MAU22C00: ASSIGNMENT 2 SOLUTIONS

1) (10 points) Let $A = \mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R},\ 1 \leq i \leq n\}$. For $x, y \in A$, $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, xQy if and only if $\forall\, i$, $1 \leq i \leq n$, $x_i = y_i$ or $\exists\, i$ with $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j$ $\forall\, j$, $j < i$. Determine:

(i) Whether or not the relation Q is *reflexive*;
(ii) Whether or not the relation Q is *symmetric*;
(iii) Whether or not the relation Q is *anti-symmetric*;
(iv) Whether or not the relation Q is *transitive*;
(v) Whether or not the relation Q is an *equivalence relation*;
(vi) Whether or not the relation Q is a *partial order*.

Justify your answers.

Solution: Note that xQy if and only if one of two mutually exclusive things happens:

(a) $\forall\, i$, $1 \leq i \leq n$, $x_i = y_i$, namely $x = y$

(b) $\exists\, i$ with $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j$ $\forall\, j$, $j < i$.

(i) The relation Q is reflexive because $\forall\, x \in \mathbb{R}^n$ xQx holds (we are in scenario (a) as $x_i = x_i$ $\forall\, i$, $1 \leq i \leq n$).

(ii) To check symmetry, assume xQy for some $x, y \in \mathbb{R}^n$. If we are in scenario (a), namely $x = y$, then clearly yQx will hold as well. If we are in scenario (b), however, it would be impossible for yQx to hold as well since $x_i < y_i$ and $y_i < x_i$ cannot hold simultaneously. To justify properly, it is best to exhibit a counterexample to symmetry by giving an example of $x, y \in \mathbb{R}^n$ such that xQy holds, but yQx is false. Indeed, take $x = (1, 0, \dots, 0)$ and $y = (2, 0, \dots, 0)$. Since $x_1 < y_1$ and the condition $x_j = y_j$ $\forall\, j$, $j < 1$ is vacuously satisfied as there are no indices j less than 1, xQy (scenario (b)). Clearly, yQx is false as both scenarios (a) and (b) fail in this case.

(iii) To check anti-symmetry, assume xQy and yQx for some $x, y \in \mathbb{R}^n$. As $x_i < y_i$ and $y_i < x_i$ cannot hold simultaneously, we conclude that xQy and yQx hold because we are in scenario (a). Therefore, $x = y$, so Q is indeed anti-symmetric.

(iv) To check transitivity, assume xQy and yQz for some $x, y, z \in \mathbb{R}^n$. There are two mutually exclusive scenarios for xQy and two mutually exclusive scenarios for yQz. Therefore, we must consider a total of four cases:

Case 1: xQy due to scenario (a) and yQz due to scenario (a). Then $x = y$ and $y = z$, so $x = z$, hence xQz via scenario (a).

Case 2: xQy due to scenario (a) and yQz due to scenario (b). Then $x = y$ and $\exists\, i$ with $1 \leq i \leq n$ such that $y_i < z_i$ and $y_j = z_j \;\forall\, j,\, j < i$. Since $x = y$, the latter implies that $x_i < z_i$ and $x_j = z_j \;\forall\, j,\, j < i$. Therefore, xQz via scenario (b).

Case 3: xQy due to scenario (b) and yQz due to scenario (a). Thus, $\exists\, i$ with $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j \;\forall\, j,\, j < i$ and $y = z$. We conclude that $x_i < z_i$ and $x_j = z_j \;\forall\, j,\, j < i$, so xQz via scenario (b).

Case 4: xQy due to scenario (b) and yQz due to scenario (b). xQy due to scenario (b) means $\exists\, i$ with $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j \;\forall\, j,\, j < i$. yQz due to scenario (b) means $\exists\, k$ with $1 \leq k \leq n$ such that $y_k < z_k$ and $y_l = z_l \;\forall\, l,\, l < k$. This case then splits into two subcases based on how i and k relate to each other:

Case 4.1: $i \leq k$. Then $x_i < y_i \leq z_i$, namely $x_i < y_i < z_i$ if $i = k$ and $x_i < y_i = z_i$ if $i < k$. Note that $x_j = y_j \;\forall\, j,\, j < i$ and $y_l = z_l \;\forall\, l,\, l < k$. Since $i \leq k$, $x_j = y_j = z_j \;\forall\, j,\, j < i \leq k$. Therefore, $x_i < z_i$ and $x_j = z_j$ $\forall\, j,\, j < i$, so xQz via scenario (b).

Case 4.2: $k < i$. Then $y_k < z_k$, but $x_k = y_k$ as $k < i$. Therefore, $x_k < z_k$. Moreover, $y_l = z_l \;\forall\, l,\, l < k$ and $x_l = y_l$ also for every l satisfying $l < k$ because $l < k < i$, hence $l < i$. We conclude that $x_l = z_l \;\forall\, l,\, l < k$ and $x_k < z_k$, so xQz via scenario (b).

