MAU22C00: ASSIGNMENT 1 SOLUTIONS

1) (10 points) Please carry out the following proof in propositional logic following the proof format in tutorial 1: Hypotheses: $P \to (Q \leftrightarrow \neg R)$, $P \vee \neg S$, $R \to S$, $\neg Q \to \neg R$. Conclusion: $\neg R$.

For each line of the proof, mention which tautology you used giving its number according to the list of tautologies posted in folder Course Documents. Solutions based on truth tables or any other method except for the one specified will be given **NO CREDIT**.

Solution: We wish to prove $\neg R$. We first list the hypotheses:

(1) $P \to (Q \leftrightarrow \neg R)$ hypothesis
(2) $P \vee \neg S$ hypothesis
(3) $R \to S$ hypothesis
(4) $\neg Q \to \neg R$ hypothesis
(5) $\neg S \vee P$ tautology #32 applied to (2)
(6) $S \to P$ tautology #21 applied to (5)
(7) $R \to P$ tautology #14 applied to (3) and (6)
(8) $R \to (Q \leftrightarrow \neg R)$ tautology #14 applied to (1) and (7)
(9) $R \to [(Q \to \neg R) \wedge (\neg R \to Q)]$ tautology #22 applied to (8)
(10) $[R \to (Q \to \neg R)] \wedge [R \to (\neg R \to Q)]$ tautology #25 applied to (9)
(11) $R \to (Q \to \neg R)$ tautology #4 applied to (10)
(12) $R \to (\neg R \to Q)$ tautology #4 applied to (10)
(13) $\neg R \to (Q \to \neg R)$ tautology #8
(14) $R \vee \neg R \to (Q \to \neg R)$ tautology #26 applied to (11) and (13)
(15) $R \vee \neg R$ tautology #1
(16) $Q \to \neg R$ modus ponens (tautology #10) applied to (14) and (15)
(17) $Q \vee \neg Q \to \neg R$ tautology #26 applied to (4) and (16)
(18) $Q \vee \neg Q$ tautology #1
(19) $\neg R$ modus ponens (tautology #10) applied to (17) and (18)

Marking rubric: 10 marks total. 4 marks for getting to a statement involving only Q and R and 6 marks for the rest of the solution.

2) (10 points) Prove the following statement: If n is any integer, then $n^2 - 3n$ must be even. (Hint: Cases come in handy here. See tautology

#26 for the basis of proofs by cases. This proof follows the format of the one given in lecture that $\sqrt{2}$ is not a rational number.)

Solution: We split the argument into cases based on whether n is even or odd:

Case 1: n is even. Therefore, $\exists\, k \in \mathbb{Z}$ such that $n = 2k$. We substitute this expression into $n^2 - 3n$ as follows: $n^2 - 3n = (2k)^2 - 3(2k) = 4k^2 - 6k = 2(2k^2 - 3k)$. Since $k \in \mathbb{Z}$, $2k^2 - 3k$ is likewise an integer since $\mathbb{Z}$ is closed under addition and multiplication. Let $p = 2k^2 - 3k$. $p \in \mathbb{Z}$ and $n^2 - 3n = 2p$. Therefore, $n^2 - 3n$ is even.

Case 2: n is odd. Therefore, $\exists\, k \in \mathbb{Z}$ such that $n = 2k + 1$. We substitute this expression into $n^2 - 3n$ as follows: $n^2 - 3n = (2k+1)^2 - 3(2k+1) = 4k^2 + 4k + 1 - 6k - 3 = 4k^2 - 2k - 2 = 2(2k^2 - k - 1)$. Once again, since $\mathbb{Z}$ is closed under addition and multiplication, $p = 2k^2 - k - 1 \in \mathbb{Z}$. Therefore, $n^2 - 3n = 2p$ for $p \in \mathbb{Z}$, so $n^2 - 3n$ is even.

We have proven that n is even $\implies$ $n^2 - 3n$ is even and that n is odd $\implies$ $n^2 - 3n$ is even. If n is an integer, n is either even or odd. Therefore, by tautology #26, we conclude that if n is any integer, $n^2 - 3n$ is even. $\square$

Marking rubric: 10 marks total, 5 marks for each case.

3) (10 points) Prove via inclusion in both directions that for any three sets A, B, and C

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C).$$

Venn diagrams, truth tables, or diagrams for simplifying statements in Boolean algebra such as Veitch diagrams are **NOT** acceptable and will not be awarded any credit.

