, $x=q$, $y=p$.

8.18 (a) $G_X(z)=e^{\mu(z-1)}$. (b) The mth cumulant is μ, for all $m\geqslant 1$. (The case $\mu=1$ is called F_∞ in (8.55).)

8.19 (a) $G_{X_1+X_2}(z)=G_{X_1}(z)G_{X_2}(z)=e^{(\mu_1+\mu_2)(z-1)}$. Hence the probability is $e^{-\mu_1-\mu_2}(\mu_1+\mu_2)^n/n!$; the sum of independent Poisson variables is Poisson. (b) In general, if K_mX denotes the mth cumulant of a random variable X, we have $K_m(aX_1+bX_2)=a^m(K_mX_1)+b^m(K_mX_2)$, when $a,b\geqslant 0$. Hence the answer is $2^m\mu_1+3^m\mu_2$.

8.20 The general pgf will be $G(z) = z^m/F(z)$, where

$$\begin{aligned} F(z) &= z^m + (1-z)\sum_{k=1}^{m} \widetilde{A}_{(k)}[A^{(k)}=A_{(k)}]z^{m-k}\,, \\ F'(1) &= m - \sum_{k=1}^{m} \widetilde{A}_{(k)}[A^{(k)}=A_{(k)}]\,, \\ F''(1) &= m(m-1) - 2\sum_{k=1}^{m}(m-k)\widetilde{A}_{(k)}[A^{(k)}=A_{(k)}]\,. \end{aligned}$$

8.21 This is $\sum_{n\geqslant 0} q_n$, where q_n is the probability that the game between Alice and Bill is still incomplete after n flips. Let p_n be the probability that the game ends at the nth flip; then $p_n + q_n = q_{n-1}$. Hence the average time to play the game is $\sum_{n\geqslant 1} np_n = (q_0 - q_1) + 2(q_1 - q_2) + 3(q_2 - q_3) + \cdots = q_0 + q_1 + q_2 + \cdots = N$, since $\lim_{n\to\infty} nq_n = 0$.

Another way to establish this answer is to replace H and T by $\frac{1}{2}z$. Then the derivative of the first equation in (8.78) tells us that $N(1) + N'(1) = N'(1) + S'_A(1) + S'_B(1)$.

By the way, $N = \frac{16}{3}$.

8.22 By definition we have $V(X|Y) = E(X^2|Y) - \bigl(E(X|Y)\bigr)^2$ and $V\bigl(E(X|Y)\bigr) = E\bigl((E(X|Y))^2\bigr) - \bigl(E\bigl(E(X|Y)\bigr)\bigr)^2$; hence $E\bigl(V(X|Y)\bigr) + V\bigl(E(X|Y)\bigr) = E\bigl(E(X^2|Y)\bigr) - \bigl(E\bigl(E(X|Y)\bigr)\bigr)^2$. But $E\bigl(E(X|Y)\bigr) = \sum_y \Pr(Y{=}y)E(x|y) = \sum_{x,y}\Pr(Y{=}y)\times \Pr\bigl((X|y){=}x\bigr) = EX$ and $E\bigl(E(X^2|Y)\bigr) = E(X^2)$, so the result is just VX.

8.23 Let $\Omega_0 = \{$ ⚀, ⚅ $\}^2$ and $\Omega_1 = \{$ ⚁, ⚂, ⚃, ⚄ $\}^2$; and let Ω_2 be the other 16 elements of Ω. Then $\Pr_{11}(\omega) - \Pr_{00}(\omega) = \frac{+20}{576}, \frac{-7}{576}, \frac{+2}{576}$ according as $\omega \in \Omega_0, \Omega_1, \Omega_2$. The events A must therefore be chosen with k_j elements from Ω_j, where (k_0, k_1, k_2) is one of the following: $(0,0,0)$, $(0,2,7)$, $(0,4,14)$, $(1,4,4)$, $(1,6,11)$, $(2,6,1)$, $(2,8,8)$, $(2,10,15)$, $(3,10,5)$, $(3,12,12)$, $(4,12,2)$, $(4,14,9)$, $(4,16,16)$. For example, there are $\binom{4}{2}\binom{16}{6}\binom{16}{1}$ events of type $(2,6,1)$. The total number of such events is $[z^0](1+z^{20})^4(1+z^{-7})^{16}(1+z^2)^{16}$, which turns out to be 1304872090. If we restrict ourselves to events that depend on S only, we get 40 solutions $S \in A$, where $A = \emptyset$, $\{{}^{\,2}_{12}, {}^{\,4}_{10}, {}^{6}_{8}\}$, $\{{}^{\,2}_{12}, 5, 9\}$, $\{2, 12, {}^{\,4}_{10}, {}^{6}_{8}, 5, 9\}$, $\{2,4,6,8,10,12\}$, $\{{}^{\,3}_{11}, 7, {}^{5}_{9}, 4, 10\}$, and the complements of these sets. (Here the notation '${}^{\,2}_{12}$' means either 2 or 12 but not both.)

8.24 (a) Any one of the dice ends up in J's possession with probability $p = \frac{1}{6} + \left(\frac{5}{6}\right)^2 p$; hence $p = \frac{6}{11}$. Let $q = \frac{5}{11}$. Then the pgf for J's total holdings is $(q+pz)^{2n+1}$, with mean $(2n+1)p$ and variance $(2n+1)pq$, by (8.61). (b) $\binom{5}{3}p^3q^2 + \binom{5}{4}p^4q + \binom{5}{5}p^5 = \frac{94176}{161051} \approx .585$.

8.25 The pgf for the current stake after n rolls is $G_n(z)$, where

$$G_0(z) = z^A;$$
$$G_n(z) = \textstyle\sum_{k=1}^{6} G_{n-1}(z^{2(k-1)/5})/6, \qquad \text{for } n > 0.$$

This problem can perhaps be solved more easily without generating functions than with them.

(The noninteger exponents cause no trouble.) It follows that $\mathrm{Mean}(G_n) = \mathrm{Mean}(G_{n-1})$, and $\mathrm{Var}(G_n) + \mathrm{Mean}(G_n)^2 = \frac{22}{15}(\mathrm{Var}(G_{n-1}) + \mathrm{Mean}(G_{n-1})^2)$. So the mean is always A, but the variance grows to $\bigl((\frac{22}{15})^n - 1\bigr)A^2$.

8.26 The pgf $F_{l,n}(z)$ satisfies $F'_{l,n}(z) = F_{l,n-l}(z)/l$; hence $\mathrm{Mean}(F_{l,n}) = F'_{l,n}(1) = [n \geqslant l]/l$ and $F''_{l,n}(1) = [n \geqslant 2l]/l^2$; the variance is easily computed. (In fact, we have

$$F_{l,n}(z) = \sum_{0 \leqslant k \leqslant n/l} \frac{1}{k!}\Bigl(\frac{z-1}{l}\Bigr)^k,$$

which approaches a Poisson distribution with mean $1/l$ as $n \to \infty$.)

8.27 $(n^2\Sigma_3 - 3n\Sigma_2\Sigma_1 + 2\Sigma_1^3)/n(n-1)(n-2)$ has the desired mean, where $\Sigma_k = X_1^k + \cdots + X_n^k$. This follows from the identities

$$\begin{aligned} E\Sigma_3 &= n\mu_3; \\ E(\Sigma_2\Sigma_1) &= n\mu_3 + n(n-1)\mu_2\mu_1; \\ E(\Sigma_1^3) &= n\mu_3 + 3n(n-1)\mu_2\mu_1 + n(n-1)(n-2)\mu_1^3. \end{aligned}$$

Incidentally, the third cumulant is $\kappa_3 = E\bigl((X-EX)^3\bigr)$, but the fourth cumulant does not have such a simple expression; we have $\kappa_4 = E\bigl((X-EX)^4\bigr) - 3(VX)^2$.

8.28 (The exercise implicitly calls for $p = q = \frac{1}{2}$, but the general answer is given here for completeness.) Replace H by pz and T by qz, getting $S_A(z) = p^2qz^3/(1-pz)(1-qz)(1-pqz^2)$ and $S_B(z) = pq^2z^3/(1-qz)(1-pqz^2)$. The pgf for the conditional probability that Alice wins at the nth flip, given that she wins the game, is

$$\frac{S_A(z)}{S_A(1)} = z^3 \cdot \frac{q}{1-pz} \cdot \frac{p}{1-qz} \cdot \frac{1-pq}{1-pqz^2}.$$

This is a product of pseudo-pgf's, whose mean is $3+p/q+q/p+2pq/(1-pq)$. The formulas for Bill are the same but without the factor $q/(1-pz)$, so Bill's mean is $3 + q/p + 2pq/(1-pq)$. When $p = q = \frac{1}{2}$, the answer in case (a) is $\frac{17}{3}$; in case (b) it is $\frac{14}{3}$. Bill wins only half as often, but when he does win he tends to win sooner. The overall average number of flips is $\frac{2}{3}\cdot\frac{17}{3} + \frac{1}{3}\cdot\frac{14}{3} = \frac{16}{3}$, agreeing with exercise 21. The solitaire game for each pattern has a waiting time of 8.

8.29 Set $\mathtt{H} = \mathtt{T} = \frac{1}{2}$ in

$$\begin{aligned} 1 + N(\mathtt{H}+\mathtt{T}) &= N + S_A + S_B + S_C \\ N\,\mathtt{HHTH} &= S_A(\mathtt{HTH}+1) + S_B(\mathtt{HTH}+\mathtt{TH}) + S_C(\mathtt{HTH}+\mathtt{TH}) \\ N\,\mathtt{HTHH} &= S_A(\mathtt{THH}+\mathtt{H}) + S_B(\mathtt{THH}+1) + S_C(\mathtt{THH}) \\ N\,\mathtt{THHH} &= S_A(\mathtt{HH}) + S_B(\mathtt{H}) + S_C \end{aligned}$$

to get the winning probabilities. In general we will have $S_A + S_B + S_C = 1$ and

$$\begin{aligned} S_A(A{:}A) + S_B(B{:}A) + S_C(C{:}A) &= S_A(A{:}B) + S_B(B{:}B) + S_C(C{:}B) \\ &= S_A(A{:}B) + S_B(B{:}C) + S_C(C{:}C)\,. \end{aligned}$$

In particular, the equations $9S_A + 3S_B + 3S_C = 5S_A + 9S_B + S_C = 2S_A + 4S_B + 8S_C$ imply that $S_A = \frac{16}{52}$, $S_B = \frac{17}{52}$, $S_C = \frac{19}{52}$.

8.30 The variance of $P(h_1,\ldots,h_n;k)\,|\,k$ is the variance of the shifted binomial distribution $\bigl((m-1+z)/m\bigr)^{k-1}z$, which is $(k-1)(\frac{1}{m})(1-\frac{1}{m})$ by (8.61). Hence the average of the variance is $\mathrm{Mean}(S)(m-1)/m^2$. The variance of the average is the variance of $(k-1)/m$, namely $\mathrm{Var}(S)/m^2$. According to (8.106), the sum of these two quantities should be VP, and it is. Indeed, we have just replayed the derivation of (8.96) in slight disguise. (See exercise 15.)

8.31 (a) A brute force solution would set up five equations in five unknowns:

$$\begin{aligned} &A = \tfrac{1}{2}zB + \tfrac{1}{2}zE\,; \quad B = \tfrac{1}{2}zC\,; \quad C = 1 + \tfrac{1}{2}zB + \tfrac{1}{2}zD\,; \\ &D = \tfrac{1}{2}zC + \tfrac{1}{2}zE\,; \quad E = \tfrac{1}{2}zD\,. \end{aligned}$$

But positions C and D are equidistant from the goal, as are B and E, so we can lump them together. If $X = B + E$ and $Y = C + D$, there are now three equations:

$$A = \tfrac{1}{2}zX\,; \quad X = \tfrac{1}{2}zY\,; \quad Y = 1 + \tfrac{1}{2}zX + \tfrac{1}{2}zY\,.$$

Hence $A = z^2/(4-2z-z^2)$; we have $\mathrm{Mean}(A) = 6$ and $\mathrm{Var}(A) = 22$. (Rings a bell? In fact, this problem is equivalent to flipping a fair coin until getting heads twice in a row: Heads means "advance toward the apple" and tails means "go back.") (b) Chebyshev's inequality says that $\Pr(S \geqslant 100) = \Pr\bigl((S-6)^2 \geqslant 94^2\bigr) \leqslant 22/94^2 \approx .0025$. (c) The second tail inequality says that $\Pr(S \geqslant 100) \leqslant 1/x^{98}(4-2x-x^2)$ for all $x \geqslant 1$, and we get the upper bound 0.00000005 when $x = (\sqrt{49001} - 99)/100$. (The actual probability is approximately 0.0000000009, according to exercise 37.)

