STU22004 – Solutions for Sample Questions 10

Q1. If $m_x(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^{12}$, find $P(X > 10)$.

This is $m_x(t)$ for $B(12, 3/4)$

$$P(X>10) = \sum_{x=11}^{12} \binom{12}{x} \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{12-x} = \dots$$

Q2. If $X_i s$ are 3 iid RVs with $f(x) = 2x$, $0 < x < 1$, find $P(min(X_i s) > 0.5)$.

$$P(\min(X_i s) > 0.5) = P(\text{all } X_i s > 0.5)$$

$$= \left[\int_{0.5}^{1} 2x\, dx\right]^3 = \left[1 - (0.5)^2\right]^3$$

Q3. In a telecom transmission channel, there are some noise pulses which occur 3 times per minute with Poisson distribution. If we would like to send a message of 10 seconds using this channel, what is the probability that it does not be disturb by those noise pulses?

$r = 3/\text{min}$

$p = P(\text{No pulse in 10 seconds})$

$t = 10 \rightarrow \lambda = rt = 3 \times \frac{1}{6} \text{min} = 1/2$

$$\Rightarrow p = \frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-1/2}\left(\frac{1}{2}\right)^0}{0!} = e^{-1/2}$$

Q4. All screws used in a machine are manufactured by a single supplier, that could be either 1 or 2 with same chance. Probability that screws are faulty are q_1 and q_2 for suppliers 1 and 2, respectively. We investigate 2 screws; if the first one is defective, what is the probability that the second one is defective as well?

$$P(F_2 | F_1) = \frac{P(F_2 \cap F_1)}{P(F_1)} = \frac{\sum_{i=1}^{2} P(F_2 \cap F_1 | \text{sup } i) P(\text{sup } i)}{\sum_{i=1}^{2} P(F_1 | \text{sup } i) P(\text{sup } i)}$$

$$= \frac{(q_1^2 + q_2^2)\cancel{0.5}}{(q_1 + q_2)\cancel{0.5}}$$

Q5. You have n coins. You flip all and X of them are head; take them and flip the rest again. What is the probability that at the end of the second round, at the latest, all coins show heads?

P(all head in Maximum two flips)

$= \sum_{x=0}^{n}$ P(x heads in the first flip $\cap$ $n-x$ heads in the second flip)

$$= \sum_{x=0}^{n} \binom{n}{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{n-x} \cdot \left(\frac{1}{2}\right)^{n-x} = \left(\frac{1}{2}\right)^{n} \sum_{x=0}^{n} \binom{n}{x} \left(\frac{1}{2}\right)^{n-x}$$

$$= \left(\frac{1}{2}\right)^{n} \sum_{x=0}^{n} \binom{n}{x} \left(\frac{1}{2}\right)^{n-x} (1)^{x} = \left(\frac{1}{2}\right)^{n} \left(1 + \frac{1}{2}\right)^{n}$$

$$= \left(\frac{3}{4}\right)^{n}$$

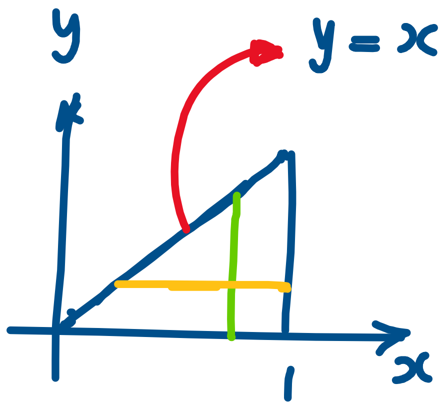
Q6. If X and Y have the following joint distribution, find $E[X\,|Y=0]$.

Y \ X	0	1	$f(y)$
0	1/8	3/8	$1/2$
1	1/3	1/6	$1/2$
$f(x)$	$\frac{11}{24}$	$\frac{13}{24}$	1

$$f(x|y=0) = \frac{f_{x,y}(x,0)}{f_y(0)} = \begin{cases} \frac{1/8}{1/2} = \frac{1}{4} & x=0 \\ \frac{3/8}{1/2} = \frac{3}{4} & x=1 \end{cases}$$

$$E[X|Y=0] = \sum x f(x|y=0) = 0 \times 1/4 + 1 \times \frac{3}{4} = \frac{3}{4}$$

Q7. For $f(x,y) = 8xy\,,\ \ 0 < x < 1,\ 0 < y < x$, find $Cov(X,Y)$.



$$f(x) = \int_0^x 8xy\,dy = 4x^3 \qquad 0 < x < 1$$

$$E[X] = \int_0^1 x \cdot 4x^3\,dx = \frac{4}{5}$$

$$f(y) = \int_y^1 8xy\,dx = 4y(1-y^2) \qquad 0 < y < 1$$

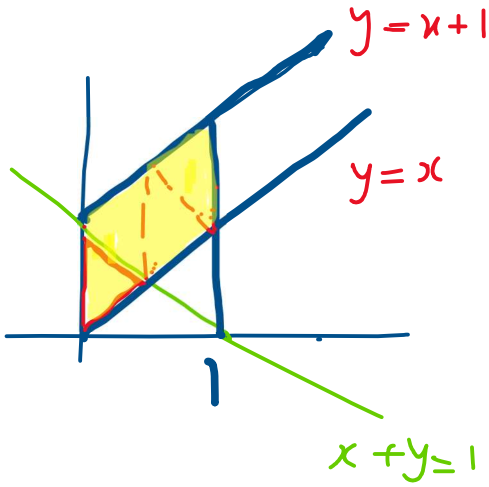
$$E[Y] = \int_0^1 y \cdot 4y(1-y^2)\,dy = \ldots$$

$$E[XY] = \int_0^1 \int_0^x xy\ 8xy\ dy\,dx = 8 \int_0^1 \int_0^x x^2 y^2\,dy\,dx$$

$$= 8 \int_0^1 x^2 \cdot \frac{x^3}{3}\,dx = \frac{4}{9}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

