Sample Questions 5 - Sol

Q1. $f(x) = 2e^{-2x} \quad x > 0$

$\text{Var}(X^2+1) = \text{var}(X^2) = E[X^4] - E^2[X^2]$

$$E[X^k] = \int x^k f(x)\,dx = 2\int_0^\infty x^k e^{-2x}\,dx = \frac{2\,k!}{2^{k+1}}$$

Then find $E[X^2]$ and $E[X^4]$

Hint : $$\int_0^\infty x^k e^{-\alpha x}\,dx = \frac{k!}{\alpha^{k+1}}$$

Q2. $X \sim U(11{:}00, 11{:}25)$

$$P(X > 11{:}05) = \frac{25-5}{25-0} = 4/5$$

Q3. $$P(X>2 \mid X>1) = \frac{P(X>2 \cap X>1)}{P(X>1)} = \frac{P(X>2)}{P(X>1)}$$

$$= \frac{1-P(X\le 2)}{1-P(X\le 1)} = \frac{1-\Phi\left(\frac{2-1}{1}\right)}{1-\Phi\left(\frac{1-1}{1}\right)} = \frac{1-\Phi(1)}{1-\Phi(0)}$$

$$= 2\left[1-\Phi(1)\right]$$

Q4. $$P(X_1 X_2 < 0) = \binom{2}{1} P(X_i > 0) P(X_j < 0)$$

$$= 2\left(\int_0^1 (ax + \tfrac{1}{2})\,dx\right)\left(1 - \int_0^1 (ax+\tfrac{1}{2})\,dx\right) = \dots$$

Q6. $\int_0^{\Phi_1} \lambda e^{-\lambda x} dx = \frac{1}{4} = 1 - e^{-\lambda \Phi_1} = \frac{1}{4} \Rightarrow e^{-\lambda \Phi_1} = 3/4$

$\int_0^{\Phi_3} \lambda e^{-\lambda x} dx = 3/4 \Rightarrow 1 - e^{-\lambda \Phi_3} = 3/4 \Rightarrow e^{-\lambda \Phi_3} = 1/4$

$\Rightarrow e^{\lambda(\Phi_3 - \Phi_1)} = \frac{3/4}{1/4} = 3$

$\Rightarrow \lambda(\Phi_3 - \Phi_1) = \ln 3 \Rightarrow \Phi_3 - \Phi = \frac{\ln 3}{\lambda}$

Q7. $r = 20/h \quad t = 5 \min \quad rt = \lambda = 20 \times \frac{1}{12} = 5/3$

$P(X=0) = \frac{e^{-5/3} (5/3)^0}{0!}$

or $P(T>5) = \int_5^{\infty} \frac{1}{3} e^{-t/3} dt = e^{-5/3}$

Q8. $F(-\infty) = 0 \Rightarrow a = 0 \quad F(+\infty) = 1 \Rightarrow b = 1$

$P(X=2) = \lim_{x \to 2^+} F(x) - \lim_{x \to 2^-} F(x) = b - \frac{2^2}{5} = 1 - \frac{4}{5} = \frac{1}{5}$

$E(X) = \int_0^2 x \cdot \frac{2x}{5} dx + (2)\frac{1}{5} = \dots$

$E(X^2) = \int_0^2 x^2 \left(\frac{2x}{5}\right) dx + 2^2 \times \frac{1}{5} = \dots$

$Var(X) = E(X^2) - E^2(X)$