MA2C03: PROBLEM SHEET

1) (From the 2016-2017 Annual Exam) Let Q denote the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xQy if and only if

$$x - y = (x - y)(x + 2y).$$

Determine the following:

(i) Whether or not the relation R is *reflexive*;
(ii) Whether or not the relation R is *symmetric*;
(iii) Whether or not the relation R is *transitive*;
(iv) Whether or not the relation R is an *equivalence relation*;
(v) Whether or not the relation R is *anti-symmetric*;
(vi) Whether or not the relation R is a *partial order*.

Justify your answers.

Solution: $x, y \in \mathbb{Z}$ satisfy xRy iff $x - y = (x - y)(x + 2y)$, which is equivalent to $(x - y)(x + 2y - 1) = 0$, i.e., $x = y$ or $x + 2y - 1 = 0$.

(i) **Reflexivity:** The relation R is reflexive because xRx holds for all $x \in \mathbb{Z}$ as $x - x = (x - x)(x + 2x) = 0$.

(ii) **Symmetry:** The relation R is not symmetric because if $x \neq y$, then xRy holds if $x + 2y = 1$, thus for yRx we would need $y + 2x = 1$, which only holds at the same time with $x+2y = 1$ when $x = y = \frac{1}{3} \notin \mathbb{Z}$.

(iii) **Anti-symmetry:** The relation R is anti-symmetric. Having xRy and yRx when $x \neq y$ would imply $x + 2y = 1$ and $y + 2x = 1$ hold simultaneously, which gives $x = y = \frac{1}{3} \notin \mathbb{Z}$. Therefore, xRy and yRx can both be true only if $x = y$.

(iv) **Transitivity:** The relation R is not transitive. Assume xRy and yRz hold for $x, y, z \in \mathbb{Z}$. There are 4 cases to consider:
Case 1: $x = y$ and $y = z$, then $x = z$, so xRz as needed.
Case 2: $x = y$ and $y + 2z = 1$, then $x + 2z = 1$, so xRz as needed.
Case 3: $x + 2y = 1$ and $y = z$, then $x + 2z = 1$, so xRz as needed.
Case 4: $x + 2y = 1$ and $y + 2z = 1$, then $x + 2(1 - 2z) = 1$, so $x + 2 - 4z = 1$, i.e., $x - 4z = -1$. This last equation is satisfied for example for $x = 3$, $z = 1$. Take $y = -1$ in order to satisfy $x + 2y = 1$. We see that $x + 2z = 3 + 2 = 5 \neq 1$, so xRz fails. We have constructed a counterexample.

(v) **Equivalence relation:** The relation R is not an equivalence relation because while reflexive, it fails to be symmetric and transitive.

(vi) **Partial order:** The relation R is not a partial order because while reflexive and anti-symmetric, it fails to be transitive.

2) (From the 2016-2017 Annual Exam) Let $f : [-2, 2] \to [-15, 1]$ be the function defined by $f(x) = x^2 + 3x - 10$ for all $x \in [-2, 2]$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers.

Injectivity: $f(x) = x^2 + 3x - 10 = (x - 2)(x - 5)$ This function is not injective on the interval $[-2, 2]$. Acceptable justifications: drawing the graph, providing two values $x_1, x_2 \in [-2, 2]$, $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$, applying Rolle's theorem (noticing that $f'(x) = 2x + 3$ so $f'\left(-\frac{3}{2}\right) = 0$, and $\frac{3}{2} \in [-2, 2]$), etc.

Surjectivity: $f(x) = x^2 + 3x - 10$ is not surjective on the interval $[-2, 2]$. Acceptable justifications: drawing the graph, providing a value in $[-15, 1]$ that $f(x)$ does not assume, showing the minimum value occurs at $\frac{3}{2}$, where $f\left(\frac{3}{2}\right) = -12.25 > -15$, etc.

