

Index

(*Graffiti have been indexed too.*)

WHEN AN INDEX ENTRY refers to a page containing a relevant exercise, the answer to that exercise (in Appendix A) might divulge further information; an answer page is not indexed here unless it refers to a topic that isn't included in the statement of the relevant exercise. Some notations not indexed here (like $x^{\underline{n}}$, $\lfloor x \rfloor$, and $\langle \frac{n}{m} \rangle$) are listed on pages x and xi, just before the table of contents.

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