Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science
School of Computer Science and Statistics

SF Integrated Computer Science **Trinity Term 2021**
SF CSL

MAU22C00: Discrete Mathematics

Thursday, May 20 **Blackboard** **12:00-18:00**

Prof. Andreea Nicoara
anicoara@maths.tcd.ie

Instructions that apply to all take-home exams:

1. This is an open-book exam. You are allowed to use your class notes, textbooks and any material which is directly linked from the module's Blackboard page or from the module's webpage, if it has one. You may not use any other resources except where your examiner has specifically indicated in the "Additional instructions" section below. Similarly, you may only use software if its use is specifically permitted in that section. You are not allowed to collaborate, seek help from others, or provide help to others.

2. If you have any questions about the content of this exam, you may seek clarification from the lecturer using the e-mail address provided. You are not allowed to discuss this exam with others. You are not allowed to send exam questions or parts of exam questions to anyone or post them anywhere.

3. Unless otherwise indicated by the lecturer in the "Additional instructions" section, solutions must be submitted through Blackboard in the appropriate section of the module webpage by the deadline listed above. You must submit a single pdf file for each exam separately and sign the following declaration in each case. It is your responsibility to check that your submission has uploaded correctly in the correct section.

Additional instructions for this particular exam:

All questions have equal weight. Solve every question. You may use software to draw finite state acceptors or Turing machine diagrams if you wish. Solutions may be typed, handwritten or a combination of the two. Submit a **single** pdf file with your solutions.

Plagiarism declaration: I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar which are available through https://www.tcd.ie/calendar.

Signature: ______________________________

1. (a) Let $f : [0,1] \to \mathbb{R}$ be the function defined by $f(x) = 2x^3 + 3$ for all $x \in [0,1]$. Determine whether or not this function is injective and whether or not it is surjective. Justify your answers.

(8 points)

(b) Let $\mathfrak{B}$ be the set of all nonempty regular languages over a finite alphabet A. For $L, M \in \mathfrak{B}$, $L \sim M$ if and only if

$$L \cap M \neq \emptyset,$$

namely if languages L and M share at least one word. Determine the following:

(i) Whether or not the relation $\sim$ is *reflexive*;

(ii) Whether or not the relation $\sim$ is *symmetric*;

(iii) Whether or not the relation $\sim$ is *anti-symmetric*;

(iv) Whether or not the relation $\sim$ is *transitive*;

(v) Whether or not the relation $\sim$ is an *equivalence relation*;

(vi) Whether or not the relation $\sim$ is a *partial order*.

Give appropriate short proofs and/or counterexamples to justify your answer.

(12 points)

(End of Question)

2. Let $\mathfrak{C}$ be the set of all regular languages over a finite alphabet A consisting only of strings of even length, namely $\forall\, L \in \mathfrak{C}$ and $\forall\, w \in L$, the length of w is even. Let the operation be concatenation. In other words, for $L, M \in \mathfrak{C}$,

$$L \circ M = \{u \circ v \mid u \in L \text{ and } v \in M\}.$$

(a) Is $(\mathfrak{C}, \circ)$ a semigroup? Justify your answer.

(5 points)

(b) Is $(\mathfrak{C}, \circ)$ a monoid? Justify your answer.

(5 points)

(c) Is $(\mathfrak{C}, \circ)$ a group? Justify your answer.

(5 points)

(d) Is $\mathfrak{C}$ finite, countably infinite, or uncountably infinite? Justify your answer.

(5 points)

(End of Question)

3. Let L be the language over the alphabet $A = \{0, 1\}$ given by

$$L = \{(01)^m \mid m \in \mathbb{N}, m \geq 0\} \cup \{(10)^p \mid p \in \mathbb{N}, p \geq 1\}.$$

(a) Draw a finite state acceptor that accepts the language L. Carefully label all the states including the starting state and the finishing states as well as all the transitions.

(5 points)

(b) Devise a regular grammar in normal form that generates the language L. Be sure to specify the start symbol, the non-terminals, and all the production rules.

(5 points)

(c) Write down a regular expression that gives the language L and justify your answer.

(5 points)

(d) Is the language L finite, countably infinite, or uncountably infinite? Justify your answer.

(5 points)

(End of Question)

4. Let (V, E) be the undirected graph with vertices a, b, c, d, e, f, g, h, i, and j, and edges ab, ac, bc, bd, de, ce, df, fg, eg, dh, ej, bh, fh, fi, gi, gj, and cj.

   (a) (i) Draw this graph.

   (ii) Does (V, E) contain a complete subgraph? Justify your answer.

   (iii) What is the minimal number of edges you would have to remove to make this graph disconnected? Justify your answer.

   (iv) What is the minimal number of edges you would have to remove to make this graph regular? Justify your answer.

   (v) Does this graph have an Eulerian trail? Justify your answer.

   (vi) Does this graph have a Hamiltonian circuit? Justify your answer.

   (vii) Is this graph a tree? Justify your answer.

(7 points)

   (b) Give two distinct examples of isomorphisms $\varphi : V \to V$ from the graph (V, E) to itself.

(4 points)

   (c) Let a cost function be given on (V, E) according to the following table:

fi	fg	ac	eg	gi	ce	dh	gj	bc
1	1	1	2	2	3	4	4	5
df	ej	ab	bh	de	bd	cj	fh	
6	7	8	9	9	10	11	12	

   Determine the minimum spanning tree generated by Prim's Algorithm, starting from the vertex e, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, write down the edge that is added.

(9 points)

(End of Question)

5. (a) Is the set of all natural numbers divisible by 4 finite, countably infinite, or uncountably infinite? Justify your answer.

(5 points)

(b) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 1\}$. Is A finite, countably infinite, or uncountably infinite? Justify your answer.

(5 points)

(c) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 25\} \cap \mathbb{Z}^2$. Is A finite, countably infinite, or uncountably infinite? Justify your answer.

(5 points)

(d) In lecture we defined the language

$$EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and are Turing machines and } L(M_1) = L(M_2)\}.$$

Is EQ_{TM} finite, countably infinite, or uncountably infinite? Justify your answer.

(5 points)

(End of Question)

6. (a) Consider the language over the binary alphabet $A = \{0, 1\}$ given by

$$L = \{01^m01^m \mid m \in \mathbb{N},\ m \geq 0\}.$$

Use the Pumping Lemma to show the language L is not regular.

(6 points)

(b) Let A be a finite alphabet, and let L_1 and L_2 be two Turing-recognisable languages over A such that L_1 is a proper subset of L_2, i.e. $L_1 \subset L_2$ but $L_1 \neq L_2$. Let a language L over the alphabet A satisfy that $L_1 \subset L \subset L_2$. Does L have to be Turing-recognisable as well? Justify your answer.

(7 points)

(c) Let L and M be two Turing-decidable languages over a finite alphabet A. Write down an enumerator that outputs the language

$$L - M = \{u \mid u \in L \text{ and } u \notin M\}.$$

Explain why the enumerator you wrote down does indeed output all strings in $L - M$ and no others.

(7 points)

(End of Question)