# MAU22C00: ASSIGNMENT 5 SOLUTIONS

Each of the following problems is worth 5 points, 2 points for the answer and 3 points for the justification:

(1) Let $A = \mathbb{N}\times\mathbb{Z}\times\mathbb{Q}\times\mathbb{C}$. Is $A$ finite, countably infinite or uncountably infinite? Justify your answer.

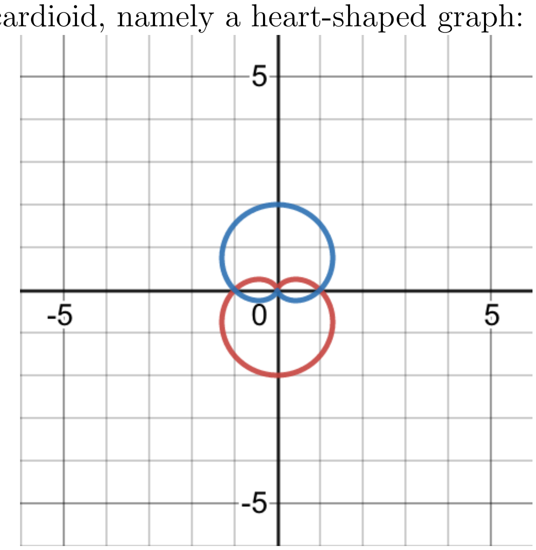
**Solution:** A is uncountably infinite since it is given by a Cartesian product, which has a factor $\mathbb{C}$ that is uncountably infinite. Here is how we prove that $\mathbb{C}$ is uncountably infinite:

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\} \sim \{(a, b) \mid a, b \in \mathbb{R}\} = \mathbb{R}^2$$

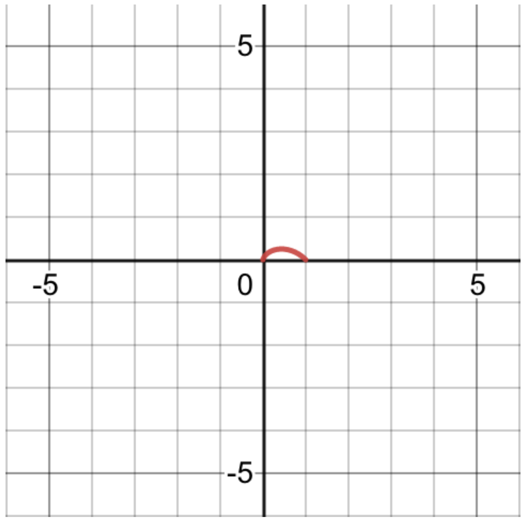
We have already shown that $\mathbb{R}^n$ for $n \geq 1$ is uncountably infinite in tutorial 18. Please refer to the argument given in the solutions to tutorial 18.

(2) Let $A$ be the the set of points in $\mathbb{R}^2$ whose polar coordinates $(r, \theta)$ satisfy the equation $r^2 = (\sin(\theta) - 1)^2$. Is $A$ finite, countably infinite or uncountably infinite? Justify your answer.

**Solution:** $A$ is uncountably infinite. $r^2 = (\sin(\theta) - 1)^2$ is equivalent to $r = \pm(\sin(\theta) - 1)$. The graph of $A$ is below, where the red graph is $r = 1 - \sin(\theta)$, and the blue graph is $r = \sin(\theta) - 1$. Each of them is a cardioid, namely a heart-shaped graph:




Let us consider only the piece of the red graph $r = 1 - \sin(\theta)$ given by restricting $\theta$ to the interval $0 \leq \theta \leq \frac{\pi}{2}$. The graph is below:




