Contents

1 Startup templates

1.1 template

```cpp
#include <vector>
#include <list>
#include <map>
#include <set>
#include <deque>
#include <stack>
#include <bitset>
#include <algorithm>
#include <functional>
#include <numeric>
#include <utility>
#include <sstream>
#include <iostream>
#include <cstdio>
#include <cmath>
#include <cstdlib>
#include <cstring>
#include <cassert>

using namespace std;

typedef long long ll;
typedef pair<int, int> pii;

template<typename T> int size(T& a){ return (int) a.size(); }
template<typename T> T sqr(T a){ return a * a; }

#define _(a, b) memset((a), (b), sizeof(a))
#define fs first
#define sc second
#define pb push_back
#define mp make_pair
#define all(a) a.begin(), a.end()
#define REP(i, a, b) for(int i = (a); i < (b); ++i)
#define REPD(i, a, b) for(int i = (b) - 1; i >= a; --i)
#define ve vector
```

1.2 gvimrc

```
set autoread
set autoindent
set autochdir
set cindent
set number
syntax on
set shiftwidth =4
set tabstop =4
colorscheme desert
set gfn =Monospace\ 12
```

2 Graph Algorithms

2.1 Dinic Max-Flow

```
// need: graph( head, nxt, to, capa, flow ), dist, q
bool bfs(int src, int dest) {
    _(dist, -1);
    dist[src] = 0;
    int H = 0;
    q[H ++] = src;
    REP(i, 0, H) {
        int cur = q[i];
        for(int e = head[cur]; e != -1; e = nxt[e]) {
            if (capa[e] > flow[e] && dist[to[e]] == -1) {
                dist[to[e]] = dist[cur] + 1;
                q[H ++] = to[e];
            }
        }
    }
    return dist[dest] >= 0;
}

int dfs(int cur, int curflow) {
    if (cur == dest) return curflow;
    int d ;
    for(int& e = work[cur]; e != -1; e = nxt[e]) {
        if (capa[e] > flow[e] && (dist[to[e]] == dist[cur] + 1) &&
                (d = dfs(to[e], min(curflow, capa[e] - flow[e])))) {
            flow[e] += d;
            flow[e ^ 1] -= d;
            return d;
        }
    }
    return 0;
}

int dinic() {
    int res = 0;
    while (bfs(src, des)) {
        int d;
        memcpy(work, head, sizeof(head));
        while (true) {
            d = dfs(src, INF);
            if (d == 0) break;
            res += d;
        }
    }
    return res;
}
```

2.2 Min-Cost Max-Flow

```
// need: graph (head, nxt, to, from, capa, cost, flow)
// pi, dist, prve
void updatePotentials() {
    memcpy(pi, dist, sizeof(int) * N);
}

bool fordBellman(int src, int dst) {
    REP(i, 0, N) dist[i] = INF;
    dist[src] = 0;
    bool changed;

    REP(phase, 0, N) {
        changed = false;
        REP(v, 0, N) {
            if(dist[v] == INF) continue;
            for(int e = head[v]; e != -1; e = nxt[e]) {
                int u = to[e];
                if(capa[e] > flow[e] && dist[u] > dist[v] + cost[e]) {
                    dist[u] = dist[v] + cost[e];
```

```
20            prve[u] = e;
21            changed = true;
22          }
23        }
24      }
25      if(!changed) break;
26    }
27    return !changed;
28 }
29
30 set< pii > q;
31 bool dijkstra(int src, int dst) {
32    REP(i, 0, N) dist[i] = INF;
33    dist[src] = 0;
34    q.insert( mp(0, 0) );
35
36    while(size(q)) {
37      pii tmp = (*q.begin());
38      int v = tmp.sc, d = tmp.fs;
39      q.erase(q.begin());
40      if(d != dist[v]) continue;
41
42      for(int e = head[v]; e != -1; e = nxt[e]) {
43        int u = to[e];
44        if(capa[e] > flow[e] && dist[u] > dist[v] + cost[e] - pi[v] + pi[u]) {
45          dist[u] = dist[v] + cost[e] - pi[v] + pi[u];
46          prve[u] = e;
47          q.insert( mp(dist[u], u) );
48        }
49      }
50    }
51    return dist[dst] != INF;
52 }
53
54 pii minCostMaxFlow(int src, int dst) {
55    if(!fordBellman(src, dst)) return mp(0, 0);
56    int sumFlow = 0, sumCost = 0;
57
58    do {
59      int curFlow = INF, curCost = 0;
60      int cur = dst;
61      while(cur != src) {
62        int e = prve[cur];
63        curFlow = min(curFlow, capa[e] - flow[e]);
64        curCost += cost[e];
65        cur = from[e];
66      }
67      cur = dst;
68      while(cur != src) {
69        int e = prve[cur];
70        flow[e] += curFlow;
71        flow[e ^ 1] -= curFlow;
72        cur = from[e];
73      }
74      sumCost += curFlow * curCost;
75      updatePotentials();
76    } while(dijkstra(src, dst));
77
78    return mp(sumFlow, sumCost);
79 }
```

2.3 Hungary Algo

```
1 // need: a[n][m], all indices start with 1
2 vector<int> u (n+1), v (m+1), p (m+1), way (m+1);
3 for (int i=1; i<=n; ++i) {
4    p[0] = i;
5    int j0 = 0;
6    vector<int> minv (m+1, INF);
7    vector<char> used (m+1, false);
8    do {
9      used[j0] = true;
```

```
10    int i0 = p[j0],  delta = INF,  j1;
11    for (int j=1; j<=m; ++j)
12      if (!used[j]) {
13        int cur = a[i0][j]-u[i0]-v[j];
14        if (cur < minv[j])
15          minv[j] = cur,  way[j] = j0;
16        if (minv[j] < delta)
17          delta = minv[j],  j1 = j;
18      }
19    for (int j=0; j<=m; ++j)
20      if (used[j])
21        u[p[j]] += delta,  v[j] -= delta;
22      else
23        minv[j] -= delta;
24    j0 = j1;
25  } while (p[j0] != 0);
26  do {
27    int j1 = way[j0];
28    p[j0] = p[j1];
29    j0 = j1;
30  } while (j0);
31 }
32 // restore ans[] -- selected column for each row
33 for (int j=1; j<=m; ++j)
34   ans[p[j]] = j;
35 // cost
36 int cost = -v[0];
```

