MAU22C00 - TUTORIAL 3 SOLUTIONS

1) (From the 2016-2017 Annual Exam) Let Q denote the relation on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xQy if and only if

$$x - y = (x - y)(x + 2y).$$

Determine the following:

(i) Whether or not the relation Q is *reflexive*;
(ii) Whether or not the relation Q is *symmetric*;
(iii) Whether or not the relation Q is *transitive*;
(iv) Whether or not the relation Q is an *equivalence relation*;
(v) Whether or not the relation Q is *anti-symmetric*;
(vi) Whether or not the relation Q is a *partial order*.

Justify your answers.

Solution: $x, y \in \mathbb{Z}$ satisfy xQy iff $x - y = (x - y)(x + 2y)$, which is equivalent to $(x - y)(x + 2y - 1) = 0$, i.e., $x = y$ or $x + 2y - 1 = 0$.

(i) **Reflexivity:** The relation Q is reflexive because xQx holds for all $x \in \mathbb{Z}$ as $x - x = (x - x)(x + 2x) = 0$.

(ii) **Symmetry:** The relation Q is not symmetric because if $x \neq y$, then xQy holds if $x + 2y = 1$, thus for yQx we would need $y + 2x = 1$, which only holds at the same time with $x+2y = 1$ when $x = y = \frac{1}{3} \notin \mathbb{Z}$.

(iii) **Anti-symmetry:** The relation Q is anti-symmetric. Having xQy and yQx when $x \neq y$ would imply $x+2y = 1$ and $y+2x = 1$ hold simultaneously, which gives $x = y = \frac{1}{3} \notin \mathbb{Z}$. Therefore, xQy and yQx can both be true only if $x = y$.

(iv) **Transitivity:** The relation Q is not transitive. Assume xQy and yQz hold for $x, y, z \in \mathbb{Z}$. There are 4 cases to consider:

Case 1: $x = y$ and $y = z$, then $x = z$, so xQz as needed.

Case 2: $x = y$ and $y + 2z = 1$, then $x + 2z = 1$, so xQz as needed.

Case 3: $x + 2y = 1$ and $y = z$, then $x + 2z = 1$, so xQz as needed.

Case 4: $x + 2y = 1$ and $y + 2z = 1$, then $x + 2(1 - 2z) = 1$, so $x + 2 - 4z = 1$, i.e., $x - 4z = -1$. This last equation is satisfied for example for $x = 3$, $z = 1$. Take $y = -1$ in order to satisfy $x + 2y = 1$. We see that $x + 2z = 3 + 2 = 5 \neq 1$, so xQz fails. We have constructed a counterexample.

(v) **Equivalence relation:** The relation Q is not an equivalence relation because while reflexive, it fails to be symmetric and transitive.

(vi) **Partial order:** The relation Q is not a partial order because while reflexive and anti-symmetric, it fails to be transitive.

2) In lecture we discussed an equivalence relation given by $f : A \to A$ for f any function on a non-empty set A with the relation R defined by $R = \{(x, y) \mid f(x) = f(y)\}$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \cos x$. What are the equivalence classes determined by the relation R on $\mathbb{R}$, namely for any $x \in \mathbb{R}$ what is $[x]_R$?

Solution: By the definition of our relation R,

$$[x]_R = \{y \in \mathbb{R} \mid \cos x = \cos y\}.$$

We know two key facts about $f(x) = \cos x$. The cosine function is periodic with period 2π, so $f(x) = f(x + 2n\pi)$ for all $n \in \mathbb{Z}$. Also, the cosine function is even, which means $\cos(x) = \cos(-x)$ for all $x \in \mathbb{R}$. We put these two properties together in order to write down the equivalence class of x, $[x]_R$:

$$[x]_R = \{x + 2n\pi \mid n \in \mathbb{Z}\} \cup \{-x + 2p\pi \mid p \in \mathbb{Z}\}.$$

Note that if $x = 0$, then $x = -x$, so $[0]_R = \{2n\pi \mid n \in \mathbb{Z}\}$.