Non-Markovian Gaussian dissipative stochastic wave vector

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(Received 1 May 2000; revised manuscript received 14 August 2000; published 8 December 2000)

We give a complete characterization of the stochastic wave formalism when color Gaussian fluctuations are assumed. It is shown that in this case there is only one possible generalization of the stochastic wave dynamics, which is given by the ''non-Markovian quantum state diffusion model'' [Phys. Rev. A **58**, 1699 (1998)]. Using functional techniques, we found an equivalent evolution in terms of the stochastic propagator. From this equation any controlled approximation can be constructed. The mean density-matrix evolution is given in a closed form. A relation with quantum master equations is established. An interaction parameter expansion and a small correlation time expansion are developed.

DOI: 10.1103/PhysRevA.63.012106 PACS number(s): 03.65.−w, 03.65.Ta, 05.40.−a, 42.50.Lc

I. INTRODUCTION

In the last years ever-increasing interest has been paid to descriptions of open quantum systems in terms of stochastic wave functions (SWF's) [1–17]. In this formalism, the state of an open quantum system is described by an ensemble of pure states $|\Psi(t)\rangle$ instead of a reduced density matrix. This method permits a description of the dynamics of continuously monitored individual quantum systems, and hence provides additional information about the state of the system, compared to the description given by a reduced density matrix. Moreover, the possibility of working out numerically with SWF's, rather than with the density matrix, clearly reduces the computational time in solving complex dissipative system dynamics. The SWF evolution is constructed in such a way that the equation of motion governing the average of the stochastic matrix,

$$\rho_{st}(t) \equiv |\Psi(t)\rangle\langle\Psi(t)|, \tag{1}$$

coincides with the evolution equation of the reduced density matrix $\rho(t)$. Thus

$$\rho(t) = \langle \rho_{st}(t) \rangle, \tag{2}$$

where $\langle \cdots \rangle$ means the average over the stochastic ensemble.

Since the SWF method originated in the context of quantum optics, most work on this subject considered the ensemble dynamics as Markovian. In contrast, the conditions required to describe an open quantum system by means of the standard Born-Markov approximation become too restrictive when they are applied to the experimental advances made at the mesoscopic scale; examples are the spontaneous decay of an atom in a photonic band-gap material, real-time spectral measurement, femtosecond techniques, ion traps, etc.

Motivated by the above comments, many studies have been devoted to extend the SWF to non-Markovian evolutions [8–17]. Any of these generalizations must respond to two central questions: which is the generalized ensemble dynamics, and which is the mean density matrix evolution. It is remarkable that with only the assumption of non-white Gaussian fluctuations, it is possible to respond to these questions and give a full characterization of the non-Markovian generalized dynamics. In fact, this is the aim of the present work. We remark here that we are not interested in establishing a relation between the SWF dynamics and some measurement procedure. The present point of view is to determine the SWF dynamics on the basis of a minimum set of required conditions, and, by using functional techniques, to characterize the SWF and the mean density-matrix evolutions.

The paper is organized as follows: In Sec. II we find the SWF evolution starting from the Schrödinger-Langevin (SL) picture. In Sec. III we give a complete characterization of the dynamics in term of time-convolution equations. In Sec. IV we analyze the time-convolutionless approach. In Sec. V we establish a relation between the present formalism and quantum master equations. In Sec. VI we give the conclusions. In Appendix A we develop an interaction parameter expansion and a small correlation time expansion.

II. SCHRÖDINGER-LANGEVIN PICTURE AND QUANTUM DIFFUSION MODEL

In this section we will give a short review of the SL picture. By generalizing this formalism we will obtain the more general Gaussian SWF dynamics.

A. Schrödinger-Langevin picture

The SL model was first introduced by van Kampen [7]. The starting point of this formalism is to postulate a stochastic multiplicative equation for the state vector of an open quantum system $\mathcal{S}$. This equation is written in terms of an unknown Hermitian linear operator U representing dissipation and a random operator $\mathcal{F}(t)$ representing the effect of fluctuations, both of them as results of interaction with the ''external world.'' The SL equation reads

$$\frac{d}{dt}|\Psi(t)\rangle = [-iH_S - \lambda(U + i\mathcal{F}(t))]|\Psi(t)\rangle. \tag{3}$$

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Here λ is a coupling parameter, and it is assumed that $\langle \mathcal{F}(t)\rangle = 0$. Thus the stochastic matrix $\rho_{st}(t)$ evolves according to

$$\frac{d}{dt}\rho_{st}(t) = -i[H_S,\rho_{st}(t)] - \lambda\{U,\rho_{st}(t)\}_+ - i\lambda(\mathcal{F}(t)\rho_{st}(t) - \rho_{st}(t)\mathcal{F}^\dagger(t)), \tag{4}$$

where $\{\cdots\}_+$ denotes an anticommutator. Assuming that the influence of the environment can be represented in a sort of ''noisy way,'' the stochastic operator $\mathcal{F}(t)$ is written as a linear combination of complex-random numbers times operators in the Hilbert space of $\mathcal{S}$,

$$\mathcal{F}(t) = \sum_{\alpha=1}^{n} l_\alpha(t)V_\alpha, \tag{5}$$

where the noise correlations are defined by

$$\chi_{\alpha\beta}(t,s) \equiv \langle\langle l_\alpha^*(t) l_\beta(s)\rangle\rangle, \quad \langle\langle l_\alpha(t) l_\beta(s)\rangle\rangle \equiv 0. \tag{6}$$

Finally, the operator U is determined by demanding averaged trace conservation: $(d/dt)\mathrm{Tr}\langle\rho_{st}(t)\rangle = 0$. Assuming uncorrelated white noises [7], it is easy to find U, and the resulting evolution of $\langle\rho_{st}(t)\rangle$ is given by a standard diagonal Kossakowsky–Lindblad generator [18].

In the case of color noises, to close the evolution it is necessary to specify the approach used to find the mean evolution of $\rho_{st}(t)$. In fact, when averaging Eq. (4) the problematic terms are the mean values $\langle \mathcal{F}(t)\rho_{st}(t)\rangle$ and $\langle\rho_{st}(t)\mathcal{F}^\dagger(t)\rangle$. In previous publications we obtained these terms in a second-order cumulant expansion [16,17]. In this context, it is possible to demonstrate that the average evolution of $\rho_{st}(t)$ can be matched in one-to-one correspondence to the dynamics coming from the quantum Born-Markov approximation [17].

A remarkable fact is that the average of the multiplicative terms can be worked in an exact way, if—in correspondence with a bosonic environment—we assume Gaussian noises. This follows from Novikov's theorem [19]. This theorem gives an exact result for the mean value of the product of a Gaussian noise $\xi(t)$ with any functional $\mathcal{M}[\xi]$ of that noise,

$$\langle \xi(t)\mathcal{M}[\xi]\rangle = \langle\xi(t)\rangle\langle\mathcal{M}[\xi]\rangle + \int_0^t ds C_2(t,s)\left\langle \frac{\delta\mathcal{M}[\xi]}{\delta\xi(s)}\right\rangle, \tag{7}$$

where $C_2(t,s)$ is the cumulant of $\xi(t)$. The extension of this result to higher dimensions is straightforward, and will be applied to our problem.