2.4 Min Cut

```
1  pair<int, ve<int> > GetMinCut(ve< ve<int> > &weights) {
2    int N = weights.size();
3    ve<int> used(N), cut, best_cut;
4    int best_weight = -1;
5
6    REPD(phase, 0, N) {
7      ve<int> w = weights[0];
8      ve<int> added = used;
9      int prev, last = 0;
10     REP(i, 0, phase) {
11       prev = last;
12       last = -1;
13       REP(j, 1, N)
14         if (!added[j] && (last == -1 || w[j] > w[last])) last = j;
15       if (i == phase-1) {
16         REP(j, 0, N) weights[prev][j] += weights[last][j];
17         REP(j, 0, N) weights[j][prev] = weights[prev][j];
18         used[last] = true;
19         cut.pb(last);
20         if (best_weight == -1 || w[last] < best_weight) {
21           best_cut = cut;
22           best_weight = w[last];
23         }
24       } else {
25         REP(j, 0, N)
26           w[j] += weights[last][j];
27         added[last] = true;
28       }
29     }
30   }
31   return mp(best_weight, best_cut);
32 }
```

2.5 Bridges & Cut Vertices & Biconnected Components

```
1  void dfs(int v, int u) {
2    timer++;
3    dfsnum[v] = lownum[v] = timer;
4    int sons = 0;
5
```

```
6    for(int e = head[v]; e != -1; e = nxt[e]) {
7       int w = to[e];
8       if(dfsnum[w] == 0) { // (v, w) is tree edge
9          st[stSize++] = e;
10         sons++;
11         dfs(w, v);
12         lownum[v] = min(lownum[v], lownum[w]);
13         if( lownum[w] >= dfsnum[v] ) { // v is root or cut point
14            if(u != -1) cutPoint[v] = true;
15
16            // All edges from stack up to edge (v, w) form
17            // new biconnected component
18         }
19      } else if(dfsnum[w] < dfsnum[v] && w != u) { // (v, w) is back edge
20         st[stSize++] = e;
21         lownum[v] = min(lownum[v], dfsnum[w]);
22      }
23   }
24
25   if(u == -1 && sons > 1) cutPoint[v] = true;
26
27   if(u != -1 && lownum[v] > dfsnum[u]) {
28      // (v, u) is bridge
29   }
30 }
```

2.6   Strongly Connected Components

```
1  void dfs(int v) {
2     timer++;
3     dfsnum[v] = lownum[v] = timer;
4     st[stSize++] = v;
5     inSt[v] = true;
6
7     for(int e = head[v]; e != -1; e = nxt[e]) {
8        int w = to[e];
9        if(dfsnum[w] == 0) {
10          dfs(w, v);
11          lownum[v] = min(lownum[v], lownum[w]);
12       } else if(inSt[w]) {
13          lownum[v] = min(lownum[v], dfsnum[w]);
14       }
15    }
16
17    if(lownum[v] == dfsnum[v]) {
18       int w;
19       do {
20          w = st[--stSize];
21          inSt[w] = false;
22          comp[w] = compID;
23       } while(w != v);
24       compID++;
25    }
26 }
```

2.7   2-SAT

Problem: $(a \lor c)$ & $(a \lor !b)$ & ...
   Edges: $(a \lor b)$ is equivalent to $(!a \Rightarrow b) \lor (!b \Rightarrow a)$
   Solution: there is no solution iff for some x $compID[x] = compID[!x]$, else see code below

```
1  // need: graph, scc
2  int main() {
3     _(used, false);
4     REP(i, 0, N)
5        if (!used[i]) dfs1(i);
6
7     _(compID, -1);
8     int id = 0;
9     REPD(i, 0, N) {
```

```
10        int v = order[i];
11        if (comp[v] == -1) dfs2(v, id++);
12    }
13
14    REP(i, 0, N)
15        if (compID[i] == compID[i^1]) {
16            puts ("NO SOLUTION");
17            return 0;
18        }
19    REP(i, 0, N) {
20        int ans = comp[i] > comp[i^1] ? i : i^1;
21        printf ("%d ", ans);
22    }
23 }
```

3 Linear Algebra

3.1 Gauss Elimination

```
1  // Ax = B. RETURN: determinant, A -> A^(-1), B -> solution
2  T GaussJordan(VVT &a, VT &b) {
3    const int n = a.size();
4    ve<int> irow(n), icol(n), ipiv(n);
5    T det = 1;
6
7    REP(i, 0, n) {
8      int pj = -1, pk = -1;
9      REP(j, 0, n) if (!ipiv[j])
10       REP(k, 0, n) if (!ipiv[k])
11         if (pj == -1 || fabs(a[j][k]) > fabs(a[pj][pk])) { pj = j; pk = k; }
12     if (fabs(a[pj][pk]) < EPS) { cerr << "Matrix is singular." << endl; exit(0); }
13     ipiv[pk]++;
14     swap(a[pj], a[pk]);
15     swap(b[pj], b[pk]);
16     if (pj != pk) det *= -1;
17     irow[i] = pj;
18     icol[i] = pk;
19
20     T c = 1.0 / a[pk][pk];
21     det *= a[pk][pk];
22     a[pk][pk] = 1.0;
23     REP(p, 0, n) a[pk][p] *= c;
24     b[pk] *= c;
25     REP(p, 0, n) if (p != pk) {
26       c = a[p][pk];
27       a[p][pk] = 0;
28       REP(q, 0, n) a[p][q] -= a[pk][q] * c;
29       b[p] -= b[pk] * c;
30     }
31   }
32
33   REPD(p, 0, n) if (irow[p] != icol[p]) {
34     REP(k, 0, n) swap(a[k][irow[p]], a[k][icol[p]]);
35   }
36
37   return det;
38 }
```

3.2 Fast-Fourier Transform

```
1  typedef complex<double> base;
2
3  void fft (vector<base> & a, bool invert) {
4    int n = (int) a.size();
5
6    for (int i = 1, j = 0; i < n; i++) {
7      int bit = n >> 1;
8      for (; j >= bit; bit >>= 1)
9        j -= bit;
10     j += bit;
11     if (i < j) swap (a[i], a[j]);
12   }
13
14   for (int len = 2; len <= n; len <<= 1) {
15     double ang = 2 * PI/len * (invert ? -1 : 1);
16     base wlen (cos(ang), sin(ang));
17     for (int i = 0; i < n; i += len) {
18       base w (1);
19       for (int j = 0; j < len/2; j++) {
20         base u = a[i+j], v = a[i+j+len/2] * w;
21         a[i+j] = u + v;
22         a[i+j+len/2] = u - v;
23         w *= wlen;
24       }
25     }
26   }
```

```
27    if (invert)
28      REP(i, 0, n) a[i] /= n;
29  }
```