In all cases and subcases, the conclusion is that xQz, so Q is indeed transitive.

(v) An equivalence relation is a relation that is reflexive, symmetric, and transitive. Q is reflexive and transitive, but it is not symmetric. We thus conclude that Q is not an equivalence relation.

(vi) A partial order is a relation that is reflexive, anti-symmetric, and transitive. Q satisfies all three properties, so it is a partial order.

Marking rubric: 10 marks total. 2 marks for each of the first four parts (1 mark for the answer and 1 mark for the justification); 1 mark for each of the last two parts.

2) (10 points) Use mathematical induction to prove that for all $n \geq 7$, $n! > 3^n$.

Solution: Note that we are asked to prove the statement $n! > 3^n$ for all $n \geq 7$, so this is an induction where the base case takes place at $n = 7$.

Base case: $n = 7$. Then $7! = 7 \cdot 6 \cdot \dots \cdot 2 \cdot 1 = 5040 > 2187 = 3^7$ as needed.

Inductive case: We assume $n! > 3^n$ and seek to prove $(n+1)! > 3^{n+1}$. Since $(n+1)! = (n+1) \cdot n!$, we multiply $n! > 3^n$ on both sides by $n+1 \geq 7+1 = 8 > 0$ to obtain that $(n+1) \cdot n! > (n+1) \cdot 3^n > 3 \cdot 3^n = 3^{n+1}$ as $n+1 > 3$. We conclude that $(n+1)! > 3^{n+1}$ as needed.

Marking rubric: 10 marks total. 1 mark for figuring out at which n the argument starts, 2 marks for the base case, and 7 marks for the inductive case.

3) (20 points) (a) Let $\{C_n\}_{n=1,2,\dots} = \{C_1, C_2, \dots\}$ be a sequence of sets satisfying that $C_n \subseteq C_{n+1}$ $\forall\, n \geq 1$. Prove by mathematical induction that $C_m \subseteq C_n$ whenever $m < n$.

(b) Recall that the graph of a function $f : A \to B$ is given by

$$\Gamma(f) = \{(x, y) \bigm| x \in A \text{ and } y = f(x)\} \subseteq A \times B.$$

Let $Funct(A, B)$ the set of all functions $f : \tilde{A} \to \tilde{B}$ such that $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$. We define a relation on $Funct(A, B)$ as follows:
$\forall f, g \in Funct(A, B)$ $f \subseteq g$ iff $\Gamma(f) \subseteq \Gamma(g)$.
Prove that this relation is a partial order on $Funct(A, B)$.

(c) Let $\{f_n\}_{n=1,2,\dots} = \{f_1, f_2, \dots\}$ be a sequence of functions in $Funct(A, B)$ satisfying that $f_n \subseteq f_{n+1}$ for every $n \geq 1$. Since functions are in one-to-one correspondence with their graphs, we identify $\bigcup_{n \in \mathbb{N}} f_n$ with $\bigcup_{n \in \mathbb{N}} \Gamma(f_n)$. Using part (a), prove that $\bigcup_{n \in \mathbb{N}} f_n$ is a function and $\bigcup_{n \in \mathbb{N}} f_n \in Funct(A, B)$.

(d) For every $f \in Funct(A, B)$, let $Dom(f)$ be the domain of f, namely if $f : \tilde{A} \to \tilde{B}$ with $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$, $Dom(f) = \tilde{A}$. Prove that $Dom\left(\bigcup_{n \in \mathbb{N}} f_n\right) = \bigcup_{n \in \mathbb{N}} Dom(f_n)$ for every sequence of functions $\{f_n\}_{n=1,2,\dots} = \{f_1, f_2, \dots\}$ in $Funct(A, B)$ satisfying that $f_n \subseteq f_{n+1}$ for every $n \geq 1$.

Solution: (a) Our induction starts at m. Therefore:

Base case: $n = m + 1$. By our hypothesis, $C_m \subseteq C_{m+1}$. Therefore, $C_m \subseteq C_n$ as needed.

Inductive case: We assume $C_m \subseteq C_n$ and seek to prove $C_m \subseteq C_{n+1}$. By our hypothesis, $C_n \subseteq C_{n+1}$. We combine this statement with the inductive hypothesis $C_m \subseteq C_n$ to conclude that $C_m \subseteq C_n \subseteq C_{n+1}$. Since inclusion of sets is transitive as we showed in the unit on set theory earlier this term, we conclude that $C_m \subseteq C_{n+1}$ as needed.