B. Functional Schrödinger-Langevin picture

Now we will handle the SL dynamics with the aid of Novikov's theorem. From this theorem it is straightforward to realize that the trace conservation condition cannot be satisfied exactly by assuming deterministic dissipation, not even if it is time dependent. Thus, in what follows, in contrast to previous works, we start by postulating the stochastic density matrix dynamics

$$\frac{d}{dt}\rho_{st}(t) = -i[H_S,\rho_{st}(t)] - \mathcal{U}[\rho_{st}] - i\lambda(\mathcal{F}(t)\rho_{st}(t) - \rho_{st}(t)\mathcal{F}^\dagger(t)), \tag{8}$$

where the dissipative contribution is represented by the unknown functional $\mathcal{U}[\rho_{st}]$. The definitions of the fluctuations operator and of the noise correlations remain the same.

The unknown functional $\mathcal{U}[\rho_{st}]$ will be determined by the following conditions:

(i) Mean trace conservation $(d/dt)\mathrm{Tr}\langle\rho_{st}(t)\rangle = 0$.
(ii) Factorization $\rho_{st}(t) \equiv |\Psi(t)\rangle\langle\Psi(t)|$.
(iii) Non-Hermitian Gaussian fluctuations $\mathcal{F}(t) \neq \mathcal{F}^\dagger(t)$.

The last condition is necessary in order to obtain an explicitly dissipative dynamics and not a stochastic Hamiltonian one [17].

By averaging Eq. (8) and imposing condition (i), we obtain

$$\mathrm{Tr}\langle\mathcal{U}[\rho_{st}]\rangle = -i\lambda\,\mathrm{Tr}[\langle\mathcal{F}(t)\rho_{st}(t)\rangle - \langle\rho_{st}(t)\mathcal{F}^\dagger(t)\rangle]. \tag{9}$$

Now, using Novikov's theorem, the fact that $\rho_{st}(t)$ is a functional of all the noises $l_\alpha(t)$ and the correlations definitions (6), we obtain

$$\langle\mathcal{F}(t)\rho_{st}(t)\rangle = \int_0^t ds\,\chi_{\alpha\beta}^*(t,s)V_\alpha\left\langle\frac{\delta\rho_{st}(t)}{\delta l_\beta^*(s)}\right\rangle,$$
$$\langle\rho_{st}(t)\mathcal{F}^\dagger(t)\rangle = \int_0^t ds\,\chi_{\alpha\beta}(t,s)\left\langle\frac{\delta\rho_{st}(t)}{\delta l_\beta(s)}\right\rangle V_\alpha^\dagger. \tag{10}$$

From now on we assume the convention of sum over repeated indices. From these last expressions, it follows that the functional $\mathcal{U}[\rho_{st}]$ must satisfy

$$\mathrm{Tr}\langle\mathcal{U}[\rho_{st}]\rangle = i\lambda\int_0^t ds\,\mathrm{Tr}\left(\chi_{\alpha\beta}(t,s)\left\langle\frac{\delta\rho_{st}(t)}{\delta l_\beta(s)}\right\rangle V_\alpha^\dagger - \mathrm{H.c.}\right). \tag{11}$$

Due to the commutativity of the trace operation $[\mathrm{Tr}(AB) = \mathrm{Tr}(BA)]$, there is not a unique choice for $\mathcal{U}[\rho_{st}]$. These alternatives can be eliminated from conditions (ii) and (iii). Using these conditions and the analyticity of the SWF, we obtain

$$\frac{\delta\rho_{st}(t)}{\delta l_\beta(s)} = \frac{\delta|\Psi(t)\rangle}{\delta l_\beta(s)}\langle\Psi(t)|, \quad \frac{\delta\rho_{st}(t)}{\delta l_\beta^*(s)} = |\Psi(t)\rangle\frac{\delta\langle\Psi(t)|}{\delta l_\beta^*(s)}. \tag{12}$$

Inserting these expressions into Eq. (11), and again using the factorization condition, it follows that the only possible choice for the dissipative functional is

$$\mathcal{U}[\rho_{st}]=i\lambda\int_0^t ds\left(\chi_{\alpha\beta}(t,s)V_\alpha^\dagger\frac{\delta\rho_{st}(t)}{\delta l_\beta(s)}-\text{H.c.}\right), \qquad (13)$$

where $\delta\rho_{st}(t)/\delta l_\beta(s)$ and $\delta\rho_{st}(t)/\delta l_\beta^*(s)$ are given by Eq. (12). Therefore from Eq. (8), the evolution of the stochastic matrix is given by

$$\frac{d}{dt}\rho_{st}(t)=-i[H_S,\rho_{st}(t)]-i\lambda(\mathcal{F}(t)\rho_{st}(t)-\rho_{st}(t)\mathcal{F}^\dagger(t))$$
$$-i\lambda\int_0^t ds\left(\chi_{\alpha\beta}(t,s)V_\alpha^\dagger\frac{\delta\rho_{st}(t)}{\delta l_\beta(s)}-\text{H.c.}\right), \qquad (14)$$

and the corresponding SWF evolution is

$$\frac{d}{dt}|\Psi(t)\rangle=-iH_S|\Psi(t)\rangle-i\lambda\mathcal{F}(t)|\Psi(t)\rangle$$
$$-i\lambda V_\alpha^\dagger\int_0^t ds\,\chi_{\alpha\beta}(t,s)\frac{\delta|\Psi(t)\rangle}{\delta l_\beta(s)}. \qquad (15)$$

We remark that this equation corresponds to the *non-Markovian linear quantum state diffusion model*. This was obtained for the first time by Diosi and Strunz [12] from the influence functional of an oscillator in contact with a bosonic thermal bath at zero temperature. The present derivation relies only on the general conditions (i)–(iii).

Finally, averaging Eq. (14) over the noises and using expressions (10), the dynamics of the mean density matrix reads

$$\frac{d}{dt}\langle\rho_{st}(t)\rangle=-i[H_S,\langle\rho_{st}(t)\rangle]-i\lambda$$
$$\times\int_0^t ds\left(\chi_{\alpha\beta}(t,s)\left[V_\alpha^\dagger,\left\langle\frac{\delta\rho_{st}(t)}{\delta l_\beta(s)}\right\rangle\right]+\text{H.c.}\right). \qquad (16)$$

III. TIME-CONVOLUTION EQUATIONS

The evolution [Eq. (15)] looks like a very formal expression. The complexity of this equation comes from the ''dissipative'' contribution. This term introduces a memory integral on the unknown functional derivative of the state vector. In this section, differently from the previous approach [13–15], we will obtain this object in an exact way. From this solution, it is possible to obtain an equivalent evolution in terms of the stochastic propagator, from which any systematical approximations can be constructed.