3.3 Simplex

```
1  // maximize c^T x
2  // Ax <= b
3  // x >= 0
4
5  struct LPSolver {
6    int m, n;
7    ve<int> B, N;
8    ve< ve<double> > D;
9
10   LPSolver(const ve< ve<double> > &A, const ve<double> &b, const ve<double> &c) :
11     m(b.size()), n(c.size()), N(n+1), B(m), D(m+2, ve<double>(n+2)) {
12       REP(i, 0, m) REP(j, 0, n) D[i][j] = A[i][j];
13       REP(i, 0, m) { B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[i]; }
14       REP(j, 0, n) { N[j] = j; D[m][j] = -c[j]; }
15       N[n] = -1; D[m+1][n] = 1;
16     }
17
18   void Pivot(int r, int s) {
19     REP(i, 0, m + 2) if (i != r)
20       REP(j, 0, n + 2) if (j != s)
21         D[i][j] -= D[r][j] * D[i][s] / D[r][s];
22     REP(j, 0, n + 2) if (j != s) D[r][j] /= D[r][s];
23     REP(i, 0, n + 2) if (i != r) D[i][s] /= -D[r][s];
24     D[r][s] = 1.0 / D[r][s];
25     swap(B[r], N[s]);
26   }
27
28   bool Simplex(int phase) {
29     int x = phase == 1 ? m+1 : m;
30     while (true) {
31       int s = -1;
32       REP(j, 0, n + 1) {
33         if (phase == 2 && N[j] == -1) continue;
34         if (s == -1 || D[x][j] < D[x][s] || D[x][j] == D[x][s] && N[j] < N[s]) s = j;
35       }
36       if (D[x][s] >= -EPS) return true;
37       int r = -1;
38       REP(i, 0, m) {
39         if (D[i][s] <= 0) continue;
40         if (r == -1 || D[i][n+1] / D[i][s] < D[r][n+1] / D[r][s] ||
41             D[i][n+1] / D[i][s] == D[r][n+1] / D[r][s] && B[i] < B[r]) r = i;
42       }
43       if (r == -1) return false;
44       Pivot(r, s);
45     }
46   }
47
48   double Solve(ve<double> &x) {
49     int r = 0;
50     REP(i, 1, m) if (D[i][n+1] < D[r][n+1]) r = i;
51     if (D[r][n+1] <= -EPS) {
52       Pivot(r, n);
53       if (!Simplex(1) || D[m+1][n+1] < -EPS) return -numeric_limits<double>::infinity();
54       REP(i, 0, m) if (B[i] == -1) {
55         int s = -1;
56         REP(j, 0, n + 1)
57           if (s == -1 || D[i][j] < D[i][s] || D[i][j] == D[i][s] && N[j] < N[s]) s = j;
58         Pivot(i, s);
59       }
60     }
61     if (!Simplex(2)) return numeric_limits<double>::infinity();
62     x = ve<double>(n);
63     REP(i, 0, m) if (B[i] < n) x[B[i]] = D[i][n+1];
64     return D[m][n+1];
65   }
66 };
```

4   String Algorithms

4.1   Suffix Array

```
1  struct entry {
2    int nr[2], p;
3  } L[MAXN], tmp[MAXN];
4  #define eq(a, b) ((a).nr[0] == (b).nr[0] && (a).nr[1] == (b).nr[1])
5  int cnt[MAXN], p[2][MAXN];
6
7  void radixPass(entry * a, int N, int pass, int K, entry * b) {
8    memset(cnt, 0, (K + 1) * sizeof(int));
9    REP(i, 0, N) cnt[a[i].nr[pass]]++;
10   int sum = 0;
11   REP(i, 0, K + 1) {
12     sum += cnt[i];
13     cnt[i] = sum - cnt[i];
14   }
15   REP(i, 0, N) b[cnt[a[i].nr[pass]]++] = a[i];
16 }
17
18 void makeSA(int * s, int N, int * suftab, int * isuftab) {
19   REP(i, 0, N) p[0][i] = s[i];
20   int k = 200;
21   for(int step = 1, len = 1; ; step ^= 1, len <<= 1) {
22     for(int i = 0, j = len; i < N; i++, j++) {
23       L[i].nr[0] = p[step ^ 1][i];
24       L[i].nr[1] = j < N ? p[step ^ 1][j] : 0;
25       L[i].p = i;
26     }
27     radixPass(L, N, 1, k, tmp);
28     radixPass(tmp, N, 0, k, L);
29     k = 1;
30     REP(i, 0, N) {
31       p[step][L[i].p] = i > 0 && eq(L[i], L[i - 1]) ?
32         p[step][L[i - 1].p] : k++;
33     }
34     if(k > N) break;
35   }
36   REP(i, 0, N) suftab[i] = L[i].p;
37   REP(i, 0, N) isuftab[suftab[i]] = i;
38 }
39
40 void makeLCP(int * s, int * suftab, int * isuftab, int N, int * lcptab) {
41   int cur = 0;
42   REP(i, 0, N) {
43     if(isuftab[i] == 0) continue;
44     int ii = i + cur, jj = suftab[isuftab[i] - 1] + cur;
45     while(ii < N && jj < N && s[ii] == s[jj]) ii++, jj++, cur++;
46     lcptab[isuftab[i]] = cur--;
47     if(cur < 0) cur = 0;
48   }
49 }
```

4.2   Suffix Tree from Suffix Array

```
1  struct Seg {
2    int lb, rb, lcp;
3    vector<Seg*> childList;
4    void init(int l, int i, int j) {
5      lb = i; rb = j; lcp = l;
6    }
7    void add(Seg * son) {
8      childList.pb(son);
9    }
10 };
11 typedef Seg* pSeg;
12
13 struct Stack {
14   pSeg segs[MAXN << 1];
15   int size;
```

```
16     void push(pSeg seg) {
17       segs[size++] = seg;
18     }
19     pSeg pop() {
20       return segs[--size];
21     }
22     pSeg top() {
23       return segs[size - 1];
24     }
25  } stack;
26
27  pSeg top() { return stack.top(); }
28  void push(pSeg seg) { stack.push(seg); }
29  pSeg pop() { return stack.pop(); }
30
31  pSeg init(int lcp, int lb, int rb) {
32     pSeg ret = new Seg;
33     ret->init(lcp, lb, rb);
34     return ret;
35  }
36
37  pSeg makeTree() {
38     stack.size = 0;
39     pSeg lastInterval = NULL;
40     stack.push(init(0, 0, -1));
41     REP(i, 1, N) {
42        int lb = i - 1;
43        pSeg singleton = init(N - suftab[i - 1] - 1, i - 1, i - 1);
44        //process(singleton);
45
46        while(lcptab[i] < top()->lcp) {
47           if(singleton != NULL) {
48              top()->add(singleton);
49              singleton = NULL;
50           }
51           top()->rb = i - 1;
52           lastInterval = pop();
53           //process(lastInterval);
54           lb = lastInterval->lb;
55           if(lcptab[i] <= top()->lcp) {
56              top()->add(lastInterval);
57              lastInterval = NULL;
58           }
59        }
60        if(lcptab[i] > top()->lcp) {
61           if(lastInterval != NULL) {
62              pSeg seg = init(lcptab[i], lb, -1);
63              seg->add(lastInterval);
64              push(seg);
65              lastInterval = NULL;
66           } else push(init(lcptab[i], lb, -1));
67        }
68        if(singleton != NULL) {
69           top()->add(singleton);
70        }
71     }
72     assert(stack.size == 1);
73     //process(top());
74     return top();
75  }
```

