CSU22012: Data Structures and Algorithms II

Merge sort and quick sort

Ivana.Dusparic@scss.tcd.ie

Divide and Conquer Sorting

- Divide problem into smaller parts
- Independently solve the parts
- Combine these solutions to get overall solution
- 2 common approaches:
  - Divide array into two halves, *recursively* sort left and right halves, then *merge* two halves → Mergesort
  - Partition array into items that are “small” and items that are “large”, then recursively sort the two sets → Quicksort

Merge vs quick

- In Java, Arrays.sort() uses **Quick**Sort for sorting primitives and **Merge**Sort for sorting Arrays of Objects.
  - Why does it matter for Objects and not for primitive data types?

Mergesort

Merge sort

- Top down merge sort
  - Recursive
  - Divide array in 2 halves, sort each array recursively, merge the arrays
- Bottom up merge sort
  - Iterative
  - Iterate through array merging subarrays of size 1, size 2, 4, 8, etc

Top down merge sort

a[]

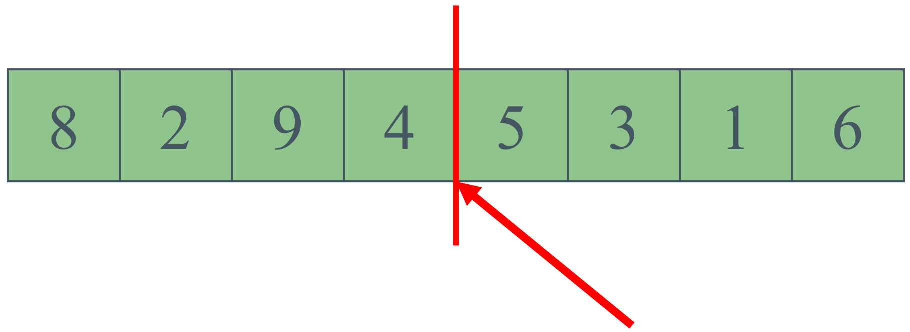
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Bottom up merge sort

a[i]

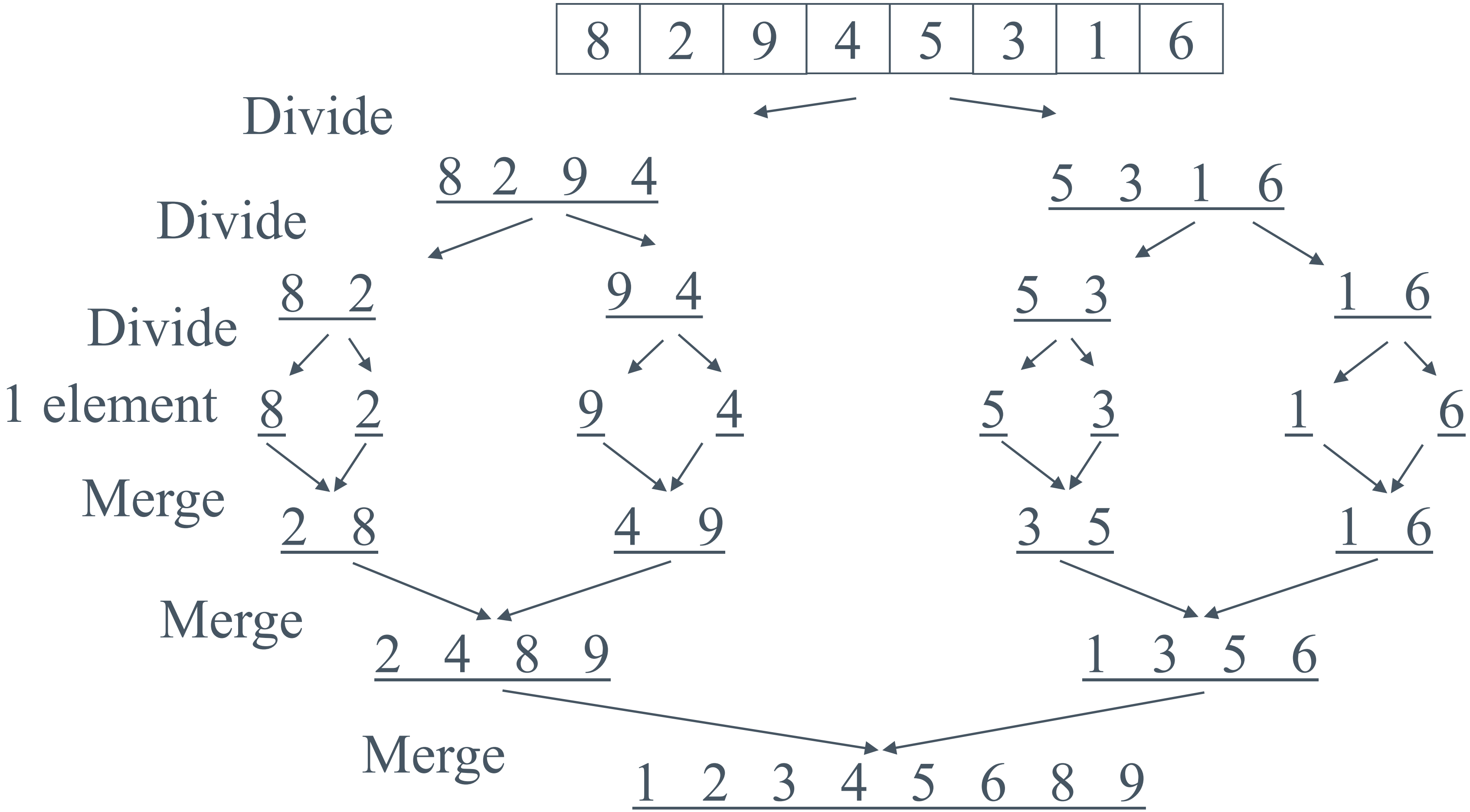
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E
E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L
E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L
E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L
E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L
E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P
E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Top down merge sort



- Divide it in two at the midpoint
- Conquer each side in turn (by recursively sorting)
- Merge two halves together