A. Response functions hierarchy

Here we want to find an expression for the ''response function'' $\delta|\Psi(t)\rangle/\delta l_\beta(s)$. In order to obtain this object, we follow Refs. [19,20]. First, to short the notation, we rewrite Eq. (15) as

$$\frac{d}{dt}|\Psi(t)\rangle=-i\mathcal{T}_{st}(t)|\Psi(t)\rangle, \qquad (17)$$

where the functional linear stochastic operator $\mathcal{T}_{st}(t)$ is given by

$$\mathcal{T}_{st}(t)=H_S+\lambda\mathcal{F}(t)+\lambda V_\alpha^\dagger\int_0^t ds\,\chi_{\alpha\beta}(t,s)\frac{\delta}{\delta l_\beta(s)}. \qquad (18)$$

Integrating Eq. (17) formally, we obtain $|\Psi(t)\rangle=-i\int_0^t du\,\mathcal{T}_{st}(u)|\Psi(u)\rangle$, and taking a functional derivative with respect to $l_{\gamma_1}(t_1)$, we obtain $(t_1<t)$

$$\frac{\delta|\Psi(t)\rangle}{\delta l_{\gamma_1}(t_1)}=-i\lambda V_{\gamma_1}|\Psi(t_1)\rangle-i\int_{t_1}^t du\,\mathcal{T}_{st}(u)\frac{\delta|\Psi(u)\rangle}{\delta l_{\gamma_1}(t_1)}, \qquad (19)$$

where we have used the commutativity of functional derivatives. Now if Eq. (19) is differentiated with respect to *t*, an equation for the first variational derivative of the SWF is obtained,

$$\frac{d}{dt}\frac{\delta|\Psi(t)\rangle}{\delta l_{\gamma_1}(t_1)}=-i\mathcal{T}_{st}(t)\frac{\delta|\Psi(t)\rangle}{\delta l_{\gamma_1}(t_1)}, \qquad (20)$$

where the initial condition of this equation follows from the first term of the right-hand side of Eq. (19). We realize that in this evolution, the operator $\mathcal{T}_{st}(t)$ introduces the second functional derivative of the state vector. Using the same procedure, it is possible to obtain the evolution of these higher derivatives. We obtain

$$\frac{d}{dt}\frac{\delta^n|\Psi(t)\rangle}{\delta l_{\gamma_n}(t_n)\cdots\delta l_{\gamma_1}(t_1)}=-i\mathcal{T}_{st}(t)\frac{\delta^n|\Psi(t)\rangle}{\delta l_{\gamma_n}(t_n)\cdots\delta l_{\gamma_1}(t_1)}, \qquad (21)$$

where now the initial condition is given by

$$\left.\frac{\delta^n|\Psi(t)\rangle}{\delta l_{\gamma_n}(t_n)\cdots\delta l_{\gamma_1}(t_1)}\right|_{t=ti}=-i\lambda V_{\gamma_i}\left.\frac{\delta^{n-1}|\Psi(t)\rangle}{\delta l_{\gamma_n}(t_n)\cdots\delta l_{\gamma_{i+1}}(t_{i+1})\delta l_{\gamma_{i-1}}(t_{i-1})\cdots\delta l_{\gamma_1}(t_1)}\right|_{t=t_i} \qquad (22)$$

and where $t_i=\max\{t_1,\ldots,t_n\}<t$. In this way, we have obtained an ''infinite hierarchy of linear coupled stochastic equations.'' The coupling is given by the third term of $\mathcal{T}_{st}(t)$ [see Eq. (18)]. This term couples any *n*-functional derivative with the $n+1$ one. Furthermore, the initial conditions couples the *n*-functional derivative with the $n-1$ one.

Although this set of equations seems very complicated, we note that all the *n*-response functions follow the same evolution, which is the SWF one. This is a consequence of the linearity of the original evolution. Thus, defining the stochastic propagator G_{st} by

$$|\Psi(t)\rangle = G_{st}(t,s)|\Psi(s)\rangle, \tag{23}$$

with

$$G_{st}(t,s) = \lceil \exp\left(-i\int_s^t du \mathcal{T}_{st}(u)\right)\rceil, \tag{24}$$

where $\lceil \cdots \rceil$ indicates time ordering, we can write

$$\frac{\delta^n |\Psi(t)\rangle}{\delta l_{\gamma_n}(t_n)\cdots\delta l_{\gamma_1}(t_1)} = -i\lambda G_{st}(t,t_i)V_{\gamma_i} \times \frac{\delta^{n-1}|\Psi(t_i)\rangle}{\delta l_{\gamma_n}(t_n)\cdots\delta l_{\gamma_{i+1}}(t_{i+1})\,\delta l_{\gamma_{i-1}}(t_{i-1})\cdots\delta l_{\gamma_1}(t_1)}. \tag{25}$$

Applying this equation iteratively, and assuming that the set of times $\{t_1, \ldots, t_n\}$ is chronologically ordered, it follows that

$$\frac{\delta^n |\Psi(t)\rangle}{\delta l_{\gamma_n}(t_n)\cdots\delta l_{\gamma_1}(t_1)} = (-i\lambda)^n G_{st}(t,t_n)V_{\gamma_n}G_{st}(t_n,t_{n-1}) \times V_{\gamma_{n-1}}\cdots G_{st}(t_2,t_1)V_{\gamma_1}|\Psi(t_1)\rangle. \tag{26}$$

Note the clear causal structure of this equation. Taking $n=1$, we obtain the desired expression for the first functional derivative of the state vector.

B. Stochastic propagator and SWF evolution

After introducing the first variational derivative in terms of the stochastic propagator into expression (15), we obtain the *exact* equivalent SWF dynamics:

$$\frac{d}{dt}|\Psi(t)\rangle = -iH_S|\Psi(t)\rangle - i\lambda\mathcal{F}(t)|\Psi(t)\rangle - \lambda^2 V_\alpha^\dagger \int_0^t ds\,\chi_{\alpha\beta}(t,s)G_{st}(t,s)V_\beta|\Psi(s)\rangle. \tag{27}$$

Note that this is a very particular evolution: $G_{st}(t,s)$ is the propagator of this equation, and it appears in the evolution equation. This fact is a mathematical consequence of the coupling between next derivatives. From a physical point of view, this equation expresses the magnitude of the dissipation at time t, as a functional of each particular previous stochastic trajectory of the system. This functional consists in the time integral of the ‘‘free evolution,’’ interrupted at arbitrary times by the environment action.

We remark that although the stochastic propagator $G_{st}(t,s)$ has the functional operator form of Eq. (24), Eq. (27) must be considered as a *nonvariational evolution*. In fact, this equation is the starting point for writing any controlled expansion: introducing an arbitrary ‘‘small parameter expansion,’’ and expressing $G_{st}(t,s)$ as a power series in this parameter, it is always possible to write—order by order—the evolution equation (27) in terms of the SWF. This comes from the fact that we can always apply expression (26) iteratively, allowing us to collect all contributions to a given order.

The election of the parameter expansion and the validity of the truncated evolutions depend on each particular problem. In Appendix A we work an interaction parameter expansion and a small correlation time expansion. Now, as simple examples of approximations of this dynamics, we will analyze the cases of white noises and the Born-Markov approximation.

1. White noises

Uncorrelated white noises are characterized by the correlations

$$\chi_{\alpha\beta}(t,s) \equiv \delta_{\alpha\beta}\delta(t-s). \tag{28}$$

Inserting this expression into Eq. (27), and using the fact that $G_{st}(t,t)=I$, we obtain

$$\frac{d}{dt}|\Psi(t)\rangle = [-iH_S - \lambda^2 V_\alpha^\dagger V_\alpha - i\lambda\mathcal{F}(t)]|\Psi(t)\rangle. \tag{29}$$

This evolution was obtained by van Kampen ($\lambda=1$) from the original SL dynamics. In this case, dissipation is represented by the deterministic operator $U=V_\alpha^\dagger V_\alpha$.

