3. Properties of Expectation
   a. $E[h(X)] = \begin{cases} \sum_{\text{all } x} h(x)f(x) & \text{(for a discrete distribution)} \\ \int_{-\infty}^{\infty} h(x)f(x)dx & \text{(for a continuous distribution)} \end{cases}$
   b. For constants $a$ and $b$, $E(aX+b) = aE(X)+b$ and $Var(aX+b) = a^2Var(X)$.
   c. $E[h(X_1,\cdots,X_n)] = \begin{cases} \sum_{x_1}\cdots\sum_{x_n} h(x_1,\cdots,x_n)f(x_1,\cdots,x_n) & \text{discrete} \\ \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(x_1,\cdots,x_n)f(x_1,\cdots,x_n)\,dx_1\cdots dx_n & \text{continuous} \end{cases}$
   d. $Cov(X_1,X_2) = E[(X_1-\mu_1)(X_2-\mu_2)]$
   e. Correlation coefficient: $\rho = Cor(X_1,X_2) = \frac{Cov(X_1,X_2)}{\sigma_{X_1}\sigma_{X_2}}$

4. Distribution of a Linear Combination of Random Variables
   a. $E(a_1X_1+\cdots+a_nX_n) = a_1E(X_1)+\cdots+a_nE(X_n)$
   b. If $X_1, X_2, \cdots, X_n$ are independent,
      $\text{Var}(a_1X_1+\cdots+a_nX_n) = a_1^2Var(X_1)+\cdots+a_n^2Var(X_n)$
   c. If $X_1, X_2, \cdots, X_n$ are independent with $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, \cdots, n$, then
      $Y = a_1X_1+\ldots+a_nX_n \sim \text{N}(a_1\mu_1+\ldots+a_n\mu_n,\ a_1^2\sigma_1^2+\ldots+a_n^2\sigma_n^2)$

## EXERCISES

**5.1** Random variables $X_1$ and $X_2$ have the following distribution.

| $f(x_1, x_2)$ | | $x_2$: 0 | 10 | 20 |
|---|---|---|---|---|
| | 5 | 0.22 | 0.10 | 0.10 |
| $x_1$ | 10 | 0.10 | 0.08 | 0.12 |
| | 20 | 0.05 | 0.05 | 0.18 |

Answer the following questions.

a. Find $f(10, 10)$.
b. Find $P(X_1 \geq 10, X_2 \leq 10)$.
c. Find $f_1(10)$.
d. Find $f_2(20)$.
e. Find the marginal distributions of $X_1$ and $X_2$.
f. Find $P(X_1 \geq 10)$.
g. Find $P(X_2 \leq 10)$.

**5.2** Let the random variables $X$ and $Y$ have joint distribution

$$f(x, y) = \frac{2x + y}{30}, \qquad x = 1, 2; \;\; y = 1, 2, 3$$

a. Find the marginal distributions $f_1(x)$ of $X$ and $f_2(y)$ of $Y$.
b. Are $X$ and $Y$ independent? Justify your answer.

**5.3** Suppose the random variables $X$ and $Y$ have joint distribution as follows:

$$f(x, y) = \frac{1}{12}, \qquad x = 1, 2, 3; \; y = 1, 2, 3, 4$$

a. Find the marginal distributions $f_1(x)$ of $X$ and $f_2(y)$ of $Y$.
b. Show that $X$ and $Y$ are independent.

**5.4** Suppose the random variables $X$ and $Y$ have joint distribution as follows:

$$f(x, y) = \frac{xy}{36}, \; x = 1, 2, 3; \;\; y = 1, 2, 3$$

a. Find $f_1(x)$.
b. Find $f_2(y)$.
c. Find $f_1(x|y)$.
d. Find $f_2(y|x)$.
e. Find $P(X \leq 2)$.
f. Find $P(Y > 1)$.
g. Find $P(X \leq 2|Y > 1)$.
h. Are $X$ and $Y$ independent?

**5.5** For the distribution given in Example 5.1, perform the following calculations.
a. Find $f_1(x_1)$.
b. Find $f_2(x_2)$.
c. Find $f_1(x_1|x_2)$.
d. Find $f_2(x_2|x_1)$.
e. Find $P(X_1 \leq 1)$.
f. Find $P(X_2 > 0)$.
g. Find $P(X_1 \leq 1, X_2 < 2)$.
h. Find $P(X_1 + X_2 \leq 1)$.