4.3 Z-function

```
1  ve<int> z_function (string s) {
2     int n = (int) s.length();
3     ve<int> z (n);
4     for (int i=1, l=0, r=0; i<n; ++i) {
5        if (i <= r)
6           z[i] = min (r-i+1, z[i-l]);
7        while (i+z[i] < n && s[z[i]] == s[i+z[i]])
8           ++z[i];
9        if (i+z[i]-1 > r)
```

```
10        l = i,   r = i+z[i]-1;
11    }
12    return z;
13 }
```

4.4 Suffix Automata

```
 1 struct state {
 2    int len, link;
 3    map<char,int> next;
 4 };
 5
 6 state st[MAXLEN*2];
 7 int sz, last;
 8
 9 void sa_init() {
10    sz = last = 0;
11    st[0].len = 0;
12    st[0].link = -1;
13    ++sz;
14 }
15
16 void sa_extend (char c) {
17    int cur = sz++;
18    st[cur].len = st[last].len + 1;
19    int p;
20    for (p=last; p!=-1 && !st[p].next.count(c); p=st[p].link)
21      st[p].next[c] = cur;
22    if (p == -1)
23      st[cur].link = 0;
24    else {
25      int q = st[p].next[c];
26      if (st[p].len + 1 == st[q].len)
27        st[cur].link = q;
28      else {
29        int clone = sz++;
30        st[clone].len = st[p].len + 1;
31        st[clone].next = st[q].next;
32        st[clone].link = st[q].link;
33        for (; p!=-1 && st[p].next[c]==q; p=st[p].link)
34          st[p].next[c] = clone;
35        st[q].link = st[cur].link = clone;
36      }
37    }
38    last = cur;
39 }
```

4.5 Palindromes

```
 1 for(i = 0; i < n; i++){
 2    if(i > r) k = 1;
 3    else k = min(d1[l + r - i], r - i);
 4
 5    while(0 <= i-k && i+k < n && s[i - k] == s[i + k]) k++;
 6    d1[i] = k;
 7    if(i + k - 1 > r)
 8      r = i + k - 1, l = i - k + 1;
 9 }
10
11 for(i = 0; i < n; i++){
12    if(i > r) k = 0;
13    else k = min(d2[l + r - i + 1], r - i + 1);
14
15    while(i + k < n && i - k - 1 >= 0 && s[i+k] == s[i - k - 1]) k++;
16    d2[i] = k;
17
18    if(i + k - 1 > r)
19      l = i - k, r = i + k - 1;
20 }
```

4.6 Lyndon decomposition & Duval

```cpp
// Lyndon decomposition
for (int i = 0; i < n;) {
    int j=i+1, k=i;
    while (j < n && s[k] <= s[j]) {
        if (s[k] < s[j])
            k = i;
        else
            ++k;
        ++j;
    }
    while (i <= k) {
        cout << s.substr (i, j-k) << ' ';
        i += j - k;
    }
}

string min_cyclic_shift (string s) {
    s += s;
    int n = (int) s.length();
    int i=0, ans=0;
    while (i < n/2) {
        ans = i;
        int j=i+1, k=i;
        while (j < n && s[k] <= s[j]) {
            if (s[k] < s[j])
                k = i;
            else
                ++k;
            ++j;
        }
        while (i <= k)  i += j - k;
    }
    return s.substr (ans, n/2);
}
```

5 Modular

```
1  // All algorithms described here work on nonnegative integers.
2
3  // return a % b (positive value)
4  int mod(int a, int b) {
5    return ((a%b)+b)%b;
6  }
7
8  // computes gcd(a,b)
9  int gcd(int a, int b) {
10   int tmp;
11   while(b){a%=b; tmp=a; a=b; b=tmp;}
12   return a;
13 }
14
15 // computes lcm(a,b)
16 int lcm(int a, int b) {
17   return a/gcd(a,b)*b;
18 }
19
20 // returns d = gcd(a,b); finds x,y such that d = ax + by
21 int extended_euclid(int a, int b, int &x, int &y) {
22   int xx = y = 0;
23   int yy = x = 1;
24   while (b) {
25     int q = a/b;
26     int t = b; b = a%b; a = t;
27     t = xx; xx = x−q*xx; x = t;
28     t = yy; yy = y−q*yy; y = t;
29   }
30   return a;
31 }
32
33 // finds all solutions to ax = b (mod n)
34 ve<int> modular_linear_equation_solver(int a, int b, int n) {
35   int x, y;
36   ve<int> solutions;
37   int d = extended_euclid(a, n, x, y);
38   if (!(b%d)) {
39     x = mod (x*(b/d), n);
40     for (int i = 0; i < d; i++)
41       solutions.push_back(mod(x + i*(n/d), n));
42   }
43   return solutions;
44 }
45
46 // computes b such that ab = 1 (mod n), returns −1 on failure
47 int mod_inverse(int a, int n) {
48   int x, y;
49   int d = extended_euclid(a, n, x, y);
50   if (d > 1) return −1;
51   return mod(x,n);
52 }
53
54 // find z such that z % x = a, z % y = b. Here, z is unique modulo M = lcm(x,y).
55 // Return (z,M). On failure, M = −1.
56 pii chinese_remainder_theorem(int x, int a, int y, int b) {
57   int s, t;
58   int d = extended_euclid(x, y, s, t);
59   if (a%d != b%d) return make_pair(0, −1);
60   return make_pair(mod(s*b*x+t*a*y,x*y)/d, x*y/d);
61 }
62
63 // Chinese remainder theorem: find z such that
64 // z % x[i] = a[i] for all i. Note that the solution is
65 // unique modulo M = lcm_i (x[i]). Return (z,M). On failure, M = −1.
66 pii chinese_remainder_theorem(const ve<int> &x, const ve<int> &a) {
67   pii ret = make_pair(a[0], x[0]);
68   for (int i = 1; i < x.size(); i++) {
69     ret = chinese_remainder_theorem(ret.second, ret.first, x[i], a[i]);
70     if (ret.second == −1) break;
71   }
72   return ret;
```

```
73 }
74
75 // computes x and y such that ax + by = c; on failure, x = y =-1
76 void linear_diophantine(int a, int b, int c, int &x, int &y) {
77     int d = gcd(a,b);
78     if (c%d) {
79         x = y = -1;
80     } else {
81         x = c/d * mod_inverse(a/d, b/d);
82         y = (c-a*x)/b;
83     }
84 }
```