2. Born-Markov approximation

As usual, to obtain the Born-Markov approximation we have to work in the interaction representation. Using the hat symbol to indicate explicitly time-dependent objects, we obtain

$$\frac{d}{dt}|\hat\Psi(t)\rangle = -i\lambda\hat{\mathcal{F}}(t)|\hat\Psi(t)\rangle - \lambda^2\int_0^t ds\,\chi_{\alpha\beta}(t,s) \times V_\alpha^\dagger(t)\hat G_{st}(t,s)V_\beta(s)|\hat\Psi(s)\rangle. \tag{30}$$

A central hypothesis in the Born-Markov approximation is that the environmental correlation time is negligibly short compared to the system's characteristic time scale. This hypothesis implies that, in comparison to the evolution given by $\hat G_{st}(t,s)$, the correlations $\chi_{\alpha\beta}(t,s)$ are rapidly decaying functions. Therefore, in the integral term in Eq. (30) we have to consider only times $s\simeq t$. Then $\hat G_{st}(t,s)\simeq I$, and we can assume that

$$[\hat G_{st}(t,s), V_\beta(s)] \simeq 0. \tag{31}$$

Thus we arrive at a local time evolution, that in the Schrödinger representation reads

$$\frac{d}{dt}|\Psi(t)\rangle=[-iH_{eff}-D-i\lambda\mathcal{F}(t)]|\Psi(t)\rangle, \tag{32}$$

where we have extended the time integral limit to infinity, and where D and the Hamiltonian shift are given by the Hermitian and anti-Hermitian parts of the integral $\int_0^\infty d\tau\chi_{\alpha\beta}(-\tau)V_\alpha^\dagger V_\beta(-\tau)$. As proved in Ref. [17], the evolution equation (32) gives a factorized stochastic representation of the Born-Markov generator for any system and system-bath interaction.

C. Density-matrix evolution

Here we will obtain the density-matrix evolution that corresponds to the SWF [Eq. (15)]. From expressions (12) and (26), it is possible to obtain

$$\frac{\delta^n\rho_{st}(t)}{\delta l_{\gamma_n}(t_n)\cdots\delta l_{\gamma_1}(t_1)} =(-i\lambda)^n G_{st}(t,t_n)V_{\gamma_n}G_{st}(t_n,t_{n-1}) \times V_{\gamma_{n-1}}\cdots G_{st}(t_2,t_1)V_{\gamma_1}\rho_{st}(t_1)G_{st}^\dagger(t,t_1), \tag{33}$$

where we have used the fact that G_{st} is the propagator of the evolution. Note that while the $|\text{bra}\rangle$ term is propagated through ''n interactions'' with the environment, the $\langle\text{ket}|$ term is propagated freely. Inserting the average of this expression ($n=1$) into Eq. (16) yields

$$\frac{d}{dt}\langle\rho_{st}(t)\rangle=-i[H_S,\langle\rho_{st}(t)\rangle]-\lambda^2\int_0^t ds\{\chi_{\alpha\beta}(t,s) \times[V_\alpha^\dagger,\langle G_{st}(t,s)V_\beta\rho_{st}(s)G_{st}^\dagger(t,s)\rangle]+\text{H.c.}\}. \tag{34}$$

This exact equation gives the solution to the problem of finding the averaged density-matrix evolution. In fact, for each proposed approximation scheme to the stochastic propagator, it is always possible to express this dynamics in terms of $\langle\rho_{st}\rangle$: with the aid of Eq. (33), the action of $G_{st}(t,s)$ and its Hermitian conjugate on ρ_{st} can always be written in terms of the latter; the posterior average can be exactly solved with a generalization of Novikov's theorem [20(c)] (see Appendix A). Furthermore, this theorem permits one to control the contribution of each term of the SWF evolution to the mean density matrix, giving the possibility of retaining only those terms that make a non-null contribution to a given order.

Following the previous examples, it is simple to see that in the case of white noises, the evolution equation (34) is reduced to a standard diagonal Kossakowsky–Lindblad generator. Furthermore, using the same approximations as in Sec. III B 2, it is possible to obtain the matrix structure of the Born-Markov approximation and its effective Hamiltonian [17].

A remarkable point is that as a consequence of the linearity of the stochastic evolution, independently of the approximation used for $G_{st}(t,s)$, it is possible to find the exact master equation that governs the mean density-matrix evolution.

1. Master equation

We now address the problem of obtaining the averaged density evolution by means of the projection operator technique [21,22]. First we rewrite Eq. (14), in the interaction representation, as

$$\frac{d}{dt}\hat{\rho}_{st}(t)=-i\lambda\mathcal{L}_{st}(t)\hat{\rho}_{st}(t), \tag{35}$$

where the linear functional super-operator $\mathcal{L}_{st}(t)$ is

$$\mathcal{L}_{st}(t)=\hat{\mathcal{T}}_{st}(t)\bullet-\bullet\hat{\mathcal{T}}_{st}^\dagger(t). \tag{36}$$

For convenience, we have extracted the parameter λ from the definition of $\hat{\mathcal{T}}_{st}(t)$. Defining the projectors $\mathcal{P}$ and $\mathcal{Q}$ as

$$\mathcal{P}\hat{\rho}_{st}(t)=\langle\hat{\rho}_{st}(t)\rangle \tag{37}$$

and $\mathcal{P}+\mathcal{Q}=I$, we can write

$$\frac{d}{dt}\mathcal{P}\hat{\rho}_{st}(t)=-i\lambda\mathcal{P}\mathcal{L}_{st}(t)\mathcal{P}\hat{\rho}_{st}(t)-i\lambda\mathcal{P}\mathcal{L}_{st}(t)\mathcal{Q}\hat{\rho}_{st}(t),$$
$$\frac{d}{dt}\mathcal{Q}\hat{\rho}_{st}(t)=-i\lambda\mathcal{Q}\mathcal{L}_{st}(t)\mathcal{P}\hat{\rho}_{st}(t)-i\lambda\mathcal{Q}\mathcal{L}_{st}(t)\mathcal{Q}\hat{\rho}_{st}(t). \tag{38}$$

It is immediate to integrate the ''irrelevant part,'' that by virtue of the initial conditions $\mathcal{Q}\rho_{st}(0)=0$ reads

$$\mathcal{Q}\hat{\rho}_{st}(t)=-i\lambda\int_0^t ds\mathcal{G}_{st}(t,s)\mathcal{Q}\mathcal{L}_{st}(s)\mathcal{P}\hat{\rho}_{st}(s), \tag{39}$$

where we have defined

$$\mathcal{G}_{st}(t,s)=\left\lceil\exp\left(-i\lambda\int_s^t du\,\mathcal{Q}\mathcal{L}_{st}(u)\right)\right\rceil. \tag{40}$$

Inserting Eq. (39) into expression (38), the exact relevant evolution follows:

$$\frac{d}{dt}\mathcal{P}\hat{\rho}_{st}(t)=-i\lambda\mathcal{P}\mathcal{L}_{st}(t)\mathcal{P}\hat{\rho}_{st}(t) -\lambda^2\int_0^t ds\mathcal{P}\mathcal{L}_{st}(t)\mathcal{G}_{st}(t,s)\mathcal{Q}\mathcal{L}_{st}(s)\mathcal{P}\hat{\rho}_{st}(s). \tag{41}$$

Using the property $\mathcal{P}\mathcal{L}_{st}(t)\mathcal{P}=0$, we finally arrive at the closed evolution

$$\frac{d}{dt}\hat{\rho}(t)=\int_0^t ds\tilde{K}(t,s)\hat{\rho}(s), \tag{42}$$

where the deterministic nonvariational memory kernel is

$$\tilde{K}(t,s) = -\lambda^2 \langle \mathcal{L}_{st}(t) \mathcal{G}_{st}(t,s) \mathcal{L}_{st}(s) \rangle. \tag{43}$$

These last two equations give the desired exact master equation for the averaged density matrix. These expressions have an obvious advantage over the previous one, because they can be immediately read independently of any approximation scheme. Nevertheless it is important to note that both evolutions are equivalent. On the other hand, the advantage of expression (34) is to permit one to work the matrix evolution through the same steps to obtain the SWF dynamics.