3) Let $A = \{3^p \mid p \in \mathbb{N}\}$ with the operation of multiplication.
(a) Is $(A, \cdot)$ a semigroup? Justify your answer.
(b) Is $(A, \cdot)$ a monoid? Justify your answer.
(c) Is $(A, \cdot)$ a group? Justify your answer.

Solution: (a) Yes, $(A, \cdot)$ is a semi-group. $A \subset \mathbb{Q}^*$, and $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ is a monoid under the operation of multiplication. We proved in lecture that if $a \in M$ for M a monoid with operation $*$ and $m, n \in \mathbb{N}$, then $a^m * a^n = a^{m+n}$. Here $a = 3$ and since addition is a binary operation on $\mathbb{N}$ as we showed in class, multiplication is a binary operation on A. The associativity of multiplication on A follows from the associativity of addition on $\mathbb{N}$ and the theorem that if $a \in M$ for M a monoid with operation $*$ and $m, n \in \mathbb{N}$, then $a^m * a^n = a^{m+n}$.

(b) Yes, $(A, \cdot)$ is a monoid. $3^0 = 1$ is the identity element on A because any $b \in A$ is of the form 3^p, so $b \cdot 1 = a^p \cdot a^0 = a^{p+0} = a^{0+p} = 1 \cdot b = a^p = b$.

(c) No, $(A, \cdot)$ is not a group. By the theorem on powers we proved in lecture, 3^{-p} would have to be the inverse of 3^p for $p \in \mathbb{N}$ because $3^{-p} \cdot 3^p = 3^p \cdot 3^{-p} = 3^{p-p} = 3^0 = 1$, but if $p \in \mathbb{N}$, then $p \geq 0$, so $-p \notin \mathbb{N}$

when p is negative. So if $p < 0$, $3^{-p} \notin A$. Therefore, the only invertible element in A is the identity element $3^0 = 1$.

4) (Slightly modified question from the annual exam 2017-2018) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 0\}$ with the operation of addition given by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

(a) Is $(A, +)$ a semigroup? Justify your answer.
(b) Is $(A, +)$ a monoid? Justify your answer.
(c) Is $(A, +)$ a group? Justify your answer.
(d) What geometric object is the set A in $\mathbb{R}^2$?

Solution: (a) Yes, $(A, +)$ is a semi-group. If $x_1 = -2y_1$ and $x_2 = -2y_2$, then $x_1 + x_2 = -2y_1 - 2y_2 = -2(y_1 + y_2)$, so $+$ is a binary operation on A. We proved in lecture that addition is an associative binary operation on $\mathbb{R}$, so $+$ is associative on A as associativity will function component by component in the vector (x, y).

(b) Yes, $(A, +)$ is a monoid. $(0, 0)$ is the identity element on A because for any $(x, y) \in A$,

$$(x, y) + (0, 0) = (x + 0, y + 0) = (0 + x, 0 + y) = (0, 0) + (x, y) = (x, y).$$

(c) Yes, $(A, +)$ is a group. For any $(x, y) \in A$, $(-x, -y)$ is its inverse because $(x, y) + (-x, -y) = (-x, -y) + (x, y) = (0, 0)$. Therefore, all elements of A are invertible.

(d) A is the line passing through the origin $(0, 0)$ and the point $(2, -1)$ as $2 + 2(-1) = 0$.

5) Let A be a finite set, and let A^* be the set of all words over the alphabet A. Consider $(A^*, \circ, \epsilon)$ with the operation of concatenation and empty word ϵ as the identity element. Let $(\mathbb{N}, +, 0)$ be the set of natural numbers with the operation of addition and 0 as the identity element. Let $f : A^* \to \mathbb{N}$ be the function that assigns to each word $w \in A^*$ its length, $f(w) = |w| \in \mathbb{N}$.

