MAU22C00 - TUTORIAL 5

1) Use mathematical induction to prove the geometric series formula, which states that for any $a, r \in \mathbb{R}$ with $r \neq 1$ and any $n \in \mathbb{N}^*$,

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{(1-r^n)}{(1-r)}.$$

2) Where is the fallacy in the following argument by induction?

Statement: If p is an even number and $p \geq 2$, then p is a power of 2.

"Proof:" We give a proof using strong induction on the even number p. Denote by $P(n)$ the statement "if n is an even number and $n \geq 2$, then $n = 2^j$, where $j \in \mathbb{N}$."

Base case: Show $P(2)$. $2 = 2^1$, so 2 is indeed a power of 2.

Inductive step: Assume $p > 2$ and that $P(n)$ is true for every n such that $2 \leq n < p$ (the strong induction hypothesis). We have to show that $P(p)$ also holds. We consider two cases:

Case 1: p is odd, then there is nothing to show.

Case 2: p is even. Since $p \geq 4$ and p is an even number, we can write $p = 2n$ with $2 \leq n < p$. By the inductive hypothesis, $P(n)$ holds, so we conclude that $n = 2^j$ for some $j \in \mathbb{N}$. Since $p = 2n = 2 \times 2^j = 2^{j+1}$, we conclude that $P(p)$ also holds.

3) In the 1730's, the "Grande Loge" of Freemasons in Paris was a highly secretive society following some rather bizarre rules. Each of the freemasons in the lodge had shaved one other member. No freemason in the lodge had ever shaved himself. Furthermore, no freemason was ever shaved by more than one member of the lodge. There was one freemason who had never been shaved by any other member of the lodge. The number and identity of the freemasons in the lodge was kept secret. One rumour circulating in Paris at that time was that there were fewer than a hundred freemasons in the "Grande Loge." Another rumour put the number at over a hundred. Which one of the two rumours is true? Justify your answer.