MAU22C00 - TUTORIAL 2 SOLUTIONS

1) Prove $A \setminus (A \setminus B) \subseteq B$.

Solution: This is done by examining where the elements lie in the set to the left of $\subseteq$ and proving they also lie in B. To this end, take $x \in A \setminus (A \setminus B)$. By the definition of $X \setminus Y = X \cap Y^c$, we have

$$x \in A \setminus (A \setminus B) \Rightarrow x \in A \cap (A \setminus B)^c \Rightarrow x \in A \text{ AND } x \in (A \setminus B)^c$$

Applying the definition of $\setminus$ again. we conclude $x \in A$ AND $x \in (A \cap B^c)^c$. Using De Morgan's laws for the later, we get $x \in A$ AND $x \in A^c \cup (B^c)^c$. Let's focus more on the later, with the knowledge that $x \in A$.

$$x \in A^c \cup (B^c)^c \Rightarrow x \in A^c \text{ OR } x \in (B^c)^c \Rightarrow x \in A^c \text{ OR } x \in B$$

Since x cannot be in both A and A^c at the same time, we conclude $x \in B$ (now ignoring A). What we have shown:

$$\forall x (x \in A \setminus (A \setminus B) \Rightarrow x \in B)$$

So $A \setminus (A \setminus B) \subseteq B$ as required. $\square$

Remark. Veitch diagrams and/or Venn diagrams will **NOT** be accepted as a form of proof in set theory. Please bear in mind that only a solution of this kind is acceptable as a proof to an assertion in set theory.

2) For each of the following statements, determine whether it is either true or false and give a brief justification for your answer:

(a) $3 \in \mathcal{P}(\mathbb{N})$

(b) $\{3\} \in \mathcal{P}(\mathbb{N})$

(c) $\{3\} \subseteq \mathcal{P}(\mathbb{N})$

(d)$\{\emptyset\} \in \mathcal{P}(\{\{\emptyset\}\})$

(e) $\mathcal{P}(\mathbb{Z} \cap (2,4)) = \{\emptyset, \{3\}\}$, where $(2,4)$ means the interval with endpoints 2 and 4 on the real line.

Solution: (a) FALSE. The power set of $\mathbb{N}$ is a set whose elements are all the subsets of $\mathbb{N}$. We know $3 \in \mathbb{N}$, so 3 is an element of $\mathbb{N}$ but NOT a subset of $\mathbb{N}$, so $3 \in \mathcal{P}(\mathbb{N})$ is false.

(b) TRUE. As we showed in (a), $\mathcal{P}(\mathbb{N})$ consists of all subsets of $\mathbb{N}$. Since $3 \in \mathbb{N}$, the set consisting of the element 3 is a subset of $\mathbb{N}$, which written symbolically as $\{3\} \subseteq \mathbb{N}$, so $\{3\} \in \mathcal{P}(\mathbb{N})$ is true.

(c) FALSE. $\{3\}$ is an element of $\mathcal{P}(\mathbb{N})$ and NOT a subset of $\mathcal{P}(\mathbb{N})$, so $\{3\} \subseteq \mathcal{P}(\mathbb{N})$ is false. $\{\{3\}\} \subseteq \mathcal{P}(\mathbb{N})$ is true. In other words, the set consisting of $\{3\}$ is an element of $\mathcal{P}(\mathbb{N})$.

(d) FALSE. $\{\{\emptyset\}\}$ is the set consisting of one element $\{\emptyset\}$, which means $\mathcal{P}(\{\{\emptyset\}\}) = \{\emptyset, \{\{\emptyset\}\}\}$. $\{\emptyset\}$ is neither of the two elements of $\mathcal{P}(\{\{\emptyset\}\})$, so the statement is false.

(e) TRUE. $(2,4) = \{x \in \mathbb{R} \mid 2 < x < 4\}$, so $\mathbb{Z} \cap (2,4) = \{3\}$. $\mathcal{P}(\{3\}) = \{\emptyset, \{3\}\}$, so the statement is true.

Moral of the story: $\in$ means 'is an element of,' whereas $\subseteq$ means 'is a subset of.' You CANNOT use them interchangeably. You will be penalised on an exam or on the homework if you do. Please DO NOT make this mistake!

3) In the country of Tannu Tuva, a valid license plate consists of any digit except 0, followed by any two letters of the English alphabet, followed by any two digits.

(a) Let D be the set of all digits and L the set of all letters. With this notation, write the set of all possible license plates as a Cartesian product.

(b) How many possible license plates are there?

Solution: (a) The first character on the license plate belongs to the set $D \setminus \{0\}$ consisting of all digits but zero. The second character belongs to L, the third also to L, the fourth to D, and the fifth to D. The set of all possible license plates is the Cartesian product of all these sets in order, namely $(D \setminus \{0\}) \times L \times L \times D \times D$.

(b) The number of possible license plates is exactly the number of elements in the Cartesian product $(D \setminus \{0\}) \times L \times L \times D \times D$ from part (a). The number of elements of a **finite** Cartesian product is thus the product of the number of elements in each of the finite sets composing the product. We'll see in Hilary term what happens when we look at Cartesian products of infinite sets. Since there are 10 digits, $\#(D) = 10$, which means $\#(D \setminus \{0\}) = 9$. There are 26 letters in the English alphabet, so $\#(L) = 26$. We conclude that

$$\#((D \setminus \{0\}) \times L \times L \times D \times D = 9 \times 26 \times 26 \times 10 \times 10 = 608,400.$$