MAU22C00 - TUTORIAL 4 SOLUTIONS

1) Let A be a set, and let $\mathcal{A} = \{A_\alpha \mid \alpha \in I\}$, where I is an indexing set, be any partition of the set A. Define a relation R on A as follows: $x, y \in A$ satisfy xRy iff $x, y \in A_\alpha$ for some $\alpha \in I$. In other words, xRy iff x and y belong to the same set of the partition. Prove that R is an equivalence relation and that the partition R defines on A is precisely the given partition $\mathcal{A}$.

(Hint: Recall we discussed in lecture the one-to-one correspondence between partitions and equivalence relations, and this is the proof direction I sketched in lecture without providing the details.)

Solution: First, let us prove R is an equivalence relation:

Reflexivity: For any $x \in A$, since $\mathcal{A} = \{A_\alpha \mid \alpha \in I\}$ is a partition of A, there exists $\alpha \in I$ such that $x \in A_\alpha$. The element x is in the same set A_α as itself, so xRx.

Symmetry: If xRy, then by definition $x, y \in A_\alpha$ for some $\alpha \in I$, i.e. x and y belong to the same set of the partition. Therefore, yRx holds as well.

Transitivity: If xRy, then by definition $x, y \in A_\alpha$ for some $\alpha \in I$. If yRz, then z belongs to the same set of the partition as y, namely $z \in A_\alpha$ for the same α. Thus, $x, y, z \in A_\alpha$, which means xRz holds as well.

The partition determined by R is exactly $\mathcal{A}$: If $x \in A_\alpha$, then the equivalence class of x given by $[x]_R = A_\alpha$ by the very definition of R. Since $\mathcal{A}$ is a partition of A and it consists of the set of equivalence classes determined by the relation R, we conclude that R determines the partition $\mathcal{A}$ as needed.

2) (From the 2016-2017 Annual Exam) Let $f : [-2, 2] \to [-15, 1]$ be the function defined by $f(x) = x^2 + 3x - 10$ for all $x \in [-2, 2]$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers.

Injectivity: $f(x) = x^2 + 3x - 10 = (x - 2)(x - 5)$ This function is not injective on the interval $[-2, 2]$. Acceptable justifications: drawing the graph, providing two values $x_1, x_2 \in [-2, 2]$, $x_1 \neq x_2$ such that

$f(x_1) = f(x_2)$, applying Rolle's theorem (noticing that $f'(x) = 2x + 3$ so $f'\left(-\frac{3}{2}\right) = 0$, and $\frac{3}{2} \in [-2, 2]$), etc.

Surjectivity: $f(x) = x^2 + 3x - 10$ is not surjective on the interval $[-2, 2]$. Acceptable justifications: drawing the graph, providing a value in $[-15, 1]$ that $f(x)$ does not assume, showing the minimum value occurs at $\frac{3}{2}$, where $f\left(\frac{3}{2}\right) = -12.25 > -15$, etc.