(a) What type of object is $(A^*, \circ, \epsilon)$ in abstract algebra (semigroup, monoid, group)? Justify your answer.
(b) What type of object is $(\mathbb{N}, +, 0)$ in abstract algebra (semigroup, monoid, group)? Justify your answer.
(c) Is f a homomorphism? Justify your answer.
(d) Is f an isomorphism? Justify your answer.

Solution: (a) $\circ$ is an associative binary operation as we proved in lecture, so $(A^*, \circ, \epsilon)$ is definitely a semigroup. As we showed in lecture,

ϵ is the identity element for $\circ$ on A^*, which means $(A^*, \circ, \epsilon)$ is a monoid. We discussed in lecture during the abstract algebra unit that ϵ is the only invertible element in A^*, so $(A^*, \circ, \epsilon)$ cannot be a group.

(b) Addition is an associative binary operation as we showed in lecture, so $(\mathbb{N}, +, 0)$ is clearly a semigroup. 0 is the identity element for addition on $\mathbb{N}$ which means $(\mathbb{N}, +, 0)$ is a monoid. Note that 0 is the only invertible element in $\mathbb{N}$ so $(\mathbb{N}, +, 0)$ cannot be a group.

(c) To show that f is a homomorphism, we need to show that for any two words $w_1, w_2 \in A^*$, $f(w_1 \circ w_2) = |w_1| + |w_2|$, but we have already showed in lecture that this property holds. Therefore, f is a homomorphism.

(d) An isomorphism is a homomorphism that is also bijective. We know f is homomorphism. Now we need to decide whether it is bijective. The function f is clearly surjective because for any length $n \in \mathbb{N}$, we can construct a word in $w \in A^*$, whose length $|w| = n$. Is f injective? Well, the answer depends whether A has one element or several. If $A = \{a\}$ has only one element, there is one and only one word $a \cdots a$ of any given length n, so f is injective. However, if A has more than one element, then there exist letters $a, b \in A$ such that $a \neq b$. Then the words ab and ba that are distinct have the same length 2, which means f cannot be injective, so it is not an isomorphism.

6) Let $(\mathbb{Z}, +, 0)$ be the set of integers with the operation of addition and 0 as the identity element. Let E be the set of even integers, $E = \{2p \mid p \in \mathbb{Z}\}$. Consider $(E, +, 0)$ the set of even integers with the operation of addition and 0 as the identity element. Let $f : \mathbb{Z} \to E$ be the function $f(n) = 2n$.

(a) What type of object is $(\mathbb{Z}, +, 0)$ in abstract algebra (semigroup, monoid, group)? Justify your answer.
(b) What type of object is $(E+, 0)$ in abstract algebra (semigroup, monoid, group)? Justify your answer.
(c) Is f a homomorphism? Justify your answer.
(d) Is f an isomorphism? Justify your answer.

Solution: (a) Addition on $\mathbb{R}$ hence on $\mathbb{Z}$ is an associative binary operation as we discussed in lecture. Therefore, $(\mathbb{Z}, +, 0)$ is a semigroup. 0 is the identity element for addition on $\mathbb{Z}$ as for any $n \in \mathbb{Z}$, $n + 0 = 0 + n = n$, so $(\mathbb{Z}, +, 0)$ is a monoid. Given any $n \in \mathbb{Z}$, $-n$ is its inverse as $n + (-n) = n - n = 0$, so every element of $(\mathbb{Z}, +, 0)$ is invertible. Therefore, $(\mathbb{Z}, +, 0)$ is a group.

