Sample Questions 3 _ Sol

Q1. $\sum_{k=0}^{\infty}\left[P(a_k)+P(b_k)\right] = 1 \Rightarrow \sum_{k=0}^{\infty} \frac{2A}{(k+1)(k+2)} = 1$

$\Rightarrow 2A \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = 2A \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) = 1$

$\Rightarrow 2A\left[\left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{\cdots}\right] = 1$

$\Rightarrow 2A(1) = 1 \quad \Rightarrow A = 1/2$

Q2. $E[XY] = E[X(aX+b)] = E[aX^2+bX]$

$= aE(X^2) + bE[X] = a\left[Var(X) + E^2[X]\right] + bE[X]$

$= a(2+1) + b(1) = 3a+b$

Q4. $F(-\infty) = 0 \Rightarrow A = 0 \qquad F(+\infty) = 1 \Rightarrow D = 1$

$P(X=0) = \lim_{x\to 0^+} F(x) - \lim_{x\to 0^-} F(x) = C - B = \frac{1}{6}$

$P\left(-\frac{1}{3} < X < 3/2\right) = P\left(-\frac{1}{2} < X \leq 3/2\right) = F\left(\frac{3}{2}\right) - F\left(-\frac{1}{2}\right)$

$= C - B = \frac{1}{6}$

Q3. $P(X=x) = f(x) = \dfrac{\binom{100}{x}}{2^{100}}$ $\quad x = 0, 1, \dots, 100$

This probability function is symmetric as

$\binom{100}{x} = \binom{100}{100-x}$

Therefore $E[X] = \dfrac{0+100}{2} = 50$

Q5. $\theta + \frac{1}{3} - \theta + p + p + p + p = 1 \Rightarrow p = \frac{1}{6}$

$E(X) = 6 \times \theta + (5+4+3+2)\frac{1}{6} + (1)(\frac{1}{3} - \theta) = \dots$

Q6. $E(X^2) = 0^2 f(0) + 1^2 f(1) + (-1)^2 f(-1) = f(1) + f(-1)$

$= 1 - f(0) = 1 - \frac{1}{4} = 3/4$

Q7. $P(X=1) = \lim_{x \to 1^+} F(x) - \lim_{x \to 1^-} F(x) =$

$= 1 - \frac{1}{2^2} + \frac{1}{2^2} - \left[1 - \frac{1}{2^2} - \frac{1}{2^1}\right] = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Q8. This is the MGF of a Binomial Dist

with $n = 12$ $p = 3/4$ $\Rightarrow Var(X) = npq = 12 \times \frac{3}{4} \times \frac{1}{4}$

$= \frac{9}{4}$

Q9. $X_i = \begin{cases} 1 & \text{if bus stops at station } i \\ 0 & \text{otherwise} \end{cases}$

$P(X_i=1) = P(\text{at least 1 passenger gets off at station } i)$

$= 1-\left(1-\frac{1}{n}\right)^k$

Total Number of bus stops $= X = X_1+X_2+\dots+X_n$

$E(X) = nE(X_i) = n\left[(1)\left(1-\left(\frac{1}{n}\right)^k\right)+0\left(1-\left(\frac{1}{n}\right)^k\right)\right]$

$= n\left[1-\left(\frac{1}{n}\right)^k\right]$

Q10. $Y = \operatorname{sgn}(X) = \begin{cases} +1 & x>0 \\ -1 & x\leq 0 \end{cases}$

$E(Y) = (-1)P(Y=-1)+(1)P(Y=+1)$

$= (-1)P(X\leq 0)+(1)\cdot P(X>0)$

$= (-1)F_X(0)+(1)\left[1-P(X\leq 0)\right]$

$= (-1)F_X(0)+\left[1-F_X(0)\right] = 1-2F_X(0)$