6 Data Structures

6.1 Treap

```
// key(l) < key, key(r) >= key
void split (pitem t, int key, pitem & l, pitem & r) {
  if (!t) l = r = NULL;
  else if (key < t->key)
    split (t->l, key, l, t->l),   r = t;
  else
    split (t->r, key, t->r, r),   l = t;
}

void insert (pitem & t, pitem it) {
  if (!t) t = it;
  else if (it->prior > t->prior)
    split (t, it->key, it->l, it->r),   t = it;
  else
    insert (it->key < t->key ? t->l : t->r, it);
}

void merge (pitem & t, pitem l, pitem r) {
  if (!l || !r) t = l ? l : r;
  else if (l->prior > r->prior)
    merge (l->r, l->r, r),   t = l;
  else
    merge (r->l, l, r->l),   t = r;
}
```

6.2 Implicit Treap

```
int cnt (pitem t) { return t ? t->cnt : 0; }

void upd_cnt (pitem t) {
  if (t) t->cnt = 1 + cnt(t->l) + cnt (t->r);
}

void merge (pitem & t, pitem l, pitem r) {
  if (!l || !r)
    t = l ? l : r;
  else if (l->prior > r->prior)
    merge (l->r, l->r, r),   t = l;
  else
    merge (r->l, l, r->l),   t = r;
  upd_cnt (t);
}
// key(l) < key, key(r) > key
void split (pitem t, pitem & l, pitem & r, int key, int add = 0) {
  if (!t)
    return void( l = r = 0 );
  int cur_key = add + cnt(t->l);
  if (key <= cur_key)
    split (t->l, l, t->l, key, add),   r = t;
  else
    split (t->r, t->r, r, key, add + 1 + cnt(t->l)),   l = t;
  upd_cnt (t);
}

pitem build(int *a, int cnt) {
  if (cnt <= 0) return NULL;
  int p = cnt/2;
  pitem root = &buf[buf_sz++];
  root->val = a[p]; // root->prior ???
  root->l = build(a, p);
  root->r = build(a + p + 1, cnt - p - 1);
  upd_cnt(root);
  return root;
}
```

6.3 Fenwick Tree

```
// 1-indexation
void set(int x, int v) {
    while(x <= N) {
        tree[x] += v;
        x += (x & -x);
    }
}
int get(int x) {
    int res = 0;
    while(x) {
        res += tree[x];
        x -= (x & -x);
    }
    return res;
}

// 0-indexation
int sum(int r) {
    int res = 0;
    for (; r >= 0; r = (r & (r+1)) - 1)
        res += t[r];
    return res;
}
void inc(int i, int val) {
    for (; i < n; i = (i | (i+1)))
        t[i] += val;
}
```

6.4 Stack with Min

```
// stack[i].second = min { stack[j].first }, j <= i

// add val
int minima = st.empty() ? val : min (val, st.top().second);
st.push ( mp(val, minima) );

// pop
int res = st.top().first;
st.pop();

// get minimum
int minima = st.top().second;
```

6.5 Queue with Min

```
deque<int> q;

// get minimum
current_minimum = q.front();

// add val
while (!q.empty() && q.back() > val)
    q.pop_back();
q.push_back (val);

// pop val
if (!q.empty() && q.front() == val)
    q.pop_front();
```

6.6 kD-tree

```
struct Node {
    Point p, v;
    Node *left, *right;
};
```

```cpp
// sX -- points sorted by cmpX, sY -- points sorted by cmpY
Node* construct(int N, Point *sX, Point *sY, int depth) {
  Node * root = allocateNode();
  if(N == 1) {
    root->p = sX[0];
  } else {

    Point * sx = (depth & 1) ? sX : sY;
    Point * sy = (depth & 1) ? sY : sX;

    bool (*cmp)(const Point &, const Point &);
    cmp = (depth & 1) ? cmpX : cmpY;

    root->v = sx[N / 2];

    int K = 0;
    REP(i, 0, N) {
      if(!cmp(sy[i], root->v)) {
        tmp[K++] = sy[i];
      } else {
        sy[i - K] = sy[i];
      }
    }
    REP(i, 0, K) sy[N - K + i] = tmp[i];

    root->left = construct(N / 2, sX, sY, depth + 1);
    root->right = construct(K, sX + N/2, sY + N/2, depth + 1);

  }
  return root;
}

void findClosest(Node * root, const Point & p, ll & minDist, int depth) {

  if(root->left == NULL) {
    if(! (root->p == p) ) {
      minDist = min(minDist, (p - root->p).len());
    }
    return;
  }

  bool (*cmp)(const Point &, const Point &);
  cmp = (depth & 1) ? cmpX : cmpY;
  ll d = (depth & 1) ? (root->v.x - p.x) : (root->v.y - p.y);

  if(cmp(p, root->v)) {
    findClosest(root->left, p, minDist, depth + 1);
  } else {
    findClosest(root->right, p, minDist, depth + 1);
  }

  if(sqr(d) < minDist) {
    if(cmp(p, root->v)) {
      findClosest(root->right, p, minDist, depth + 1);
    } else {
      findClosest(root->left, p, minDist, depth + 1);
    }
  }
}
```