(b) $E \subset \mathbb{Z}$. Since addition is associative on $\mathbb{Z}$, it is also associative on E. We do, however, have to prove it is a binary operation, i.e. closed. Consider $m, n \in E$. Thus, there exist $p, s \in \mathbb{Z}$ such that $m = 2p$ and $n = 2s$ by the definition of E. Then $m + n = 2p + 2s = 2(p + s)$. Since addition is a binary operation on $\mathbb{Z}$ hence closed, it follows

$$p, s \in \mathbb{Z} \implies p + s \in \mathbb{Z}.$$

Thus, $m + n \in E$, and addition is indeed closed on E. We conclude $(E+, 0)$ is a semigroup. Since $E \subset \mathbb{Z}$, the fact that 0 is the identity element for addition on $\mathbb{Z}$ carries over to E, so E has 0 as its identity element. Therefore, $(E+, 0)$ is a monoid. Now let $n \in E$. We know from part (a) that $-n \in \mathbb{Z}$ is the inverse of n under addition. We just have to prove $-n \in E$. Since $n \in E$, there exists $p \in \mathbb{Z}$ such that $n = 2p$. Therefore, $-n = -2p = 2(-p)$, so $-n \in E$ as needed, which means every element in E is invertible. Therefore, $(E+, 0)$ is a group.

(c) To show that f is a homomorphism, we need to show that for any two integers $p, s \in \mathbb{Z}$, $f(p + s) = f(p) + f(s)$. We apply the definition of f as follows: $f(p) + f(s) = 2p + 2s = 2(p + s) = f(p + s)$. Therefore, f is a homomorphism.

(d) An isomorphism is a homomorphism that is also bijective. We know f is homomorphism. We need to figure out whether it is bijective. The function f is clearly surjective by the definition of E because for every $n \in E$, there exists $p \in \mathbb{Z}$ such that $n = 2p = f(p)$. To show injectivity, assume there exist $p, s \in \mathbb{Z}$ such that $f(p) = f(s)$. Then by the definition of f, $2p = 2s \iff p = s$. Therefore, f is indeed injective. We have shown f is bijective hence an isomorphism from $(\mathbb{Z}, +, 0)$ to $(E, +, 0)$.

7) Describe the formal language over the alphabet $\{a, b, c\}$ generated by the context-free grammar whose only non-terminal is $\langle S \rangle$, whose start symbol is $\langle S \rangle$, and whose production rules are the following:

(1) $\langle S \rangle \to b$
(2) $\langle S \rangle \to c$
(3) $\langle S \rangle \to a\langle S \rangle$

In other words, describe the structure of the strings generated by this grammar.

Solution: This context-free grammar produces strings of the type $a^m b$ or $a^m c$ for $m \geq 0$.

8) Let L be the language over the alphabet $\{0,1\}$ consisting of all words where the string 00 occurs as a substring. Prove from the definition of a regular language that the language L is regular.

Solution: Let the alphabet $A = \{0,1\}$. Recall that the definition of a regular language allows for finite subsets of A^*, the Kleene star, concatenations, and unions. Note that

$$L = \{w \in A^* \mid w = u \circ 00 \circ v \quad u, v \in A^*\}.$$

Therefore, we can let $L_1 = \{00\}$ be the language consisting of just the string 00 of interest. L_1 is a finite set, so it is allowed in the definition of a regular language. Let $L_2 = \{0,1\}$. L_2 is finite, hence likewise allowed. Let $L_3 = L_2^*$, the Kleene star applied to L_2. The language $L_3 = A^*$, i.e. it is the set of all words that can be formed over the alphabet $A = \{0,1\}$. Set $L_4 = L_3 \circ L_1$, and then $L_5 = L_4 \circ L_3$. Note that the words in L_5 have exactly the structure of the words in L, and in fact, $L = L_5$. Note also that the solution here is by no means unique. No two of you will necessarily have arrived at the same exact expression, order or labelling of the intermediate languages L_i that come into the definition of a regular language as applied to L.

9) Let L be the language over the alphabet $\{0,1\}$ consisting of all words where the string 00 occurs as a substring.

(a) Draw a finite state acceptor that accepts the language L. Carefully label all the states including the starting state and the finishing states as well as all the transitions. Make sure you justify it accepts all strings in the language L and no others.
(b) Write down the transition mapping of the finite state acceptor you drew in the previous part of the problem.
(c) Devise a regular grammar in normal form that generates the language L. Be sure to specify the start symbol, the non-terminals, and all the production rules.
(d) Write down a regular expression that gives the language L and justify your answer.