IV. TIME-CONVOLUTIONLESS APPROACH

In Sec. III we successfully characterized the SWF and density-matrix evolutions in terms of the stochastic propagator. As expected—working with non-Markovian processes—these equations are nonlocal in time. In this situation, a common attitude is to try to find equivalent evolutions which are local in time [21–23]. Now we will show that this objective can be achieved by introducing the inverse stochastic propagator.

A. SWF evolution

The inverse stochastic propagator is defined by

$$|\Psi(s)\rangle = G_{st}^{-1}(t,s) |\Psi(t)\rangle, \tag{44}$$

with

$$G_{st}^{-1}(t,s) = \left\lfloor \exp\left(+i \int_s^t du \mathcal{T}_{st}(u) \right) \right\rfloor, \tag{45}$$

where $\lfloor \cdots \rfloor$ indicates antichronological time ordering. In this manner, from Eq. (26), one can immediately express the n-functional response function as

$$\frac{\delta^n |\Psi(t)\rangle}{\delta l_{\gamma_n}(t_n) \cdots \delta l_{\gamma_1}(t_1)} = (-i\lambda)^n U_{\gamma_n}(t,t_n) U_{\gamma_{n-1}}(t,t_{n-1}) \cdots U_{\gamma_1}(t,t_1) |\Psi(t)\rangle, \tag{46}$$

where we have defined the ''channel propagators''

$$U_\beta(t,s) = G_{st}(t,s) V_\beta G_{st}^{-1}(t,s). \tag{47}$$

As a consequence, we can express Eq. (27) as the equivalent *exact* time-convolutionless evolution:

$$\frac{d}{dt} |\Psi(t)\rangle = -iH_S |\Psi(t)\rangle - i\lambda \mathcal{F}(t) |\Psi(t)\rangle - \lambda^2 V_\alpha^\dagger \int_0^t ds \chi_{\alpha\beta}(t,s) U_\beta(t,s) |\Psi(t)\rangle. \tag{48}$$

In this manner, it is possible to characterize the time-convolutionless approach in terms of the stochastic propagator $G_{st}(t,s)$ and its inverse. As in the time-convolution case, we can consider the previous expression as a nonvariational evolution. In Appendix A we work the interaction parameter expansion and the small correlation time expansion.

1. Previous approach

In contrast to the time-convolution case, we note that by means of Eq. (46), the action of $U_\beta(t,s)$ on $|\Psi(t)\rangle$ can always be written in terms of the SWF valued at a unique time: the present time t. This fact permits us to write

$$\frac{\delta |\Psi(t)\rangle}{\delta l_\beta(s)} = -i\lambda O_\beta(t,s) |\Psi(t)\rangle, \tag{49}$$

where $O_\beta(t,s)$ is a stochastic nonfunctional operator. This equation is the starting point of the proposals given in Refs. [13–15]. There the authors determined the operators $O_\beta(t,s)$ using the consistency relation

$$\frac{d}{dt} \frac{\delta |\Psi(t)\rangle}{\delta l_\beta(s)} = \frac{\delta}{\delta l_\beta(s)} \frac{d}{dt} |\Psi(t)\rangle.$$

This procedure is equivalent to obtaining $O_\beta(t,s)$ from its time evolution, which can be obtained introducing Eq. (49) into Eq. (20). These evolutions consist of a set of coupled *nonlinear* functional stochastic equations. Therefore, no advantage is obtained with this approach. Nevertheless, in some particular cases this set of equations admits nonstochastic solutions, which can be found in an exact—nonapproximate—way [13].

B. Density-matrix evolution

In a direct way, it is possible to find for the n-functional derivative

$$\frac{\delta^n \rho_{st}(t)}{\delta l_{\gamma_n}(t_n) \cdots \delta l_{\gamma_1}(t_1)} = (-i\lambda)^n U_{\gamma_n}(t,t_n) U_{\gamma_{n-1}}(t,t_{n-1}) \cdots U_{\gamma_1}(t,t_1) \rho_{st}(t). \tag{50}$$

Using this result, we obtain the exact equivalent time-convolutionless evolution

$$\frac{d}{dt} \langle \rho_{st}(t) \rangle = -i[H_S, \langle \rho_{st}(t) \rangle] - \lambda^2 \int_0^t ds \{ \chi_{\alpha\beta}(t,s) \times [V_\alpha^\dagger, \langle U_\beta(t,s) \rho_{st}(t) \rangle] + \text{H.c.} \}. \tag{51}$$

These expressions give the solution to finding a density-matrix evolution that is local in time (see Appendix A). Nevertheless, as in Sec. III C, they must be worked for each proposed expansion of the stochastic propagator $G_{st}(t,s)$. Thus, in what follows we will find, independently of the scheme of approximation, the exact time-convolutionless master equation.

1. Time-convolutionless master equation

Here we will follow the procedure introduced in Ref. [22]. As with the SWF evolution, the basic idea is to express $\hat{\rho}_{st}(s)$ in the form

$$\hat{\rho}_{st}(s)=\Lambda^{-1}(t,s)\hat{\rho}_{st}(t), \tag{52}$$

where $\Lambda^{-1}(t,s)$ is the inverse stochastic density-matrix propagator:

$$\Lambda^{-1}(t,s)=\left\lfloor \exp\left(+i\lambda\int_s^t du\mathcal{L}_{st}(u)\right)\right\rfloor. \tag{53}$$

Introducing this expression into Eq. (39), and using that $\mathcal{P}+\mathcal{Q}=I$, it is possible to express the irrelevant part as

$$\mathcal{Q}\hat{\rho}_{st}(t)=[1-\Sigma(t)]^{-1}\Sigma(t)\mathcal{P}\hat{\rho}_{st}(t), \tag{54}$$

where

$$\Sigma(t)=-i\lambda\int_0^t ds\mathcal{G}(t,s)\mathcal{Q}\mathcal{L}_{st}(s)\mathcal{P}\Lambda^{-1}(t,s), \tag{55}$$

which, substituted into Eq. (38), gives rise to the exact time-convolutionless master equation

$$\frac{d}{dt}\hat{\rho}(t)=K(t)\hat{\rho}(t)=-i\lambda\langle\mathcal{L}_{st}(t)[1-\Sigma(t)]^{-1}\rangle\hat{\rho}(t). \tag{56}$$

At this point, the existence of $K(t)$ can be guaranteed, assuming the analyticity of $[1-\Sigma(t)]^{-1}$. Therefore, we can express this term as a geometric series, giving rise to

$$K(t)=-i\lambda\sum_{n=0}^{\infty}\langle\mathcal{L}_{st}(t)\Sigma(t)^n\rangle. \tag{57}$$

As claimed in Ref. [11], this expansion is only valid in an intermediate coupling regime, where non-Markovian effects are significant and where $K(t)$ exists for all time. The same restriction is valid to the SWF evolution equation (48), and it must be assumed in any scheme of approximation.