8.32 By symmetry, we can reduce each month's situation to one of four possibilities:

"Toto, I have a feeling we're not in Kansas anymore."
— Dorothy

> D, the states are diagonally opposite;
> A, the states are adjacent and not Kansas;
> K, the states are Kansas and one other;
> S, the states are the same.

Considering the Markovian transitions, we get four equations

$$\begin{aligned} D &= 1 + z(\tfrac{2}{9}D + \tfrac{2}{12}K) \\ A &= z(\tfrac{4}{9}A + \tfrac{4}{12}K) \\ K &= z(\tfrac{4}{9}D + \tfrac{4}{9}A + \tfrac{4}{12}K) \\ S &= z(\tfrac{3}{9}D + \tfrac{1}{9}A + \tfrac{2}{12}K) \end{aligned}$$

whose sum is $D + K + A + S = 1 + z(D + A + K)$. The solution is

$$S = \frac{81z - 45z^2 - 4z^3}{243 - 243z + 24z^2 + 8z^3},$$

but the simplest way to find the mean and variance may be to write $z = 1+w$ and expand in powers of w, ignoring multiples of w^2:

$$\begin{aligned} D &= \tfrac{27}{16} + \tfrac{1593}{512}w + \cdots; \\ A &= \tfrac{9}{8} + \tfrac{2115}{256}w + \cdots; \\ K &= \tfrac{15}{8} + \tfrac{2661}{256}w + \cdots. \end{aligned}$$

Now $S'(1) = \frac{27}{16} + \frac{9}{8} + \frac{15}{8} = \frac{75}{16}$, and $\frac{1}{2}S''(1) = \frac{1593}{512} + \frac{2115}{256} + \frac{2661}{256} = \frac{11145}{512}$. The mean is $\frac{75}{16}$ and the variance is $\frac{105}{4}$. (Is there a simpler way?)

8.33 First answer: Clearly yes, because the hash values $h_1, \ldots, h_n$ are independent. Second answer: Certainly no, even though the hash values $h_1, \ldots, h_n$ are independent. We have $\Pr(X_j = 0) = \sum_{k=1}^{n} s_k\bigl([j \ne k](m-1)/m\bigr) = (1 - s_j)(m-1)/m$, but $\Pr(X_1 = X_2 = 0) = \sum_{k=1}^{n} s_k[k > 2](m-1)^2/m^2 = (1 - s_1 - s_2)(m-1)^2/m^2 \ne \Pr(X_1 = 0)\Pr(X_2 = 0)$.

8.34 Let $[z^n]\,S_m(z)$ be the probability that Gina has advanced $< m$ steps after taking n turns. Then $S_m(1)$ is her average score on a par-m hole; $[z^m]\,S_m(z)$ is the probability that she loses such a hole against a steady player; and $1 - [z^{m-1}]\,S_m(z)$ is the probability that she wins it. We have the recurrence

$$\begin{aligned} S_0(z) &= 0; \\ S_m(z) &= \bigl(1 + pzS_{m-2}(z) + qzS_{m-1}(z)\bigr)/(1 - rz), \qquad \text{for } m > 0. \end{aligned}$$

To solve part (a), it suffices to compute the coefficients for $m, n \leqslant 4$; it is convenient to replace z by $100w$ so that the computations involve nothing but integers. We obtain the following tableau of coefficients:

S_0	0	0	0	0	0
S_1	1	4	16	64	256
S_2	1	95	744	4432	23552
S_3	1	100	9065	104044	819808
S_4	1	100	9975	868535	12964304

Therefore Gina wins with probability $1 - .868535 = .131465$; she loses with probability $.12964304$. (b) To find the mean number of strokes, we compute

$$S_1(1) = \tfrac{25}{24}\,; \quad S_2(1) = \tfrac{4675}{2304}\,; \quad S_3(1) = \tfrac{667825}{221184}\,; \quad S_4(1) = \tfrac{85134475}{21233664}\,.$$

(Incidentally, $S_5(1) \approx 4.9995$; she wins with respect to both holes and strokes on a par-5 hole, but loses either way when par is 3.)

8.35 The condition will be true for all n if and only if it is true for $n = 1$, by the Chinese remainder theorem. One necessary and sufficient condition is the polynomial identity

$$\begin{aligned}\bigl(p_2{+}p_4{+}p_6 + (p_1{+}p_3{+}p_5)w\bigr)&\bigl(p_3{+}p_6 + (p_1{+}p_4)z + (p_2{+}p_5)z^2\bigr)\\ &= \bigl(p_1wz + p_2z^2 + p_3w + p_4z + p_5wz^2 + p_6\bigr)\,,\end{aligned}$$

but that just more-or-less restates the problem. A simpler characterization is

$$(p_2 + p_4 + p_6)(p_3 + p_6) = p_6\,, \qquad (p_1 + p_3 + p_5)(p_2 + p_5) = p_5\,,$$

which checks only two of the coefficients in the former product. The general solution has three degrees of freedom: Let $a_0 + a_1 = b_0 + b_1 + b_2 = 1$, and put $p_1 = a_1b_1,\ p_2 = a_0b_2,\ p_3 = a_1b_0,\ p_4 = a_0b_1,\ p_5 = a_1b_2,\ p_6 = a_0b_0$.

8.36 (a) ⚀ ⚁ ⚁ ⚂ ⚂ ⚃. (b) If the kth die has faces with $s_1, \ldots, s_6$ spots, let $p_k(z) = z^{s_1} + \cdots + z^{s_6}$. We want to find such polynomials with $p_1(z)\ldots p_n(z) = (z + z^2 + z^3 + z^4 + z^5 + z^6)^n$. The irreducible factors of this polynomial with rational coefficients are $z^n(z+1)^n \times (z^2+z+1)^n(z^2-z+1)^n$; hence $p_k(z)$ must be of the form $z^{a_k}(z+1)^{b_k} \times (z^2+z+1)^{c_k}(z^2-z+1)^{d_k}$. We must have $a_k \geqslant 1$, since $p_k(0) = 0$; and in fact $a_k = 1$, since $a_1 + \cdots + a_n = n$. Furthermore the condition $p_k(1) = 6$ implies that $b_k = c_k = 1$. It is now easy to see that $0 \leqslant d_k \leqslant 2$, since $d_k > 2$ gives negative coefficients. When $d = 0$ and $d = 2$, we get the two dice in part (a); therefore the only solutions have k pairs of dice as in (a), plus $n - 2k$ ordinary dice, for some $k \leqslant \frac{1}{2}n$.

8.37 The number of coin-toss sequences of length n is F_{n-1}, for all $n > 0$, because of the relation between domino tilings and coin flips. Therefore the probability that exactly n tosses are needed is $F_{n-1}/2^n$, when the coin is fair. Also $q_n = F_{n+1}/2^{n-1}$, since $\sum_{k\geqslant n} F_k z^k = (F_n z^n + F_{n-1}z^{n+1})/(1-z-z^2)$. (A systematic solution via generating functions is, of course, also possible.)

8.38 When k faces have been seen, the task of rolling a new one is equivalent to flipping coins with success probability $p_k = (m-k)/m$. Hence the pgf is $\prod_{k=0}^{l-1} p_k z/(1-q_k z) = \prod_{k=0}^{l-1}(m-k)z/(m-kz)$. The mean is $\sum_{k=0}^{l-1} p_k^{-1} = m(H_m - H_{m-l})$; the variance is $m^2\bigl(H_m^{(2)} - H_{m-l}^{(2)}\bigr) - m(H_m - H_{m-l})$; and equation (7.47) provides a closed form for the requested probability, namely $m^{-n}m!\bigl\{{n-1 \atop l-1}\bigr\}/(m-l)!$. (The problem discussed in this exercise is traditionally called "coupon collecting.")

8.39 $E(X) = P(-1)$; $V(X) = P(-2) - P(-1)^2$; $E(\ln X) = -P'(0)$.

8.40 (a) We have $\kappa_m = n\bigl(0!\{{m \atop 1}\}p - 1!\{{m \atop 2}\}p^2 + 2!\{{m \atop 3}\}p^3 - \cdots\bigr)$, by (7.49). Incidentally, the third cumulant is $npq(q-p)$ and the fourth is $npq(1-6pq)$. The identity $q+pe^t = (p+qe^{-t})e^t$ shows that $f_m(p) = (-1)^m f_m(q) + [m=1]$; hence we can write $f_m(p) = g_m(pq)(q-p)^{[m\ \mathrm{odd}]}$, where g_m is a polynomial of degree $\lfloor m/2 \rfloor$, whenever $m > 1$. (b) Let $p = \frac{1}{2}$ and $F(t) = \ln(\frac{1}{2} + \frac{1}{2}e^t)$. Then $\sum_{m\geqslant 1} \kappa_m t^{m-1}/(m-1)! = F'(t) = 1 - 1/(e^t+1)$, and we can use exercise 6.23.

8.41 If $G(z)$ is the pgf for a random variable X that assumes only positive integer values, then $\int_0^1 G(z)\,dz/z = \sum_{k\geqslant 1} \Pr(X=k)/k = E(X^{-1})$. If X is the distribution of the number of flips to obtain $n+1$ heads, we have $G(z) = \bigl(pz/(1-qz)\bigr)^{n+1}$ by (8.59), and the integral is

$$\int_0^1 \left(\frac{pz}{1-qz}\right)^{n+1} \frac{dz}{z} = \int_0^1 \frac{w^n\,dw}{1+(q/p)w}$$

if we substitute $w = pz/(1-qz)$. When $p = q$ the integrand can be written $(-1)^n\bigl((1+w)^{-1} - 1 + w - w^2 + \cdots + (-1)^n w^{n-1}\bigr)$, so the integral is $(-1)^n\bigl(\ln 2 - 1 + \frac{1}{2} - \frac{1}{3} + \cdots + (-1)^n/n\bigr)$. We have $H_{2n} - H_n = \ln 2 - \frac{1}{4}n^{-1} + \frac{1}{16}n^{-2} + O(n^{-4})$ by (9.28), and it follows that $E(X_{n+1}^{-1}) = \frac{1}{2}n^{-1} - \frac{1}{4}n^{-2} + O(n^{-4})$.

8.42 Let $F_n(z)$ and $G_n(z)$ be pgf's for the number of employed evenings, if the man is initially unemployed or employed, respectively. Let $q_h = 1 - p_h$ and $q_f = 1 - p_f$. Then $F_0(z) = G_0(z) = 1$, and

$$\begin{aligned} F_n(z) &= p_h z G_{n-1}(z) + q_h F_{n-1}(z)\,; \\ G_n(z) &= p_f F_{n-1}(z) + q_f z G_{n-1}(z)\,. \end{aligned}$$

The solution is given by the super generating function

$$G(w,z) \;=\; \sum_{n\geqslant 0} G_n(z)w^n \;=\; A(w)/\bigl(1-zB(w)\bigr)\,,$$

where $B(w)=w\bigl(q_f-(q_f-p_h)w\bigr)/(1-q_hw)$ and $A(w)=\bigl(1-B(w)\bigr)/(1-w)$. Now $\sum_{n\geqslant 0} G'_n(1)w^n = \alpha w/(1-w)^2+\beta/(1-w)-\beta/\bigl(1-(q_f-p_h)w\bigr)$ where

$$\alpha \;=\; \frac{p_h}{p_h+p_f}\,, \qquad \beta \;=\; \frac{p_f(q_f-p_h)}{(p_h+p_f)^2}\,;$$

hence $G'_n(1)=\alpha n+\beta\bigl(1-(q_f-p_h)^n\bigr)$. (Similarly $G''_n(1)=\alpha^2n^2+O(n)$, so the variance is $O(n)$.)