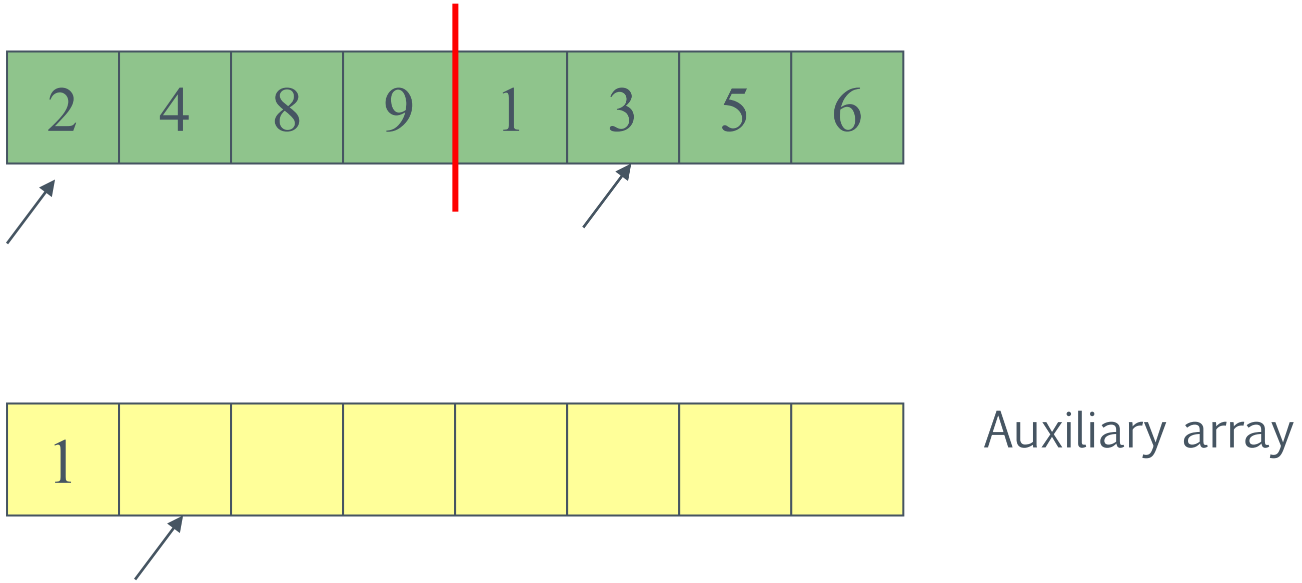
Top down merge sort



Top down merge sort

› The merging requires an auxiliary array

  – Requires extra space



Top down merge sort Java implementation

- What methods do we need?

- public method that passes in array to be sorted

public static void sort (Comparable [] a)

- Recursive method with original and auxiliary arrays, and indices of the subarray to be sorted

private static void sort (Comparable [] a, Comparable [] aux, int lo, int hi)

- Merge method, to merge sorted subarrays, with the 2 arrays to be merged, lowest, highest and midpoint indices

private static void merge (Comparable [] a, Comparable [] aux, int lo, int mid, int hi)

Top down merge sort

- Create an auxiliary array of the same size as the original one
- Kick off recursion by passing in 0 and array length-1 as indices (ie the full original array)

```
public static void sort(Comparable[] a)
{
   Comparable[] aux = new Comparable[a.length];
   sort(a, aux, 0, a.length - 1);
}
```

Top down merge sort

- Recursive method
  - Repeat until lo and hi are equal, ie get to array of length 1
  - Note: mid = lo+ (hi-lo)/2 to avoid integer overflow

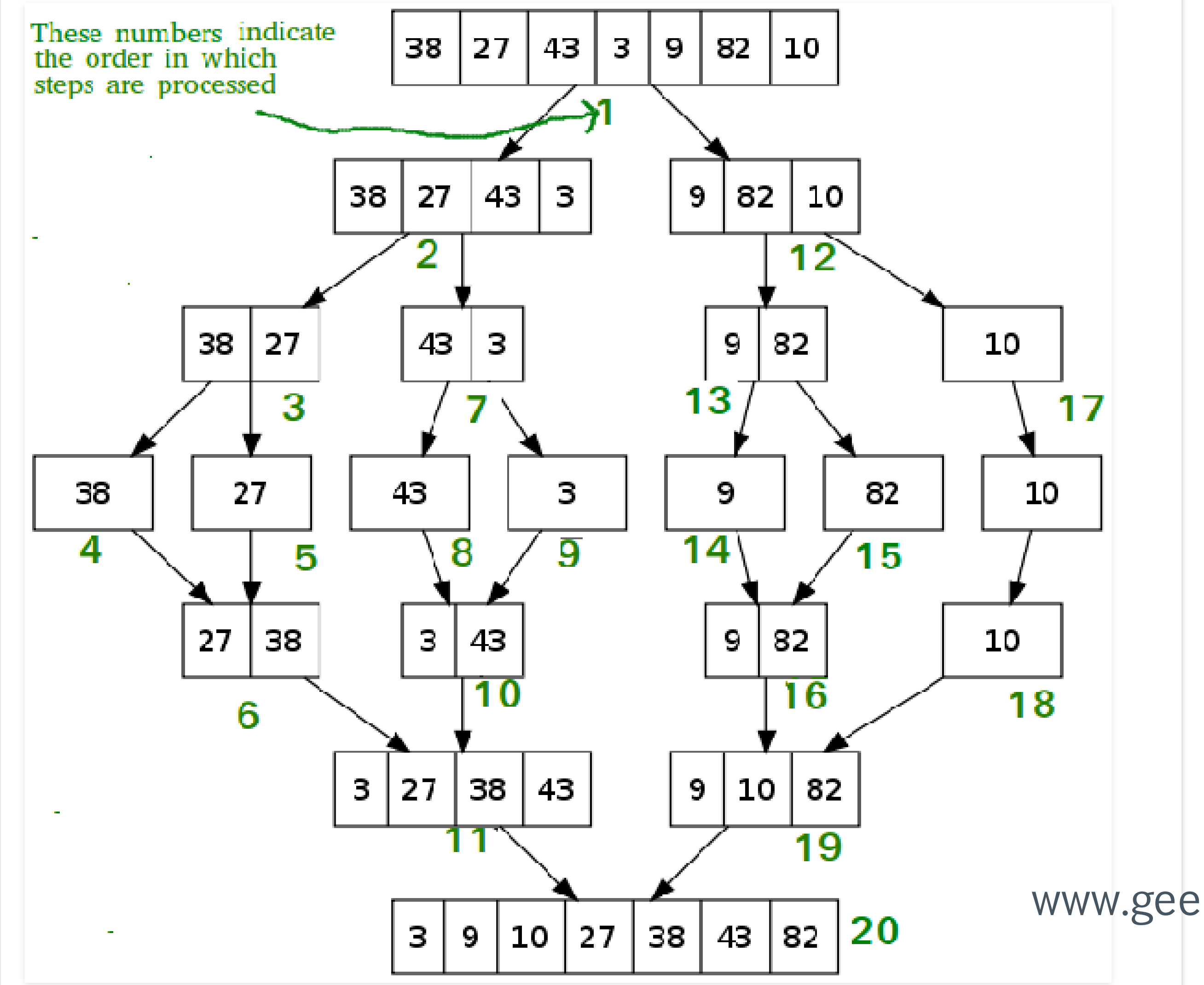
```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort(a, aux, lo, mid);
   sort(a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

Top down merge sort

- Merge method
- Copy the original array into auxiliary one, and then merge elements back into the original one in sorted order

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{

   for (int k = lo; k <= hi; k++)                                         copy
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];                     merge
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }
}
```