V. QUANTUM MASTER EQUATIONS AND QUANTUM DIFFUSION MODEL

In the previous sections we completely characterized the SWF formalism when color Gaussian fluctuations are assumed. At this point, this technique must be considered as a phenomenological approach to the description of an open quantum system. This comes from the fact that we have a total degree of freedom in the choice of the fluctuations operator properties. In this section we will show the conditions under which this formalism can be associated with a quantum microscopic description.

We assume that the total closed dynamics of an open system $\mathcal{S}$ interacting with a thermal bath $\mathcal{B}$ is given by

$$H_T=H_S+H_B+\lambda H_I, \tag{58}$$

where H_B represents the bath Hamiltonian, and where H_I is the system-bath interaction. In the interaction representation, the total density matrix $W(t)$ follows the reversible dynamics

$$\frac{d}{dt}W(t)=-i\lambda L(t)W(t), \tag{59}$$

with

$$L(t)=H_I(t)\bullet-\bullet H_I(t), \tag{60}$$

and where $H_I(t)$ corresponds to the interaction representation of H_I. Splitting the total dynamics into relevant and irrelevant parts through the quantum projector $\tilde{\mathcal{P}}$,

$$\tilde{\mathcal{P}}W(t)=\mathrm{Tr}_R\{W(t)\}\otimes\rho_R\equiv\rho(t)\otimes\rho_R, \tag{61}$$

where ρ_R is a stationary reference bath state, it is possible to obtain the marginal evolution of system $\mathcal{S}$, which is described by a quantum master equation [24]. The generator obtained is exactly the same as in Eq. (41), after doing the following changes:

$$\begin{gathered}\hat{\rho}_{st}(t)\to W(t),\\ \mathcal{L}_{st}(t)\to L(t),\\ \langle\cdots\rangle\to\mathrm{Tr}_R\{\cdots\}\otimes\rho_R.\end{gathered} \tag{62}$$

Thus at this point a natural question arises: Does there exist some set of conditions such that both generators are equal? Expanding expression (41) in parameter λ, one can immediately see that a necessary and sufficient condition is that all "m-time-ordered moments" of $L(t)$ and $\mathcal{L}_{st}(t)$ must be equal:

$$\mathrm{Tr}_R\{L(t_m)\cdots L(t_1)\rho_R\}\overset{?}{=}\langle\mathcal{L}_{st}(t_m)\cdots\mathcal{L}_{st}(t_1)\rangle. \tag{63}$$

Assuming that the interaction Hamiltonian has the form

$$H_I=\sum_{\alpha=1}^{n}V_\alpha\otimes B_\alpha, \tag{64}$$

where B_α are boson bath operators and $\{V_\alpha\}_1^n$ is the same set of operators appearing in the definition of the fluctuations operator $\mathcal{F}(t)$, in Appendix B we demonstrate that equality (63) can be satisfied if

$$\chi_{\alpha\beta}(t,s)=\langle\langle l_\alpha^*(t)l_\beta(s)\rangle\rangle=\mathrm{Tr}_R\{\rho_R B_\alpha^\dagger(t)B_\beta(s)\}. \tag{65}$$

As proved in Ref. [17], this correlation mapping can always be consistently satisfied if $\mathcal{F}(t)$ is a non-Hermitian operator. Thus we have proved that the general evolution equation

(15), under condition (65), gives a factorized Gaussian stochastic representation of an open quantum system, whose marginal evolution is obtained with the projector [Eq. (61)] from the microscopic dynamics [Eq. (59)].

Furthermore, we remark that the previous conditions also guarantee the equality of the time-convolutionless master equations (56). This comes from the fact that these equations are also characterized in terms of the m-time-ordered moments of $L(t)$ and $\mathcal{L}_{st}(t)$.

VI. SUMMARY AND CONCLUSIONS

In this paper we have completely characterized the linear [25] non-Markovian SWF generalization in the case of nonwhite Gaussian fluctuations. Since we worked in an exact way, we have proved that the unique SWF dynamics that satisfies the minimum requirements (i)–(iii) is the evolution equation (15). Any other stochastic Gaussian evolution must be an approximate solution of this dynamics [26]. These are, for example, the cases of white noises and the Born-Markov approximation, where the dissipation functional reduces to a deterministic operator.

After solving the response function hierarchy in terms of the stochastic propagator G_{st}, we have found the equivalent equation (27). From this equation, any approximate SWF evolution can be consistently developed both for time-convolution and time-convolutionless dynamics. Furthermore, through the same steps used to find the SWF evolution, it is possible to give the corresponding mean density-matrix evolution.

As examples of controlled approximations, in Appendix A we work with an expansion in the interaction parameter and a small time correlation expansion. From these expansions we realize that higher contributions to the Born-Markov approximation consist of a complex structure of fluctuation terms plus the corresponding dissipative effect. This structure is highly dependent on the chosen parameter expansion.

An interesting advance in the theory of open quantum systems is given by the proof of the equivalence between the evolution corresponding to the non-Markovian Gaussian diffusion model and that coming from quantum master equations. In fact, assuming the mapping between noises and bath operators correlations, Eq. (65), the SWF evolution worked out in this paper, must be considered as a factorized Gaussian stochastic representation of the dynamics obtained with quantum projection operator methods from a microscopic closed description.

Finally, we remark that the results of the present work provide insight into the structure of dissipation and fluctuations in an open quantum system. We hope that these results and the techniques used in the present work may be a starting point toward investigating other approximations and non-Markovian SWF dynamics.

ACKNOWLEDGMENTS

I would like to thank Gil E. Schieber for a critical reading of this manuscript. This work was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

APPENDIX A: CONSISTENT APPROXIMATE EVOLUTIONS

In this appendix we will show the steps from which consistent systematic expansions can be constructed. By ‘‘consistent’’ we mean that we can control the contribution of each term in the SWF evolution to the density-matrix evolution. This is possible, because the averaged density evolution can be worked with the same approximations used to obtain the SWF evolution. This follows from the expression

$$\begin{aligned}\langle\mathcal{M}[\xi]\mathcal{N}[\xi]\rangle=&\langle\mathcal{M}[\xi]\rangle\langle\mathcal{N}[\xi]\rangle\\&+\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^t\cdots\int_0^t\prod_{i=1}^{n}dt_i ds_i C_2(t_i,s_i)\\&\times\left\langle\frac{\delta^n\mathcal{M}[\xi]}{\delta\xi(t_1)\cdots\delta\xi(t_n)}\right\rangle\left\langle\frac{\delta^n\mathcal{N}[\xi]}{\delta\xi(s_1)\cdots\delta\xi(s_n)}\right\rangle.\end{aligned}\tag{A1}$$

This generalized Novikov theorem [20(c)] gives an exact expression for the averaged product of two functionals of a Gaussian noise, as a function of the averaged variational derivative of each functional. The extension of this result to the present problem is immediate. It will be used to obtain the average of products of ρ_{st} with stochastic terms that will appear in each proposed expansion. The variational derivatives of the stochastic density matrix can be found recursively with the aid of the expressions given in Secs. III and IV.