Solution: We first prove "$\subseteq$:" $\forall x \in A \cap (B \setminus C)$, $x \in A$ and $x \in B \setminus C$. Therefore, $x \in A$ and $x \in B$ and $x \notin C$. Since $x \in A$ and $x \in B$, $x \in A \cap B$. Since $x \in A$ and $x \notin C$, we conclude $x \notin A \cap C$. Therefore, $x \in A \cap B$ and $x \notin A \cap C$. We conclude that $x \in (A \cap B) \setminus (A \cap C)$.

We now prove "$\supseteq$:" $\forall x \in (A \cap B) \setminus (A \cap C)$, by definition, $x \in A \cap B$ and $x \notin A \cap C$. Since $x \in A \cap B$, $x \in A$ and $x \in B$. Since $x \notin A \cap C$ but $x \in A$, it must be the case that $x \notin C$. Therefore, $x \in B$ and $x \notin C$, so $x \in B \setminus C$, but $x \in A$. Therefore, $x \in A \cap (B \setminus C)$ as needed.

Marking rubric: 10 marks total, 5 marks for each inclusion.

4) (10 points) Let $\mathbb{N} \times \mathbb{N}$ be the Cartesian product of the set of natural numbers with itself consisting of all ordered pairs (x_1, x_2) such that $x_1 \in \mathbb{N}$ and $x_2 \in \mathbb{N}$. We define a relation on its power set $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ as follows: $\forall A, B \in \mathcal{P}(\mathbb{N} \times \mathbb{N})$ $A \sim B$ iff $(A \setminus B) \cup (B \setminus A) = C$ and C

is a finite set. Determine whether or not $\sim$ is an equivalence relation and justify your answer by checking each of the three properties in the definition of an equivalence relation. Please note that a set C is finite if it has finitely many elements. In particular, the empty set $\emptyset$ has zero elements and is thus finite.

Solution: (a) We check whether the relation $\sim$ is reflexive. $(A \setminus A) \cup (A \setminus A) = A \setminus A = \emptyset$. Since $\emptyset$ has zero elements, it is a finite set. We conclude $A \sim A$ is true for all $A \in \mathcal{P}(\mathbb{N} \times \mathbb{N})$, so the relation $\sim$ is reflexive.

(b) We check whether the relation $\sim$ is symmetric. $\forall A, B \in \mathcal{P}(\mathbb{N}\times\mathbb{N})$ if $A \sim B$, then $(A \setminus B) \cup (B \setminus A) = C$ and C is a finite set. Note that $\cup$ is commutative, so $(A \setminus B) \cup (B \setminus A) = (B \setminus A) \cup (A \setminus B) = C$. Since C is finite, $B \sim A$ holds as well, so the relation $\sim$ is symmetric.

(c) We check whether the relation $\sim$ is transitive. $\forall A, B, C \in \mathcal{P}(\mathbb{N} \times \mathbb{N})$ if $A \sim B$ and $B \sim C$, then $(A \setminus B) \cup (B \setminus A) = D$ and D is a finite set and $(B \setminus C) \cup (C \setminus B) = E$ and E is a finite set. We need to check whether $(A \setminus C) \cup (C \setminus A)$ is a finite set. I claim that $(A \setminus C) \cup (C \setminus A) \subseteq D \cup E$. It suffices to prove that $A \setminus C \subseteq D \cup E$ and $C \setminus A \subseteq D \cup E$. To prove $A \setminus C \subseteq D \cup E$, we consider $\forall x \in A \setminus C$. From the definition, $x \in A$ and $x \notin C$. The argument splits into two cases:

Case 1: $x \in B$. Then $x \in B$ and $x \notin C$, so $x \in B \setminus C$, but $(B \setminus C) \cup (C \setminus B) = E$. Therefore, $x \in E$.

Case 2: $x \notin B$. Then $x \in A$ and $x \notin B$, so $x \in A \setminus B$, but $(A \setminus B) \cup (B \setminus A) = D$. Therefore, $x \in D$.

We conclude that if $x \in A \setminus C$, then $x \in D \cup E$. Therefore, $A \setminus C \subseteq D \cup E$. We carry out the same argument with C and A exchanged to conclude that $C \setminus A \subseteq D \cup E$. Putting the two together, we have that $(A \setminus C) \cup (C \setminus A) \subseteq D \cup E$. Now, we know $(A \setminus C) \cup (C \setminus A) = F$ for some $F \in \mathcal{P}(\mathbb{N} \times \mathbb{N})$. So we have shown that $F \subseteq D \cup E$, but we know both D and E are finite sets, which means their union $D \cup E$ must also be a finite set of size at most the sum of the sizes of D and E. Therefore, F is a subset of $D \cup E$, which is a finite set, so F itself must be finite. We conclude that $A \sim C$.

Marking rubric: 10 marks total, 2 marks for proving reflexivity, 2 marks for proving symmetry, and 6 marks for proving transitivity.