8.43 $G_n(z) \;=\; \sum_{k\geqslant 0}{n\brack k}z^k/n! \;=\; z^{\overline{n}}/n!$, by (6.11). This is a product of binomial pgf's, $\prod_{k=1}^n\bigl((k-1+z)/k\bigr)$, where the kth has mean $1/k$ and variance $(k-1)/k^2$; hence $\mathrm{Mean}(G_n)=H_n$ and $\mathrm{Var}(G_n)=H_n-H_n^{(2)}$.

8.44 (a) The champion must be undefeated in n rounds, so the answer is p^n. (b,c) Players x_1, …, x_{2^k} must be "seeded" (by chance) in distinct subtournaments and they must win all $2^k(n-k)$ of their matches. The 2^n leaves of the tournament tree can be filled in $2^n!$ ways; to seed it we have $2^k!(2^{n-k})^{2^k}$ ways to place the top 2^k players, and $(2^n-2^k)!$ ways to place the others. Hence the probability is $(2p)^{2^k(n-k)}/\binom{2^n}{2^k}$. When $k=1$ this simplifies to $(2p^2)^{n-1}/(2^n-1)$. (d) Each tournament outcome corresponds to a permutation of the players: Let y_1 be the champ; let y_2 be the other finalist; let y_3 and y_4 be the players who lost to y_1 and y_2 in the semifinals; let $(y_5,\ldots,y_8)$ be those who lost respectively to $(y_1,\ldots,y_4)$ in the quarterfinals; etc. (Another proof shows that the first round has $2^n!/2^{n-1}!$ essentially different outcomes; the second round has $2^{n-1}!/2^{n-2}!$; and so on.) (e) Let S_k be the set of 2^{k-1} potential opponents of x_2 in the kth round. The conditional probability that x_2 wins, given that x_1 belongs to S_k, is

$$\begin{aligned}&\Pr(x_1\text{ plays }x_2)\cdot p^{n-1}(1-p) \;+\; \Pr(x_1\text{ doesn't play }x_2)\cdot p^n\\ &\qquad=\; p^{k-1}p^{n-1}(1-p) \;+\; (1-p^{k-1})p^n\,.\end{aligned}$$

The chance that $x_1\in S_k$ is $2^{k-1}/(2^n-1)$; summing on k gives the answer:

$$\sum_{k=1}^n \frac{2^{k-1}}{2^n-1}\bigl(p^{k-1}p^{n-1}(1-p)+(1-p^{k-1})p^n\bigr) \;=\; p^n-\frac{(2p)^n-1}{2^n-1}\,p^{n-1}\,.$$

(f) Each of the $2^n!$ tournament outcomes has a certain probability of occurring, and the probability that x_j wins is the sum of these probabilities over all $(2^n-1)!$ tournament outcomes in which x_j is champion. Consider interchanging x_j with x_{j+1} in all those outcomes; this change doesn't affect the probability if x_j and x_{j+1} never meet, but it multiplies the probability by $(1-p)/p < 1$ if they do meet.

8.45 (a) $A(z) = 1/(3-2z)$; $B(z) = zA(z)^2$; $C(z) = z^2A(z)^3$. The pgf for sherry when it's bottled is $z^3A(z)^3$, which is z^3 times a negative binomial distribution with parameters $n = 3$, $p = \frac{1}{3}$. (b) $\mathrm{Mean}(A) = 2$, $\mathrm{Var}(A) = 6$; $\mathrm{Mean}(B) = 5$, $\mathrm{Var}(B) = 2\,\mathrm{Var}(A) = 12$; $\mathrm{Mean}(C) = 8$, $\mathrm{Var}(C) = 18$. The sherry is nine years old, on the average. The fraction that's 25 years old is $\binom{-3}{22}(-2)^{22}3^{-25} = \binom{24}{22}2^{22}3^{-25} = 23\cdot(\frac{2}{3})^{24} \approx .00137$. (c) Let the coefficient of w^n be the pgf for the beginning of year n. Then

$$\begin{aligned} A &= \bigl(1+\tfrac{1}{3}w/(1-w)\bigr)/(1-\tfrac{2}{3}zw)\,;\\ B &= \bigl(1+\tfrac{1}{3}zwA\bigr)/(1-\tfrac{2}{3}zw)\,;\\ C &= \bigl(1+\tfrac{1}{3}zwB\bigr)/(1-\tfrac{2}{3}zw)\,. \end{aligned}$$

Differentiate with respect to z and set $z = 1$; this makes

$$C' = \frac{8}{1-w} - \frac{1/2}{(1-\frac{2}{3}w)^3} - \frac{3/2}{(1-\frac{2}{3}w)^2} - \frac{6}{1-\frac{2}{3}w}\,.$$

The average age of bottled sherry n years after the process started is 1 greater than the coefficient of w^{n-1}, namely $9-(\frac{2}{3})^n(3n^2+21n+72)/8$. (This already exceeds 8 when $n = 11$.)

8.46 (a) $P(w,z) = 1 + \frac{1}{2}\bigl(wP(w,z) + zP(w,z)\bigr) = \bigl(1-\frac{1}{2}(w+z)\bigr)^{-1}$, hence $p_{mn} = 2^{-m-n}\binom{m+n}{n}$. (b) $P_k(w,z) = \frac{1}{2}(w^k + z^k)P(w,z)$; hence

$$p_{k,m,n} = 2^{k-1-m-n}\left(\binom{m+n-k}{m} + \binom{m+n-k}{n}\right).$$

(c) $\sum_k kp_{k,n,n} = \sum_{k=0}^{n} k2^{k-2n}\binom{2n-k}{n} = \sum_{k=0}^{n}(n-k)2^{-n-k}\binom{n+k}{n}$; this can be summed using (5.20):

$$\begin{aligned} \sum_{k=0}^{n} 2^{-n-k}&\left((2n+1)\binom{n+k}{n} - (n+1)\binom{n+1+k}{n+1}\right)\\ &= (2n+1) - (n+1)2^{-n}\left(2^{n+1} - 2^{-n-1}\binom{2n+2}{n+1}\right)\\ &= \frac{2n+1}{2^{2n}}\binom{2n}{n} - 1\,. \end{aligned}$$

(The methods of Chapter 9 show that this is $2\sqrt{n/\pi} - 1 + O(n^{-1/2})$.)

8.47 After n irradiations there are $n+2$ equally likely receptors. Let the random variable X_n denote the number of diphages present; then $X_{n+1} =$

$X_n + Y_n$, where $Y_n = -1$ if the $(n+1)$st particle hits a diphage receptor (conditional probability $2X_n/(n+2)$) and $Y_n = +2$ otherwise. Hence

$$EX_{n+1} = EX_n + EY_n = EX_n - 2EX_n/(n+2) + 2\bigl(1 - 2EX_n/(n+2)\bigr).$$

The recurrence $(n+2)EX_{n+1} = (n-4)EX_n + 2n + 4$ can be solved if we multiply both sides by the summation factor $(n+1)^{\underline{5}}$; or we can guess the answer and prove it by induction: $EX_n = (2n+4)/7$ for all $n > 4$. (Incidentally, there are always two diphages and one triphage after five steps, regardless of the configuration after four.)

8.48 (a) The distance between frisbees (measured so as to make it an even number) is either 0, 2, or 4 units, initially 4. The corresponding generating functions A, B, C (where, say, $[z^n]\,C$ is the probability of distance 4 after n throws) satisfy

$$A = \tfrac14 zB\,, \quad B = \tfrac12 zB + \tfrac14 zC\,, \quad C = 1 + \tfrac14 zB + \tfrac34 zC\,.$$

It follows that $A = z^2/(16 - 20z + 5z^2) = z^2/F(z)$, and we have $\mathrm{Mean}(A) = 2 - \mathrm{Mean}(F) = 12$, $\mathrm{Var}(A) = -\mathrm{Var}(F) = 100$. (A more difficult but more amusing solution factors A as follows:

$$A = \frac{p_1 z}{1 - q_1 z} \cdot \frac{p_2 z}{1 - q_2 z} = \frac{p_2}{p_2 - p_1}\frac{p_1 z}{1 - q_1 z} + \frac{p_1}{p_1 - p_2}\frac{p_2 z}{1 - q_2 z}\,,$$

where $p_1 = \phi^2/4 = (3 + \sqrt5\,)/8$, $p_2 = \hat\phi^2/4 = (3 - \sqrt5\,)/8$, and $p_1 + q_1 = p_2 + q_2 = 1$. Thus, the game is equivalent to having two biased coins whose heads probabilities are p_1 and p_2; flip the coins one at a time until they have both come up heads, and the total number of flips will have the same distribution as the number of frisbee throws. The mean and variance of the waiting times for these two coins are respectively $6 \mp 2\sqrt5$ and $50 \mp 22\sqrt5$, hence the total mean and variance are 12 and 100 as before.)

(b) Expanding the generating function in partial fractions makes it possible to sum the probabilities. (Note that $\sqrt5/(4\phi) + \phi^2/4 = 1$, so the answer can be stated in terms of powers of ϕ.) The game will last more than n steps with probability $5^{(n-1)/2}4^{-n}(\phi^{n+2} - \phi^{-n-2})$; when n is even this is $5^{n/2}4^{-n}F_{n+2}$. So the answer is $5^{50}4^{-100}F_{102} \approx .00006$.

8.49 (a) If $n > 0$, $P_N(0,n) = \frac12[N\!=\!0] + \frac14 P_{N-1}(0,n) + \frac14 P_{N-1}(1,n-1)$; $P_N(m,0)$ is similar; $P_N(0,0) = [N\!=\!0]$. Hence

$$\begin{aligned} g_{m,n} &= \tfrac14 z g_{m-1,n+1} + \tfrac12 z g_{m,n} + \tfrac14 z g_{m+1,n-1}\,; \\ g_{0,n} &= \tfrac12 + \tfrac14 z g_{0,n} + \tfrac14 g_{1,n-1}\,; \quad \text{etc.} \end{aligned}$$

(b) $g'_{m,n} = 1 + \frac14 g'_{m-1,n+1} + \frac12 g'_{m,n} + \frac14 g'_{m+1,n-1}$; $g'_{0,n} = \frac12 + \frac14 g'_{0,n} + \frac14 g'_{1,n-1}$; etc. By induction on m, we have $g'_{m,n} = (2m+1)g'_{0,m+n} - 2m^2$ for all

$m, n \geqslant 0$. And since $g'_{m,0} = g'_{0,m}$, we must have $g'_{m,n} = m + n + 2mn$. (c) The recurrence is satisfied when $mn > 0$, because

$$\sin(2m+1)\theta \;=\; \frac{1}{\cos^2\theta}\biggl(\frac{\sin(2m-1)\theta}{4} + \frac{\sin(2m+1)\theta}{2} + \frac{\sin(2m+3)\theta}{4}\biggr);$$

this is a consequence of the identity $\sin(x-y) + \sin(x+y) = 2\sin x\cos y$. So all that remains is to check the boundary conditions.

8.50 (a) Using the hint, we get

$$\begin{aligned} &3(1-z)^2 \sum_k \binom{1/2}{k}\Bigl(\frac{8}{9}z\Bigr)^k (1-z)^{2-k} \\ &\qquad = 3(1-z)^2 \sum_k \binom{1/2}{k}\Bigl(\frac{8}{9}\Bigr)^k \sum_j \binom{k+j-3}{j} z^{j+k}; \end{aligned}$$

now look at the coefficient of z^{3+l}. (b) $H(z) = \frac{2}{3} + \frac{5}{27}z + \frac{1}{2}\sum_{l\geqslant 0} c_{3+l}z^{2+l}$. (c) Let $r = \sqrt{(1-z)(9-z)}$. One can show that $(z-3+r)(z-3-r) = 4z$, and hence that $\bigl(r/(1-z)+2\bigr)^2 = (13-5z+4r)/(1-z) = \bigl(9-H(z)\bigr)/\bigl(1-H(z)\bigr)$. (d) Evaluating the first derivative at $z = 1$ shows that $\mathrm{Mean}(H) = 1$. The second derivative diverges at $z = 1$, so the variance is infinite.