Top down merge sort



www.geeksforgeeks.org/merge-sort/

Top down merge sort

lo, hi	a[] 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 0, 1, 3)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 4, 5)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 6, 6, 7)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 5, 7)	E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
merge(a, aux, 0, 3, 7)	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
merge(a, aux, 8, 8, 9)	E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
merge(a, aux, 10, 10, 11)	E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
merge(a, aux, 8, 9, 11)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, aux, 12, 12, 13)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, aux, 14, 14, 15)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
merge(a, aux, 12, 13, 15)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, aux, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge(a, aux, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Merge sort exercise

Illustrate merge sort by showing table partitioning and numbering the order in which steps happen (like in the lecture notes example)

7	8	2	4	5	4	3	1

Merge sort exercise

Illustrate merge sort by showing table partitioning and numbering the order in which steps happen (like in the lecture notes example)

7	8	2	4	5	4	3	1

Merge sort running time

Running time estimates:

- Laptop executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort ($N \log N$)		
computer	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min
super	instant	1 second	1 week	instant	instant	instant

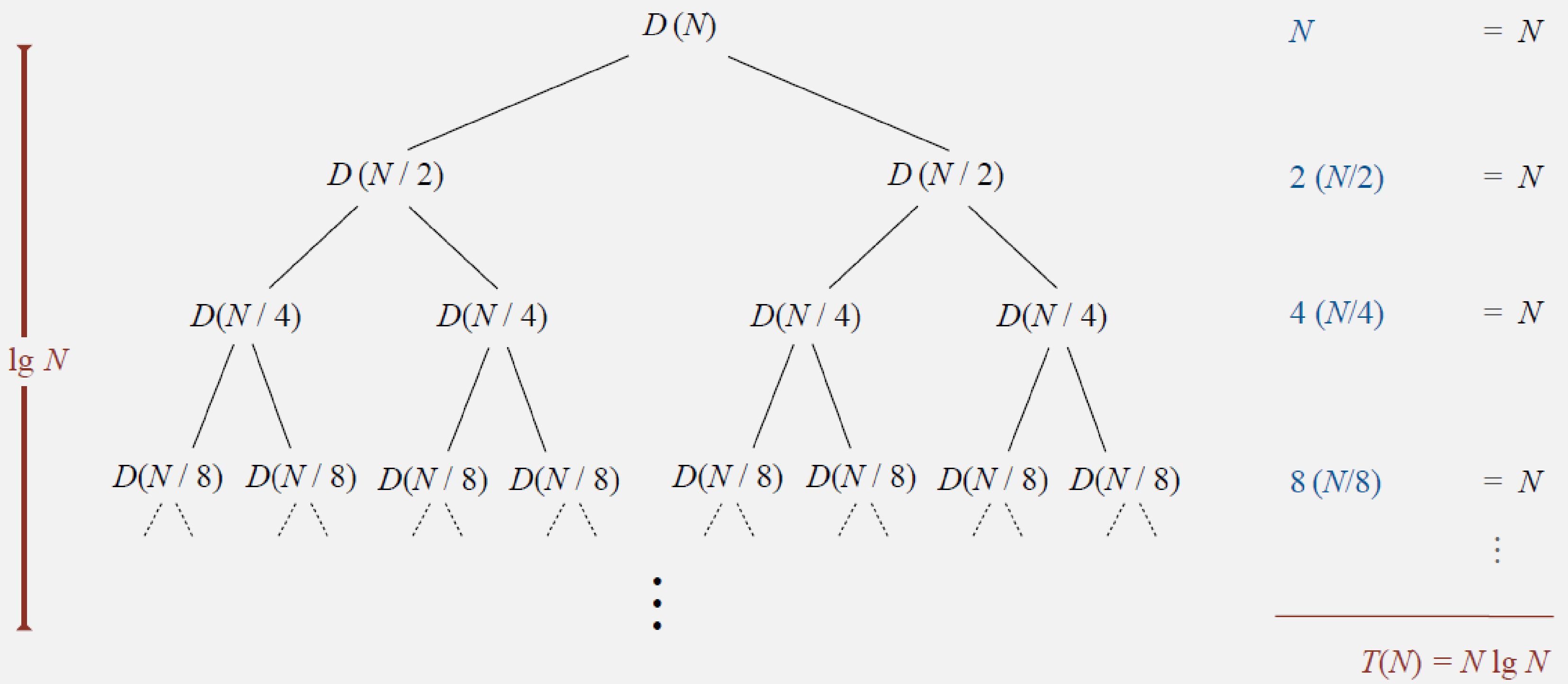
Merge sort

- Number of compares < N lg N
  - Linearithmic
  - Both average and worst
  - Stable – use "less than" - favours left hand value to right hand one even when they're equal
- Number of array accesses < 6 N lg N
- Memory use – auxiliary array of size N
- Proofs in Sedgewick

Merge sort

Proposition. If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 1. [assuming N is a power of 2]



Merge sort

Key point. Any algorithm with the following structure takes $N \log N$ time:

```
public static void linearithmic(int N)
{
    if (N == 0) return;
    linearithmic(N/2);   ← solve two problems
    linearithmic(N/2);   ← of half the size
    linear(N);           ← do a linear amount of work
}
```

