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Perspective

Exact classical noise master equations: Applications and connections

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Abstract – All quantum systems are subject to noise and imperfections due to stray fields, inhomogeneities or drifting experimental controls. An understanding of the effects of noise and decoherence is critical to the progress towards fully functional quantum devices. In this perspective, we focus on noise in quantum systems which are modelled by a dynamic stochastic parameter in the Hamiltonian. We will outline exact evolution equations describing the ensemble average dynamics for a variety of common noise types and their connections. We will also highlight an approximate evolution equation valid in the weak noise limit for an arbitrary classical stochastic field. This framework should serve as a starting point for identifying signatures of differing noise types and optimisation of robust control protocols.

Introduction. – All quantum mechanical systems experience incoherent effects due to the surrounding environment. There are many different strategies for modelling these effects. A standard approach is to model the system and environment as a large quantum system and trace out the effects of the environment (usually within the Born-Markov approximation) leaving a description of the system of interest alone [1]. Another possibility is that the system experiences repeated discrete time interactions with fresh identical independent components of the large environment. This repeated-interactions approach (also often referred to as collision models [2]) in some case reduces to a Lindblad master equation [3] in the limit of continuous time [4].

In this perspective, we will focus on an alternative approach of classical noise, whereby the Hamiltonian contains a dynamical stochastic variable. This provides a model for the effect of an additional unwanted fluctuating field in the experimental setup. In particular we will review how to derive the exact master equations for noise sources with different statistical properties. This stochastic variation has previously arisen when considering collapse models [5], clock timing errors [6] and quantum monitoring [7]. Such random fluctuations in quantum critical systems have also been shown to have universal dynamics [8]. Its consequences in quantum systems has been investigated in a variety of contexts such as quantum correlations [9–11], transport [12,13] and solid-state qubits [14]. Classical noise has also been used theoretically [15–17] and experimentally [18] as a source of heat for a quantum heat engine. Finally, this approach can often reproduce standard open system dynamics [19,20] and can be used to emulate the environmental effects [21,22].

Such stochastic variations greatly reduce the performance of quantum state manipulation. However, these unwanted losses have been countered using a combination of counterdiabatic driving and Floquet-engineering [23], numerical optimisation (using gradient ascent pulse engineering) [24], composite pulses [25] and shortcuts to adiabaticity [26] with perturbation theory [27–29].

Therefore, there are clearly several motivations for the development of classical noise master equations which capture the impact of such fluctuations. Firstly, for the noise types considered, the master equations are exact and do not assume any specific Hamiltonian structure. Second, the use of these exact master equations avoids both the numerical generation of the noise source [30] and the many iterations required to obtain accurate average dynamics. Finally, it is hoped that an accurate description of the detrimental effects of classical noise through evolution equations will help contribute towards the cre-

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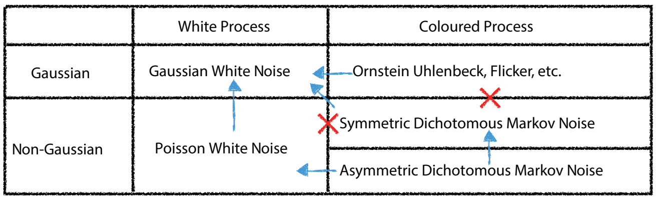

Fig. 1: Stochastic processes discussed. Blue arrows indicate limiting case. Red cross signifies the absence of a limit.

ation of robust control strategies which mitigate any significant incoherent deviation from the desired ensemble averaged system dynamics. This could be done by building on previous analytical approaches using dissipationless solutions [31], noise resistant paths [32] and perturbation theory [33].

We start with a broad overview of the general setup considered. From there, we will briefly review the derivation, properties and limiting case of several different noise types (see fig. 1). We will begin with the case for Gaussian coloured noise (including the specific case of Gaussian white noise). From there we move on to the more complex non-Gaussian cases of Poisson white noise and dichotomous Markov noise (also known as random telegraph noise). Finally, we will conclude with an approximate evolution equation accounting for incoherent effects to second order in the noise strength and a discussion of future perspectives.