**5.6** Suppose the random variables $X$, $Y$, and $Z$ have joint distribution as follows:

$$f(x, y, z) = \frac{xy^2z}{180}, \qquad x = 1, 2, 3;\ y = 1, 2;\ z = 1, 2, 3$$

a. Find the two-dimensional marginal distributions $f_{1,2}(x,y)$, $f_{1,3}(x,z)$ and $f_{2,3}(y,z)$.
b. Find the marginal distributions $f_1(x)$, $f_2(y)$, and $f_3(z)$.
c. Find $P(Y = 2 | X = 1, Z = 3)$.
d. Find $P(X \geq 2, Y = 2 | Z = 2)$.
e. Are $X$, $Y$, and $Z$ independent?

**5.7** Let $X$ denote the sum of the points in two tosses of a fair die.
a. Find the probability distribution and events corresponding to the values of $X$.
b. Obtain the cdf $F(x)$ of $X$.
c. Find $P(3 < X \leq 6)$.

**5.8** If $X$ and $Y$ are independent exponential random variables with pdf $f(x) = e^{-x}$, $x > 0$, find $P(X > 2, Y > 1)$.

**5.9** Let the random variables $X$ and $Y$ have joint cdf as follows:

$$F(x, y) = \begin{cases} 1 - e^{-2x} - e^{-3y} + e^{-2x-3y}, & x > 0,\ y > 0 \\ 0, & \text{otherwise} \end{cases}$$

a. Find the joint pdf $f(x, y)$ of $X$ and $Y$.
b. Are $X$ and $Y$ independent?
c. Find $f_1(x|y)$.
d. Find $P(X \leq 2 \text{ and } Y \geq 3)$.
e. Find the means of the two variables.
f. Find the variances of the two variables.
g. Find the correlation coefficient of $X$ and $Y$.

**5.10** Let the random variables $X$ and $Y$ have joint pdf given as

$$f(x, y) = 12x(1-x)y, \quad 0 < x < 1, 0 < y < 1$$

a. Find the marginal pdf $f_1(x)$ of $X$.
b. Find the marginal pdf $f_2(y)$ of $Y$.
c. Find the conditional pdf $f_1(x|y)$.
d. Are $X$ and $Y$ independent?
e. Find $P(X \leq 0.5|Y = 0.5)$.

**5.11** The joint pdf of $X$ and $Y$ is given as

$$f(x, y) = \frac{2(x+2y)}{3}, \ 0 < x < a, \ 0 < y < 1,$$

where $a$ is a constant.

a. Find the value of $a$.
b. Using the value of $a$ obtained in part (a), determine the marginal pdf's of $X$ and $Y$.
c. Determine the conditional pdf $f_1(x|y)$.
d. Find $P(X \leq 1/2|Y = 1/2)$.

**5.12** Suppose the random variables $X$ and $Y$ have joint pdf as follows:

$$f(x, y) = 15xy^2, \quad 0 < y < x < 1$$

a. Find the marginal pdf $f_1(x)$ of $X$.
b. Find the conditional pdf $f_2(y \mid x)$.
c. Find $P(Y > 1/3 \mid X = x)$ for any $1/3 < x < 1$.
d. Are $X$ and $Y$ independent? Justify your answer.

**5.13** Let the random variables $X$ and $Y$ have joint pdf as follows:

$$f(x, y) = \frac{4}{7}\left(x^2 + \frac{xy}{3}\right), \qquad 0 < x < 1, 0 < y < 3$$

a. Find the marginal densities of $X$ and $Y$.
b. Find the cdf of $X$ and cdf of $Y$.
c. Find $P(Y < 2)$.
d. Find $P(X > \frac{1}{2}, Y < 1)$.
e. Find $P(X > \frac{1}{2})$ and $P(Y < 1)$.
f. Are $X$ and $Y$ independent?
g. Determine the conditional pdf of $Y$ given $X = x$.
h. Find $P(1 < Y < 2 | X = \frac{1}{2})$.

**5.14** Let the joint pdf of $X$ and $Y$ be $f(x, y) = 12e^{-4x-3y}$, $x > 0$, $y > 0$.
a. Find the marginal pdf's of $X$ and $Y$.
b. Are $X$ and $Y$ independent?
c. Find the conditional pdf $f_1(x|y)$.
d. Find the marginal cdf's of $X$ and $Y$.
e. Find $P(1 < Y < 3)$.
f. Find $P(1 < Y < 3 | X = 3)$.
g. Find $P(X > 2, 1 < Y < 3)$.
h. Find $E(X)$ and $E(Y)$.
i. Find $Var(X)$ and $Var(Y)$.