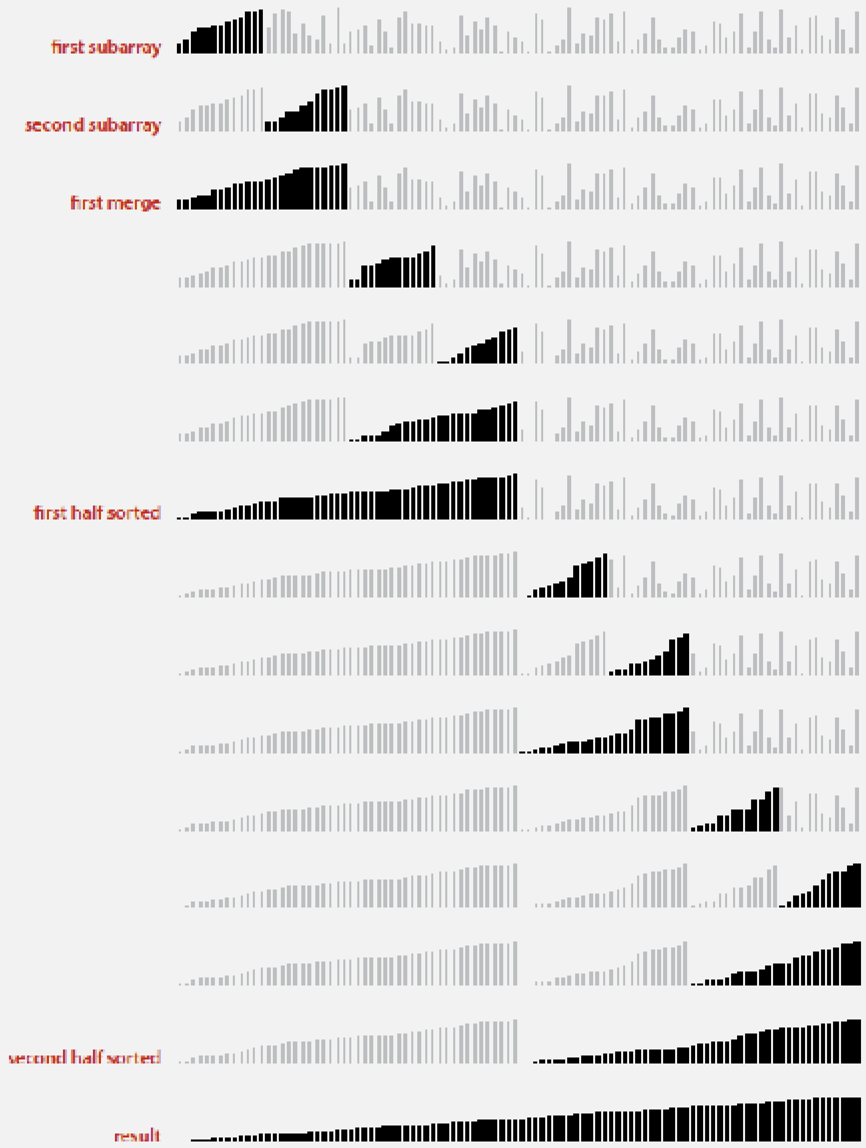
Merge sort improvement

› Too much overhead for small subarrays

› Cut off to insertion sort for ~10 items

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

Mergesort with cutoff to insertion sort: visualization



Merge sort further improvements

› Stop if already sorted
  – Is largest item in first half smaller than smallest in second half

A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V
A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   if (!less(a[mid+1], a[mid])) return;
   merge(a, aux, lo, mid, hi);
}
```

Merge sort further improvements

- Eliminate the time (but not the space) taken to copy to the auxiliary array used for merging
- Use two invocations of the sort method
  - one that takes its input from the given array and puts the sorted output in the auxiliary array
  - the other takes its input from the auxiliary array and puts the sorted output in the given array.

Merge sort further improvements

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)          aux[k] = a[j++];
      else if (j > hi)           aux[k] = a[i++];
      else if (less(a[j], a[i])) aux[k] = a[j++];
      else                       aux[k] = a[i++];
   }
}

private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (aux, a, lo, mid);
   sort (aux, a, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

merge from a[] to aux[]

assumes aux[] is initialize to a[] once, before recursive calls

switch roles of aux[] and a[]

Bottom up mergesort

Merge sort bottom up

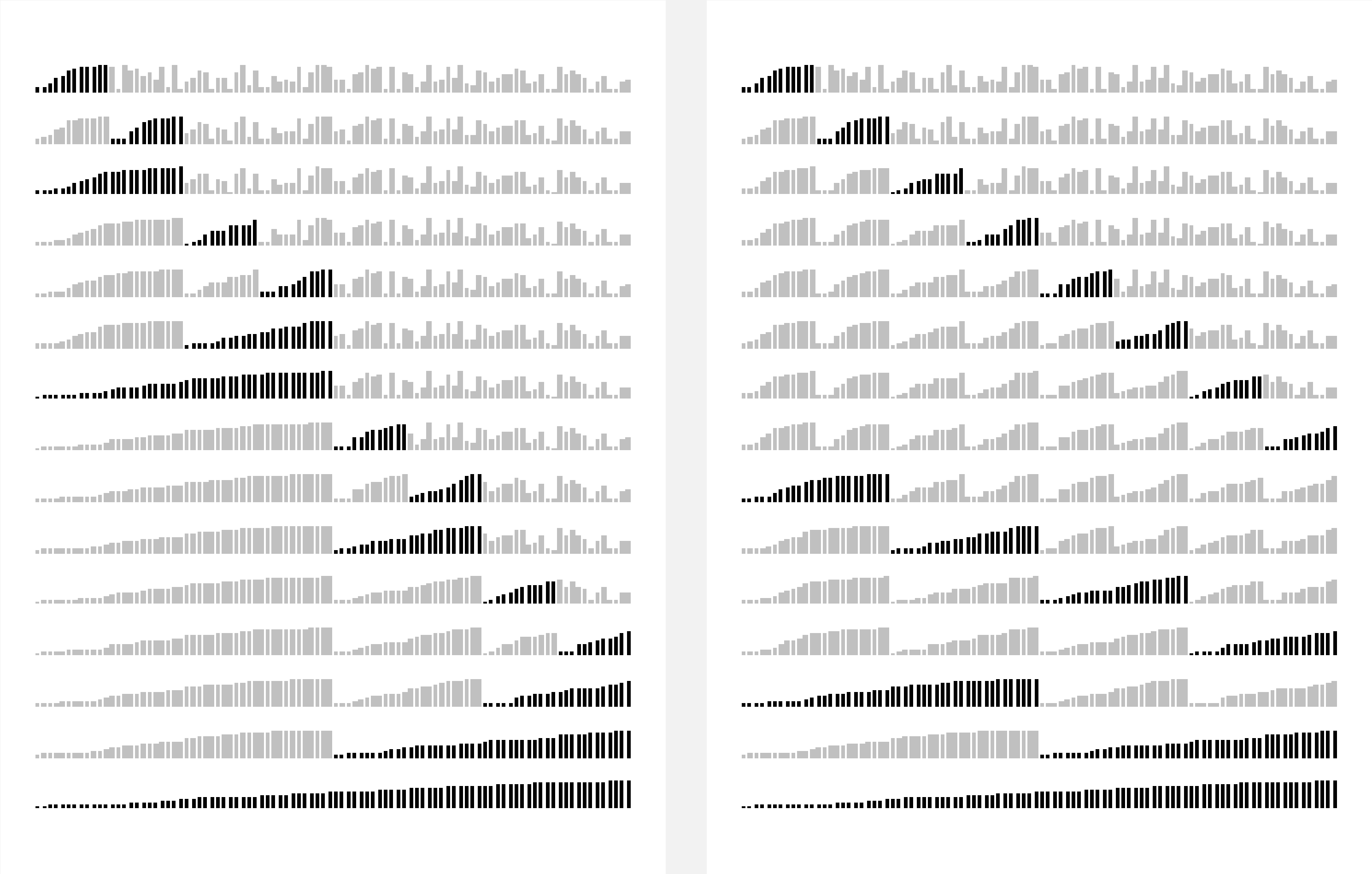
› Pass through array merging subarrays of size 1, 2, 4, etc