Q8. If $X \sim U(0,1)$ and $Y|x \sim U(x, x+1)$, find $P(X+Y<1)$.



$$f(x) = \frac{1}{1-0} = 1 \qquad 0 < x < 1$$

$$f(y|x) = \frac{1}{(x+1)-x} = 1 \qquad 0 < y < 1$$

$$f(x,y) = f(y|x)\, f(x) = 1 \times 1 = 1$$

$\rightarrow$ uniform in yellow region

$\Rightarrow P(X+Y<1) = P(\text{The red triangle}) = \frac{1}{4}$

Q9. If $X \sim U(0,1)$ and $Y|x \sim B(n, x)$, find $E[Y]$ and $Var[Y]$.

$$E[Y] = E[E(Y|X)] = E[nX] = nE(X) = \frac{n}{2}$$

$$Var(Y) = E[Var(Y|X)] + Var(E[Y|X])$$

$$= E[nX(1-X)] + var[nX]$$

$$= nE[X] - nE[X^2] + n^2 var(X)$$

$$= nE[X] - n[var(X) + E^2[X]] + n^2 var[X]$$

$$= n\left(\frac{1}{2}\right) - n\left[\frac{1}{12} + \left(\frac{1}{2}\right)^2\right] + n^2\left(\frac{1}{12}\right) = \ldots$$

Q10. If $E[Y|x] = 7 - \frac{1}{4}x$ and $E[X\,|y] = 10 - y$, find $Corr(X,Y)$.

$$E[Y|x] = E[Y] + \rho_{x,y}\frac{\sigma_y}{\sigma_x}(x - E(x))$$

$$E[X|y] = E[X] + \rho_{x,y}\frac{\sigma_x}{\sigma_y}(y - E[y])$$

$$\rho_{x,y}\frac{\sigma_y}{\sigma_x} \cdot \rho_{x,y}\frac{\sigma_x}{\sigma_y} = \rho_{x,y}^2 = \left(-\frac{1}{4}\right)(-1) = 1/4$$

$$\Rightarrow \rho_{x,y} = -1/2$$

Q11. A system has two components, main and spare ones, both with Exponential lifetime with parameter λ. What is the expected value of percentage of the time that the system works with the spare component?

The question is asking for $E\left[\frac{X}{X+Y}\right]$.

See the solution for Q4, Sample questions 7.

$$\frac{X}{X+Y} \sim U(0,1) \rightarrow E\left(\frac{X}{X+Y}\right) = 1/2$$

Q12. You roll two dice repeatedly until getting their sum equal 7. What is the probability that the required number of rolls is odd?

$$P(7) = P\big((1,6), (6,1), (5,2), (2,5), (4,3), (3,4)\big) = \frac{1}{6}$$

X : Number of rolls

$$f(x) = \left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right) \qquad x = 1, 2, 3, \ldots$$

$$P(X = \text{odd}) = \sum_{k=0}^{\infty} \left(\frac{5}{6}\right)^{(2k+1)-1} \left(\frac{1}{6}\right)$$

$$= \left(\frac{1}{6}\right) \sum_{k=0}^{\infty} \left(\frac{25}{36}\right)^{k} = \frac{1/6}{1 - \frac{25}{36}} = \frac{6}{11}$$

Q13. If $X_i s$ are iid Geometric RVs, find the probability distribution of $Y = min(X_i s)$.

$$f_Y(y) = P(Y = y) = P(\min(X_i s) = y)$$

$$= \sum_{j=1}^{n} \binom{n}{j} (pq^{y})^{j} \left(\sum_{x=y+1}^{\infty} pq^{x}\right)^{n-j}$$

$$= \sum_{j=1}^{n} \binom{n}{j} (pq^{y})^{j} \left(\frac{\cancel{p}q^{y+1}}{\cancel{1-q}}\right)^{n-j}$$

$$= q^{ny} \underbrace{\sum_{j=1}^{n} \binom{n}{j} p^{j} q^{n-j}}_{1-q^{n}} = (q^{n})^{y}(1-q^{n}) = Q^{y}(1-Q)$$

Geometric

Q14. A box contains 5 red and 10 black chips. We take 4 chips randomly and without replacement. If the number of taken red and black chips are shown by U and V, find $Corr(X, Y)$.

$U + V = 4$

By having $U = u$, you can precisely tell me $V = 4 - u$. This means 100% correlation.

Now, if $u \uparrow$ $v \downarrow$ $\Rightarrow \rho_{X,Y} = -1$

Q15. Mary flips 3 fair coins. What is the probability that she gets 3 heads for the second time in the 5th flip?

$P(3\text{ heads}) = \frac{1}{8}$ 3heads : S

$\boxed{1S, 3F}$ S $\Rightarrow \binom{4}{1}\left(\frac{1}{8}\right)^1\left(\frac{7}{8}\right)^3 \cdot \left(\frac{1}{8}\right)$