Since $\sin(\theta)$ is a bijective function for $\theta \in \left[0, \frac{\pi}{2}\right]$, this red piece as a set in $\mathbb{R}^2$ is in bijective correspondence to $\left[0, \frac{\pi}{2}\right]$, hence with the closed interval $[0, 1]$ (use the bijection $f : \left[0, \frac{\pi}{2}\right] \to [0, 1]$ given by $f(x) = \frac{2}{\pi} x$.) The closed interval $[0, 1]$ contains as a proper set the open interval $(0, 1)$. We proved in lecture that $(0, 1)$ is uncountably infinite. Therefore, $[0, 1]$ must be uncountably infinite as it has an uncountably infinite subset. Thus, $\left[0, \frac{\pi}{2}\right]$ is uncountably infinite, hence that red piece of the cardioid is uncountably infinite. That red piece is a subset of $A$, however, which means that $A$ has an uncountably infinite subset. We conclude that $A$ must be uncountably infinite.

(3) Let $A = \{(x, y) \in \mathbb{C}^2 \mid x^6 - 3x^2 + 1 = 0\}$. Is $A$ finite, countably infinite or uncountably infinite? Justify your answer.

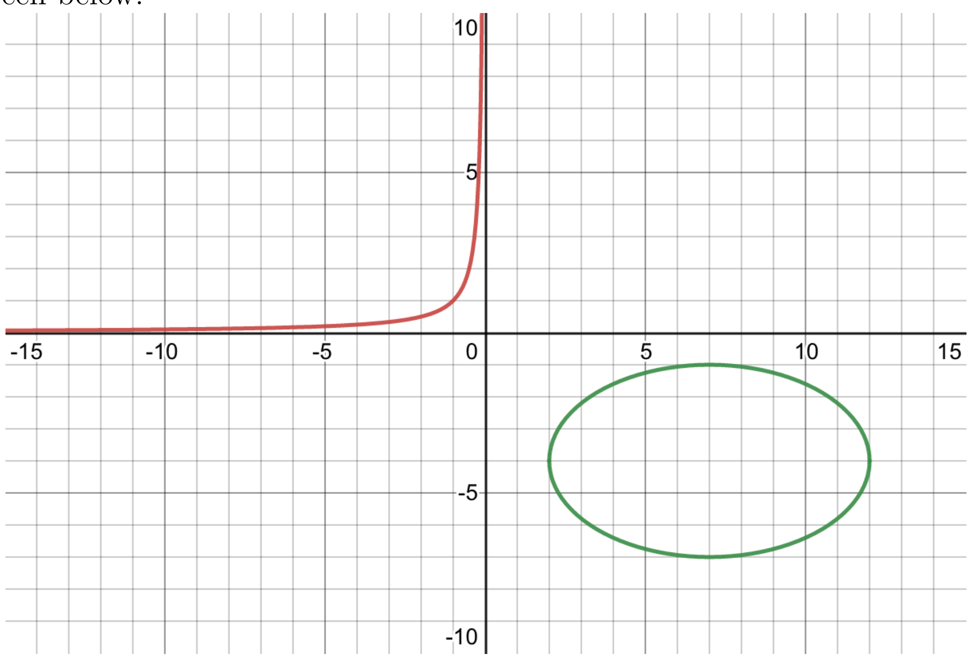
**Solution:** $A$ is uncountably infinite. By the Fundamental Theorem of Algebra, there are at most six $x \in \mathbb{C}$ that satisfy the equation $x^6 - 3x^2 + 1 = 0$. Recall that the Fundamental Theorem of Algebra tells us that the number of roots of a polynomial equation counted with multiplicity equals the degree of the polynomial, which is 6 here. There is no condition on $y$, however, which means $A$ is the Cartesian product of a finite set with $\mathbb{C}$. As explained in the solution of problem (1), $\mathbb{C}$ is uncountably infinite, and since $A$ is the Cartesian product containing one uncountably infinite factor, it is itself uncountably infinite.

