MAU22C00: ASSIGNMENT 2
DUE BY THURSDAY, NOVEMBER 12 BEFORE MIDNIGHT
UPLOAD SOLUTION ON BLACKBOARD

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1) (10 points) Let $A = \mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R},\ 1 \leq i \leq n\}$. For $x, y \in A$, $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, xQy if and only if $\forall\, i$, $1 \leq i \leq n$, $x_i = y_i$ or $\exists\, i$ with $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j$ $\forall\, j$, $j < i$. Determine:

(i) Whether or not the relation Q is *reflexive*;
(ii) Whether or not the relation Q is *symmetric*;
(iii) Whether or not the relation Q is *anti-symmetric*;
(iv) Whether or not the relation Q is *transitive*;
(v) Whether or not the relation Q is an *equivalence relation*;
(vi) Whether or not the relation Q is a *partial order*.

Justify your answers.

2) (10 points) Use mathematical induction to prove that for all $n \geq 7$, $n! > 3^n$.

3) (20 points) (a) Let $\{C_n\}_{n=1,2,\dots} = \{C_1, C_2, \dots\}$ be a sequence of sets satisfying that $C_n \subseteq C_{n+1}$ $\forall\, n \geq 1$. Prove by mathematical induction that $C_m \subseteq C_n$ whenever $m < n$.

(b) Recall that the graph of a function $f : A \to B$ is given by

$$\Gamma(f) = \{(x, y) \mid x \in A \text{ and } y = f(x)\} \subseteq A \times B.$$

Let $Funct(A, B)$ the set of all functions $f : \tilde{A} \to \tilde{B}$ such that $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$. We define a relation on $Funct(A, B)$ as follows:

$\forall f, g \in Funct(A, B)$ $f \subseteq g$ iff $\Gamma(f) \subseteq \Gamma(g)$.

Prove that this relation is a partial order on $Funct(A, B)$.

(c) Let $\{f_n\}_{n=1,2,\dots} = \{f_1, f_2, \dots\}$ be a sequence of functions in $Funct(A, B)$ satisfying that $f_n \subseteq f_{n+1}$ for every $n \geq 1$. Since functions are in one-to-one correspondence with their graphs, we identify $\bigcup_{n\in\mathbb{N}} f_n$ with $\bigcup_{n\in\mathbb{N}} \Gamma(f_n)$. Using part (a), prove that $\bigcup_{n\in\mathbb{N}} f_n$ is a function and $\bigcup_{n\in\mathbb{N}} f_n \in Funct(A,B)$.

(d) For every $f \in Funct(A,B)$, let $Dom(f)$ be the domain of f, namely if $f : \tilde{A} \to \tilde{B}$ with $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$, $Dom(f) = \tilde{A}$. Prove that $Dom\left(\bigcup_{n\in\mathbb{N}} f_n\right) = \bigcup_{n\in\mathbb{N}} Dom(f_n)$ for every sequence of functions $\{f_n\}_{n=1,2,\dots} = \{f_1, f_2, \dots\}$ in $Funct(A,B)$ satisfying that $f_n \subseteq f_{n+1}$ for every $n \geq 1$.

4) (10 points) Let $\mathbb{R}[x]$ be the set of all polynomials in variable x with coefficients in $\mathbb{R}$. In other words,

$$\mathbb{R}[x] = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mid n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in \mathbb{R}\}.$$

(a) Give three examples of elements of $\mathbb{R}[x]$.

(b) Prove that $(\mathbb{R}[x], +)$, $\mathbb{R}[x]$ with addition as the operation, is a semigroup.

(c) Is $(\mathbb{R}[x], +)$ a monoid? Justify your answer.

(d) Does $(\mathbb{R}[x], +)$ have invertible elements? If so, which of its elements are invertible? Justify your answer.