MAU22C00 - TUTORIAL 6 SOLUTIONS

1) Let $A = \{3^p \mid p \in \mathbb{N}\}$ with the operation of multiplication.

(a) Is $(A, \cdot)$ a semigroup? Justify your answer.
(b) Is $(A, \cdot)$ a monoid? Justify your answer.
(c) Does $(A, \cdot)$ have invertible elements? If so, which of its elements are invertible? Justify your answer.

Solution: (a) Yes, $(A, \cdot)$ is a semi-group. $A \subset \mathbb{Q}^*$, and $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ is a monoid under the operation of multiplication. We proved in lecture that if $a \in M$ for M a monoid with operation $*$ and $m, n \in \mathbb{N}$, then $a^m * a^n = a^{m+n}$. Here $a = 3$ and since addition is a binary operation on $\mathbb{N}$ as we showed in class, multiplication is a binary operation on A. The associativity of multiplication on A follows from the associativity of addition on $\mathbb{N}$ and the theorem that if $a \in M$ for M a monoid with operation $*$ and $m, n \in \mathbb{N}$, then $a^m * a^n = a^{m+n}$.

(b) Yes, $(A, \cdot)$ is a monoid. $3^0 = 1$ is the identity element on A because any $b \in A$ is of the form 3^p, so $b \cdot 1 = a^p \cdot a^0 = a^{p+0} = a^{0+p} = 1 \cdot b = a^p = b$.

(c) By the theorem on powers proved in lecture that was quoted above, $3^m \cdot 3^n = 3^{m+n}$. The condition for invertibility $3^{m+n} = 3^0$ for $m \geq 0$ and $n \geq 0$ is only satisfied if $m = n = 0$. Therefore, $3^0 = 1$ is the only invertible element of A.

2) (Slightly modified question from the annual exam 2017-2018) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 0\}$ with the operation of addition given by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

(a) Is $(A, +)$ a semigroup? Justify your answer.
(b) Is $(A, +)$ a monoid? Justify your answer.
(c) Does $(A, +)$ have invertible elements? If so, which of its elements are invertible? Justify your answer.
(d) What geometric object is the set A in $\mathbb{R}^2$?

Solution: (a) Yes, $(A, +)$ is a semi-group. If $x_1 = -2y_1$ and $x_2 = -2y_2$, then $x_1 + x_2 = -2y_1 - 2y_2 = -2(y_1 + y_2)$, so $+$ is a binary operation on A. We proved in lecture that addition is an associative binary operation on $\mathbb{R}$, so $+$ is associative on A as associativity will function component by component in the vector (x, y).

(b) Yes, $(A,+)$ is a monoid. $(0,0)$ is the identity element on A because for any $(x,y) \in A$,

$$(x,y)+(0,0)=(x+0,y+0)=(0+x,0+y)=(0,0)+(x,y)=(x,y).$$

(c) For any $(x,y) \in A$, $(-x,-y)$ is its inverse because $(x,y)+(-x,-y) = (-x,-y)+(x,y)=(0,0)$. Therefore, all elements of A are invertible.

(d) A is the line passing through the origin $(0,0)$ and the point $(2,-1)$ as $2+2(-1)=0$.