```
public class MergeBU
{
   private static void merge(...)
   {  /* as before */  }

   public static void sort(Comparable[] a)
   {
      int N = a.length;
      Comparable[] aux = new Comparable[N];
      for (int sz = 1; sz < N; sz = sz+sz)
         for (int lo = 0; lo < N-sz; lo += sz+sz)
            merge(a, aux, lo, lo+sz-1, Math.min(lo+sz+sz-1, N-1));
   }
}
```

Merge sort bottom up

	a[i] 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sz = 1																
merge(a, aux, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 4, 5)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 6, 6, 7)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 8, 8, 9)	E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E
merge(a, aux, 10, 10, 11)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, aux, 12, 12, 13)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, aux, 14, 14, 15)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L
sz = 2																
merge(a, aux, 0, 1, 3)	E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L
merge(a, aux, 4, 5, 7)	E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L
merge(a, aux, 8, 9, 11)	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L
merge(a, aux, 12, 13, 15)	E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P
sz = 4																
merge(a, aux, 0, 3, 7)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, aux, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
sz = 8																
merge(a, aux, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Merge sort top down vs bottom up



top-down mergesort (cutoff = 12)   bottom-up mergesort (cutoff = 12)

Timsort

- adaptive sort, combination of
  - natural merge sort – exploit pre-existing order by identifying naturally occurring non-descending sequences (so ascending or equal) - Look for at least 2 elements
  - Use insertion sort to make initial runs
- Java 7 (for non primitive data types), Python, Android

Quicksort

Quicksort

- One of top 10 algorithms of 20^{th} century in science and engineering
  - "the greatest influence on the development and practice of science and engineering in the 20th century"
  - https://www.computer.org/csdl/mags/cs/2000/01/c1022.html
  - "one of the best practical sorting algorithm for general inputs"
  - inspiration for developing general algorithm techniques for various applications

Quicksort

- Invented by Tony Hoare in 1959
  - Visiting student in Russia, needed to sort the words before looking them up in dictionary
  - Insert sort was too slow so he developed quicksort, but couldn't implement it until learnt ALGOL and its ability to do recursion
- Further improvements
  - Sedgwick, Bentley, Yaroslavskiy
  - Dual-pivot implementation in 2009, now implemented in Java 7 onwards

Quicksort

1. Shuffle the array a[] (we'll talk later why)
2. Partition the array so that, for some j
   - a[j] is in place (called pivot)
   - There is nothing larger than a[j] to the left of it
   - There is nothing smaller to the right of it

   (where does equal go?)
3. Sort each subarray recursively

Quicksort

- To sort an array **S**
  1. If the number of elements in **S** is 0 or 1, then return. The array is sorted.
  2. Pick an element *v* in **S**. This is the *pivot* value.
  3. Partition **S**-{*v*} into two disjoint subsets, $\mathbf{S}_1$ = {all values $x \leq v$}, and $\mathbf{S}_2$ = {all values $x \geq v$}.
  4. Return QuickSort($\mathbf{S}_1$), *v*, QuickSort($\mathbf{S}_2$)

Quicksort example

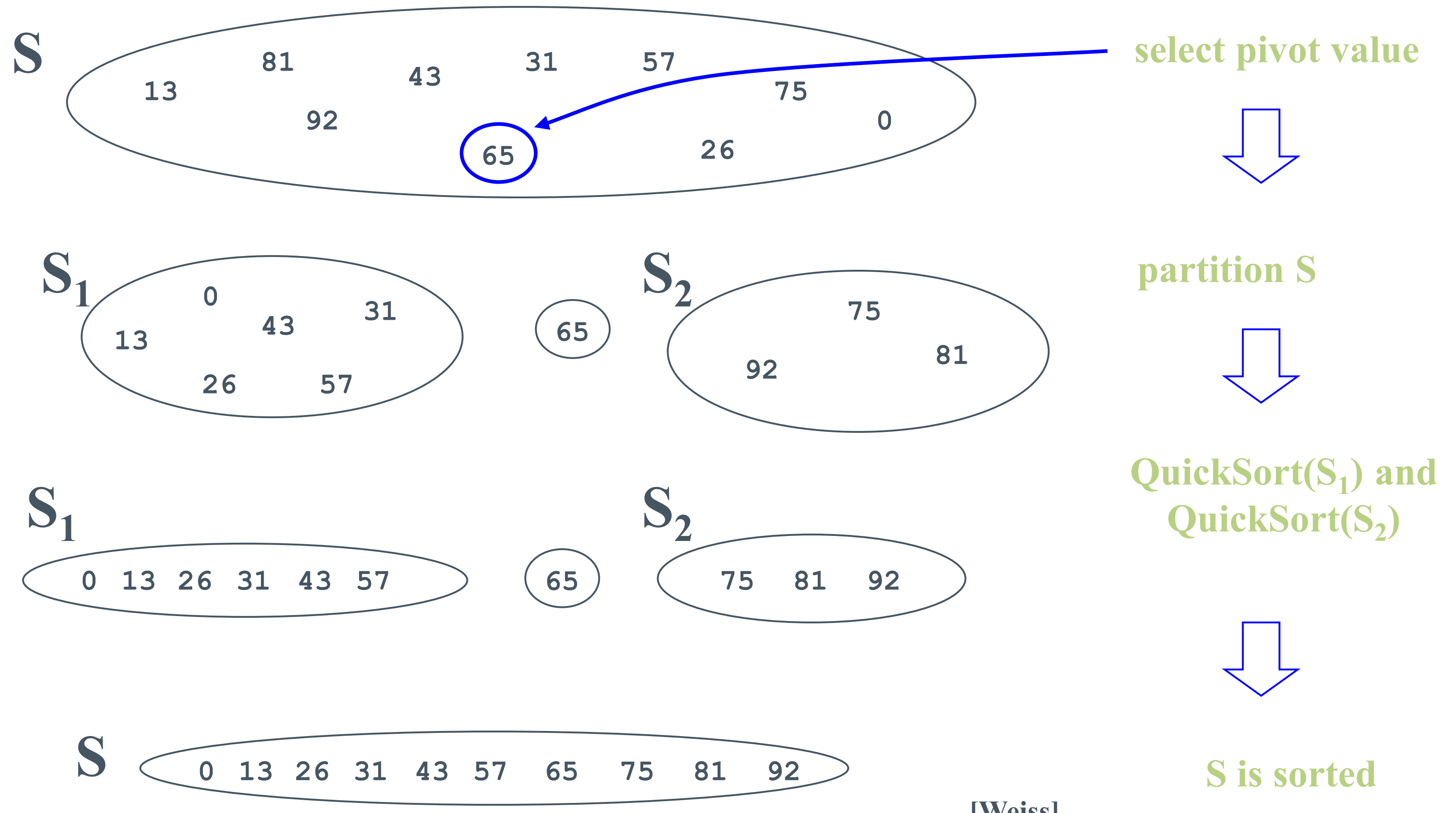
input	Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
shuffle	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
partition	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
sort left	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
sort right	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

partitioning item

not greater

not less

Quicksort example



[Weiss]

Quicksort - details

- Implement partitioning
  - recursive
- Pick a pivot
  - want a value that will cause $|S_1|$ and $|S_2|$ to be non-zero, and close to equal in size if possible

– Dealing with cases where the element equals the pivot

Quicksort – partitioning

- Need to partition the array into left and right sub-arrays
  - the elements in left sub-array are ≤ pivot
  - elements in right sub-array are ≥ pivot
- How do the elements get to the correct partition?
  - Choose an element from the array as the pivot
  - Make one pass through the rest of the array and swap as needed to put elements in partitions

Quicksort – picking a pivot

– Ideally median value
  › Expensive, calculating median
  › Approximate: choose a median of first, middle and last values
– Choose pivot randomly
  › Need a random number generator
– Choose the first element
  › Ok if array shuffled, bad if array sorted – worst case for quicksort

Quicksort – in-place partitioning

- If we use an extra array, partitioning is easy to implement, but not so much easier that it is worth the extra cost of copying the partitioned version back into the original.
- Partition in-place

Quicksort – in-place partitioning example

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.

K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
↑ lo	↑ i														↑ j

Quicksort – in-place partitioning example

v a[]

	i	j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values	0	16	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
scan left, scan right	1	12	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
exchange	1	12	K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
scan left, scan right	3	9	K	C	A	T	E	L	E	P	U	I	M	Q	R	X	O	S
exchange	3	9	K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
scan left, scan right	5	6	K	C	A	I	E	L	E	P	U	T	M	Q	R	X	O	S
exchange	5	6	K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
scan left, scan right	6	5	K	C	A	I	E	E	L	P	U	T	M	Q	R	X	O	S
final exchange	6	5	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
result		5	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S

Partitioning trace (array contents before and after each exchange)

Quicksort – in-place partitioning example

Repeat until `i` and `j` pointers cross.

- Scan `i` from left to right so long as `(a[i] < a[lo])`.
- Scan `j` from right to left so long as `(a[j] > a[lo])`.
- Exchange `a[i]` with `a[j]`.

When pointers cross.

- Exchange `a[lo]` with `a[j]`.

E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
↑ lo					↑ j										↑ hi

Quick sort exercise

A single pass of sorting (no need to continue recursively)

Show pivot value

Show values of I, j, and status of array on each pass

Array:

I L O V E A L G O R I T H M S

Quicksort – partition code