Setting. – To begin, we assume a general form

$$H(t) = H_0(t) + z(t)H_1(t), \tag{1}$$

for the time-dependent system Hamiltonian $H(t)$ where $H_0(t)$ is the noise-free Hamiltonian, $H_1(t)$ is the "noisy" Hamiltonian and $z(t)$ is a real function for a given realisation of the noise. This general form is motivated by the assumption that only linear corrections from a weak stochastic variation are relevant. However, the results presented could be adapted to other cases.

This leads to the following von Neuman equation:

$$\dot{\rho}_z = -\frac{i}{\hbar}[H, \rho_z], \tag{2}$$

for a singular realisation of the noise function. Consider now the dynamics of an observable of interest A. The expectation value for a single realisation is simply $\mathrm{tr}(A\rho_z)$. However, one is typically unconcerned about or does not have access to individual realisations. It is much more natural to consider the ensemble average expectation value over many realisations $\langle \mathrm{tr}(A\rho_z)\rangle = \mathrm{tr}(A\rho)$, where the average density matrix $\rho = \langle \rho_z \rangle$ emerges as the relevant quantity of interest.

The averaged density matrix evolves as $\rho(t) = \langle U_z(t,0)\rho_0 U_z^\dagger(t,0)\rangle$, where $U_z(t,0) = \hat{T}\exp\left[-\frac{i}{\hbar}\int_0^t \mathrm{d}s H(s)\right]$ is the time evolution operator for a given realisation, $\hat{T}$ is the Dyson time-ordering operator and $\rho_z(0) = \rho_0$ is the noise-free initial condition for all realisations. Since the averaging $\langle \ldots \rangle$ is linear, this dynamical map clearly preserves the trace, hermiticity and positivity. The map is also unital (*i.e.*, the maximally mixed state is a fixed point). Note that the deviations from the average can be captured by the variance $\langle(\rho_z - \rho)^2\rangle$ which in the case of a pure initial state (*i.e.*, $\rho_0^2 = \rho_0$) simplifies to $\rho - \rho^2$.

Our goal then becomes to derive the master equation for the averaged density matrix for a classical noise source acting on a Hamiltonian. Averaging over all realisations, eq. (2), of the noise gives

$$\dot{\rho} = -\frac{i}{\hbar}[H_0, \rho] - \frac{i}{\hbar}[H_1, \langle z\rho_z\rangle]. \tag{3}$$

This is now the starting point for all the cases considered. In what follows, we will demonstrate how to determine $\langle z\rho_z\rangle$ for different noise types.

Gaussian coloured noise. – We will begin with the case of Gaussian coloured noise. We assume that the noise has zero mean $\langle z(t)\rangle = 0$ and a given correlation function $\langle z(t)z(s)\rangle = C(t,s)$ which is stationary $C(t,s) = C(t-s)$. Approximate master equations have been previously derived by expanding the ratio between the noise correlation time and the typical system time scale [34]. This approximate description has been used to reduce the effect of noise on fast ion shuttling [28,29].

In order to derive the corresponding exact master equation it is useful to recall Novikov's theorem [35,36] (valid only for Gaussian noise)

$$\langle z(t)M_t[z]\rangle = \langle z(t)\rangle\langle M_t[z]\rangle + \int_0^t \mathrm{d}s \langle z(t)z(s)\rangle \left\langle \frac{\delta M_s[z]}{\delta z(s)} \right\rangle, \tag{4}$$

where $M_t[z]$ is a time-dependent functional of $z(t)$. Setting $M_t[z] = \rho_z(t)$, the functional derivative evaluates to

$$\begin{aligned} \frac{\delta \rho_z}{\delta z} &= \frac{\partial \dot{\rho}_z}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \dot{\rho}_z}{\partial \dot{z}} \\ &= -\frac{i}{\hbar}[H_1, \rho_z], \end{aligned} \tag{5}$$

where we have used eq. (2). Inserting this result into eq. (3), we arrive at the exact master equation for Gaussian coloured noise

$$\begin{aligned} \dot{\rho} &= -\frac{i}{\hbar}[H_0, \rho] \\ &\quad -\frac{1}{\hbar^2}\left[H_1(t), \int_0^t \mathrm{d}s\, C(t-s)\,[H_1(s), \rho(s)]\right]. \end{aligned} \tag{6}$$

Master equations for Gaussian coloured noise have previously been derived in [37,38].