(4) Let

$$A = \{(x,y) \in \mathbb{R}\times\mathbb{R}^+ \big| 1+xy = 0\} \cap \left\{(x,y) \in \mathbb{R}^2 \;\Big|\; \frac{(x-7)^2}{25} + \frac{(y+4)^2}{9} = 1\right\}.$$

$\mathbb{R}^+$ stands for all positive real numbers. Consider $\mathcal{P}(A)$, the power set of $A$. Is $\mathcal{P}(A)$ finite, countably infinite or uncountably infinite? Justify your answer.

**Solution:** $\{(x,y) \in \mathbb{R}\times\mathbb{R}^+ \mid 1+xy = 0\}$ is the part of the hyperbola $1+xy = 0$ satisfying that $y \geq 0$, namely the branch of the hyperbola in the second quadrant. This graph is red below. $\dfrac{(x-7)^2}{25} + \dfrac{(y+4)^2}{9} = 1$ is an ellipse with center at the point with coordinates $(7,-4)$, horizontal radius 5 (the square root of 25), and vertical radius 3 (the square root of 9). This ellipse is entirely inside the fourth quadrant. Its graph is green below:




As you can see, the intersection of the red graph with the green one is empty, so $A = \emptyset$. Therefore, the power set of it $\mathcal{P}(A)$ must contain exactly one element, namely $\{\emptyset\}$, the set containing the empty set as we saw in the unit on set theory. Thus, $\mathcal{P}(A)$ is finite.

(5) Let $A$ consist of all $2\times 2$ matrices with entries in the real numbers $\mathbb{R}$ and determinant equal to 1. Is $A$ finite, countably infinite or uncountably infinite? Justify your answer.

**Solution:** $A$ consists of elements that are matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfying that $ad - bc = 1$ (this is the condition that the determinant is 1 as you hopefully remember from linear algebra, which you studied with Meriel last year). In particular, every element of the form

$$\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$$

for $a \in \mathbb{R} \setminus \{0\}$ is in $A$. Let the set of all such elements be called B. Note that B is in one-to-one correspondence with $\mathbb{R} \setminus \{0\}$, but the open interval $(0, 1)$, which we proved to be uncountably infinite in lecture is a subset of $\mathbb{R} \setminus \{0\}$. Therefore, $\mathbb{R} \setminus \{0\}$ is uncountably infinite, and likewise B is uncountably infinite. Therefore, since $B \subset A$, A contains an uncountably infinite subset and hence must be uncountably infinite itself.

(6)] Let $A = \{(x, y, z) \in \mathbb{R}^3 \mid 3x - y + 2z = 0 \quad \text{and} \quad x + 2y + 3z = 0\}$. Is $A$ finite, countably infinite or uncountably infinite? Justify your answer.

**Solution:** As you hopefully remember from Meriel's linear algebra, $3x-y+2z = 0$ determines a plane perpendicular to the vector $(3, -1, 2)$, and $x + 2y + 3z = 0$ determines a plane perpendicular to the vector $(1, 2, 3)$. As these two vectors are linearly independent, the two planes are not parallel but rather must intersect in a line. Therefore, $A$ is a line in $\mathbb{R}^3$, which is in bijective correspondence to any other line in $\mathbb{R}^3$. Consider the x-axis in $\mathbb{R}^3$ (which is a line), and remember that it contains the interval $(0, 1)$, which we proved to be uncountably infinite, as a subset. Therefore, the x-axis is uncountably infinite, and hence the line $A$ is likewise uncountably infinite.

(7) Let $A = \{0, 1\}$. Is the language $[(0 \cup \epsilon)^* \circ (1 \cup \epsilon)] \cap (A \circ A)^*$ finite, countably infinite, or uncountably infinite? Justify your answer.

**Solution:** As you showed in a tutorial, any regular expression gives a regular language that is either finite or countably infinite, and since our language is given by a regular expression, it can only be finite or countably infinite. Note that

$$(0^* \circ \epsilon) \cap (A \circ A)^* = 0^* \cap (A \circ A)^* = \{0^{2m} \mid m \in \mathbb{N}\}$$

is a sublanguage of our language, which is countably infinite because it can be put into one-to-one correspondence with $\mathbb{N}$, which is countably infinite, via the bijection $f(m) = 0^{2m}$. We conclude that the language given by $[(0 \cup \epsilon)^* \circ (1 \cup \epsilon)] \cap (A \circ A)^*$, which is at most countably infinite, contains a countably infinite sublanguage, so it must be countably infinite.