```java
private int partition(Comparable[] numbers, int lo, int hi) {
    int i = lo;
    int j = hi+1;
    Comparable pivot =  numbers[lo];
    while(true) {
        while((numbers[++i].compareTo(pivot) < 0)) {
            if(i == hi) break;
        }
        while((pivot.compareTo(numbers[--j]) < 0)) {
            if(j == lo) break;
        }
        if(i >= j) break;
        Comparable temp = numbers[i];
        numbers[i] = numbers[j];
        numbers[j] = temp;
    }
    numbers[lo] = numbers[j];
    numbers[j] = pivot;
    return j;
}
```

Quicksort – example

› Partitioning one array – need to do this recursively on the array left of j and right of j

	lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values				Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
random shuffle				K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
	0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
	0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	1		1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	4		4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	7	7	8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	8		8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	10	13	15	A	C	E	E	I	K	L	M	O	P	S	Q	R	T	U	X
	10	12	12	A	C	E	E	I	K	L	M	O	P	R	Q	S	T	U	X
	10	11	11	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	10		10	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	14	14	15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	15		15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result				A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

no partition for subarrays of size 1

Quicksort – recursive code

```java
public void sort(Comparable[] numbers) {
   recursiveQuick(numbers, 0, numbers.length-1);
}

public void recursiveQuick(Comparable[] numbers, int lo, int hi) {
   if(hi <= lo) {
      return;
   }
   int pivotPos = partition(numbers, lo, hi);
   recursiveQuick(numbers, lo, pivotPos-1);
   recursiveQuick(numbers, pivotPos+1, hi);
}
```

Quicksort – iterative version?

- With the help of auxiliary stack

Quicksort – performance

- How many compares to partition the array of length N?
- How many recursive calls? – depth of recursion
- Best case analysis – for shuffled elements?
- Worst case analysis – for sorted elements?

- Polls

Quicksort – best case analysis

What is the number of compares?

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0		0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
2		2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4		4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
6		6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8		8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
10		10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12		12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort – worst case analysis

What is the number of compares?

			a[ ]														
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort

- Make sure to always avoid worst case performance by shuffling the array at the start!
- Alternatively – pick a random pivot in each subarray
- Quicksort is therefore a randomized algorithm
  - Uses random numbers to decide what to do next somewhere in its logic

Quicksort – performance

- Home PC executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (n^2)			mergesort (n log n)			quicksort (n log n)		
computer	thousand	million	billion	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min	instant	0.6 sec	12 min