Correlation function. We will now outline common correlation functions of Gaussian noise. The correlation function and the spectral power density are related by the Wiener-Khinchin theorem,

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(t)\cos(\omega t)\mathrm{d}t, \tag{7}$$

$$C(t) = \int_{-\infty}^{\infty} S(\omega)\cos(\omega t)\mathrm{d}\omega. \tag{8}$$

The most common case is Gaussian white noise (see fig. 1) which is completely uncorrelated in time. This is defined by a correlation function $C(t) = \alpha\,\delta(t)$ with a corresponding constant power spectrum $S(\omega) = \frac{\alpha}{2\pi}$. Hence the master equation (6) simplifies to

$$\dot{\rho} = -\frac{i}{\hbar}\,[H_0, \rho] - \frac{\alpha}{2\hbar^2}\,[H_1(t), [H_1(t), \rho(t)]]\,, \qquad (9)$$

where the factor of 1/2 results from evaluating the Dirac delta function at the edge of the integral and α characterises the noise strength. This is seen clearly in the short-time expansion of the purity, $\approx 1 - \frac{\alpha}{2\hbar^2}\mathrm{tr}(\rho_0[H_1(0), [H_1(0), \rho_0]])t$ for a pure initial state.

Another common case is Ornstein-Uhlenbeck noise which has the correlation function $C(t) = \frac{\beta}{2\tau}e^{-|t|/\tau}$, where β is the noise strength and τ is a correlation time. The corresponding power spectrum has the form of a Lorentzian

$$S(\omega) = \frac{\beta}{2\pi(1+\omega^2\tau^2)}. \qquad (10)$$

This returns to white noise in the limit as τ goes to zero. The effect of this noise on the energy levels (due to solvent fluctuations for example) of a system undergoing stimulated Raman adiabatic passage has been previously investigated in [39–41].

A more complex extension of this noise type is flicker noise [42,43]. Flicker noise can be thought of as an average over many Ornstein-Uhlenbeck noises with a range of correlation times $[\tau_1, \tau_2]$. The correlation function is then given by

$$C(t) = \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} \frac{1}{2\tau} e^{-|t|/\tau}\mathrm{d}\tau. \qquad (11)$$

From eq. (10), the corresponding power spectrum is

$$S(\omega) = \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} \frac{1}{2\pi(1+\omega^2\tau^2)}\mathrm{d}\tau \qquad (12)$$

$$= \frac{\arctan(\omega\tau_2) - \arctan(\omega\tau_1)}{2\pi(\tau_2 - \tau_1)\omega}. \qquad (13)$$

To characterise this spectrum we define frequency cutoffs $\omega_{1,2} = (2\pi)/\tau_{1,2}$. The power spectrum is white if the frequency is below ω_2 and decreases as $1/\omega^2$ above ω_1. In the range $\omega_2 \ll \omega \ll \omega_1$, the spectrum varies as $1/\omega$. Therefore, in the frequency range $[\omega_2, \omega_1]$ it behaves as $1/f$ noise.

There are many other possible correlation functions including square exponential noise $C(t) = \frac{\gamma}{\sqrt{\pi}\tau}e^{-(t/\tau)^2}$ and power law noise $C(t) = \frac{\eta(\delta-1)}{2\tau[|t|/\tau+1]^\delta}$ for $\delta > 2$ [38,44].