**5.15** Random variables $X$ and $Y$ have the following joint probability distribution.

| | | $x$ | | |
|---|---|---|---|---|
| $f(x, y)$ | | 1 | 2 | 3 |
| $y$ | 1 | 0.1 | 0.3 | 0.2 |
| | 2 | 0.2 | 0.15 | 0.05 |

a. Find $P(X + Y > 3)$.
b. Find the marginal probability distributions $f_1(x)$ and $f_2(y)$.
c. Find $f_1(x|y = 2)$.
d. Are $X$ and $Y$ independent?
e. Find $E(X)$ and $E(Y)$.
f. Find $Var(X)$ and $Var(Y)$.
g. Find the correlation coefficient of $X$ and $Y$.

**5.16** Let random variables $X$ and $Y$ have the joint distribution given in the following table:

| $f(x, y)$ | | $y$: 0 | 1 | 2 |
|---|---|---|---|---|
| $x$ | 0 | 1/12 | 1/12 | 1/6 |
| | 1 | 0 | 1/3 | 1/4 |
| | 2 | 0 | 0 | 1/12 |

Answer the following questions:

a. Find the marginal probability distributions $f_1(x)$ and $f_2(y)$.
b. Find $E(X)$ and $E(Y)$.
c. Find $Var(X)$ and $Var(Y)$.
d. Find the correlation coefficient of $X$ and $Y$.

**5.17** Random variables $X$ and $Y$ have the following joint probability distribution.

| $f(x, y)$ | | $x$: 1 | 2 | 3 |
|---|---|---|---|---|
| $y$ | 1 | 0.1 | 0.1 | 0.15 |
| | 2 | 0.05 | 0.05 | 0.2 |
| | 3 | 0.15 | 0.1 | 0.1 |

a. Find $P(X + Y \leq 4)$.
b. Find the marginal probability distributions $f_1(x)$ and $f_2(y)$.
c. Find $P(X < 2 | Y = 2)$.
d. Are $X$ and $Y$ independent?
e. Find $E(X)$ and $E(Y)$.
f. Find $Var(X)$ and $Var(Y)$.
g. Find the correlation coefficient of $X$ and $Y$.

**5.18** Let the random variables $X$ and $Y$ have joint pdf as follows:

$$f(x, y) = cx + y, \qquad 0 < x < \frac{1}{2}, 0 < y < 1$$

a. Find the value of constant $c$.
b. Find the marginal densities of $X$ and $Y$.
c. Are the random variables $X$ and $Y$ independent?
d. Find the conditional pdf's $f_1(x|y)$ and $f_2(y|x)$.
e. Find $E(X)$ and $E(Y)$.
f. Find $Var(X)$ and $Var(Y)$.
g. Find $Cov(X, Y)$.
h. Find the correlation coefficient of $X$ and $Y$.

**5.19** Suppose the random variables $X$ and $Y$ have joint pdf $f(x, y) = 6y, 0 < y < x < 1$.

a. Find the marginal pdf of $X$ and marginal pdf of $Y$.
b. Find the conditional pdf of $X$ given $Y = y$.
c. Find $P(X > \frac{1}{2}|Y = \frac{1}{3})$.
d. Find $E(X)$ and $E(Y)$.
e. Find $Var(X)$ and $Var(Y)$.
f. Find $Cov(X, Y)$.
g. Find the correlation coefficient of $X$ and $Y$.

**5.20** Suppose the random variables $X$ and $Y$ have joint pdf $f(x, y) = \frac{1}{2}, 0 < y < x < 2$.

a. Find the marginal pdf of $X$ and marginal pdf of $Y$.
b. Find the conditional pdf of $Y$ given $X = x$.
c. Find the conditional pdf of $X$ given $Y = y$.
d. Find $P(X > \frac{3}{4}|Y = \frac{1}{3})$.
e. Find $E(X)$ and $E(Y)$.
f. Find $E(Y|X = x)$ and $E(X|Y = y)$.
g. Find $Var(X)$ and $Var(Y)$.
h. Find $Cov(X, Y)$.
i. Find the correlation coefficient of $X$ and $Y$.

**5.21** Let the random variables $X$ and $Y$ have joint pdf as follows:

$$f(x, y) = \frac{3}{4}(x^2 + 3y^2), \qquad 0 < x < 1, 0 < y < 1$$

a. Find the marginal densities of $X$ and $Y$.
b. Find the cdf of $X$ and cdf of $Y$.
c. Determine the conditional pdf's $f_1(x|y)$ and $f_2(y|x)$.
d. Find $E(X)$ and $E(Y)$.
e. Find $Var(X)$ and $Var(Y)$.
f. Find $Cov(X, Y)$.
g. Find $P(Y < 1/3|X = 1/3)$.
h. Find $E(Y|X = \frac{1}{2})$.

**5.22** Let $X$ have the following distribution.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f(x)$ | 0.1 | 0.2 | 0.4 | 0.3 |

a. Find $E(X)$.
b. Find the standard deviation of $X$.
c. Find $E(X^2 - 2)$.