super	instant	1 second	1 week	instant	instant	instant	instant	instant	instant

Quicksort - performance

Average case. Expected number of compares is $\sim 1.39\, n \lg n$.

- 39% more compares than mergesort.
- Faster than mergesort in practice because of less data movement.

Maths in Sedgwick

Quicksort – properties summary

- Not stable because of long distance swapping.
- No iterative version (without using a stack).
- Pure quicksort not good for small arrays.
- “In-place”, but uses auxiliary storage because of recursive call (O(logn) space).
- $O(n \log n)$ average case performance, but $O(n^2)$ worst case performance.

Quicksort improvements

- Use insertion sort for small arrays
  - Cut off to insertion sort at subarray size ~10
- Use median for pivot value (median of 3 random items, ie first, last, middle)
- 3-way quicksort, dual pivot, 3-pivot

Quicksort – all items the same

What is the result of partitioning the following array (stop on equal keys)?

scan until ≥ A → ← scan until ≤ A

A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

A.

A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

B.

A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	**A**

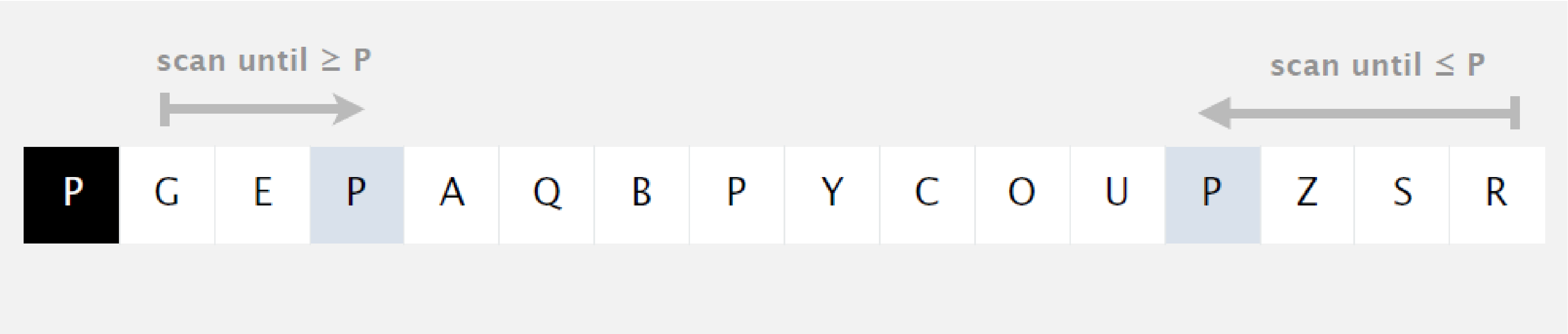
C.

A	A	A	A	A	A	A	A	**A**	A	A	A	A	A	A	A

D. *I don't know.*

Quicksort – stop at equal keys

- qsort() in C bug reported in 1991 – "unbearably slow" for organ-pipe inputs (eg "01233210")
  - In implementations and textbooks until then
- N^2 time to sort organ-pipe inputs, and random arrays of 0s and 1s
- Improvement now: stop scanning if keys are equal



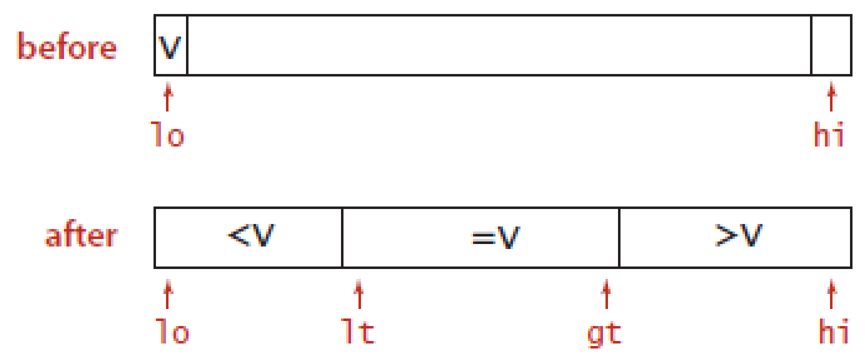
Quicksort – stop at equal keys

- Problem – if all items equal to pivot are moved to one side of it
  - Consequence ~1/2 n^2 compares when all keys are equal
- Stop when keys are equal
  - If all keys are equal, divides the array exactly
  - Why not put all items that are the same as partition item in place? 3-way partitioning

3-way partitioning

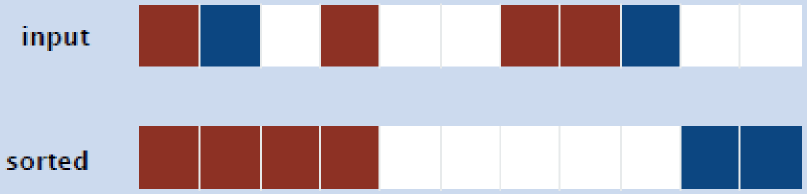
Goal. Partition array into three parts so that:

- Entries between `lt` and `gt` equal to the partition item.
- No larger entries to left of `lt`.
- No smaller entries to right of `gt`.



Dutch national flag problem

Problem. [Edsger Dijkstra] Given an array of n buckets, each containing a red, white, or blue pebble, sort them by color.



Operations allowed.

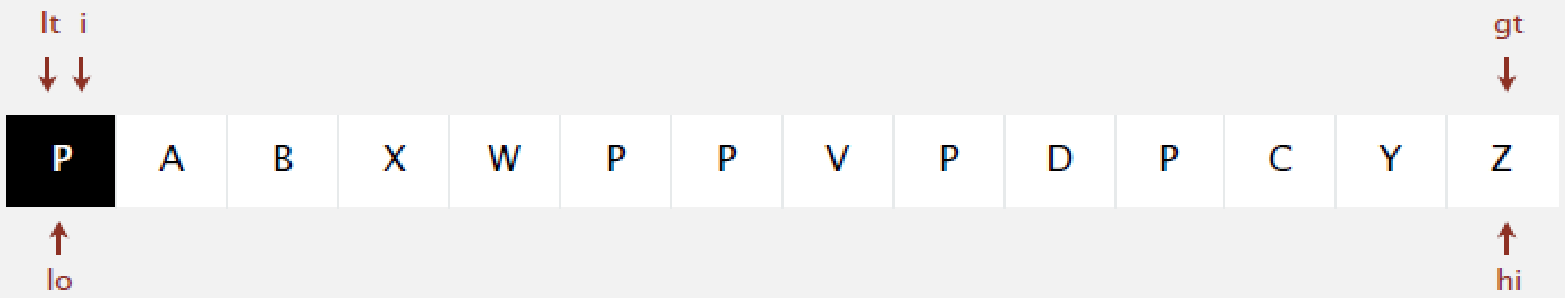
- $swap(i, j)$: swap the pebble in bucket i with the pebble in bucket j.
- $color(i)$: color of pebble in bucket i.

Requirements.

- Exactly n calls to $color()$.
- At most n calls to $swap()$.
- Constant extra space.

3-way partitioning

- Let v be partitioning item `a[lo]`.
- Scan `i` from left to right.
  - (`a[i]  < v`): exchange `a[lt]` with `a[i]`; increment both `lt` and `i`
  - (`a[i]  > v`): exchange `a[gt]` with `a[i]`; decrement `gt`
  - (`a[i] == v`): increment `i`

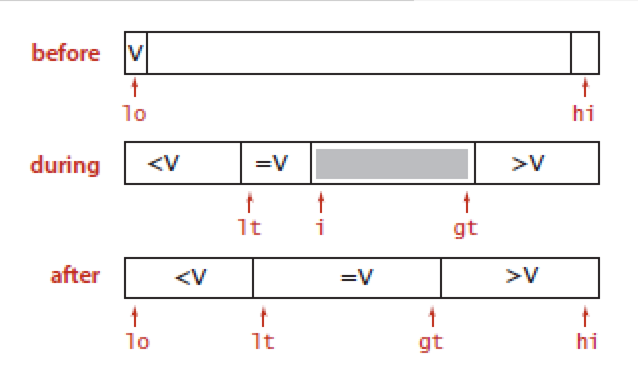


[Demo https://algs4.cs.princeton.edu/lectures/23DemoPartitioning.pdf](https://algs4.cs.princeton.edu/lectures/23DemoPartitioning.pdf)

3-way partitioning

› Improves quick sort when there are duplicate keys

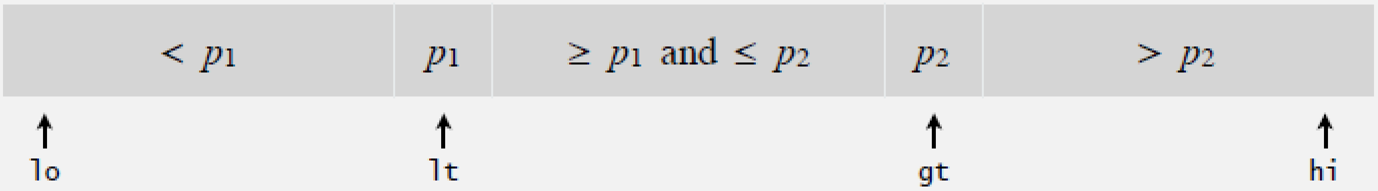
```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;
   int lt = lo, gt = hi;
   Comparable v = a[lo];
   int i = lo;
   while (i <= gt)
   {
      int cmp = a[i].compareTo(v);
      if      (cmp < 0) exch(a, lt++, i++);
      else if (cmp > 0) exch(a, i, gt--);
      else              i++;
   }