8.51 (a) Let $H_n(z)$ be the pgf for your holdings after n rounds of play, with $H_0(z) = z$. The distribution for n rounds is

$$H_{n+1}(z) \;=\; H_n\bigl(H(z)\bigr),$$

so the result is true by induction (using the amazing identity of the preceding problem). (b) $g_n = H_n(0) - H_{n-1}(0) = 4/n(n+1)(n+2) = 4(n-1)^{\underline{-3}}$. The mean is 2, and the variance is infinite. (c) The expected number of tickets you buy on the nth round is $\mathrm{Mean}(H_n) = 1$, by exercise 15. So the total expected number of tickets is infinite. (Thus, you almost surely lose eventually, and you expect to lose after the second game, yet you also expect to buy an infinite number of tickets.) (d) Now the pgf after n games is $H_n(z)^2$, and the method of part (b) yields a mean of $16 - \frac{4}{3}\pi^2 \approx 2.8$. (The sum $\sum_{k\geqslant 1} 1/k^2 = \pi^2/6$ shows up here.)

8.52 If ω and ω' are events with $\Pr(\omega) > \Pr(\omega')$, then a sequence of n independent experiments will encounter ω more often than ω', with high probability, because ω will occur very nearly $n\Pr(\omega)$ times. Consequently, as $n \to \infty$, the probability approaches 1 that the median or mode of the values of X in a sequence of independent trials will be a median or mode of the random variable X.

8.53 We can disprove the statement, even in the special case that each variable is 0 or 1. Let $p_0 = \Pr(X=Y=Z=0)$, $p_1 = \Pr(X=Y=\overline{Z}=0)$, …, $p_7 = \Pr(\overline{X}=\overline{Y}=\overline{Z}=0)$, where $\overline{X} = 1-X$. Then $p_0+p_1+\cdots+p_7 = 1$, and the variables are independent in pairs if and only if we have

$$\begin{aligned}(p_4+p_5+p_6+p_7)(p_2+p_3+p_6+p_7) &= p_6+p_7\,,\\ (p_4+p_5+p_6+p_7)(p_1+p_3+p_5+p_7) &= p_5+p_7\,,\\ (p_2+p_3+p_6+p_7)(p_1+p_3+p_5+p_7) &= p_3+p_7\,.\end{aligned}$$

But $\Pr(X+Y=Z=0) \ne \Pr(X+Y=0)\Pr(Z=0) \iff p_0 \ne (p_0+p_1)(p_0+p_2+p_4+p_6)$. One solution is

$$p_0 = p_3 = p_5 = p_6 = 1/4\,; \qquad p_1 = p_2 = p_4 = p_7 = 0\,.$$

This is equivalent to flipping two fair coins and letting X = (the first coin is heads), Y = (the second coin is heads), Z = (the coins differ). Another example, with all probabilities nonzero, is

$$\begin{aligned}&p_0 = 4/64\,, \quad p_1 = p_2 = p_4 = 5/64\,,\\ &p_3 = p_5 = p_6 = 10/64\,, \quad p_7 = 15/64\,.\end{aligned}$$

For this reason we say that n variables X_1, …, X_n are independent if

$$\Pr(X_1=x_1 \text{ and } \cdots \text{ and } X_n=x_n) = \Pr(X_1=x_1)\ldots\Pr(X_n=x_n)\,;$$

pairwise independence isn't enough to guarantee this.

8.54 (See exercise 27 for notation.) We have

$$\begin{aligned}E(\Sigma_2^2) &= n\mu_4 + n(n-1)\mu_2^2\,;\\ E(\Sigma_2\Sigma_1^2) &= n\mu_4 + 2n(n-1)\mu_3\mu_1 + n(n-1)\mu_2^2 + n(n-1)(n-2)\mu_2\mu_1^2\,;\\ E(\Sigma_1^4) &= n\mu_4 + 4n(n-1)\mu_3\mu_1 + 3n(n-1)\mu_2^2\\ &\qquad + 6n(n-1)(n-2)\mu_2\mu_1^2 + n(n-1)(n-2)(n-3)\mu_1^4\,;\end{aligned}$$

it follows that $V(\widehat{V}X) = \kappa_4/n + 2\kappa_2^2/(n-1)$.

8.55 There are $A = \frac{1}{17}\cdot 52!$ permutations with $X = Y$, and $B = \frac{16}{17}\cdot 52!$ permutations with $X \ne Y$. After the stated procedure, each permutation with $X = Y$ occurs with probability $\frac{1}{17}\big/\bigl((1-\frac{16}{17}p)A\bigr)$, because we return to step S1 with probability $\frac{16}{17}p$. Similarly, each permutation with $X \ne Y$ occurs with probability $\frac{16}{17}(1-p)\big/\bigl((1-\frac{16}{17}p)B\bigr)$. Choosing $p = \frac{1}{4}$ makes $\Pr(X=x \text{ and } Y=y) = \frac{1}{169}$ for all x and y. (We could therefore make two flips of a fair coin and go back to S1 if both come up heads.)

8.56 If m is even, the frisbees always stay an odd distance apart and the game lasts forever. If $m = 2l + 1$, the relevant generating functions are

$$\begin{aligned} G_m &= \tfrac{1}{4}zA_1\,; \\ A_1 &= \tfrac{1}{2}zA_1 + \tfrac{1}{4}zA_2\,, \\ A_k &= \tfrac{1}{4}zA_{k-1} + \tfrac{1}{2}zA_k + \tfrac{1}{4}zA_{k+1}\,, \qquad \text{for } 1 < k < l, \\ A_l &= \tfrac{1}{4}zA_{l-1} + \tfrac{3}{4}zA_l + 1\,. \end{aligned}$$

(The coefficient $[z^n]\,A_k$ is the probability that the distance between frisbees is $2k$ after n throws.) Taking a clue from the similar equations in exercise 49, we set $z = 1/\cos^2\theta$ and $A_1 = X\sin 2\theta$, where X is to be determined. It follows by induction (not using the equation for A_l) that $A_k = X\sin 2k\theta$. Therefore we want to choose X such that

$$\Bigl(1 - \frac{3}{4\cos^2\theta}\Bigr) X\sin 2l\theta \;=\; 1 + \frac{1}{4\cos^2\theta}\, X\sin(2l-2)\theta\,.$$

It turns out that $X = 2\cos^2\theta/\sin\theta\cos(2l+1)\theta$, hence

$$G_m \;=\; \frac{\cos\theta}{\cos m\theta}\,.$$

The denominator vanishes when θ is an odd multiple of $\pi/(2m)$; thus $1-q_kz$ is a root of the denominator for $1 \leqslant k \leqslant l$, and the stated product representation must hold. To find the mean and variance we can write

Trigonometry wins again. Is there a connection with pitching pennies along the angles of the m-gon?

$$\begin{aligned} G_m &= \bigl(1 - \tfrac{1}{2}\theta^2 + \tfrac{1}{24}\theta^4 - \cdots\bigr)/\bigl(1 - \tfrac{1}{2}m^2\theta^2 + \tfrac{1}{24}m^4\theta^4 - \cdots\bigr) \\ &= 1 + \tfrac{1}{2}(m^2-1)\theta^2 + \tfrac{1}{24}(5m^4 - 6m^2 + 1)\theta^4 + \cdots \\ &= 1 + \tfrac{1}{2}(m^2-1)(\tan\theta)^2 + \tfrac{1}{24}(5m^4 - 14m^2 + 9)(\tan\theta)^4 + \cdots \\ &= 1 + G_m'(1)(\tan\theta)^2 + \tfrac{1}{2}G_m''(1)(\tan\theta)^4 + \cdots\,, \end{aligned}$$

because $\tan^2\theta = z - 1$ and $\tan\theta = \theta + \frac{1}{3}\theta^3 + \cdots$. So we have $\mathrm{Mean}(G_m) = \frac{1}{2}(m^2-1)$ and $\mathrm{Var}(G_m) = \frac{1}{6}m^2(m^2-1)$. (Note that this implies the identities

$$\begin{aligned} \frac{m^2-1}{2} &= \sum_{k=1}^{(m-1)/2} \frac{1}{p_k} \;=\; \sum_{k=1}^{(m-1)/2} \Bigl(1\Big/\sin\frac{(2k-1)\pi}{2m}\Bigr)^2\,; \\ \frac{m^2(m^2-1)}{6} &= \sum_{k=1}^{(m-1)/2} \Bigl(\cot\frac{(2k-1)\pi}{2m}\Big/\sin\frac{(2k-1)\pi}{2m}\Bigr)^2\,. \end{aligned}$$

The third cumulant of this distribution is $\frac{1}{30}m^2(m^2-1)(4m^2-1)$; but the pattern of nice cumulant factorizations stops there. There's a much simpler way to derive the mean: We have $G_m + A_1 + \cdots + A_l = z(A_1 + \cdots + A_l) + 1$, hence when $z = 1$ we have $G'_m = A_1 + \cdots + A_l$. Since $G_m = 1$ when $z = 1$, an easy induction shows that $A_k = 4k$.)

8.57 We have $A{:}A \geqslant 2^{l-1}$ and $B{:}B < 2^{l-1} + 2^{l-3}$ and $B{:}A \geqslant 2^{l-2}$, hence $B{:}B - B{:}A \geqslant A{:}A - A{:}B$ is possible only if $A{:}B > 2^{l-3}$. This means that $\overline{\tau}_2 = \tau_3$, $\tau_1 = \tau_4$, $\tau_2 = \tau_5$, ..., $\tau_{l-3} = \tau_l$. But then $A{:}A \approx 2^{l-1} + 2^{l-4} + \cdots$, $A{:}B \approx 2^{l-3} + 2^{l-6} + \cdots$, $B{:}A \approx 2^{l-2} + 2^{l-5} + \cdots$, and $B{:}B \approx 2^{l-1} + 2^{l-4} + \cdots$; hence $B{:}B - B{:}A$ is less than $A{:}A - A{:}B$ after all. (Sharper results have been obtained by Guibas and Odlyzko [168], who show that Bill's chances are always maximized with one of the two patterns $\mathrm{H}\tau_1 \ldots \tau_{l-1}$ or $\mathrm{T}\tau_1 \ldots \tau_{l-1}$. Bill's winning strategy is, in fact, unique; see the following exercise.)

8.58 (Solution by J. Csirik.) If A is H^l or T^l, one of the two sequences matches A and cannot be used. Otherwise let $\hat{A} = \tau_1 \ldots \tau_{l-1}$, $H = \mathrm{H}\hat{A}$, and $T = \mathrm{T}\hat{A}$. It is not difficult to verify that $H{:}A = T{:}A = \hat{A}{:}\hat{A}$, $H{:}H + T{:}T = 2^{l-1} + 2(\hat{A}{:}\hat{A}) + 1$, and $A{:}H + A{:}T = 1 + 2(A{:}A) - 2^l$. Therefore the equation

$$\frac{H{:}H - H{:}A}{A{:}A - A{:}H} = \frac{T{:}T - T{:}A}{A{:}A - A{:}T}$$

implies that both fractions equal

$$\frac{H{:}H - H{:}A + T{:}T - T{:}A}{A{:}A - A{:}H + A{:}A - A{:}T} = \frac{2^{l-1} + 1}{2^l - 1}.$$

Then we can rearrange the original fractions to show that

$$\frac{H{:}H - H{:}A}{T{:}T - T{:}A} = \frac{A{:}A - A{:}H}{A{:}A - A{:}T} = \frac{p}{q},$$

where $p \perp q$. And $(p+1)\backslash \gcd(2^{l-1} + 1, 2^l - 1) = \gcd(3, 2^l - 1)$; so we may assume that l is even and that $p = 1$, $q = 2$. It follows that $A{:}A - A{:}H = (2^l - 1)/3$ and $A{:}A - A{:}T = (2^{l+1} - 2)/3$, hence $A{:}H - A{:}T = (2^l - 1)/3 \geqslant 2^{l-2}$. We have $A{:}H \geqslant 2^{l-2}$ if and only if $A = (\mathrm{TH})^{l/2}$. But then $H{:}H - H{:}A = A{:}A - A{:}H$, so $2^{l-1} + 1 = 2^l - 1$ and $l = 2$.

(Csirik [69] goes on to show that, when $l \geqslant 4$, Alice can do no better than to play $\mathrm{HT}^{l-3}\mathrm{H}^2$. But even with this strategy, Bill wins with probability nearly $\frac{2}{3}$.)

8.59 According to (8.82), we want $B{:}B - B{:}A > A{:}A - A{:}B$. One solution is $A = \mathrm{TTHH}$, $B = \mathrm{HHH}$.