1. Interaction parameter expansion

As mentioned above, in order to give an expanded evolution it is only necessary to define the parameter expansion. The most simple choice is the interaction parameter λ. Thus we write

$$\hat{G}_{st}(t,s)=\sum_{n=0}^{\infty}\lambda^n\hat{G}_{st}^{(n)}(t,s).\tag{A2}$$

Here each term $\hat{G}_{st}^{(n)}(t,s)$ follows immediately from the stochastic propagator definition. Nevertheless note that each term in the series will generate contributions at all order in λ.

a. Time-convolution λ expansion

Inserting expansion (A2) into evolution equation (27), and after iteratively applying expression (26), the SWF evolution up to the fourth order reads

$$\frac{d}{dt}|\hat{\Psi}(t)\rangle = -i\lambda\hat{\mathcal{F}}(t)|\hat{\Psi}(t)\rangle - \lambda^2\int_0^t ds\chi_{\alpha\beta}(t,s)V_\alpha^\dagger(t)V_\beta(s)|\hat{\Psi}(s)\rangle + i\lambda^3\int_0^t ds\chi_{\alpha\beta}(t,s)V_\alpha^\dagger(t)\int_s^t du_1\hat{\mathcal{F}}(u_1)V_\beta(s)|\hat{\Psi}(s)\rangle$$
$$+\lambda^4\int_0^t ds\chi_{\alpha\beta}(t,s)V_\alpha^\dagger(t)\int_s^t du_2\int_s^{u_2}du_1\chi_{\mu\nu}(u_2,u_1)V_\mu^\dagger(u_2)V_\nu(u_1)V_\beta(s)|\hat{\Psi}(s)\rangle$$
$$+\lambda^4\int_0^t ds\chi_{\alpha\beta}(t,s)V_\alpha^\dagger(t)\int_s^t du_2\int_0^s du_1\chi_{\mu\nu}(u_2,u_1)V_\mu^\dagger(u_2)V_\beta(s)V_\nu(u_1)|\hat{\Psi}(u_1)\rangle + \vartheta\langle\lambda^6\rangle, \tag{A3}$$

where $\vartheta\langle\lambda^6\rangle$ means that any other term will make a contribution of order λ^6 in the mean density-matrix evolution. The density matrix was obtained by introducing the same expansion [Eq. (A2)] into evolution equation (34), which gives

$$\frac{d}{dt}\hat{\rho}(t) = -\lambda^2\int_0^t ds\chi_{\alpha\beta}(t,s)[V_\alpha^\dagger(t),V_\beta(s)\hat{\rho}(s)] + \lambda^4\int_0^t ds\chi_{\alpha\beta}(t,s)[V_\alpha^\dagger(t),\mathcal{R}_\beta(t,s)[\hat{\rho}]] + \text{H.c.}, \tag{A4}$$

where the superoperators $\mathcal{R}_\beta(t,s)[\hat{\rho}]$ are defined by

$$\mathcal{R}_\beta(t,s)[\hat{\rho}] = +\int_s^t du_2\int_0^s du_1\chi_{\mu\nu}(u_1,u_2)V_\beta(s)\hat{\rho}(u_1)V_\mu^\dagger(u_1)V_\nu(u_2) - \int_s^t du_2\int_s^t du_1\chi_{\mu\nu}(u_2,u_1)V_\nu(u_1)V_\beta(s)\hat{\rho}(s)V_\mu^\dagger(u_2)$$
$$+\int_s^t du_2\int_0^s du_1\chi_{\mu\nu}(u_2,u_1)V_\mu^\dagger(u_2)V_\beta(s)V_\nu(u_1)\hat{\rho}(u_1). \tag{A5}$$

In obtaining this result, we have iteratively applied expression (33), and used the generalized Novikov's theorem [Eq. (A1)].

b. Time-convolutionless $\boldsymbol{\lambda}$ *expansion*

In this case, the procedure is exactly the same as in the previous case. The inverse stochastic propagator can be expanded in the same form as the stochastic propagator. Using the expressions of Sec. IV, we give

$$\frac{d}{dt}|\hat{\Psi}(t)\rangle = -i\lambda\hat{\mathcal{F}}(t)|\hat{\Psi}(t)\rangle - \lambda^2\int_0^t ds\,\chi_{\alpha\beta}(t,s)V_\alpha^\dagger(t)V_\beta(s)|\hat{\Psi}(t)\rangle + i\lambda^3\int_0^t ds\,\chi_{\alpha\beta}(t,s)V_\alpha^\dagger(t)\left(\int_s^t du_1[\hat{\mathcal{F}}(u_1),V_\beta(s)]\right)|\hat{\Psi}(t)\rangle$$
$$+\lambda^4\int_0^t ds\,\chi_{\alpha\beta}(t,s)V_\alpha^\dagger(t)\left(\int_s^t du_2\int_s^{u_2}du_1\chi_{\mu\nu}(u_2,u_1)[V_\mu^\dagger(u_2)V_\nu(u_1),V_\beta(s)]\right)|\hat{\Psi}(t)\rangle$$
$$+\lambda^4\int_0^t ds\,\chi_{\alpha\beta}(t,s)V_\alpha^\dagger(t)\left(\int_s^t du_2\int_0^s du_1\chi_{\mu\nu}(u_2,u_1)[V_\mu^\dagger(u_2),V_\beta(s)]V_\nu(u_1)\right)|\hat{\Psi}(t)\rangle + \vartheta\langle\lambda^6\rangle. \tag{A6}$$

The density-matrix dynamics results in

$$\frac{d}{dt}\hat{\rho}(t) = -\lambda^2\int_0^t ds\,\chi_{\alpha\beta}(t,s)[V_\alpha^\dagger(t),V_\beta(s)\hat{\rho}(t)] + \lambda^4\int_0^t ds\,\chi_{\alpha\beta}(t,s)[V_\alpha^\dagger(t),\mathcal{S}_\beta(t,s)[\hat{\rho}(t)]] + \text{H.c.}, \tag{A7}$$

where the second-order contribution corresponds to the Born-Markov approximation, and

$$\mathcal{S}_\beta(t,s)[\bullet] = -\int_0^t du_2\int_s^t du_1\chi_{\mu\nu}(u_2,u_1)[V_\nu(u_1),V_\beta(s)]\bullet V_\mu^\dagger(u_2) + \int_s^t du_2\int_s^{u_2}du_1\chi_{\mu\nu}(u_2,u_1)[V_\mu^\dagger(u_2)V_\nu(u_1),V_\beta(s)]\bullet$$
$$+\int_s^t du_2\int_0^s du_1\chi_{\mu\nu}(u_2,u_1)[V_\mu^\dagger(u_2),V_\beta(s)]V_\nu(u_1)\bullet. \tag{A8}$$

Consistently, transforming all time integrals to the form $\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3$, it is possible to show that the evolution equation (A7) coincides with that obtained from the master equation (56). Furthermore, assuming the correlation mapping Eq. (65), the evolution obtained is exactly the same as that of Eq. (29) in Ref. [11].

2. Small-correlation-time expansion

If the noises $\{l_\alpha\}_1^n$ are close to the white-noise limit (i.e., colored noise with very small correlation time) it seems appropriate to expand the stochastic propagator around its Markovian value which it attains for δ-correlated noises [20]. Thus we expand $\hat{G}_{st}(t,s)$ into a Taylor series around the Markovian end point t, i.e.,

$$\hat{G}_{st}(t,s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial^n s}\hat{G}_{st}(t,s)\big|_{s=t}(t-s)^n. \quad \text{(A9)}$$

It is important to note that this type of expansion has sense only if it is realized around the final point t [15]. On the other hand, different evolutions are obtained if this expansion is realized in the Schrödinger or interaction representations.