Solution: (a) See diagram of finite state acceptor on the next page.

We can use three states $\{i, A, B\}$, where i is the initial state. Since we must ensure the word contains the string 00, when 1 is the input, we stay in the initial state i. For input 0, we move to a new state A. A is not an accepting state as we have so far only half of the string 00, the first zero. If we get input 1, we have to restart the process of capturing the string 00, so we get back to the initial state i. If we get input 0, then we will have received the second zero we want, so we'll move to a new state B, which is an accepting state. Once we have the substring 00, we don't care what follows, so the transitions for both 0 and 1 out of state B are back into B itself.

(b)

$$\begin{aligned} t(i,0) &= A \qquad & t(i,1) &= i \\ t(A,0) &= B \qquad & t(A,1) &= i \\ t(B,0) &= B \qquad & t(B,1) &= B \end{aligned}$$

(c) We shall use the algorithm discussed in lecture in order to generate the regular grammar in normal form corresponding to the finite state acceptor constructed above. The finite state acceptor had three states $\{i, A, B\}$, where i was the initial state. Correspondingly, we use three non-terminals in our regular grammar: the start symbol $\langle S \rangle$ corresponding to the initial state i, $\langle A \rangle$ corresponding to state A, and $\langle B \rangle$ corresponding to state B. We first write the production rules corresponding to the transitions out of the initial state i :

(1) $\langle S \rangle \to 1\langle S \rangle$.
(2) $\langle S \rangle \to 0\langle A \rangle$.

Next, we write the production rules corresponding to the transitions out of state A :

(3) $\langle A \rangle \to 1\langle S \rangle$.
(4) $\langle A \rangle \to 0\langle B \rangle$.

Finally, we write the production rules corresponding to the transitions out of state B :

(5) $\langle B \rangle \to 1\langle B \rangle$.
(6) $\langle B \rangle \to 0\langle B \rangle$.

Rules (1)-(6) are of type (i). For each accepting state, we will write down a rule of type (iii). Since there is only one accepting state, B, we have only one such rule:

(7) $\langle B \rangle \to \epsilon$.

(d) Recall that

$$L = \{w \in A^* \mid w = u \circ 00 \circ v \quad u, v \in A^*\}.$$

Therefore, $L = A^* \circ 00 \circ A^*$, and we have obtained the regular expression giving us the language L.

10) (Annual Exam 2017) Consider the language L over the alphabet $A = \{a, l, p\}$ consisting of all words of the form $a^m l^{2m} p^m$ for $m \in \mathbb{N}^*$. Use the Pumping Lemma to show the language L is not regular.

Solution: Assume L is regular. Then it must have a pumping length T. We will now choose a string in terms of T that is particularly easy to analyse in the setting of the Pumping Lemma. Let this string be $w = a^T l^{2T} p^T$. By the Pumping Lemma, we can break w into three components: x, u, and y, with $u \neq \epsilon$ and $|xu| \leq T$.

Note that if $|xu| \leq T$, then xu must consist of a's as the first T characters in w are a's. Also if $u \neq \epsilon$, we must conclude $u = a^k$ for some $k \geq 1$.

Therefore, by the Pumping Lemma, for all n, $xu^n y \in L$. By choosing $n = 2$, however, we obtain a string $xu^2 y = a^q l^{2T} p^T$ with $q > T$. This string cannot be in L. We thus have obtained the needed contradiction showing that the language L cannot be regular.

11) Let M be the language

$$\{0101,\ 001001,\ 00010001,\ 0000100001,\ \ldots\}$$

whose words consist of some positive number n of occurrences of the digit 0, followed by the digit 1, followed by n further occurrences of the digit 0, and followed by the digit 1. (In particular, the number of occurrences of 0 preceding the first 1 is equal to the number of occurrences of 0 preceding the second 1.)