8.60 (a) Two cases arise depending on whether $h_k \neq h_n$ or $h_k = h_n$:

$$G(w,z) = \frac{m-1}{m}\Bigl(\frac{m-2+w+z}{m}\Bigr)^{k-1} w \Bigl(\frac{m-1+z}{m}\Bigr)^{n-k-1} z$$

$$+\frac{1}{m}\Big(\frac{m-1+wz}{m}\Big)^{k-1}wz\Big(\frac{m-1+z}{m}\Big)^{n-k-1}z\,.$$

(b) We can either argue algebraically, taking partial derivatives of $G(w,z)$ with respect to w and z and setting $w = z = 1$; or we can argue combinatorially: Whatever the values of $h_1, \ldots, h_{n-1}$, the expected value of $P(h_1,\ldots,h_{n-1},h_n;n)$ is the same (averaged over h_n), because the hash sequence $(h_1,\ldots,h_{n-1})$ determines a sequence of list sizes $(n_1,n_2,\ldots,n_m)$ such that the stated expected value is $\big((n_1{+}1)+(n_2{+}1)+\cdots+(n_m{+}1)\big)/m = (n-1+m)/m$. Therefore the random variable $EP(h_1,\ldots,h_n;n)$ is independent of $(h_1,\ldots,h_{n-1})$, hence independent of $P(h_1,\ldots,h_n;k)$.

8.61 If $1 \leqslant k < l \leqslant n$, the previous exercise shows that the coefficient of $s_k s_l$ in the variance of the average is zero. Therefore we need only consider the coefficient of s_k^2, which is

$$\sum_{1\leqslant h_1,\ldots,h_n\leqslant m}\frac{P(h_1,\ldots,h_n;k)^2}{m^n} - \Bigg(\sum_{1\leqslant h_1,\ldots,h_n\leqslant m}\frac{P(h_1,\ldots,h_n;k)}{m^n}\Bigg)^2,$$

the variance of $\big((m-1+z)/m\big)^{k-1}z$; and this is $(k-1)(m-1)/m^2$ as in exercise 30.

8.62 The pgf $D_n(z)$ satisfies the recurrence

$$\begin{aligned} D_0(z) &= z\,;\\ D_n(z) &= z^2D_{n-1}(z) + 2(1-z^3)D'_{n-1}(z)/(n+1)\,, \qquad \text{for } n > 0. \end{aligned}$$

We can now derive the recurrence

$$D''_n(1) = (n-11)D''_{n-1}(1)/(n+1) + (8n-2)/7\,,$$

which has the solution $\frac{2}{637}(n{+}2)(26n{+}15)$ for all $n \geqslant 11$ (regardless of initial conditions). Hence the variance comes to $\frac{108}{637}(n+2)$ for $n \geqslant 11$.

8.63 (Another question asks if a given sequence of purported cumulants comes from any distribution whatever; for example, κ_2 must be nonnegative, and $\kappa_4 + 3\kappa_2^2 = E\big((X-\mu)^4\big)$ must be at least $\big(E\big((X-\mu)^2\big)\big)^2 = \kappa_2^2$, etc. A necessary and sufficient condition for this other problem was found by Hamburger [6], [175].)

9.1 True if the functions are all positive. But otherwise we might have, say, $f_1(n) = n^3 + n^2$, $f_2(n) = -n^3$, $g_1(n) = n^4 + n$, $g_2(n) = -n^4$.

9.2 (a) We have $n^{\ln n} \prec c^n \prec (\ln n)^n$, since $(\ln n)^2 \prec n\ln c \prec n\ln\ln n$. (b) $n^{\ln\ln\ln n} \prec (\ln n)! \prec n^{\ln\ln n}$. (c) Take logarithms to show that $(n!)!$ wins. (d) $F^2_{\lceil H_n\rceil} \asymp \phi^{2\ln n} = n^{2\ln\phi}$; $H_{F_n} \sim n\ln\phi$ wins because $\phi^2 = \phi + 1 < e$.

9.3 Replacing kn by $O(n)$ requires a different C for each k; but each O stands for a single C. In fact, the context of this O requires it to stand for a set of functions of two variables k and n. It would be correct to write $\sum_{k=1}^{n} kn = \sum_{k=1}^{n} O(n^2) = O(n^3)$.

9.4 For example, $\lim_{n\to\infty} O(1/n) = 0$. On the left, $O(1/n)$ is the set of all functions $f(n)$ such that there are constants C and n_0 with $|f(n)| \leqslant C/n$ for all $n \geqslant n_0$. The limit of all functions in that set is 0, so the left-hand side is the singleton set $\{0\}$. On the right, there are no variables; 0 represents $\{0\}$, the (singleton) set of all "functions of no variables, whose value is zero." (Can you see the inherent logic here? If not, come back to it next year; you probably can still manipulate O-notation even if you can't shape your intuitions into rigorous formalisms.)

9.5 Let $f(n) = n^2$ and $g(n) = 1$; then n is in the left set but not in the right, so the statement is false.

9.6 $n \ln n + \gamma n + O(\sqrt{n} \ln n)$.

9.7 $(1 - e^{-1/n})^{-1} = nB_0 - B_1 + B_2 n^{-1}/2! + \cdots = n + \frac{1}{2} + O(n^{-1})$.

9.8 For example, let $f(n) = \lfloor n/2 \rfloor !^2 + n$, $g(n) = \bigl(\lceil n/2 \rceil - 1\bigr)!\, \lceil n/2 \rceil ! + n$. These functions, incidentally, satisfy $f(n) = O\bigl(ng(n)\bigr)$ and $g(n) = O\bigl(nf(n)\bigr)$; more extreme examples are clearly possible.

9.9 (For completeness, we assume that there is a side condition $n \to \infty$, so that two constants are implied by each O.) Every function on the left has the form $a(n) + b(n)$, where there exist constants m_0, B, n_0, C such that $|a(n)| \leqslant B|f(n)|$ for $n \geqslant m_0$ and $|b(n)| \leqslant C|g(n)|$ for $n \geqslant n_0$. Therefore the left-hand function is at most $\max(B, C)\bigl(|f(n)| + |g(n)|\bigr)$, for $n \geqslant \max(m_0, n_0)$, so it is a member of the right side.

9.10 If $g(x)$ belongs to the left, so that $g(x) = \cos y$ for some y, where $|y| \leqslant C|x|$ for some C, then $0 \leqslant 1 - g(x) = 2\sin^2(y/2) \leqslant \frac{1}{2}y^2 \leqslant \frac{1}{2}C^2x^2$; hence the set on the left is contained in the set on the right, and the formula is true.

9.11 The proposition is true. For if, say, $|x| \leqslant |y|$, we have $(x+y)^2 \leqslant 4y^2$. Thus $(x+y)^2 = O(x^2) + O(y^2)$. Thus $O(x+y)^2 = O\bigl((x+y)^2\bigr) = O\bigl(O(x^2) + O(y^2)\bigr) = O\bigl(O(x^2)\bigr) + O\bigl(O(y^2)\bigr) = O(x^2) + O(y^2)$.

9.12 $1 + 2/n + O(n^{-2}) = (1 + 2/n)\bigl(1 + O(n^{-2})/(1 + 2/n)\bigr)$ by (9.26), and $1/(1 + 2/n) = O(1)$; now use (9.26).

9.13 $n^n\bigl(1 + 2n^{-1} + O(n^{-2})\bigr)^n = n^n \exp\bigl(n\bigl(2n^{-1} + O(n^{-2})\bigr)\bigr) = e^2 n^n + O(n^{n-1})$.

9.14 It is $n^{n+\beta} \exp\bigl((n+\beta)\bigl(\alpha/n - \frac{1}{2}\alpha^2/n^2 + O(n^{-3})\bigr)\bigr)$.

9.15 $\ln\binom{3n}{n,n,n} = 3n\ln 3 - \ln n + \frac{1}{2}\ln 3 - \ln 2\pi + \left(\frac{1}{36} - \frac{1}{4}\right)n^{-1} + O(n^{-3})$, so the answer is

$$\frac{3^{3n+1/2}}{2\pi n}\left(1 - \tfrac{2}{9}n^{-1} + \tfrac{2}{81}n^{-2} + O(n^{-3})\right).$$

(It's interesting to compare this formula with the corresponding result for the middle binomial coefficient, exercise 9.60.)

9.16 If l is any integer in the range $a \leqslant l < b$ we have

$$\begin{aligned}\int_0^1 B(x)f(l+x)\,dx &= \int_{1/2}^1 B(x)f(l+x)\,dx - \int_0^{1/2} B(1-x)f(l+x)\,dx\\ &= \int_{1/2}^1 B(x)\big(f(l+x) - f(l+1-x)\big)\,dx.\end{aligned}$$

Since $l + x \geqslant l + 1 - x$ when $x \geqslant \frac{1}{2}$, this integral is positive when $f(x)$ is nondecreasing.

9.17 $\sum_{m\geqslant 0} B_m(\frac{1}{2})z^m/m! = ze^{z/2}/(e^z - 1) = z/(e^{z/2} - 1) - z/(e^z - 1)$.

9.18 The text's derivation for the case $\alpha = 1$ generalizes to give

$$b_k(n) = \frac{2^{(2n+1/2)\alpha}}{(2\pi n)^{\alpha/2}}e^{-k^2\alpha/n}, \quad c_k(n) = 2^{2n\alpha}\,n^{-(1+\alpha)/2+3\epsilon}e^{-k^2\alpha/n};$$

the answer is $2^{2n\alpha}(\pi n)^{(1-\alpha)/2}\alpha^{-1/2}\big(1 + O(n^{-1/2+3\epsilon})\big)$.

9.19 $H_{10} = 2.928968254 \approx 2.928968256$; $10! = 3628800 \approx 3628712.4$; $B_{10} = 0.075757576 \approx 0.075757494$; $\pi(10) = 4 \approx 10.0017845$; $e^{0.1} = 1.10517092 \approx 1.10517083$; $\ln 1.1 = 0.0953102 \approx 0.0953083$; $1.1111111 \approx 1.1111000$; $1.1^{0.1} = 1.00957658 \approx 1.00957643$. (The approximation to $\pi(n)$ gives more significant figures when n is larger; for example, $\pi(10^9) = 50847534 \approx 50840742$.)

9.20 (a) Yes; the left side is $o(n)$ while the right side is equivalent to $O(n)$. (b) Yes; the left side is $e\cdot e^{O(1/n)}$. (c) No; the left side is about $\sqrt{n}$ times the bound on the right.

9.21 We have $P_n = m = n\big(\ln m - 1 - 1/\ln m + O(1/\log n)^2\big)$, where

$$\begin{aligned}\ln m &= \ln n + \ln\ln m - 1/\ln n + \ln\ln n/(\ln n)^2 + O(1/\log n)^2;\\ \ln\ln m &= \ln\ln n + \frac{\ln\ln n}{\ln n} - \frac{(\ln\ln n)^2}{2(\ln n)^2} + \frac{\ln\ln n}{(\ln n)^2} + O(1/\log n)^2.\end{aligned}$$

It follows that

$$\begin{aligned}P_n = n\bigg(&\ln n + \ln\ln n - 1\\ &+ \frac{\ln\ln n - 2}{\ln n} - \frac{\frac{1}{2}(\ln\ln n)^2 - 3\ln\ln n}{(\ln n)^2} + O(1/\log n)^2\bigg).\end{aligned}$$

hat does a drowning analytic number theorist say?
log log log ...

(A slightly better approximation replaces this $O(1/\log n)^2$ by the quantity $-5/(\ln n)^2 + O(\log\log n/\log n)^3$; then we estimate $P_{1000000} \approx 15483612.4$.)

9.22 Replace $O(n^{-2k})$ by $-\frac{1}{12}n^{-2k} + O(n^{-4k})$ in the expansion of H_{n^k}; this replaces $O\bigl(\Sigma_3(n^2)\bigr)$ by $-\frac{1}{12}\Sigma_3(n^2) + O\bigl(\Sigma_3(n^4)\bigr)$ in (9.53). We have

$$\Sigma_3(n) \;=\; \tfrac{3}{4}n^{-1} + \tfrac{5}{36}n^{-2} + O(n^{-3})\,,$$

hence the term $O(n^{-2})$ in (9.54) can be replaced by $-\frac{19}{144}n^{-2} + O(n^{-3})$.