7   Geometry

7.1   Point 2D

```
 1 | // point projection on line (A, B)
 2 | pt getH(const pt & A, const pt & B) const {
 3 |   pt C = *this;
 4 |   pt v = B − A;
 5 |   pt u = C − A;
 6 |   double k = v ^ u / v.len();
 7 |   v = v.norm(k);
 8 |   pt H = A + v;
 9 |   return H;
10 | }
11 |
12 | // Intersection of lines (A, B) & (C, D)
13 | int getIntersection(const pt & A, const pt & B, const pt & C, const pt & D, pt & O) {
14 |   pt v = B − A;
15 |   double s1 = (C − A) * (D − A);
16 |   double s2 = (D − B) * (C − B);
17 |   double s = s1 + s2;
18 |   if(eq(s, 0) ) {
19 |     if(!A.isOnLine(C, D) ) {
20 |       return 0;
21 |     }
22 |     return 2;
23 |   }
24 |   v = v / s;
25 |   v = v * s1;
26 |   O = A + v;
27 |   return 1;
28 | }
29 |
30 | // Intersection of circles (A, rA) & (B, rB)
31 | int getIntersection(const pt & A, double rA, const pt & B, double rB, pt & M, pt & N) {
32 |   double d = A.distTo(B);
33 |   if(ls(rA + rB, d) || ls(d, fabs(rA − rB)) ) {
34 |     return 0;
35 |   }
36 |   double a = (sqr(rA) − sqr(rB) + sqr(d) ) / 2 / d;
37 |   double h = sqrt(sqr(rA) − sqr(a));
38 |   pt v = B − A;
39 |   pt u = v.rotate();
40 |   v = v.norm(a);
41 |   u = u.norm(h);
42 |   pt H = A + v;
43 |   M = H + u;
44 |   N = H − u;
45 |   if(u.isZero()) return 1;
46 |   return 2;
47 | }
48 |
49 | // Intersection of line (A, B) & circle (O, r)
50 | int getIntersection(const pt & A, const pt & B, const pt & O, double r, pt & M, pt & N) {
51 |   double h = O.distTo(A, B);
52 |   if(ls(r, h) ) {
53 |     return 0;
54 |   }
55 |   pt H = O.getH(A, B);
56 |   pt v = B − A;
57 |   double k = sqrt(sqr(r) − sqr(h) );
58 |   v = v.norm(k);
59 |   M = H + v;
60 |   N = H − v;
61 |   if(v.isZero() ) return 1;
62 |   return 2;
63 | }
64 |
65 | // Tangent lines through point A to circle (O, r)
66 | int getTangent(const pt & A, const pt & O, double r, pt & M, pt & N) {
67 |   pt v = O − A;
68 |   double d = v.len();
69 |   if(ls(d, r) ) return 0;
```

```
70    double alpha = asin(r / d);
71    double L = sqrt(sqr(d) - sqr(r));
72   v = v.norm(L);
73   M = A + v.rotate(alpha);
74   N = A + v.rotate(-alpha);
75    if(eq(r, d)) return 1;
76    return 2;
77 }
78
79 // Outer tangent lines between circles (A, rA) & (B, rB)
80 void getOutTangent(pt A, double rA, pt B, double rB, pair<pt, pt> & P, pair<pt, pt> & Q) {
81    if(rA > rB) {
82      swap(rA, rB);
83      swap(A, B);
84    }
85    double d = (A - B).len();
86    double r = rB - rA;
87    pt u = (A - B).rotate(asin(r / d)).rotate().norm(rA);
88    P.first = A + u;
89    Q.first = A - u;
90    pt T1, T2;
91    getTangent(A, B, rB - rA, T1, T2);
92    P.second = T1 + u;
93    Q.second = T2 - u;
94 }
95
96 // Inner tangent lines between circles (A, rA) & (B, rB)
97 void getInTangent(pt A, double rA, pt B, double rB, pair<pt, pt> & P, pair<pt, pt> & Q) {
98    pt I = (A * rB / (rA + rB)) + (B * rA / (rA + rB));
99    pt M1, N1, M2, N2;
100   getTangent(I, A, rA, M1, N1);
101   getTangent(I, B, rB, M2, N2);
102   if(I.isOnLine(M1, M2)) P = mp(M1, M2), Q = mp(N1, N2);
103   else P = mp(M1, N2), Q = mp(N1, M2);
104 }
```

7.2 Point 3D

```
1 pt operator*(const pt & p) const {
2    return pt( y * p.z - z * p.y,
3             z * p.x - x * p.z,
4             x * p.y - y * p.x  );
5 }
6
7 // Projection on line (A, B)
8 pt getH(const pt & A, const pt & B) const {
9    pt C = *this;
10   pt v = B - A;
11   pt u = C - A;
12   double k = v ^ u / v.length();
13   v = v.norm(k);
14   pt H = A + v;
15   return H;
16 }
17
18 // Rotation
19 pt rotate(pt normal) const { return *this * normal; }
20
21 pt rotate(double alpha, const pt & normal) const {
22   return rotate(cos(alpha), sin(alpha), normal );
23 }
24
25 pt rotate(double cosa, double sina, const pt & normal) const {
26   pt v = *this;
27   pt u = v.rotate(normal);
28   pt w = v * cosa + u * sina;
29   return w;
30 }
31
32 // Undirected angle
33 double getAngle(pt u) const {
34   pt v = *this;
```

```cpp
35      return atan2((v * u).length() , v ^ u);
36  }
37
38  bool isOnPlane(const pt & A, const pt & B, const pt & C) const {
39      return doubleEqual( (A - *this) * (B - *this) ^ (C - *this), 0);
40  }
41
42  // Intersection of lines (A, B) & (C, D)
43  int getIntersection(const pt & A, const pt & B, const pt & C, const pt & D, pt & O) {
44      if( !doubleEqual( (B - A) * (C - A) ^ (D - A), 0)   ) {
45          throw "It's not plane";
46      }
47      if( doubleEqual( ( (A - B) * (C - D) ).length(), 0)) {
48          if(A.isOnLine(C, D) ) return 2;
49          return 0;
50      }
51      pt normal = ( (A - B) * (C - B) ).norm();
52      pt v = B - A;
53      double s1 = (C - A) * (D - A) ^ normal;
54      double s2 = (D - B) * (C - B) ^ normal;
55      double s = s1 + s2;
56      v = v / s;
57      v = v * s1;
58      O = A + v;
59      return 1;
60  }
61
62  // Intersection of line (A, B) & plane (C, D, E)
63  int getIntersection(const pt & A, const pt & B, const pt & C, const pt & D, const pt & E, pt & O) {
64      pt v = B - A;
65      double V1 = (C - A) * (D - A) ^ (E - A); // tetrahedra (A, C, D, E) volume
66      double V2 = (D - B) * (C - B) ^ (E - B);
67      double V = V1 + V2;
68      v = v / V;
69      if(doubleEqual(V, 0) ) {
70          if(A.isOnPlane(C, D, E) ) return 2;
71          return 0;
72      }
73      v = v * V1;
74      O = A + v;
75      return 1;
76  }
77
78  // Intersection of planes (A, nA) & (B, nB)
79  bool getIntersection(const pt & A, const pt & nA, const pt & B, const pt & nB, pt & P, pt & Q) {
80      pt n = nA * nB;
81      if(n.isZero() ) return false;
82      pt v = n * nA;
83      double k = (B - A) ^ nB / (v ^ nB);
84
85      v = v * k;
86      P = A + v;
87      Q = P + n;
88      return true;
89  }
```

8   Misc

8.1   Dates

```
string dayOfWeek[] = {"Mon", "Tue", "Wed", "Thu", "Fri", "Sat", "Sun"};

// converts Gregorian date to integer (Julian day number)
int dateToInt (int m, int d, int y){
  return
    1461 * (y + 4800 + (m - 14) / 12) / 4 +
    367 * (m - 2 - (m - 14) / 12 * 12) / 12 -
    3 * ((y + 4900 + (m - 14) / 12) / 100) / 4 +
    d - 32075;
}

// converts integer (Julian day number) to Gregorian date: month/day/year
void intToDate (int jd, int &m, int &d, int &y){
  int x, n, i, j;

  x = jd + 68569;
  n = 4 * x / 146097;
  x -= (146097 * n + 3) / 4;
  i = (4000 * (x + 1)) / 1461001;
  x -= 1461 * i / 4 - 31;
  j = 80 * x / 2447;
  d = x - 2447 * j / 80;
  x = j / 11;
  m = j + 2 - 12 * x;
  y = 100 * (n - 49) + i + x;
}

// converts integer (Julian day number) to day of week
string intToDay (int jd){
  return dayOfWeek[jd % 7];
}
```