Poisson white noise. – We now move on to non-Gaussian noises, starting with the case of Poisson white noise [45–47] (sometimes called white shot noise). It can be understood as a sequence of independent random impulses with exponential inter-arrival times. This is often used to describe noisy processes resulting from a few discrete events such as in electrical current [48,49] and has been proposed as a power source in quantum heat engines and refrigerators [15,16]. It has also been used to model random phase fluctuations in the Rabi frequency during stimulated Raman adiabatic passage [50].

The noise function has the form $z(t) = \sum_{i=1}^{N(t)} \xi_i \delta(t - t_i)$. The number of impulses after a time t is $N(t)$, which has a value of n with a probability given by a Poissonian counting process $Q[N(t) = n] = (\nu t)^n e^{-\nu t}/n!$. The random times t_i are uniformly distributed on the interval $(0, t)$ and are statistically independent of the impulses strengths ξ_i, drawn from a probability density $P(\xi)$. The average and correlation function are given by $\langle z(t)\rangle = \nu\langle\xi\rangle$ and

$$\langle z(t)z(s)\rangle - \langle z(t)\rangle\langle z(s)\rangle = \nu\langle\xi^2\rangle\delta(t-s), \qquad (14)$$

where ν is the average frequency of the noise impulses.

Master equation. In order to derive the exact master equation for this noise type, we start again from eq. (3) and apply the Klyatskin-Tatarsky formula [36,51]. For a Poisson process this has the form

$$\langle z(t)M_t[z]\rangle = \nu\int_{-\infty}^{\infty}\mathrm{d}\xi P(\xi)\int_0^\xi \mathrm{d}\eta\langle\exp\left[\eta\frac{\delta}{\delta z(t)}\right]M_t[z]\rangle, \qquad (15)$$

where $M_t[z]$ is some functional of $z(t)$. As in the previous case we choose $M_t[z] = \rho_z$. From eq. (2) and the previous result in eq. (5) we get $\exp[\eta\frac{\delta}{\delta z(t)}]\rho_z(t) = A_\eta\rho_z(t)A_\eta^\dagger$, where $A_\eta = e^{-i\eta H_1(t)/\hbar}$. Combining this all together we arrive at the master equation,

$$\dot{\rho} = -\frac{i}{\hbar}[H_0(t), \rho] + \nu\int_{-\infty}^{\infty}\mathrm{d}\xi\,P(\xi)\left(A_\xi\rho A_\xi^\dagger - \rho\right), \qquad (16)$$

where the identity $\int_0^\xi \mathrm{d}\eta[H_1, A_\eta\rho A_\eta^\dagger] = i\hbar(A_\xi\rho A_\xi^\dagger - \rho)$ has been applied.

Applying the Hadamard lemma [52], the master equation can be written as a sum of nested commutators

$$\dot{\rho} = -\frac{i}{\hbar}[H_0, \rho] + \nu\sum_{s=1}^{\infty}\frac{1}{s!}\left(-\frac{i}{\hbar}\right)^s\langle\xi^s\rangle[H_1, \rho]_s, \qquad (17)$$

where $[H_1, \rho]_s = [H_1, [H_1, \rho]]_{s-1}$ and $[H_1, \rho]_0 = \rho$. The detailed properties of this master equation in terms of the characteristic function of $P(\xi)$ and the spectrum of H_0 and H_1 are examined in [50].

For a two-state system, the master equation (17) reduces to the same structural form as Gaussian white noise [50]. More broadly, the nested commutator structure simplifies for Hamiltonians spanned by generators from a low-dimensional Lie algebra [15]. Gaussian white noise is recovered in general (see fig. 1) if one takes the limit $\nu \to \infty$ such that $\nu\langle\xi\rangle \to 0$, $\nu\langle\xi^2\rangle \to \alpha$, a positive constant, and $\nu\langle\xi^s\rangle \to 0\,\forall s > 2$ [45]. In this limit, it is clear to see that eq. (17) reduces to eq. (9). The physical interpretation of the limit is that the intensity of the impulses gets weaker as they become more and more frequent.