(b) To prove that the relation $\subseteq$ on $Funct(A, B)$ is a partial order, we must show it is reflexive, anti-symmetric, and transitive. We do so as follows:

Reflexivity: $\forall f \in Funct(A, B)$ $\Gamma(f) = \Gamma(f)$ from the reflexivity property of the equality of sets. Clearly, $\Gamma(f) = \Gamma(f) \implies \Gamma(f) \subseteq \Gamma(f)$ as $\subseteq$ permits equality of the two sets involved. We conclude that $f \subseteq f$.

Anti-symmetry: For $f, g \in Funct(A, B)$ assume that $f \subseteq g$ and $g \subseteq f$. $f \subseteq g \implies \Gamma(f) \subseteq \Gamma(g)$. $g \subseteq f \implies \Gamma(g) \subseteq \Gamma(f)$. Therefore, both $\Gamma(f) \subseteq \Gamma(g)$ and $\Gamma(g) \subseteq \Gamma(f)$, which is exactly the criterion for equality of sets (inclusion in both directions). We conclude that $\Gamma(f) = \Gamma(g)$. Since there is a one-to-one correspondence between a function in $Funct(A, B)$ and its graph, we conclude that $f = g$.

Transitivity: For $f, g, h \in Funct(A, B)$ assume that $f \subseteq g$ and $g \subseteq h$. $f \subseteq g \implies \Gamma(f) \subseteq \Gamma(g)$. $g \subseteq h \implies \Gamma(g) \subseteq \Gamma(h)$. Therefore, both $\Gamma(f) \subseteq \Gamma(g)$ and $\Gamma(g) \subseteq \Gamma(h)$ hold, and inclusion of sets is transitive as we showed in the unit on set theory earlier this term. We conclude that $\Gamma(f) \subseteq \Gamma(h)$, so $f \subseteq h$ as needed.

(c) We prove that $\bigcup_{n \in \mathbb{N}} f_n$ is a function by contradiction. Assume that $\bigcup_{n \in \mathbb{N}} f_n$ is not a function, hence it fails the definition of a function. Therefore, $\exists\, (x, y_1), (x, y_2) \in \Gamma\left(\bigcup_{n \in \mathbb{N}} f_n\right)$ with $y_1 \neq y_2$. $(x, y_1) \in \Gamma\left(\bigcup_{n \in \mathbb{N}} f_n\right) \implies \exists\, m$ such that $(x, y_1) \in \Gamma(f_m)$. $(x, y_2) \in \Gamma\left(\bigcup_{n \in \mathbb{N}} f_n\right) \implies \exists\, p$ such that $(x, y_2) \in \Gamma(f_p)$. We now distinguish three cases based on how $p, m \in \mathbb{N}$ related to each other:

Case 1: $p = m$. Then $(x, y_1), (x, y_2) \in \Gamma(f_p)$, but $y_1 \neq y_2$. Therefore, f_p is not a function. That is clearly a CONTRADICTION to our hypotheses.

Case 2: $m < p$. By part (a), $\Gamma(f_m) \subseteq \Gamma(f_p)$ as $f_n \subseteq f_{n+1}$ for every $n \geq 1$, hence $\Gamma(f_n) \subseteq \Gamma(f_{n+1})$. Therefore, $(x, y_1), (x, y_2) \in \Gamma(f_p)$, but $y_1 \neq y_2$. Therefore, f_p is not a function. That is clearly a CONTRADICTION to our hypotheses.

Case 3: $p < m$. By part (a), $\Gamma(f_p) \subseteq \Gamma(f_m)$ as $f_n \subseteq f_{n+1}$ for every $n \geq 1$, hence $\Gamma(f_n) \subseteq \Gamma(f_{n+1})$. Therefore, $(x, y_1), (x, y_2) \in \Gamma(f_m)$, but $y_1 \neq y_2$. Therefore, f_m is not a function. Once again, this is a CONTRADICTION to our hypotheses.

In all three cases, we have obtained a contradiction, which completes the proof that $\bigcup_{n\in\mathbb{N}} f_n$ is a function. To prove that $\bigcup_{n\in\mathbb{N}} f_n \in Funct(A, B)$, we must show that the domain of $\bigcup_{n\in\mathbb{N}} f_n$ is a subset of A and that the codomain of $\bigcup_{n\in\mathbb{N}} f_n$ is a subset of B. This assertion is easier to prove than it looks a priori. $\forall (x, y) \in \Gamma\left(\bigcup_{n\in\mathbb{N}} f_n\right)$, $\exists m \in \mathbb{N}^*$ such that $(x, y) \in \Gamma(f_m)$, but $f_m \in Funct(A, B)$, so $x \in A$ and $y \in B$. Clearly, $\bigcup_{n\in\mathbb{N}} f_n \in Funct(A, B)$.