8.2   Long Long Multiplication

```
long long mul( long long a, long long b, long long m ) {
  long long q = (long long)((long double)a * (long double)b / (long double)m);
  long long r = a * b - q * m;
  return (r + 5 * m) % m;
}
```

8.3   Gray Code

```
int g (int n) {
  return n ^ (n >> 1);
}
int rev_g (int g) {
  int n = 0;
  for (; g; g>>=1)
    n ^= g;
  return n;
}
```

Theoretical Computer Science Cheat Sheet	
Definitions	Series

Definitions

$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n) \ \forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0 \ \forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n \to \infty} f(n)/g(n) = 0$.
$\lim_{n \to \infty} a_n = a$	iff $\forall \epsilon > 0, \exists n_0$ such that $\|a_n - a\| < \epsilon, \ \forall n \ge n_0$.
$\sup S$	least $b \in \mathbb{R}$ such that $b \ge s, \ \forall s \in S$.
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \le s, \ \forall s \in S$.
$\liminf_{n \to \infty} a_n$	$\lim_{n \to \infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\limsup_{n \to \infty} a_n$	$\lim_{n \to \infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle\begin{smallmatrix} n \\ k \end{smallmatrix}\right\rangle$	1st order Eulerian numbers: Permutations $\pi_1 \pi_2 \dots \pi_n$ on $\{1, 2, \dots, n\}$ with k ascents.
$\left\langle\!\!\left\langle\begin{smallmatrix} n \\ k \end{smallmatrix}\right\rangle\!\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n}\frac{1}{i}, \qquad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \quad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

1. $\binom{n}{k} = \dfrac{n!}{(n-k)!k!},$ **2.** $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n,$ **3.** $\binom{n}{k} = \binom{n}{n-k},$

4. $\binom{n}{k} = \dfrac{n}{k}\binom{n-1}{k-1},$ **5.** $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$

6. $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k},$ **7.** $\displaystyle\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n},$

8. $\displaystyle\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1},$ **9.** $\displaystyle\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n},$

10. $\binom{n}{k} = (-1)^k\binom{k-n-1}{k},$ **11.** $\left\{\begin{matrix} n \\ 1 \end{matrix}\right\} = \left\{\begin{matrix} n \\ n \end{matrix}\right\} = 1,$

12. $\left\{\begin{matrix} n \\ 2 \end{matrix}\right\} = 2^{n-1} - 1,$ **13.** $\left\{\begin{matrix} n \\ k \end{matrix}\right\} = k\left\{\begin{matrix} n-1 \\ k \end{matrix}\right\} + \left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\},$

14. $\left[\begin{matrix} n \\ 1 \end{matrix}\right] = (n-1)!,$ **15.** $\left[\begin{matrix} n \\ 2 \end{matrix}\right] = (n-1)!H_{n-1},$ **16.** $\left[\begin{matrix} n \\ n \end{matrix}\right] = 1,$ **17.** $\left[\begin{matrix} n \\ k \end{matrix}\right] \ge \left\{\begin{matrix} n \\ k \end{matrix}\right\},$

18. $\left[\begin{matrix} n \\ k \end{matrix}\right] = (n-1)\left[\begin{matrix} n-1 \\ k \end{matrix}\right] + \left[\begin{matrix} n-1 \\ k-1 \end{matrix}\right],$ **19.** $\left\{\begin{matrix} n \\ n-1 \end{matrix}\right\} = \left[\begin{matrix} n \\ n-1 \end{matrix}\right] = \binom{n}{2},$ **20.** $\displaystyle\sum_{k=0}^{n}\left[\begin{matrix} n \\ k \end{matrix}\right] = n!,$ **21.** $C_n = \dfrac{1}{n+1}\binom{2n}{n},$

22. $\left\langle\begin{matrix} n \\ 0 \end{matrix}\right\rangle = \left\langle\begin{matrix} n \\ n-1 \end{matrix}\right\rangle = 1,$ **23.** $\left\langle\begin{matrix} n \\ k \end{matrix}\right\rangle = \left\langle\begin{matrix} n \\ n-1-k \end{matrix}\right\rangle,$ **24.** $\left\langle\begin{matrix} n \\ k \end{matrix}\right\rangle = (k+1)\left\langle\begin{matrix} n-1 \\ k \end{matrix}\right\rangle + (n-k)\left\langle\begin{matrix} n-1 \\ k-1 \end{matrix}\right\rangle,$

25. $\left\langle\begin{matrix} 0 \\ k \end{matrix}\right\rangle = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases}$ **26.** $\left\langle\begin{matrix} n \\ 1 \end{matrix}\right\rangle = 2^n - n - 1,$ **27.** $\left\langle\begin{matrix} n \\ 2 \end{matrix}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2},$

28. $x^n = \displaystyle\sum_{k=0}^{n}\left\langle\begin{matrix} n \\ k \end{matrix}\right\rangle\binom{x+k}{n},$ **29.** $\left\langle\begin{matrix} n \\ m \end{matrix}\right\rangle = \displaystyle\sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k,$ **30.** $m!\left\{\begin{matrix} n \\ m \end{matrix}\right\} = \displaystyle\sum_{k=0}^{n}\left\langle\begin{matrix} n \\ k \end{matrix}\right\rangle\binom{k}{n-m},$

31. $\left\langle\begin{matrix} n \\ m \end{matrix}\right\rangle = \displaystyle\sum_{k=0}^{n}\left\{\begin{matrix} n \\ k \end{matrix}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!,$ **32.** $\left\langle\!\!\left\langle\begin{matrix} n \\ 0 \end{matrix}\right\rangle\!\!\right\rangle = 1,$ **33.** $\left\langle\!\!\left\langle\begin{matrix} n \\ n \end{matrix}\right\rangle\!\!\right\rangle = 0 \ \text{ for } n \ne 0,$

