MAU22C00: TUTORIAL 15 PROBLEMS
MINIMAL SPANNING TREES

1) (Annual Exam Trinity Term 2018) Consider the connected undirected graph with vertices A, B, C, D, E, F, G, H, I, J, K, and L, and with edges listed with associated costs in the following table:

CF	JK	IJ	AD	CH	EI	BL	CE	HG	BH
2	2	3	3	3	4	5	6	6	7
EF	FJ	GK	CD	DE	HL	AC	FH	EJ	AB
8	8	9	9	10	10	10	11	12	14

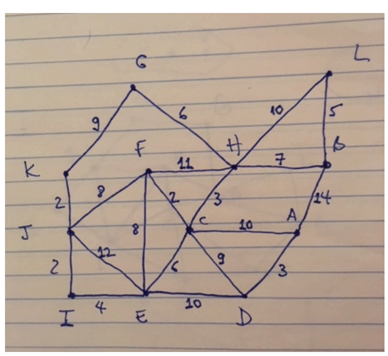
(a) Draw the graph and label each edge with its cost.
(b) Determine the minimum spanning tree generated by Kruskal's Algorithm, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, write down the edge that is added.

2) In the previous problem, how many distinct ways can the edges of the graph be ordered in non-decreasing order of cost, i.e. how many different non-decreasing queues are there for the edges of the graph? Justify your answer.

3) Would every non-decreasing queue from problem 2 give a different minimal spanning tree when Kruskal's Algorithm is applied? Justify your answer by either a proof or a counterexample.

4) Prove that every nontrivial tree is a bipartite graph.

Solution: 1)(a) Here is the graph:



(b) The edges are added in the following order: CF, JK, IJ, AD, CH, EI, BL, CE, HG, BH, and CD.

2) Two edges have cost 2, so they can be reshuffled (for a total of 2! possibilities), three edges have cost 3, two edges have cost 6, two edges have cost 8, two edges have cost 9, and three edges have cost 10. We thus have

$$2! \times 3! \times 2! \times 2! \times 2! \times 3! = 36 \times 16 = 576$$

ways of obtaining a non-decreasing queue of edges.

3) Not necessarily! Here is a counterexample: Queue CF, JK, IJ, AD, CH, EI, BL, CE, HG, BH, EF, FJ, GH, CD, DE, HL, AC, FH, EJ, AB and queue CF, JK, IJ, AD, CH, EI, BL, CE, HG, BH, FJ, EF, GH, CD, DE, HL, AC, FH, EJ, AB with edges EF and FJ exchanged give the same minimal spanning tree.

4) We need to prove that the vertices V of a tree (V, E) can be partitioned into two sets V_1 and V_2 with $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$, and every edge in E has a vertex in V_1 and a vertex in V_2. We do so by induction:

Base Case: The smallest nontrivial tree has 2 vertices. We call them v and w. Since it is a tree, there is an edge between v and w; otherwise, the graph would not be connected. We place vertex v in set V_1 and vertex w in set V_2. Since the only edge vw goes from a vertex in V_1 to a vertex in V_2, this tree with 2 vertices is indeed a bipartite graph as needed.

Inductive Step: Assume that every tree with n vertices is a bipartite graph. We seek to prove the same for every tree with $n+1$ vertices. Let (V, E) be a tree with $n+1$ vertices. By a theorem proven in lecture 38, (V, E) has at least one pendant vertex. Let v be such a pendant vertex. Since v has degree 1, there is only one edge incident to v. Let us call that edge vw, where w is the other endpoint of this edge. Consider the subgraph (V', E') of (V, E) obtained by deleting vertex v and edge vw. In other words, $V' = V \setminus \{v\}$ and $E' = E \setminus \{vw\}$. (V', E') is clearly still connected and acyclical hence a tree because v was a pendant vertex of the original graph and vw the only edge incident to it. The inductive hypothesis thus applies to (V', E'), which is therefore bipartite. Without loss of generality, assume $w \in V_1$. Place v in V_2. Since vw goes from a vertex in V_1 to a vertex in V_2 and (V', E') is itself bipartite, we conclude (V, E) must be bipartite as well.