9.23 $nh_n = \sum_{0\leqslant k<n} h_k/(n-k) + 2cH_n/(n+1)(n+2)$. Choose $c = e^{\pi^2/6} = \sum_{k\geqslant 0} g_k$ so that $\sum_{k\geqslant 0} h_k = 0$ and $h_n = O(\log n)/n^3$. The expansion of $\sum_{0\leqslant k<n} h_k/(n-k)$ as in (9.60) now yields $nh_n = 2cH_n/(n+1)(n+2) + O(n^{-2})$, hence

$$g_n \;=\; e^{\pi^2/6}\left(\frac{n + 2\ln n + O(1)}{n^3}\right).$$

9.24 (a) If $\sum_{k\geqslant 0}\bigl|f(k)\bigr| < \infty$ and if $f(n-k) = O\bigl(f(n)\bigr)$ when $0 \leqslant k \leqslant n/2$, we have

$$\sum_{k=0}^{n} a_k b_{n-k} \;=\; \sum_{k=0}^{n/2} O\bigl(f(k)\bigr)O\bigl(f(n)\bigr) + \sum_{k=n/2}^{n} O\bigl(f(n)\bigr)O\bigl(f(n-k)\bigr)\,,$$

which is $2O\bigl(f(n)\sum_{k\geqslant 0}\bigl|f(k)\bigr|\bigr)$, so this case is proved. (b) But in this case if $a_n = b_n = \alpha^{-n}$, the convolution $(n+1)\alpha^{-n}$ is not $O(\alpha^{-n})$.

9.25 $S_n/\binom{3n}{n} = \sum_{k=0}^{n} n^{\underline{k}}/(2n+1)^{\overline{k}}$. We may restrict the range of summation to $0 \leqslant k \leqslant (\log n)^2$, say. In this range $n^{\underline{k}} = n^k\bigl(1 - \binom{k}{2}/n + O(k^4/n^2)\bigr)$ and $(2n+1)^{\overline{k}} = (2n)^k\bigl(1 + \binom{k+1}{2}/2n + O(k^4/n^2)\bigr)$, so the summand is

$$\frac{1}{2^k}\left(1 - \frac{3k^2-k}{4n} + O\Bigl(\frac{k^4}{n^2}\Bigr)\right).$$

Hence the sum over k is $2 - 4/n + O(1/n^2)$. Stirling's approximation can now be applied to $\binom{3n}{n} = (3n)!/(2n)!\,n!$, proving (9.2).

9.26 The minimum occurs at a term $B_{2m}/(2m)(2m-1)n^{2m-1}$ where $2m \approx 2\pi n + \frac{3}{2}$, and this term is approximately equal to $1/(\pi e^{2\pi n}\sqrt{n}\,)$. The absolute error in $\ln n!$ is therefore too large to determine $n!$ exactly by rounding to an integer, when n is greater than about $e^{2\pi+1}$.

9.27 We may assume that $\alpha \neq -1$. Let $f(x) = x^\alpha$; the answer is

$$\sum_{k=1}^{n} k^\alpha \;=\; C_\alpha + \frac{n^{\alpha+1}}{\alpha+1} + \frac{n^\alpha}{2} + \sum_{k=1}^{m}\frac{B_{2k}}{2k}\binom{\alpha}{2k-1}n^{\alpha-2k+1} + O(n^{\alpha-2m-1})\,.$$

(The constant C_α turns out to be $\zeta(-\alpha)$, which is in fact *defined* by this formula when $\alpha > -1$.)

In particular, $\zeta(0) = -1/2$, and $\zeta(-n) = -B_{n+1}/(n+1)$ for integer $n > 0$.

9.28 In general, suppose $f(x) = x^\alpha \ln x$ in Euler's summation formula, when $\alpha \neq -1$. Proceeding as in the previous exercise, we find

$$\begin{aligned}\sum_{k=1}^{n} k^\alpha \ln k \;&=\; C'_\alpha + \frac{n^{\alpha+1}\ln n}{\alpha+1} - \frac{n^{\alpha+1}}{(\alpha+1)^2} + \frac{n^\alpha \ln n}{2}\\ &\qquad + \sum_{k=1}^{m} \frac{B_{2k}}{2k}\binom{\alpha}{2k-1} n^{\alpha-2k+1}(\ln n + H_\alpha - H_{\alpha-2k+1})\\ &\qquad + O(n^{\alpha-2m-1}\log n)\,;\end{aligned}$$

the constant C'_α can be shown [74, §3.7] to be $-\zeta'(-\alpha)$. (The $\log n$ factor in the O term can be removed when α is a positive integer $\leqslant 2m$; in that case we also replace the kth term of the right sum by $B_{2k}\alpha!\,(2k-2-\alpha)!\times(-1)^\alpha n^{\alpha-2k+1}/(2k)!$ when $\alpha < 2k-1$.) To solve the stated problem, we let $\alpha = 1$ and $m = 1$, taking the exponential of both sides to get

$$Q_n \;=\; A\cdot n^{n^2/2+n/2+1/12}e^{-n^2/4}\bigl(1+O(n^{-2})\bigr)\,,$$

where $A = e^{1/12-\zeta'(-1)} \approx 1.2824271291$ is "Glaisher's constant."

9.29 Let $f(x) = x^{-1}\ln x$. A slight modification of the calculation in the previous exercise gives

$$\begin{aligned}\sum_{k=1}^{n}\frac{\ln k}{k} \;&=\; \frac{(\ln n)^2}{2} + \gamma_1 + \frac{\ln n}{2n}\\ &\qquad - \sum_{k=1}^{m}\frac{B_{2k}}{2k} n^{-2k}(\ln n - H_{2k-1}) + O(n^{-2m-1}\log n)\,,\end{aligned}$$

where $\gamma_1 \approx -0.07281584548367672486$ is a "Stieltjes constant" (see the answer to 9.57). Taking exponentials gives

$$e^{\gamma_1}\sqrt{n^{\ln n}}\left(1 + \frac{\ln n}{2n} + O\Bigl(\Bigl(\frac{\log n}{n}\Bigr)^2\Bigr)\right).$$

9.30 Let $g(x) = x^l e^{-x^2}$ and $f(x) = g(x/\sqrt{n}\,)$. Then $n^{-1/2}\sum_{k\geqslant 0} k^l e^{-k^2/n}$ is

$$\begin{aligned}&\int_0^\infty f(x)\,dx - \sum_{k=1}^{m}\frac{B_k}{k!}f^{(k-1)}(0) - (-1)^m\int_0^\infty \frac{B_m\bigl(\{x\}\bigr)}{m!}f^{(m)}(x)\,dx\\ &\qquad = \; n^{1/2}\int_0^\infty g(x)\,dx - \sum_{k=1}^{m}\frac{B_k}{k!}n^{(k-1)/2}g^{(k-1)}(0) + O(n^{-m/2})\,.\end{aligned}$$

Since $g(x) = x^l - x^{2+l}/1! + x^{4+l}/2! - x^{6+l}/3! + \cdots$, the derivatives $g^{(m)}(x)$ obey a simple pattern, and the answer is

$$\frac{1}{2}n^{(l+1)/2}\Gamma\Bigl(\frac{l+1}{2}\Bigr) - \frac{B_{l+1}}{(l+1)!\,0!} + \frac{B_{l+3}n^{-1}}{(l+3)!\,1!} - \frac{B_{l+5}n^{-2}}{(l+5)!\,2!} + O(n^{-3})\,.$$

9.31 The somewhat surprising identity $1/(c^{m-k}+c^m) + 1/(c^{m+k}+c^m) = 1/c^m$ makes the terms for $0 \leqslant k \leqslant 2m$ sum to $(m+\frac{1}{2})/c^m$. The remaining terms are

$$\begin{aligned}\sum_{k\geqslant 1}\frac{1}{c^{2m+k}+c^m} &= \sum_{k\geqslant 1}\Bigl(\frac{1}{c^{2m+k}} - \frac{1}{c^{3m+2k}} + \frac{1}{c^{4m+3k}} - \cdots\Bigr)\\ &= \frac{1}{c^{2m+1}-c^{2m}} - \frac{1}{c^{3m+2}-c^{3m}} + \cdots\,,\end{aligned}$$

and this series can be truncated at any desired point, with an error not exceeding the first omitted term.

9.32 $H_n^{(2)} = \pi^2/6 - 1/n + O(n^{-2})$ by Euler's summation formula, since we know the constant; and H_n is given by (9.89). So the answer is

$$ne^{\gamma+\pi^2/6}\bigl(1 - \tfrac{1}{2}n^{-1} + O(n^{-2})\bigr)\,.$$

The world's top three constants, (e, π, γ), all appear in this answer.

9.33 We have $n^{\underline{k}}/n^{\overline{k}} = 1 - k(k-1)n^{-1} + \frac{1}{2}k^2(k-1)^2n^{-2} + O(k^6n^{-3})$; dividing by $k!$ and summing over $k \geqslant 0$ yields $e - en^{-1} + \frac{7}{2}en^{-2} + O(n^{-3})$.

9.34 $A = e^\gamma$; $B = 0$; $C = -\frac{1}{2}e^\gamma$; $D = \frac{1}{2}e^\gamma(1-\gamma)$; $E = \frac{1}{8}e^\gamma$; $F = \frac{1}{12}e^\gamma(3\gamma+1)$.

9.35 Since $1/k\bigl(\ln k + O(1)\bigr) = 1/k\ln k + O\bigl(1/k(\log k)^2\bigr)$, the given sum is $\sum_{k=2}^{n} 1/k\ln k + O(1)$. The remaining sum is $\ln\ln n + O(1)$ by Euler's summation formula.

9.36 This works out beautifully with Euler's summation formula:

$$\begin{aligned}S_n &= \sum_{0\leqslant k<n}\frac{1}{n^2+k^2} + \frac{1}{n^2+x^2}\bigg|_0^n\\ &= \int_0^n \frac{dx}{n^2+x^2} + \frac{1}{2}\frac{1}{n^2+x^2}\bigg|_0^n + \frac{B_2}{2!}\frac{-2x}{(n^2+x^2)^2}\bigg|_0^n + O(n^{-5})\,.\end{aligned}$$

Hence $S_n = \frac{1}{4}\pi n^{-1} - \frac{1}{4}n^{-2} - \frac{1}{24}n^{-3} + O(n^{-5})$.

9.37 This is

$$\sum_{k,q\geqslant 1}(n-qk)\bigl[n/(q+1) < k \leqslant n/q\bigr]$$

$$= n^2 - \sum_{q\geqslant 1} q\left(\binom{\lfloor n/q\rfloor+1}{2} - \binom{\lfloor n/(q+1)\rfloor+1}{2}\right)$$
$$= n^2 - \sum_{q\geqslant 1} \binom{\lfloor n/q\rfloor+1}{2}.$$

The remaining sum is like (9.55) but without the factor $\mu(q)$. The same method works here as it did there, but we get $\zeta(2)$ in place of $1/\zeta(2)$, so the answer comes to $\bigl(1-\frac{\pi^2}{12}\bigr)n^2 + O(n\log n)$.

9.38 Replace k by $n-k$ and let $a_k(n) = (n-k)^{n-k}\binom{n}{k}$. Then $\ln a_k(n) = n\ln n - \ln k! - k + O(kn^{-1})$, and we can use tail-exchange with $b_k(n) = n^n e^{-k}/k!$, $c_k(n) = kb_k(n)/n$, $D_n = \{\, k \mid k \leqslant \ln n \,\}$, to get $\sum_{k=0}^{n} a_k(n) = n^n e^{1/e}\bigl(1+O(n^{-1})\bigr)$.

9.39 Tail-exchange with $b_k(n) = (\ln n - k/n - \frac12 k^2/n^2)(\ln n)^k/k!$, $c_k(n) = n^{-3}(\ln n)^{k+3}/k!$, $D_n = \{\, k \mid 0 \leqslant k \leqslant 10\ln n \,\}$. When $k \approx 10\ln n$ we have $k! \asymp \sqrt{k}\,(10/e)^k(\ln n)^k$, so the kth term is $O(n^{-10\ln(10/e)}\log n)$. The answer is $n\ln n - \ln n - \frac12(\ln n)(1+\ln n)/n + O\bigl(n^{-2}(\log n)^3\bigr)$.