As an explicit example of this, consider a simple distribution for the intensity of the impulses $P(\xi) = [\delta(\xi + \xi_0) + \delta(\xi - \xi_0)]/2$ where only two values $\pm\xi_0$ are possible. The odd moments are clearly zero from symmetry and the even moments are $\langle \xi^{2n} \rangle = \xi_0^{2n}$. Setting $\xi_0 = \sqrt{\alpha/\nu}$ gives $\nu\langle \xi^2 \rangle = \alpha$ and $\nu\langle \xi^{2n} \rangle = \alpha^n \nu^{1-n}$ which clearly vanishes in the limit of large ν for $n > 1$.

Dichotomous Markov noise. – We now consider the case of dichotomous Markov noise (DMN) (also termed random telegraph noise) [53], which is an example of non-Gaussian coloured noise. It has been used to model effects in a diverse range of settings such as fluorescence of molecules [54], electron transport in molecular junctions [55], resonant activation in dissipative spin systems [56], quantum dots [57,58] and tunnelling dynamics [59] in the presence of a stochastic barrier. Its effects on atom laser interactions (arising from phase fluctuations in the laser) have also been investigated theoretically [60] and experimentally [61]. Dynamics arising from DMN has also been compared to interaction with a quantum environment in [19,62].

The non-Markovianity of the dynamics due to DMN has also be thoroughly demonstrated [63,64]. The effect of DMN on the non-Markovianity of a spin embedded in a bosonic bath was demonstrated in [65]. In particular, it was shown that classical noise can create strong non-Markovianity when the dynamics of the noise-free system is Markovian.

The effects of DMN on the dynamics of continuous-time quantum walks have also been examined in detail. In particular, the different transport regimes (localised, diffusive) have been characterised [66] by the noise correlation time (*i.e.*, the switching rate). Furthermore, the effects on spatial search of a complete graph (equivalent to the Grover algorithm [67]) [68] have been investigated and strong memory effects in spatially correlated noisy lattices [69] have been shown.

Here, we will consider symmetric DMN $z(t)$ so that the mean $\langle z(t) \rangle = 0$. It is a stochastic process which jumps between two equally likely values $\pm A$. This process is non-Gaussian and coloured with correlation function

$$\langle z(t)z(s) \rangle = A^2 \exp\left(-\frac{|t-s|}{\tau_c}\right), \tag{18}$$

where τ_c is the characteristic time. Note that all higher-order correlation functions can be factorised in terms of the average and correlation function [70,71].

A combination of statistically independent DMNs with a broad distributions of characteristic times gives rise to a $1/f$ power spectrum [72]. This approach is often used to model noise in solid-state qubits [73–75].

Master equation. For simplicity we will now consider a single DMN source. In the following, we outline the derivation of the exact master equation for DMN which is more involved than the previous two cases. Starting once again from eq. (3), it is useful to recall the Shapiro-Loginov formula [55,76]

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z(t) M_t[z] \rangle = \left\langle z(t) \frac{\mathrm{d}}{\mathrm{d}t} M_t[z] \right\rangle - \lambda \langle z(t) M_t[z] \rangle, \tag{19}$$

where $\lambda = 1/\tau_c$. We now choose $M_t[z] = \rho_z(t)$ as before and combine this with eq. (2) to give

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} + \lambda\right] \langle z\rho_z \rangle = -\frac{i}{\hbar}([H_0, \langle z\rho_z \rangle] + A^2[H_1, \rho]), \tag{20}$$

where we have used the Hamiltonian equation (1) and $\langle z^2 \rho_z \rangle = A^2 \rho$ since DMN can only take values $\pm A$. We now integrate both sides to get the formal solution of this equation for $\langle z\rho_z \rangle$,

$$\begin{aligned}\langle z(t)\rho_z(t) \rangle &= \langle z(0)\rho_z(0) \rangle \\ &\quad + \int_0^t \mathrm{d}s \Big\{ -\frac{i}{\hbar}([H_0(s), \langle z(s)\rho_z(s) \rangle] \\ &\quad + A^2[H_1(s), \rho(s)]) - \lambda \langle z(s)\rho_z(s) \rangle \Big\}. \end{aligned} \tag{21}$$