**5.23** Let $X$ be a continuous random variable with pdf

$$f(x) = 3x^2, \ 0 < x < 1.$$

a. Find $E(X)$.
b. Find $E(4X + 5X^2)$.

**5.24** Let $X$ and $Y$ be independent random variables representing the lifetime (in 100 hours) of type A and type B lightbulbs, respectively. Both variables have exponential distributions, and the mean of $X$ is 2 and the mean of $Y$ is 3.

a. Find the joint pdf $f(x, y)$ of $X$ and $Y$.
b. Find the conditional pdf $f_2(y|x)$ of $Y$ given $X = x$.
c. Find the probability that a type A bulb lasts at least 300 hours and a type B bulb lasts at least 400 hours.
d. Given that a type B bulb fails at 300 hours, find the probability that a type A bulb lasts longer than 300 hours.
e. What is the expected total lifetime of two type A bulbs and one type B bulb?
f. What is the variance of the total lifetime of two type A bulbs and one type B bulb?

**5.25** Suppose $X$ and $Y$ are independent random variables such that $E(X) = 4$, $Var(X) = 9$, $E(Y) = 5$, $Var(Y) = 25$. Find $E(U)$ and $Var(U)$ where $U = 3X - Y + 2$.

**5.26** Suppose $X$ and $Y$ are independent random variables such that $E(X) = 5$, $Var(X) = 8$, $E(Y) = 3$, $Var(Y) = 5$. Find $E(V)$ and $Var(V)$ where $V = 2X - 3Y - 1$.

**5.27** Let $X_1$ and $X_2$ be independent normal random variables, distributed as $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, respectively. Consider a random variable $Y = 3X_1 - 2X_2$.

a. Find $E(Y)$.
b. Find $Var(Y)$.
c. Find the distribution of $Y$.

**5.28** Let $X_1, X_2, X_3$ be three independent normal random variables with expected values $\mu_1, \mu_2, \mu_3$ and variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$, respectively.

a. If $\mu_1 = \mu_2 = \mu_3 = 50$ and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 27$, find $P(\bar{X} < 45)$.

b. If $\mu_1 = \mu_2 = \mu_3 = 60$ and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 27$, find $P(X_1 + X_2 + X_3 > 170)$.

c. If $\mu_1 = \mu_2 = \mu_3 = 60$ and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 4$, find $P(-20 < 5X_1 - 3X_2 - 2X_3 < 20)$.

d. If $\mu_1 = 40,\ \mu_2 = 50,\ \mu_3 = 60$ and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 27$, find $P(130 < X_1 + X_2 + X_3 < 160)$.

e. If $\mu_1 = 40,\ \mu_2 = 50,\ \mu_3 = 60$ and $\sigma_1^2 = 4, \sigma_2^2 = 9, \sigma_3^2 = 16$, find $P\left(-10 < 4X_1 - 2X_2 - X_3 < 10\right)$.

**5.29** Let $X_1, \cdots, X_4$ be independent normal random variables and $X_i$ be distributed as $N(\mu_i, \sigma_i^2)$ for $i = 1, \cdots, 4$.

a. Find $P(35 < X_1 + \cdots + X_4 < 45)$ when $\mu_1 = \cdots = \mu_4 = 10$ and $\sigma_1^2 = \cdots = \sigma_4^2 = 2$.

b. Find $P(\bar{X} < 11)$ when $\mu_1 = \cdots = \mu_4 = 10$ and $\sigma_1^2 = \cdots = \sigma_4^2 = 2$.

c. Find $P(4X_1 + X_2 - X_3 - 4X_4 < 3)$ when $\mu_1 = \cdots = \mu_4 = 10$ and $\sigma_1^2 = \cdots = \sigma_4^2 = 2$.

d. Find $P(35 < X_1 + \cdots + X_4 < 45)$ when $\mu_1 = \mu_2 = 8,\ \mu_3 = \mu_4 = 12$, $\sigma_1^2 = \sigma_2^2 = 1$ and $\sigma_3^2 = \sigma_4^2 = 2$.

**5.30** In a certain city, the mean price of a two-liter bottle of soda is \$1.50 and the standard deviation is \$0.20 at various grocery stores. The mean price of one gallon of milk is \$3.80 and the standard deviation is \$0.30. Ten tourists came to the city together. They separately went to randomly chosen grocery stores, and each person bought one item. Six of them bought two-liter bottles of soda, and each of the remaining four bought one gallon of milk.

a. What is the expected total amount of money that the 10 people spent?

b. If the prices of individual items are independent and normally distributed, what is the probability that the total amount of money the 10 people spent is at least \$25?