(8) Let $A$ be a countably infinite alphabet. Is $A^*$ finite, countably infinite or uncountably infinite? Justify your answer.

**Solution:** $A^* = \bigcup_{j=0}^{\infty} A^j$, where $A^j$ is the set of words of length $j$ over the alphabet $A$. Note that $A^j$ is in one-to-one correspondence with the Cartesian product of $A$ with itself $j$ times. When $j = 0$, $A^0 = \{\epsilon\}$, the set containing just the empty word, which is finite, but if $j \geq 1$, $A^j$ is the Cartesian product of the countably infinite set $A$ with itself $j$ times, which is countably infinite by a theorem proven in lecture. Therefore, $A^* = \{\epsilon\} \cup \bigcup_{j=1}^{\infty} A^j$ is the union of one element with a countably infinite union of countably infinite sets, which is countably infinite by the theorem we proved in lecture. We conclude that $A^*$ is countably infinite.

(9) Let $A = \{0, 1, 2, 3, 4, 5\}$. Let the language $L$ consist of all even length strings containing at least three odd letters. Is $L$ finite, countably infinite or uncountably infinite? Justify your answer.

**Solution:** Since $A$ is finite, $A^*$ is countably infinite as proven in lecture. $L \subset A^*$, so $L$ is either finite or countably infinite. We will prove that $L$ is countably infinite by showing it contains a countably infinite subset. This is once again a sandwich argument. Consider $L' = \{1^{2m}3^{2m}5^{2m} \mid m \in \mathbb{N}^*\}$. Clearly, $L' \subset L$, and $L'$ is in one-to-one correspondence with the countably infinite set $\mathbb{N}^*$ via the bijection $f : \mathbb{N}^* \to L'$ given by $f(m) = 1^{2m}3^{2m}5^{2m}$. Therefore, $L$ contains a countably infinite subset and hence must be countably infinite.

(10) Does there exist a sequence $\{x_1, x_2, x_3, \dots\}$ of languages over a finite alphabet $A$ such that $x_i$ is not a regular language $\forall\, i \geq 1$? Justify your answer.

**Solution:** Yes, such a sequence exists, and in fact, uncountably many such sequences exist. Some of you constructed the sequence explicitly. I will give here a formal proof for the existence of such a sequence. Let $B$ be the set of all languages over $A$. $B$ is uncountably infinite as proven in lecture. Let $C$ be the set of all regular languages over $A$. $C$ is countably infinite as proven in lecture. As we showed in lecture in the proof that $\mathbb{R}$ is uncountably infinite, the set difference between an uncountably infinite set and a countably infinite one must be uncountably infinite. Therefore, the set $B \setminus C$ of non-regular languages over the alphabet $A$ must be uncountably infinite. Since $B \setminus C$ is uncountably infinite, it is clearly non-empty. Therefore, $\exists\, x_1 \in B \setminus C$. Now, consider $B \setminus (C \cup \{x_1\})$. We have taken an element out of an uncountably infinite set. Therefore, $B \setminus (C \cup \{x_1\})$ is uncountably infinite, hence non-empty. Therefore, $\exists\, x_2 \in B \setminus (C \cup \{x_1\})$ and so on. Inductively, $B \setminus (C \cup \{x_1, x_2, \ldots, x_{n-1}\})$ is uncountably infinite, hence non-empty. Therefore, $\exists\, x_n \in B \setminus (C \cup \{x_1, x_2, \ldots, x_{n-1}\})$. By construction, each $x_i$ is not a regular language. Therefore, $\{x_1, x_2, x_3, \ldots\}$ is a sequence of the kind we were trying to construct, so such a sequence exists.