(a) Use the Pumping Lemma to show this language is not regular.
(b) Write down the production rules of a context-free grammar that generates exactly M. Justify your answer.

Solution: (a) If M is regular, then it has a pumping length p. Consider $w = 0^p 1 0^p 1 \in M$ and the decomposition $w = xuy$ with $|u| \geq 1$ and $|xu| \leq p$. Since $|xu| \leq p$, u can only consist of zeroes. Let $u = 0^{n_1}$, for some $n_1 \geq 1$. Clearly, $xu^2 y \notin M$ as $xu^2 y = 0^{p+n_1} 1 0^p 1$, so the length of the first sequence of zeroes is greater than that of the second sequence of zeroes violating the pattern of the language.

(b) Consider the following production rules:

(1) $\langle S\rangle \to 0\langle A\rangle 01$,
(2) $\langle A\rangle \to 0\langle A\rangle 0$,
(3) $\langle A\rangle \to 1$.

We can show by induction that a string w generated by these production rules is of is of one of the following forms:

- $w = \langle S\rangle$,
- $w = 0^n\langle A\rangle 0^n 1$,
- $w = 0^n 1 0^n 1$.

Here $n \geq 1$. These rules will then generate exactly M. Note how these rules differ from the production rules of a regular grammar as non-terminals occur on both sides of the non-terminal in the first two production rules.

12) Let (V, E) be the graph with vertices a, b, c, d, and e and edges ab, bd, be, ac, cd, and ae.

(a) Draw this graph. Write down its incidence table and its incidence matrix.
(b) Write down this graph's adjacency table and its adjacency matrix.
(c) Is this graph complete? Justify your answer.
(d) Is this graph bipartite? Justify your answer.
(e) Is this graph regular? Justify your answer.
(f) Does this graph have any regular subgraph? Justify your answer.
(g) Give an example of an isomorphism from the graph (V, E) specified at the beginning of this problem to the graph (V', E') with vertices p, q, r, s, and t, and edges pq, ps, rt, st, rs, and rq.

Solution: Let (V, E) be the graph with vertices a, b, c, d, and e and edges ab, bd, be, ac, cd, and ae.

(a) The graph is drawn at the end of the solutions. If we keep the same order of the vertices and edges given in the statement of the problem, the incidence table is:

	ab	bd	be	ac	cd	ae
a	1	0	0	1	0	1
b	1	1	1	0	0	0
c	0	0	0	1	1	0
d	0	1	0	0	1	0
e	0	0	1	0	0	1

The corresponding incidence matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

(b) If we keep the same order of the vertices given in the statement of the problem, the adjacency table is:

	a	b	c	d	e
a	0	1	1	0	1
b	1	0	0	1	1
c	1	0	0	1	0
d	0	1	1	0	0
e	1	1	0	0	0

The corresponding adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

(c) No, as for example edge bc does not belong to the graph, so not every vertex is connected to every other vertex.

(d) No, as the graph contains the complete subgraph $V' = \{a, b, e\}$ and $E' = \{ab, ae, be\}$, which cannot be partitioned.

(e) No, as vertices a and b have degree 3, whereas the other vertices have degree 2.

(f) Any two vertices that have an edge between them taken with that edge form a regular subgraph (1-regular) as do $\{a, b, e\}$ and the edges between them (2-regular) and $\{a, b, c, d\}$ and the edges between them (2-regular).

(g) It does NOT suffice to show the two graphs have the same structure. An isomorphism is a MAP, so you must provide the map on vertices. Two possible isomorphisms are $\varphi(a) = s$, $\varphi(b) = r$, $\varphi(c) = p$, $\varphi(d) = q$, and $\varphi(e) = t$ or the following: $\varphi(a) = r$, $\varphi(b) = s$, $\varphi(c) = q$, $\varphi(d) = p$, and $\varphi(e) = t$.