We assume that the initial state is noise free so that $\langle z(0)\rho_z(0) \rangle = 0$ (since $z(t)$ has zero mean). We now want to simplify the term inside the integral. To do this we define $R = \langle z\rho_z \rangle$ and a transformed version $\tilde{R}$ by

$$R(t) = U_0(t,0)\tilde{R}(t)U_0^\dagger(t,0)e^{-\lambda t}, \tag{22}$$

where $U_0(t,0) = \hat{T}\exp[-\frac{i}{\hbar}\int_0^t \mathrm{d}s H_0(s)]$ is the noise-free time evolution operator. Taking the derivative of R gives

$$\frac{\mathrm{d}R}{\mathrm{d}t} + \lambda R = -\frac{i}{\hbar}[H_0, R] + U_0(t,0)\frac{\mathrm{d}\tilde{R}}{\mathrm{d}t}U_0^\dagger(t,0)e^{-\lambda t}. \tag{23}$$

Comparing this result with eq. (20), it follows that

$$\frac{\mathrm{d}\tilde{R}}{\mathrm{d}t} = -\frac{i}{\hbar}A^2 e^{\lambda t}U_0^\dagger(t,0)\,[H_1,\rho]\,U_0(t,0). \tag{24}$$

Solving this equation for $\tilde{R}$ gives

$$\begin{aligned}&\tilde{R}(t) = \\ &-\frac{i}{\hbar}A^2\int_0^t \mathrm{d}s\, e^{\lambda s}U_0^\dagger(s,0)[H_1(s),\rho(s)]U_0(s,0)\end{aligned} \tag{25}$$

and using eq. (22) gives us the solution for R,

$$R(t) = -\frac{i}{\hbar}A^2\int_0^t \mathrm{d}s\, e^{-\lambda(t-s)}U_0(t,s)[H_1(s),\rho(s)]U_0^\dagger(t,s). \tag{26}$$

We now return to the starting point, eq. (3), and use our expression for $R(t)$,

$$\begin{aligned}\dot{\rho} &= -\frac{i}{\hbar}[H_0,\rho] \\ &\quad -\frac{A^2}{\hbar^2}\Big[H_1(t), \int_0^t \mathrm{d}s\, e^{-\lambda(t-s)}U_0(t,s) \\ &\quad \times[H_1(s),\rho(s)]U_0^\dagger(t,s)\Big]. \end{aligned} \tag{27}$$

Equation (27) can be expressed in a simpler form by defining a transformed density matrix $\sigma(t) = U_0^\dagger(t,0)\rho(t)U_0(t,0)$ and Hamiltonian $\tilde{H}_1(t) = U_0^\dagger(t,0)H_1(t)U_0(t,0)$. The master equation then simplifies to

$$\dot{\sigma}(t) = -\frac{A^2}{\hbar^2}[\tilde{H}_1(t), \int_0^t \mathrm{d}s\, e^{-\lambda(t-s)}[\tilde{H}_1(s), \sigma(s)]]. \tag{28}$$

The master equation for DMN has been previously shown in the time-independent case [77,78].

The limit of Gaussian white noise is $A \to \infty$ and $\tau_c \to 0$ such that $2\tau_c A^2 = \alpha$ is finite [53,79]. Therefore, in this limit $A^2 \exp(-\frac{|t-s|}{\tau_c}) \to \alpha\, \delta(t-s)$, which simplifies eq. (27) to eq. (9).

Here we have considered the case of symmetric DMN. However, the more general asymmetric case (where the two possible values of $z(t)$ are not equal in magnitude) can be shown to reduce to Poisson white noise [45,79] in the appropriate limit (see fig. 1). Deriving a master equation in this more general case could be done by transforming to a symmetric DMN noise and using an adapted Shapiro-Loginov formula [80]. Note also that it is impossible to obtain Ornstein-Uhlenbeck noise as a limit of DMN [53].