9.40 Combining terms two by two, we find that $H_{2k}^m - (H_{2k} - \frac{1}{2k})^m = \frac{m}{2k}H_{2k}^{m-1}$ plus terms whose sum over all $k \geqslant 1$ is $O(1)$. Suppose n is even. Euler's summation formula implies that

$$\sum_{k=1}^{n/2} \frac{H_{2k}^{m-1}}{k} = \sum_{k=1}^{n/2} \frac{(\ln 2e^\gamma k)^{m-1} + O(1/k)}{k} + O(1)$$
$$= \frac{(\ln e^\gamma n)^m}{m} + O(1);$$

hence the sum is $\frac12 H_n^m + O(1)$. In general the answer is $\frac12(-1)^n H_n^m + O(1)$.

9.41 Let $\alpha = \hat\phi/\phi = -\phi^{-2}$. We have

$$\sum_{k=1}^{n} \ln F_k = \sum_{k=1}^{n} \bigl(\ln\phi^k - \ln\sqrt5 + \ln(1-\alpha^k)\bigr)$$
$$= \frac{n(n+1)}{2}\ln\phi - \frac{n}{2}\ln 5 + \sum_{k\geqslant 1}\ln(1-\alpha^k) - \sum_{k>n}\ln(1-\alpha^k).$$

The latter sum is $\sum_{k>n} O(\alpha^k) = O(\alpha^n)$. Hence the answer is

$$\phi^{n(n+1)/2}5^{-n/2}C + O(\phi^{n(n-3)/2}5^{-n/2}),$$

where $C = (1-\alpha)(1-\alpha^2)(1-\alpha^3)\ldots \approx 1.226742$.

9.42 The hint follows since $\binom{n}{k-1}/\binom{n}{k} = \frac{k}{n-k+1} \leqslant \frac{\alpha n}{n-\alpha n+1} < \frac{\alpha}{1-\alpha}$. Let $m = \lfloor \alpha n \rfloor = \alpha n - \epsilon$. Then

$$\begin{aligned}\binom{n}{m} &< \sum_{k\leqslant m}\binom{n}{k}\\ &< \binom{n}{m}\biggl(1+\frac{\alpha}{1-\alpha}+\Bigl(\frac{\alpha}{1-\alpha}\Bigr)^2+\cdots\biggr) \;=\; \binom{n}{m}\frac{1-\alpha}{1-2\alpha}.\end{aligned}$$

So $\sum_{k\leqslant\alpha n}\binom{n}{k} = \binom{n}{m}O(1)$, and it remains to estimate $\binom{n}{m}$. By Stirling's approximation we have $\ln\binom{n}{m} = -\frac{1}{2}\ln n - (\alpha n-\epsilon)\ln(\alpha-\epsilon/n) - \bigl((1-\alpha)n+\epsilon\bigr)\times \ln(1-\alpha+\epsilon/n) + O(1) = -\frac{1}{2}\ln n - \alpha n\ln\alpha - (1-\alpha)n\ln(1-\alpha) + O(1)$.

9.43 The denominator has factors of the form $z-\omega$, where ω is a complex root of unity. Only the factor $z-1$ occurs with multiplicity 5. Therefore by (7.31), only one of the roots has a coefficient $\Omega(n^4)$, and the coefficient is $c = 5/(5!\cdot 1\cdot 5\cdot 10\cdot 25\cdot 50) = 1/1500000$.

9.44 Stirling's approximation says that $\ln\bigl(x^{-\alpha}x!/(x-\alpha)!\bigr)$ has an asymptotic series

$$\begin{aligned}-\alpha-(x+\tfrac{1}{2}-\alpha)\ln(1-\alpha/x)-\frac{B_2}{2\cdot 1}\bigl(x^{-1}-(x-\alpha)^{-1}\bigr)&\\ -\frac{B_4}{4\cdot 3}\bigl(x^{-3}-(x-\alpha)^{-3}\bigr)-\cdots&\end{aligned}$$

in which each coefficient of x^{-k} is a polynomial in α. Hence $x^{-\alpha}x!/(x-\alpha)! = c_0(\alpha)+c_1(\alpha)x^{-1}+\cdots+c_n(\alpha)x^{-n}+O(x^{-n-1})$ as $x\to\infty$, where $c_n(\alpha)$ is a polynomial in α. We know that $c_n(\alpha) = \left[{\alpha \atop \alpha-n}\right](-1)^n$ whenever α is an integer, and $\left[{\alpha \atop \alpha-n}\right]$ is a polynomial in α of degree $2n$; hence $c_n(\alpha) = \left[{\alpha \atop \alpha-n}\right](-1)^n$ for all real α. In other words, the asymptotic formulas

(See [220] for further discussion.)

$$\begin{aligned}x^{\underline{\alpha}} &= \sum_{k=0}^{n}\left[{\alpha \atop \alpha-k}\right](-1)^k x^{\alpha-k} + O(x^{\alpha-n-1}),\\ x^{\overline{\alpha}} &= \sum_{k=0}^{n}\left[{\alpha \atop \alpha-k}\right]x^{\alpha-k} + O(x^{\alpha-n-1})\end{aligned}$$

generalize equations (6.13) and (6.11), which hold in the all-integer case.

9.45 Let the partial quotients of α be $\langle a_1, a_2, \ldots\rangle$, and let α_m be the continued fraction $1/(a_m+\alpha_{m+1})$ for $m\geqslant 1$. Then $D(\alpha,n) = D(\alpha_1,n) < D\bigl(\alpha_2,\lfloor\alpha_1 n\rfloor\bigr)+a_1+3 < D\bigl(\alpha_3,\lfloor\alpha_2\lfloor\alpha_1 n\rfloor\rfloor\bigr)+a_1+a_2+6 < \cdots < D\bigl(\alpha_{m+1}, \lfloor\alpha_m\lfloor\ldots\lfloor\alpha_1 n\rfloor\ldots\rfloor\rfloor\bigr)+a_1+\cdots+a_m+3m < \alpha_1\ldots\alpha_m n+a_1+\cdots+a_m+3m$, for all m. Divide by n and let $n \to \infty$; the limit is less than $\alpha_1 \ldots \alpha_m$ for all m. Finally we have

$$\alpha_1 \ldots \alpha_m \;=\; \frac{1}{K(a_1, \ldots, a_{m-1}, a_m + \alpha_m)} \;<\; \frac{1}{F_{m+1}}\,.$$

9.46 For convenience we write just m instead of $m(n)$. By Stirling's approximation, the maximum value of $k^n/k!$ occurs when $k \approx m \approx n/\ln n$, so we replace k by $m + k$ and find that

$$\begin{aligned}\ln\frac{(m+k)^n}{(m+k)!} \;&=\; n\ln m - m\ln m + m - \frac{\ln 2\pi m}{2}\\ &\qquad\qquad - \frac{(m+n)k^2}{2m^2} + O(k^3 m^{-2}\log n)\,.\end{aligned}$$

Actually we want to replace k by $\lfloor m\rfloor + k$; this adds a further $O(km^{-1}\log n)$. The tail-exchange method with $|k| \leqslant m^{1/2+\epsilon}$ now allows us to sum on k, giving a fairly sharp asymptotic estimate in terms of the quantity Θ in (9.93):

A truly Bell-shaped summand.

$$\begin{aligned}\varpi_n \;&=\; \frac{e^{m-1}m^{n-m}}{\sqrt{2\pi m}}\bigl(\Theta_{2m^2/(m+n)} + O(1)\bigr)\\ &=\; e^{m-n-1/2}m^n\sqrt{\frac{m}{m+n}}\biggl(1 + O\Bigl(\frac{\log n}{n^{1/2}}\Bigr)\biggr)\,.\end{aligned}$$

The requested formula follows, with relative error $O(\log\log n/\log n)$.

9.47 Let $\log_m n = l + \theta$, where $0 \leqslant \theta < 1$. The floor sum is $l(n+1) + 1 - (m^{l+1} - 1)/(m-1)$; the ceiling sum is $(l+1)n - (m^{l+1} - 1)/(m-1)$; the exact sum is $(l+\theta)n - n/\ln m + O(\log n)$. Ignoring terms that are $o(n)$, the difference between ceiling and exact is $\bigl(1 - f(\theta)\bigr)n$, and the difference between exact and floor is $f(\theta)n$, where

$$f(\theta) \;=\; \frac{m^{1-\theta}}{m-1} + \theta - \frac{1}{\ln m}\,.$$

This function has maximum value $f(0) = f(1) = m/(m-1) - 1/\ln m$, and its minimum value is $\ln\ln m/\ln m + 1 - \bigl(\ln(m-1)\bigr)/\ln m$. The ceiling value is closer when n is nearly a power of m, but the floor value is closer when θ lies somewhere between 0 and 1.

9.48 Let $d_k = a_k + b_k$, where a_k counts digits to the left of the decimal point. Then $a_k = 1 + \lfloor\log H_k\rfloor = \log\log k + O(1)$, where 'log' denotes $\log_{10}$. To estimate b_k, let us look at the number of decimal places necessary to distinguish y from nearby numbers $y - \epsilon$ and $y + \epsilon'$: Let $\delta = 10^{-b}$ be the length of the interval of numbers that round to $\hat{y}$. We have $|y - \hat{y}| \leqslant \frac{1}{2}\delta$; also $y - \epsilon < \hat{y} - \frac{1}{2}\delta$ and $y + \epsilon' > \hat{y} + \frac{1}{2}\delta$. Therefore $\epsilon + \epsilon' > \delta$. And if $\delta < \min(\epsilon, \epsilon')$, the rounding does distinguish $\hat{y}$ from both $y - \epsilon$ and $y + \epsilon'$. Hence $10^{-b_k} < 1/(k-1) + 1/k$ and $10^{1-b_k} \geqslant 1/k$; we have $b_k = \log k + O(1)$. Finally, therefore, $\sum_{k=1}^{n} d_k = \sum_{k=1}^{n} (\log k + \log\log k + O(1))$, which is $n \log n + n \log\log n + O(n)$ by Euler's summation formula.

9.49 We have $H_n > \ln n + \gamma + \frac{1}{2}n^{-1} - \frac{1}{12}n^{-2} = f(n)$, where $f(x)$ is increasing for all $x > 0$; hence if $n \geqslant e^{\alpha-\gamma}$ we have $H_n \geqslant f(e^{\alpha-\gamma}) > \alpha$. Also $H_{n-1} < \ln n + \gamma - \frac{1}{2}n^{-1} = g(n)$, where $g(x)$ is increasing for all $x > 0$; hence if $n \leqslant e^{\alpha-\gamma}$ we have $H_{n-1} \leqslant g(e^{\alpha-\gamma}) < \alpha$. Therefore $H_{n-1} \leqslant \alpha \leqslant H_n$ implies that $e^{\alpha-\gamma} + 1 > n > e^{\alpha+\gamma} - 1$. (Sharper results have been obtained by Boas and Wrench [33].)

9.50 (a) The expected return is $\sum_{1 \leqslant k \leqslant N} k/(k^2 H_N^{(2)}) = H_N/H_N^{(2)}$, and we want the asymptotic value to $O(N^{-1})$:

$$\frac{\ln N + \gamma + O(N^{-1})}{\pi^2/6 - N^{-1} + O(N^{-2})} = \frac{6 \ln 10}{\pi^2} n + \frac{6\gamma}{\pi^2} + \frac{36 \ln 10}{\pi^4} \frac{n}{10^n} + O(10^{-n}).$$

The coefficient $(6 \ln 10)/\pi^2 \approx 1.3998$ says that we expect about 40% profit.

(b) The probability of profit is $\sum_{n < k \leqslant N} 1/(k^2 H_N^{(2)}) = 1 - H_n^{(2)}/H_N^{(2)}$, and since $H_n^{(2)} = \frac{\pi^2}{6} - n^{-1} + \frac{1}{2}n^{-2} + O(n^{-3})$ this is

$$\frac{n^{-1} - \frac{1}{2}n^{-2} + O(n^{-3})}{\pi^2/6 + O(N^{-1})} = \frac{6}{\pi^2} n^{-1} - \frac{3}{\pi^2} n^{-2} + O(n^{-3}),$$

actually *decreasing* with n. (The expected value in (a) is high because it includes payoffs so huge that the entire world's economy would be affected if they ever had to be made.)

