

Proof of a Possible Form of Green Function on a lattice system

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I. STATEMENT

For a lattice system of any dimension with total size (number of lattice point) L , define annihilation operator as C_i and creation operator as C_i^\dagger for the i -th point. For a single particle, there are L independent eigenstate $|E_k\rangle$, with corresponding eigenenergy E_k . We can write

$$|E_k\rangle = U^{-1} |i\rangle, |i\rangle = U |E_k\rangle, \quad (1)$$

where $|n\rangle$ denotes the occupied state at n -th point. Assume the unitary transformation matrix U takes the form

$$U = \begin{pmatrix} a_{k_1 1} & a_{k_2 1} & \cdots & a_{k_L 1} \\ a_{k_1 2} & a_{k_2 2} & \cdots & a_{k_L 2} \\ \vdots & \ddots & & \\ a_{k_1 L} & a_{k_2 L} & \cdots & a_{k_L L} \end{pmatrix}, \quad (2)$$

and

$$|E_k\rangle = C_k^\dagger |0\rangle, |i\rangle = C_i^\dagger |0\rangle, \quad (3)$$

then the transformations of the annihilation operator and creation operator are

$$C_i = \sum_k a_{ki} C_k, C_i^\dagger = \sum_k a_{ki}^* C_k^\dagger \quad (4)$$

Generally, the Green function for the correlation between the i -th and j -th point can be written as

$$G_{ij} = \langle C_i^\dagger C_j \rangle. \quad (5)$$

We can PROVE that for an N -particle ground states ($N \leq L$), a possible form of the Green function is

$$G = \begin{pmatrix} a_{k_1 1}^* & a_{k_2 1}^* & \cdots & a_{k_N 1}^* \\ a_{k_1 2}^* & a_{k_2 2}^* & \cdots & a_{k_N 2}^* \\ \vdots & \ddots & & \\ a_{k_1 L}^* & a_{k_2 L}^* & \cdots & a_{k_N L}^* \end{pmatrix}_{L \times N} \begin{pmatrix} a_{k_1 1} & a_{k_1 2} & \cdots & a_{k_1 L} \\ a_{k_2 1} & a_{k_2 2} & \cdots & a_{k_2 L} \\ \vdots & \ddots & & \\ a_{k_N 1} & a_{k_N 2} & \cdots & a_{k_N L} \end{pmatrix}_{N \times L}. \quad (6)$$

II. LEMMA

To prove the statement above, we introduce the Wick's theorem as follow.

$$\langle C_1 C_2 \cdots C_{2N-1} C_{2N} \rangle = \sum_P \text{sgn}(P) \langle C_i C_j \rangle \langle C_k C_l \rangle \cdots \langle C_m C_n \rangle, \quad (7)$$

where N is an integer and the right-hand-side of the equation takes the sum of all combinations of expectation value for two operators. $\{i, j, k, l, \cdots, m, n\}$ is any permutation of $\{1, 2, \cdots, 2N-1, 2N\}$, and $\text{sgn}(P)$ denotes the sign of the permutation, while the relative position between any annihilation and creation operator pair should be preserved. Here the operators can be annihilation or creation. Obviously for a non-zero result the left-hand-side should be both of number N and any $\langle C_i C_j \rangle$ should contains one annihilation and one creation operator.

For example, if the state is a vacuum state and $N = 2$, the Wick's theorem shows that

$$\langle C_1 C_2 C_3^\dagger C_4^\dagger \rangle = -\langle C_1 C_3^\dagger \rangle \langle C_2 C_4^\dagger \rangle + \langle C_1 C_4^\dagger \rangle \langle C_2 C_3^\dagger \rangle, \langle C_1 C_2^\dagger C_3 C_4^\dagger \rangle = \langle C_1 C_2^\dagger \rangle \langle C_3 C_4^\dagger \rangle. \quad (8)$$

III. PROOF

The $N = 1$ case can be easily proved using relations of annihilation and creation operators, without using Wick's theorem. Now we prove the case for $N = 2$. A two-particle eigenstate can be written as

$$|E\rangle = C_{k_1}^\dagger C_{k_2}^\dagger |0\rangle. \quad (9)$$

Then the Green function can be written as

$$G_{ij} = \langle 0 | C_{k_2} C_{k_1} C_i^\dagger C_j C_{k_1}^\dagger C_{k_2}^\dagger | 0 \rangle. \quad (10)$$

After performing unitary transformation on C_i^\dagger and C_j , that is

$$C_j = \sum_{k'} a_{k'j} C_{k'}, C_i^\dagger = \sum_k a_{ki}^* C_k^\dagger, \quad (11)$$

the Green function becomes

$$G_{ij} = \sum_{kk'} a_{ki}^* a_{k'j} \langle 0 | C_{k_2} C_{k_1} C_k^\dagger C_{k'} C_{k_1}^\dagger C_{k_2}^\dagger | 0 \rangle. \quad (12)$$

Use Wick's theorem and notice the fact that

$$\langle 0 | C_m C_n^\dagger | 0 \rangle = \delta_{mn}, \quad (13)$$

which leads to two zero terms, so

$$\begin{aligned} & \langle 0 | C_{k_2} C_{k_1} C_k^\dagger C_{k'} C_{k_1}^\dagger C_{k_2}^\dagger | 0 \rangle \\ &= \langle 0 | C_{k_2} C_k^\dagger | 0 \rangle \langle 0 | C_{k_1} C_{k_1}^\dagger | 0 \rangle \langle 0 | C_{k'} C_{k_2}^\dagger | 0 \rangle \\ & \quad + \langle 0 | C_{k_2} C_{k_2}^\dagger | 0 \rangle \langle 0 | C_{k_1} C_k^\dagger | 0 \rangle \langle 0 | C_{k'} C_{k_1}^\dagger | 0 \rangle \\ &= \delta_{k_2k} \delta_{k'k_2} + \delta_{k_1k} \delta_{k'k_1}. \end{aligned} \quad (14)$$

Thus the Green function becomes

$$G_{ij} = a_{k_1i}^* a_{k_1j} + a_{k_2i}^* a_{k_2j}, \quad (15)$$

which can be also written as

$$G = \begin{pmatrix} a_{k_11}^* & a_{k_21}^* \\ a_{k_12}^* & a_{k_22}^* \\ \vdots & \vdots \\ a_{k_1L}^* & a_{k_2L}^* \end{pmatrix}_{L \times 2} \begin{pmatrix} a_{k_11} & a_{k_12} & \cdots & a_{k_1L} \\ a_{k_21} & a_{k_22} & \cdots & a_{k_2L} \end{pmatrix}_{2 \times L} (QED.). \quad (16)$$

For the cases of $N > 2$, similar process can be followed.