Time-convolutionless expansion

At first order in $(t-s)$, in the interaction representation, we find

$$\frac{d}{dt}|\hat{\Psi}(t)\rangle=-i\lambda\hat{\mathcal{F}}(t)|\hat{\Psi}(t)\rangle-\lambda^2\int_0^t ds\,\chi_{\alpha\beta}(t,s)V_\alpha^\dagger(t)V_\beta(s)|\hat{\Psi}(t)\rangle+i\lambda^3\int_0^t ds\,\chi_{\alpha\beta}(t,s)(t-s)V_\alpha^\dagger(t)[\hat{\mathcal{F}}(t),V_\beta(s)]|\hat{\Psi}(t)\rangle$$
$$+\lambda^4\int_0^t ds\int_0^t du\,\chi_{\alpha\beta}(t,s)\chi_{\mu\nu}(t,u)(t-s)V_\alpha^\dagger(t)[V_\mu^\dagger(t),V_\beta(s)]V_\nu(u)|\hat{\Psi}(t)\rangle+\vartheta\langle(t-s)^2\rangle, \quad \text{(A10)}$$

where we have used the fact that $(\partial/\partial s)\hat{G}_{st}(t,s)|_{s=t}=i\lambda\hat{\mathcal{T}}_{st}(t)$. Note that higher orders can be worked without appealing to a noise functional expansion [15]. For the density-matrix result,

$$\frac{d}{dt}\hat{\rho}(t)=-\lambda^2\int_0^t ds\,\chi_{\alpha\beta}(t,s)[V_\alpha^\dagger(t),V_\beta(s)\hat{\rho}(t)]-\lambda^4\int_0^t ds\int_0^t du\,\chi_{\alpha\beta}(t,s)\chi_{\mu\nu}^*(t,u)(t-s)[V_\alpha^\dagger(t),[V_\mu(t),V_\beta(s)]\hat{\rho}(t)V_\nu^\dagger(u)]$$
$$+\lambda^4\int_0^t ds\int_0^t du\,\chi_{\alpha\beta}(t,s)\chi_{\mu\nu}(t,u)(t-s)[V_\alpha^\dagger(t),[V_\mu^\dagger(t),V_\beta(s)]V_\nu(u)\hat{\rho}(t)]+\text{H.c.} \quad \text{(A11)}$$

APPENDIX B: EQUIVALENCE BETWEEN MASTER EQUATIONS

Now we will sketch the demonstration of the equality of all the m-time-ordered moment of the superoperators $\mathcal{L}_{st}(t)$ and $L(t)$. Using explicit forms [Eqs. (36) and (60)], we can obtain the equivalent condition

$$\mathrm{Tr}_R\{H_I(t_r)\cdots H_I(t_1)\bullet\rho_R H_I(\tau_1)\cdots H_I(\tau_s)\}$$
$$\stackrel{?}{=}\langle\hat{\mathcal{T}}_{st}(t_r)\cdots\hat{\mathcal{T}}_{st}(t_1)\bullet\hat{\mathcal{T}}_{st}^\dagger(\tau_1)\cdots\hat{\mathcal{T}}_{st}^\dagger(\tau_s)\rangle, \quad \text{(B1)}$$

where the set of times $\{t_i\}_1^r$ and $\{\tau_i\}_1^s$ are chronologically ordered, and r and s are arbitrary natural numbers, including zero: $r,s\in\{N\cup 0\}$. From Eq. (64), it is easy to obtain the first term structure, which consists of a sum over all pairwise combination of bosonic bath correlations, times the ordered product of the corresponding operators V_α. Now we will study the second term structure. Using the stochastic functional form of $\hat{\mathcal{T}}_{st}(t)$, it is possible to obtain

$$\hat{\mathcal{T}}_{st}(t_r)\cdots\hat{\mathcal{T}}_{st}(t_1)[\bullet]$$
$$=\sum_k\sum_{\{t_u,t_\nu\}}\prod_{i=1}^{k}\chi(t_{\mu_i},t_{\nu_i})\Xi(t_r)\cdots\Xi(t_1)[\bullet], \quad \text{(B2)}$$

where we have assumed that $[\bullet]$ is a deterministic object or has a dependence on the set of complex conjugate noises $\{l_\alpha^*\}_1^n$. The parameter k runs from zero to $r/2$ if r is even, and to $(r-1)/2$ if r is odd. When $k=0$, no correlations are present. The second sum implies an addition over all possible pairwise combinations of the set $\{t_i\}_1^r$, subject to the condition $t_{\mu_i}>t_{\nu_i}$. The operators $\Xi(t_i)$ are defined by

$$\Xi(t_i)=\begin{cases} V^{\dagger}(t_i) & \text{if } t_i\in\{t_\mu\} \\ V(t_i) & \text{if } t_i\in\{t_\nu\} \\ \hat{\mathcal{F}}(t_i) & \text{if } t_i\notin\{t_\mu,t_\nu\}. \end{cases} \tag{B3}$$

Furthermore, to simplify the notation, the sum over the set $\{V_\alpha\}_1^n$, previously indicated with repeated Greek indices, now is translated to summation over repeated times.

Averaging Eq. (B2), it is simple to see the validity of Eq. (B1) in the case of arbitrary r (s) and $s=0$ ($r=0$): from the noise correlation definition, it follows that the only term that survives is the deterministic one, which results to be the same as the one coming from Eq. (B1), after using that $H_I(t)=H_I^{\dagger}(t)$ and the mapping correlation [Eq. (65)]. In general, we obtain

$$\begin{aligned}\langle \hat{\mathcal{T}}_{st}(t_r)\cdots\hat{\mathcal{T}}_{st}(t_1)\bullet\hat{\mathcal{T}}_{st}^{\dagger}(\tau_1)\cdots\hat{\mathcal{T}}_{st}^{\dagger}(\tau_s)\rangle \\ =\sum_k\sum_l\sum_{\{t_\mu,t_\nu\}}\sum_{\{\tau_\alpha,\tau_\beta\}}\prod_{i=1}^{k}\chi(t_{\mu_i},t_{\nu_i})\prod_{j=1}^{l}\chi^*(\tau_{\alpha_j},\tau_{\beta_j}) \\ \times\langle\Xi(t_r)\cdots\Xi(t_1)\bullet\Xi^{\dagger}(\tau_1)\cdots\Xi^{\dagger}(\tau_s)\rangle.\end{aligned} \tag{B4}$$

Assuming that the noises are Gaussian, the last term will generate all correlations of the type $\chi(t_\gamma,\tau_\gamma)$, where (t_γ,τ_γ) are all pairwise of times not present in the correlations $\chi(t_{\mu_i},t_{\nu_i})$ and $\chi^*(\tau_{\alpha_j},\tau_{\beta_j})$. In this manner a sum over all pairwise combination of correlation time is generated. Using that on the left-hand side of expression (B1) $H_I(t)=H_I^{\dagger}(t)$, and the mapping correlation [Eq. (65)], the equality of both terms follows.

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[25] We have worked with an evolution that does not preserve the SWF norm in each realization of the noises. This property is necessary to work out numerically, and can be achieved by means of a Girsanov transformation of the noises [13], which leads to a nonlinear SWF evolution.

[26] When the operator $\mathcal{F}(t)$ is Hermitian, the SWF dynamics reduces to a stochastic Hamiltonian one. In this case most of the expressions obtained along the paper remain valid if we omit the ''dissipative terms.'' In particular, the infinite hierarchy of equations result uncoupled, and the stochastic propagator loses its functional character. The expressions for the mean density matrix evolution remain the same.