9.51 Strictly speaking, this is false, since the function represented by $O(x^{-2})$ might not be integrable. (It might be '$[x \in S]/x^2$', where S is not a measurable set.) But if we stipulate that $f(x)$ is an integrable function such that $f(x) = O(x^{-2})$ as $x \to \infty$, then $\left|\int_n^\infty f(x)\,dx\right| \leqslant \int_n^\infty \left|f(x)\right|\,dx \leqslant \int_n^\infty Cx^{-2}\,dx = Cn^{-1}$.

(As opposed to an execrable function.)

9.52 In fact, the stack of n's can be replaced by any function $f(n)$ that approaches infinity, however fast. Define the sequence $\langle m_0, m_1, m_2, \ldots \rangle$ by setting $m_0 = 0$ and letting m_k be the least integer $> m_{k-1}$ such that

$$\left(\frac{k+1}{k}\right)^{m_k} \geqslant f(k+1)^2.$$

Now let $A(z) = \sum_{k \geqslant 1} (z/k)^{m_k}$. This power series converges for all z, because the terms for $k > |z|$ are bounded by a geometric series. Also $A(n+1) \geqslant \left((n+1)/n\right)^{m_n} \geqslant f(n+1)^2$, hence $\lim_{n\to\infty} f(n)/A(n) = 0$.

9.53 By induction, the O term is $(m-1)!^{-1}\int_0^x t^{m-1}f^{(m)}(x-t)\,dt$. Since $f^{(m+1)}$ has the opposite sign to $f^{(m)}$, the absolute value of this integral is bounded by $\bigl|f^{(m)}(0)\bigr|\int_0^x t^{m-1}\,dt$; so the error is bounded by the absolute value of the first discarded term.

Sounds like a nasty theorem.

9.54 Let $g(x) = f(x)/x^{\alpha}$. Then $g'(x) \sim -\alpha g(x)/x$ as $x \to \infty$. By the mean value theorem, $g(x-\frac12) - g(x+\frac12) = -g'(y) \sim \alpha g(y)/y$ for some y between $x-\frac12$ and $x+\frac12$. Now $g(y) = g(x)\bigl(1+O(1/x)\bigr)$, so $g(x-\frac12)-g(x+\frac12) \sim \alpha g(x)/x = \alpha f(x)/x^{1+\alpha}$. Therefore

$$\sum_{k\geqslant n}\frac{f(k)}{k^{1+\alpha}} \;=\; O\Bigl(\sum_{k\geqslant n}\bigl(g(k-\tfrac12)-g(k+\tfrac12)\bigr)\Bigr) \;=\; O\bigl(g(n-\tfrac12)\bigr)\,.$$

9.55 The estimate of $(n+k+\frac12)\ln(1+k/n) + (n-k+\frac12)\ln(1-k/n)$ is extended to $k^2/n + k^4/6n^3 + O(n^{-3/2+5\epsilon})$, so we apparently want to have an extra factor $e^{-k^4/6n^3}$ in $b_k(n)$, and $c_k(n) = 2^{2n}n^{-2+5\epsilon}e^{-k^2/n}$. But it turns out to be better to leave $b_k(n)$ untouched and to let

$$c_k(n) \;=\; 2^{2n}n^{-2+5\epsilon}e^{-k^2/n} + 2^{2n}n^{-5+5\epsilon}k^4e^{-k^2/n}\,,$$

thereby replacing $e^{-k^4/6n^3}$ by $1+O(k^4/n^3)$. The sum $\sum_k k^4e^{-k^2/n}$ is $O(n^{5/2})$, as shown in exercise 30.

9.56 If $k \leqslant n^{1/2+\epsilon}$ we have $\ln(n^{\underline{k}}/n^k) = -\frac12k^2/n + \frac12k/n - \frac16k^3/n^2 + O(n^{-1+4\epsilon})$ by Stirling's approximation, hence

$$n^{\underline{k}}/n^k \;=\; e^{-k^2/2n}\bigl(1 + k/2n - \tfrac23k^3/(2n)^2 + O(n^{-1+4\epsilon})\bigr)\,.$$

Summing with the identity in exercise 30, and remembering to omit the term for $k=0$, gives $-1 + \Theta_{2n} + \Theta_{2n}^{(1)} - \frac23\Theta_{2n}^{(3)} + O(n^{-1/2+4\epsilon}) = \sqrt{\pi n/2} - \frac13 + O(n^{-1/2+4\epsilon})$.

9.57 Using the hint, the given sum becomes $\int_0^\infty ue^{-u}\zeta(1+u/\ln n)\,du$. The zeta function can be defined by the series

$$\zeta(1+z) \;=\; z^{-1} + \sum_{m\geqslant0}(-1)^m\gamma_m z^m/m!\,,$$

where $\gamma_0 = \gamma$ and γ_m is the Stieltjes constant [341, 201]

$$\lim_{n\to\infty}\Bigl(\sum_{k=1}^{n}\frac{(\ln k)^m}{k} \;-\; \frac{(\ln n)^{m+1}}{m+1}\Bigr)\,.$$

Hence the given sum is

$$\ln n + \gamma - 2\gamma_1(\ln n)^{-1} + 3\gamma_2(\ln n)^{-2} - \cdots\,.$$

9.58 Let $0 \leqslant \theta \leqslant 1$ and $f(z) = e^{2\pi i z\theta}/(e^{2\pi i z} - 1)$. We have

$$\begin{aligned} \bigl|f(z)\bigr| &= \frac{e^{-2\pi y\theta}}{1+e^{-2\pi y}} \leqslant 1\,, && \text{when } x \bmod 1 = \tfrac{1}{2}; \\ \bigl|f(z)\bigr| &\leqslant \frac{e^{-2\pi y\theta}}{|e^{-2\pi y} - 1|} \leqslant \frac{1}{1-e^{-2\pi\epsilon}}\,, && \text{when } |y| \geqslant \epsilon. \end{aligned}$$

Therefore $\bigl|f(z)\bigr|$ is bounded on the contour, and the integral is $O(M^{1-m})$. The residue of $2\pi i f(z)/z^m$ at $z = k \neq 0$ is $e^{2\pi i k\theta}/k^m$; the residue at $z = 0$ is the coefficient of z^{-1} in

$$\frac{e^{2\pi i z\theta}}{z^{m+1}}\Bigl(B_0 + B_1\frac{2\pi i z}{1!} + \cdots\Bigr) = \frac{1}{z^{m+1}}\Bigl(B_0(\theta) + B_1(\theta)\frac{2\pi i z}{1!} + \cdots\Bigr)\,,$$

namely $(2\pi i)^m B_m(\theta)/m!$. Therefore the sum of residues inside the contour is

$$\frac{(2\pi i)^m}{m!}B_m(\theta) \;+\; 2\sum_{k=1}^{M} e^{\pi i m/2}\frac{\cos(2\pi k\theta - \pi m/2)}{k^m}\,.$$

This equals the contour integral $O(M^{1-m})$, so it approaches zero as $M \to \infty$.

9.59 If $F(x)$ is sufficiently well behaved, we have the general identity

$$\sum_k F(k+t) = \sum_n G(2\pi n)e^{2\pi i n t}\,,$$

where $G(y) = \int_{-\infty}^{+\infty} e^{-iyx}F(x)\,dx$. (This is "Poisson's summation formula," which can be found in standard texts such as Henrici [182, Theorem 10.6e].)

9.60 The stated formula is equivalent to

$$n^{\overline{1/2}} = n^{1/2}\Bigl(1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} - \frac{21}{32768n^4} + O(n^{-5})\Bigr)$$

by exercise 5.22. Hence the result follows from exercises 6.64 and 9.44.

9.61 The idea is to make α "almost" rational. Let $a_k = 2^{2^{2^k}}$ be the kth partial quotient of α, and let $n = \frac{1}{2}a_{m+1}q_m$, where $q_m = K(a_1, \ldots, a_m)$ and m is even. Then $0 < \{q_m\alpha\} < 1/K(a_1, \ldots, a_{m+1}) < 1/(2n)$, and if we take $\nu = a_{m+1}/(4n)$ we get a discrepancy $\geqslant \frac{1}{4}a_{m+1}$. If this were less than $n^{1-\epsilon}$ we would have $a_{m+1}^{\epsilon} = O(q_m^{1-\epsilon})$; but in fact $a_{m+1} > q_m^{2^m}$.

9.62 See Canfield [48]; see also David and Barton [71, Chapter 16] for asymptotics of Stirling numbers of both kinds.

9.63 Let $c = \phi^{2-\phi}$. The estimate $cn^{\phi-1}+o(n^{\phi-1})$ was proved by Fine [150]. Ilan Vardi observes that the sharper estimate stated can be deduced from the fact that the error term $e(n) = f(n) - cn^{\phi-1}$ satisfies the approximate recurrence $c^\phi n^{2-\phi} e(n) \approx -\sum_k e(k)[1 \leqslant k < cn^{\phi-1}]$. The function

$$\frac{n^{\phi-1}u(\ln\ln n/\ln\phi)}{\ln n}$$

satisfies this recurrence asymptotically, if $u(x+1) = -u(x)$. (Vardi conjectures that

$$f(n) \;=\; n^{\phi-1}\biggl(c + u\Bigl(\frac{\ln\ln n}{\ln\phi}\Bigr)(\ln n)^{-1} + O\bigl((\log n)^{-2}\bigr)\biggr)$$

for some such function u.) Calculations for small n show that $f(n)$ equals the nearest integer to $cn^{\phi-1}$ for $1 \leqslant n \leqslant 400$ except in one case: $f(273) = 39 > c \cdot 273^{\phi-1} \approx 38.4997$. But the small errors are eventually magnified, because of results like those in exercise 2.36. For example, $e(201636503) \approx 35.73$; $e(919986484788) \approx -1959.07$.

"The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact."
—A. N. Whitehead [372]

9.64 (From this identity for $B_2(x)$ we can easily derive the identity of exercise 58 by induction on m.) If $0 < x < 1$, the integral $\int_x^{1/2} \sin N\pi t\,dt/\sin\pi t$ can be expressed as a sum of N integrals that are each $O(N^{-2})$, so it is $O(N^{-1})$; the constant implied by this O may depend on x. Integrating the identity $\sum_{n=1}^{N}\cos 2n\pi t = \Re\bigl(e^{2\pi i t}(e^{2N\pi i t}-1)/(e^{2\pi i t}-1)\bigr) = -\frac{1}{2}+\frac{1}{2}\sin(2N+1)\pi t/\sin\pi t$ and letting $N \to \infty$ now gives $\sum_{n\geqslant 1}(\sin 2n\pi x)/n = \frac{\pi}{2} - \pi x$, a relation that Euler knew ([107] and [110, part 2, §92]). Integrating again yields the desired formula. (This solution was suggested by E. M. E. Wermuth [367]; Euler's original derivation did not meet modern standards of rigor.)

9.65 Since $a_0+a_1n^{-1}+a_2n^{-2}+\cdots = 1+(n-1)^{-1}(a_0+a_1(n-1)^{-1}+a_2(n-1)^{-2}+\cdots)$, we obtain the recurrence $a_{m+1} = \sum_k \binom{m}{k}a_k$, which matches the recurrence for the Bell numbers. Hence $a_m = \varpi_m$.

A slightly longer but more informative proof can be based on the fact that $1/(n-1)\ldots(n-m) = \sum_k {k \brace m}/n^k$, by (7.47).

9.66 The expected number of distinct elements in the sequence 1, $f(1)$, $f(f(1))$, $\ldots$, when f is a random mapping of $\{1,2,\ldots,n\}$ into itself, is the function $Q(n)$ of exercise 56, whose value is $\frac{1}{2}\sqrt{2\pi n}+O(1)$; this might account somehow for the factor $\sqrt{2\pi n}$.

9.67 It is known that $\ln\chi_n \sim \frac{3}{2}n^2\ln\frac{4}{3}$; the constant $e^{-\pi/6}$ has been verified empirically to eight significant digits.

9.68 This would fail if, for example, $e^{n-\gamma} = m+\frac{1}{2}+\epsilon/m$ for some integer m and some $0 < \epsilon < \frac{1}{8}$; but no counterexamples are known.