   sort(a, lo, lt - 1);
   sort(a, gt + 1, hi);
}
```



2-pivot quick sort

Use two partitioning items p_1 and p_2 and partition into three subarrays:

- Keys less than p_1.
- Keys between p_1 and p_2.
- Keys greater than p_2.



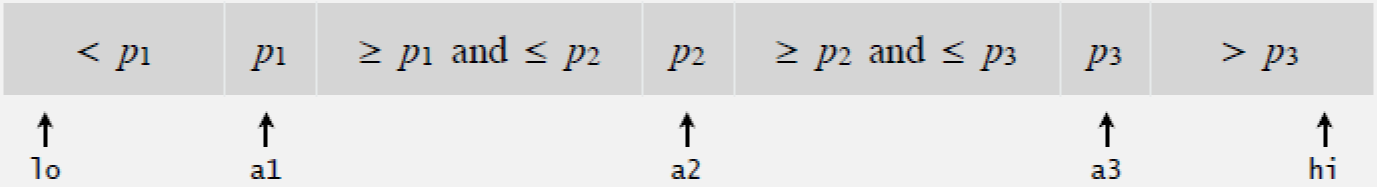
Recursively sort three subarrays.

3-pivot quick sort

Three-pivot quicksort

Use three partitioning items p_1, p_2, and p_3 and partition into four subarrays:

- Keys less than p_1.
- Keys between p_1 and p_2.
- Keys between p_2 and p_3.
- Keys greater than p_3.



Demos

- https://algs4.cs.princeton.edu/lectures/23DemoPartitioning.pdf

- Quicksort
- 3-way partitioning
- Dual pivot partitioning

Quicksort – cache improvements

- Principle of locality
  - the same values, or related storage locations, are frequently accessed
  - Temporal locality
    - If at one point a particular memory location is referenced, then it is likely that the same location will be referenced again in the near future
  - Spatial locality
    - If a particular storage location is referenced at a particular time, then it is likely that nearby memory locations will be referenced in the near future -> pre-fetch arrays
  - Predictability of memory access
  - Implications for caching
    - cache – stores data "nearer" to processor so that it can be accessed quicker in the future
- 2-pivot and 3-pivot have smaller number of cache misses and smaller number of recursive calls to a subproblem larger than the size of a cache block
- Multi-Pivot Quicksort: Theory and Experiments - by Kushagra, López-Oritz, Munro, and Qiao
  - Original paper - http://epubs.siam.org/doi/pdf/10.1137/1.9781611973198.6
  - Discussion: https://cs.stanford.edu/~rishig/courses/ref/l11a.pdf

Caching improvements

Why do 2-pivot (and 3-pivot) quicksort perform better than 1-pivot?

A. ~~Fewer compares.~~

B. ~~Fewer exchanges.~~

C. Fewer cache misses.

entries scanned is a good proxy for cache performance when comparing quicksort variants

partitioning	compares	exchanges	entries scanned
1-pivot	$2\, n \ln n$	$0.333\, n \ln n$	$2\, n \ln n$
median-of-3	$1.714\, n \ln n$	$0.343\, n \ln n$	$1.714\, n \ln n$
2-pivot	$1.9\, n \ln n$	$0.6\, n \ln n$	$1.6\, n \ln n$
3-pivot	$1.846\, n \ln n$	$0.616\, n \ln n$	$1.385\, n \ln n$

Reference: Analysis of Pivot Sampling in Dual-Pivot Quicksort by Wild-Nebel-Martínez

Bottom line. Caching can have a significant impact on performance.

Merge vs quick

- In Java, Arrays.sort() uses **Quick**Sort for sorting primitives and **Merge**Sort for sorting Arrays of Objects. This is because, merge sort is stable, so it won't reorder elements that are equal.
  - Why does it matter for Objects and not for primitive data types?
- QuickSort in java
  - 2-pivot since 2009
- MergeSort in java
  - Timsort

Sort algorithms summary

- Use system sort - Arrays.sort(); - usually good enough
- What to consider when picking an algorithm?

Compare performance to system sort in your assignment?

Comparator interface

- Comparable interface
  - Uses natural order to compare things
  - Can override method compareTo() if want custom-defined criteria
- But what if we have Objects we want to compare according to multiple custom-defined criteria?
- Comparator interface
  - Can create multiple classes implementing Comparator and override compare method
  - Custom ordering
  - To use with system sort, pass as a second argument to Array.sort(a, new MyCustomOrder());

Sorting algorithms summary

	inplace?	stable?	best	average	worst	remarks
selection	✔		$\frac{1}{2} n^2$	$\frac{1}{2} n^2$	$\frac{1}{2} n^2$	n exchanges
insertion	✔	✔	n	$\frac{1}{4} n^2$	$\frac{1}{2} n^2$	use for small n or partially ordered
shell	✔		$n \log_3 n$	?	$c\, n^{3/2}$	tight code; subquadratic
merge		✔	$\frac{1}{2} n \lg n$	$n \lg n$	$n \lg n$	$n \log n$ guarantee; stable
timsort		✔	n	$n \lg n$	$n \lg n$	improves mergesort when preexisting order
quick	✔		$n \lg n$	$2 n \ln n$	$\frac{1}{2} n^2$	$n \log n$ probabilistic guarantee; fastest in practice
3-way quick	✔		n	$2 n \ln n$	$\frac{1}{2} n^2$	improves quicksort when duplicate keys
heap	✔		$3 n$	$2 n \lg n$	$2 n \lg n$	$n \log n$ guarantee; in-place
?	✔	✔	n	$n \lg n$	$n \lg n$	holy sorting grail

Quick algorithms exercise

› **Which algorithm would work best to sort data as it arrives, one piece at a time, perhaps from a network?**

1. Mergesort
2. Selection sort
3. Quicksort
4. Insertion sort

Another quick question

› **Which algorithm would you use to sort 1 million of 32-bit integers?**

1. Mergesort
2. Selection sort
3. Quicksort
4. Insertion sort
5. None of the above

https://www.youtube.com/watch?v=k4RRi_ntQc8

Quick sort exercise

Illustrate quick sort by showing at each pass through the array the swaps that happen and circle the final location of the pivot element

Pivot = array [0]

i ++ until find element > than pivot

j - - until find element < than pi

5	4	7	2	2	8	1	9

Quick sort exercise

Illustrate quick sort by showing at each pass through the array the swaps that happen and circle the final location of the pivot element

Pivot = array [0]

i ++ until find element > than pivot j - - until find element < than pi

2	2	2	2	2	2	2	2

Quick sort exercise 2

Illustrate quick sort by showing at each pass through the array the swaps that happen and circle the final location of the pivot element

Pivot = array [0]

i ++ until find element >= than pivot

j - - until find element =< than pi

2	2	2	2	2	2	2	2