(d) For every sequence of functions $\{f_n\}_{n=1,2,\ldots} = \{f_1, f_2, \ldots\}$ in $Funct(A, B)$ satisfying that $f_n \subseteq f_{n+1}$ for every $n \geq 1$, let the domain of f_n be some A_n such that $A_n \subseteq A$. We will prove that $Dom\left(\bigcup_{n\in\mathbb{N}} f_n\right) = \bigcup_{n\in\mathbb{N}} Dom(f_n)$ via inclusion in both directions.

"$\subseteq$" For all $\forall x \in Dom\left(\bigcup_{n\in\mathbb{N}} f_n\right) \subseteq A$, $\exists! y \in B$ (recall the $\exists!$ the "one and only one quantifier from the beginning of the term) such that $(x, y) \in \Gamma\left(\bigcup_{n\in\mathbb{N}} f_n\right)$ because we have proven in part (c) that $\bigcup_{n\in\mathbb{N}} f_n$ is a function and $\bigcup_{n\in\mathbb{N}} f_n \in Funct(A, B)$. $(x, y) \in \Gamma\left(\bigcup_{n\in\mathbb{N}} f_n\right) \implies \exists m$ such that $(x, y) \in \Gamma(f_m)$, but f_m is a function, so $x \in Dom(f_m) = A_m$. Therefore, $x \in \bigcup_{n\in\mathbb{N}} Dom(f_n)$ as needed.

"$\supseteq$" $\forall x \in \bigcup_{n\in\mathbb{N}} Dom(f_n)$, $\exists m \in \mathbb{N}^*$ and $\exists! y \in B$ such that $(x,y) \in \Gamma(f_m)$, but then $(x,y) \in \Gamma\left(\bigcup_{n\in\mathbb{N}} f_n\right)$. $\bigcup_{n\in\mathbb{N}} f_n$ is a function by part (c). Therefore, $x \in Dom\left(\bigcup_{n\in\mathbb{N}} f_n\right)$ as needed.

Marking rubric: 20 marks total. 4 marks for part (a) (1 mark for the base case and 3 marks for the inductive case). 6 marks for part (b) (2 marks for each of the three properties to be proven). 6 marks for part (c) (3 marks for proving that it is a function and 3 marks for proving that it is in $Funct(A,B)$). 4 marks for part (d) (2 marks for each direction of the inclusion).

4) (10 points) Let $\mathbb{R}[x]$ be the set of all polynomials in variable x with coefficients in $\mathbb{R}$. In other words,

$$\mathbb{R}[x] = \{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \mid n \in \mathbb{N} \text{ and } a_0, \ldots, a_n \in \mathbb{R}\}.$$

(a) Give three examples of elements of $\mathbb{R}[x]$.

(b) Prove that $(\mathbb{R}[x], +)$, $\mathbb{R}[x]$ with addition as the operation, is a semi-group.

(c) Is $(\mathbb{R}[x], +)$ a monoid? Justify your answer.

(d) Does $(\mathbb{R}[x], +)$ have invertible elements? If so, which of its elements are invertible? Justify your answer.

Solution: (a) Many examples can be given. Here are three: 0, $\pi x^2 + 1$, and $\sqrt{2}x - 5$.

(b) Adding two polynomials $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and $b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0$ in $\mathbb{R}[x]$ gives us another polynomial in $\mathbb{R}[x]$ because we factor out x^j for every j with $0 \leq j \leq \max\{n,m\}$ and its coefficient will be a_j+b_j, which is a real number for every j, since $a_j, b_j \in \mathbb{R}$ and $(\mathbb{R}, +)$ is a semigroup as proven in lecture. Therefore, $+$ on $\mathbb{R}[x]$ is a binary operation. Since $+$ is associative on $\mathbb{R}$ and we gather all coefficients of each x^j in turn, we conclude that $+$ is associative on $\mathbb{R}[x]$. Therefore, $(\mathbb{R}[x], +)$ is a semi-group.

(c) Clearly, $0 \in \mathbb{R}[x]$, and $\forall p \in \mathbb{R}[x]$, $p + 0 = 0 + p = p$. Therefore, the zero polynomial 0 is the identity element of $(\mathbb{R}[x], +)$. We conclude that $(\mathbb{R}[x], +)$ is a monoid.

(d) $\forall\, p \in \mathbb{R}[x]$, $p = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$. Take $q = -a_nx^n - a_{n-1}x^{n-1} - \cdots - a_1x - a_0$, where we reverse the signs of all coefficients of p unless they are equal to zero. Clearly, $p + q = 0$, so every element of $\mathbb{R}[x]$ is invertible.

Marking rubric: 10 marks total. 1 mark for part (a) and 3 marks for each of the parts (b)-(d).