Approximate classical noise master equation. – In this final section we will derive an approximate master equation valid for any type of classical noise source. It will follow a similar structure to the so-called "coupled-disorder-channel equations" [81,82] outlined for modelling static disorder. Since only contributions that are second order in the noise strength are accounted for, the only knowledge of the noise required is the two-point correlation function $\langle z(t)z(s)\rangle$.

To begin the derivation we define the fluctuations in ρ_z, $\Delta\rho_z = \rho_z - \rho$. Using this definition and averaging eq. (2) we get the following exact dynamical equation for ρ:

$$\dot{\rho} = -\frac{i}{\hbar}[H_0, \rho] - \frac{i}{\hbar}\langle[z(t)H_1(t), \Delta\rho_z(t)]\rangle. \tag{29}$$

In this last step we have used $\langle\Delta\rho_z\rangle = 0$ and assumed $\langle z(t)\rangle = 0$ for simplicity. We now need to understand the dynamics of the fluctuations. From its definition, $\partial_t\Delta\rho_z = \dot{\rho}_z - \dot{\rho}$, which when combined with eqs. (2) and (29) gives

$$\begin{aligned}\partial_t\Delta\rho_z + \frac{i}{\hbar}[H, \Delta\rho_z] = \\ -\frac{i}{\hbar}[z(t)H_1(t), \rho] + \frac{i}{\hbar}\langle[z(t)H_1(t), \Delta\rho_z(t)]\rangle.\end{aligned} \tag{30}$$

Following a similar procedure used in the previous section to solve eq. (20), the formal solution of eq. (30) can be written as

$$\begin{aligned}\Delta\rho_z(t) = -\frac{i}{\hbar}\int_0^t \mathrm{d}s\, U_z(t,s)\{[z(s)H_1(s), \rho(s)] \\ -\langle[z(s)H_1(s), \Delta\rho_z(s)]\rangle\}U_z^\dagger(t,s).\end{aligned} \tag{31}$$

At this stage, eq. (31) could be inserted into eq. (29) recursively to generate an exact master equation. Instead we will expand eq. (31) to first order in H_1 as

$$\Delta\rho_z(t) \approx -\frac{i}{\hbar}\int_0^t \mathrm{d}s\, U_0(t,s)[z(s)H_1(s), \rho(s)]U_0^\dagger(t,s). \tag{32}$$

Inserting this approximation into the exact master equation (29) now gives an approximate master equation which is second order in H_1,

$$\begin{aligned}\dot{\rho} = -\frac{i}{\hbar}[H_0, \rho] \\ -\frac{1}{\hbar^2}\left[H_1(t), \int_0^t \mathrm{d}s\, C(t,s)U_0(t,s)[H_1(s), \rho(s)]U_0^\dagger(t,s)\right],\end{aligned} \tag{33}$$

where $C(t,s) = \langle z(t)z(s)\rangle$. The accuracy of this approximation could be improved by considering higher-order terms in eq. (32), at the expense of a more complex evolution equation and a required knowledge of higher-order correlation functions of the noise. Note that for the case of dichotomous Markov noise this equation is exact (see eq. (27)).

Conclusion and outlook. – In this perspective, we have outlined a versatile approach to account for incoherent effects by modelling the macroscopic control functions as noisy or fluctuating quantities. This incoherent modulation of the system parameters is observed in a variety of contexts [83]. We have provided an overview of how to derive the exact master equations for ensemble averaged dynamics for common types of classical noise. These evolution equations clearly separate the coherent and decoherent effects of the noise in terms of its statistical noise properties. This is presented as a natural tool to model quantum systems with a noisy Hamiltonian.

Going forward, one could extend these exactly averaged master equations to other cases, *e.g.*, a combination of DMN noises with a wide distribution of switching rates [84], Poisson coloured noise [76] or compound dichotomic Markov process [79]. A broad knowledge of these different evolution types could help inference of the underlying noise properties in an experimental setting (such as in trapped ions [85,86]). Using this framework, it would be interesting to investigate the dynamics in the weak and strong noise regimes [50] and potential stochastic resonance effects [87,88].

* * *

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