13) Let (V, E) be the graph with vertices a, b, c, d, and e and edges ab, bd, be, ac, cd, and ae. Does this graph have a Hamiltonian circuit? Justify your answer.

14) For what type of n does the complete graph K_n have an Eulerian circuit? Justify your answer.

15) For what type of n does the complete graph K_n have an Eulerian trail? Justify your answer.

16) For what type of n does the complete graph K_n have a Hamiltonian circuit? Justify your answer.

17) For what type of p and q does the complete bipartite graph $K_{p,q}$ have an Eulerian circuit? Justify your answer.

18) For what type of p and q does the complete bipartite graph $K_{p,q}$ have an Eulerian trail? Justify your answer.
19) For what type of p and q does the complete bipartite graph $K_{p,q}$ have a Hamiltonian circuit? Justify your answer.

20) Consider the language over the binary alphabet $A = \{0, 1\}$ given by $L = \{0^m 1^{2m} \mid m \in \mathbb{N}\}$.

(a) Use the Pumping Lemma to show L is not a regular language.

(b) Write down the algorithm of a Turing machine that recognizes L. Process the following strings according to your algorithm: ϵ, 01, 011, and 010.

(c) Write down the transition diagram of the Turing machine from part (a) carefully labelling the initial state, the accept state, the reject state, and all the transitions specified in your algorithm.

(d) Is the language L finite, countably infinite, or uncountably infinite? Justify your answer.

Solution: (a) If L is a regular language, then it has a pumping length p. In order to consider just one case, we work with $w = 0^p 1^{2p} \in L$. According to the Pumping Lemma, w is to be decomposed as xuy, where $|u| \geq 1$ and $|xu| \leq p$. Since $|xu| \leq p$, u can only consist of zeroes. Let $u = 0^{n_1}$, for some $n_1 \geq 1$. Clearly, $xu^2y \notin L$ as $xu^2y = 0^{p+n_1}1^{2p}$, so the length of the first sequence of zeroes is not one half that of the second sequence of zeroes violating the pattern of the language.

(b) Here is the algorithm for recognising $L = \{0^m 1^{2m} : m \in \mathbb{N}\}$.

(1) If there is a blank in the first cell, ACCEPT. If there is anything else, apart from 0, then REJECT.
(2) If 0 is in the current cell, delete it, then move right to the first 1.
(3) If there is no first 1, REJECT. Otherwise change 1 to x.
(4) Move to the leftmost non blank symbol. If 0, go to step 2. If 1, REJECT. If x, go to step 5. If y, go to step 6.
(5) Delete x, move right to the nearest 1. If none, REJECT. Otherwise change it to y and go to step 4.
(6) Move right to the rightmost non blank character. If anything but y is found, REJECT. Otherwise, ACCEPT.

Here is how the following strings are treated:

- ϵ is accepted immediately.
- $01 \to \sqcup 1 \to \sqcup x \to \sqcup\sqcup \to$ REJECT.

- $011 \to \llcorner 11 \to \llcorner x1 \to \llcorner\llcorner 1 \to \llcorner\llcorner y \to$ ACCEPT.
- $010 \to \llcorner 10 \to \llcorner x0 \to \llcorner\llcorner 0 \to$ REJECT.

(c) The transition diagram for

$$T = (\{i, s_1, s_2, s_3, s_4, s_5, s_{\text{acc}}, s_{\text{rej}}\}, \{0, 1\}, \{0, 1, x, y, \llcorner\}, t, i, s_{\text{acc}}, s_{\text{rej}})$$

is at the end of the solution set, along with an example of an accepted string.

(d) The language L is countably infinite. Consider the function $f : \mathbb{N} \to L$ given by $f(m) = 0^m 1^{2m}$. It is easy to see that f is both injective and surjective hence bijective. Therefore, L is in one-to-one correspondence with $\mathbb{N}$, hence L is countably infinite.