34. $\left\langle\!\!\left\langle\begin{matrix} n \\ k \end{matrix}\right\rangle\!\!\right\rangle = (k+1)\left\langle\!\!\left\langle\begin{matrix} n-1 \\ k \end{matrix}\right\rangle\!\!\right\rangle + (2n-1-k)\left\langle\!\!\left\langle\begin{matrix} n-1 \\ k-1 \end{matrix}\right\rangle\!\!\right\rangle,$ **35.** $\displaystyle\sum_{k=0}^{n}\left\langle\!\!\left\langle\begin{matrix} n \\ k \end{matrix}\right\rangle\!\!\right\rangle = \dfrac{(2n)^{\underline{n}}}{2^n},$

36. $\left\{\begin{matrix} x \\ x-n \end{matrix}\right\} = \displaystyle\sum_{k=0}^{n}\left\langle\!\!\left\langle\begin{matrix} n \\ k \end{matrix}\right\rangle\!\!\right\rangle\binom{x+n-1-k}{2n},$ **37.** $\left\{\begin{matrix} n+1 \\ m+1 \end{matrix}\right\} = \displaystyle\sum_{k}\binom{n}{k}\left\{\begin{matrix} k \\ m \end{matrix}\right\} = \displaystyle\sum_{k=0}^{n}\left\{\begin{matrix} k \\ m \end{matrix}\right\}(m+1)^{n-k},$

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Identities Cont.

38. $\left[\begin{matrix} n+1 \\ m+1 \end{matrix}\right] = \sum_k \left[\begin{matrix} n \\ k \end{matrix}\right]\binom{k}{m} = \sum_{k=0}^{n} \left[\begin{matrix} k \\ m \end{matrix}\right] n^{\underline{n-k}} = n! \sum_{k=0}^{n} \frac{1}{k!} \left[\begin{matrix} k \\ m \end{matrix}\right],$ **39.** $\left[\begin{matrix} x \\ x-n \end{matrix}\right] = \sum_{k=0}^{n} \left\langle\!\!\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle\!\!\right\rangle \binom{x+k}{2n},$

40. $\left\{\begin{matrix} n \\ m \end{matrix}\right\} = \sum_k \binom{n}{k}\left\{\begin{matrix} k+1 \\ m+1 \end{matrix}\right\}(-1)^{n-k},$ **41.** $\left[\begin{matrix} n \\ m \end{matrix}\right] = \sum_k \left[\begin{matrix} n+1 \\ k+1 \end{matrix}\right]\binom{k}{m}(-1)^{m-k},$

42. $\left\{\begin{matrix} m+n+1 \\ m \end{matrix}\right\} = \sum_{k=0}^{m} k\left\{\begin{matrix} n+k \\ k \end{matrix}\right\},$ **43.** $\left[\begin{matrix} m+n+1 \\ m \end{matrix}\right] = \sum_{k=0}^{m} k(n+k)\left[\begin{matrix} n+k \\ k \end{matrix}\right],$

44. $\binom{n}{m} = \sum_k \left\{\begin{matrix} n+1 \\ k+1 \end{matrix}\right\}\left[\begin{matrix} k \\ m \end{matrix}\right](-1)^{m-k},$ **45.** $(n-m)!\binom{n}{m} = \sum_k \left[\begin{matrix} n+1 \\ k+1 \end{matrix}\right]\left\{\begin{matrix} k \\ m \end{matrix}\right\}(-1)^{m-k},$ for $n \geq m,$

46. $\left\{\begin{matrix} n \\ n-m \end{matrix}\right\} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left[\begin{matrix} m+k \\ k \end{matrix}\right],$ **47.** $\left[\begin{matrix} n \\ n-m \end{matrix}\right] = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left\{\begin{matrix} m+k \\ k \end{matrix}\right\},$

48. $\left\{\begin{matrix} n \\ \ell+m \end{matrix}\right\}\binom{\ell+m}{\ell} = \sum_k \left\{\begin{matrix} k \\ \ell \end{matrix}\right\}\left\{\begin{matrix} n-k \\ m \end{matrix}\right\}\binom{n}{k},$ **49.** $\left[\begin{matrix} n \\ \ell+m \end{matrix}\right]\binom{\ell+m}{\ell} = \sum_k \left[\begin{matrix} k \\ \ell \end{matrix}\right]\left[\begin{matrix} n-k \\ m \end{matrix}\right]\binom{n}{k}.$

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \ldots, d_n$:
$$\sum_{i=1}^{n} 2^{-d_i} \leq 1,$$
and equality holds only if every internal node has 2 sons.

Recurrences

Master method:
$$T(n) = aT(n/b) + f(n), \quad a \geq 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then
$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then
$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \leq cf(n)$ for large n, then
$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence
$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have
$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get
$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find
$$u_{i+1} = \tfrac{1}{2} + u_i, \qquad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$. Summing factors (example): Consider the following recurrence
$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side
$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$1\big(T(n) - 3T(n/2) = n\big)$$
$$3\big(T(n/2) - 3T(n/4) = n/2\big)$$
$$\vdots \quad \vdots \quad \vdots$$
$$3^{\log_2 n - 1}\big(T(2) - 3T(1) = 2\big)$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get
$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\tfrac{3}{2}\right)^i.$$

Let $c = \tfrac{3}{2}$. Then we have
$$n \sum_{i=0}^{m-1} c^i = n\left(\frac{c^m - 1}{c - 1}\right)$$
$$= 2n(c^{\log_2 n} - 1)$$
$$= 2n(c^{(k-1)\log_c n} - 1)$$
$$= 2n^k - 2n,$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider
$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that
$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j.$$

Subtracting we find
$$T_{i+1} - T_i = 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j$$
$$= T_i.$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:
1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i.

Example:
$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:
$$\sum_{i \geq 0} g_{i+1} x^i = \sum_{i \geq 0} 2g_i x^i + \sum_{i \geq 0} x^i.$$

We choose $G(x) = \sum_{i \geq 0} x^i g_i$. Rewrite in terms of $G(x)$:
$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \geq 0} x^i.$$

Simplify:
$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:
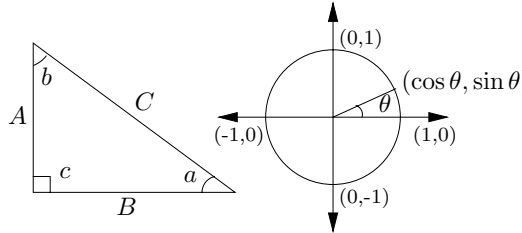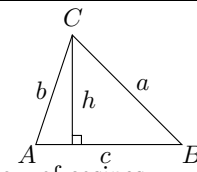$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:
$$G(x) = x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right)$$
$$= x\left(2\sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} x^i\right)$$
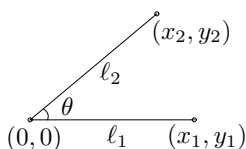$$= \sum_{i \geq 0}(2^{i+1} - 1)x^{i+1}.$$

So $g_i = 2^i - 1$.

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$\pi \approx 3.14159,$ $\qquad e \approx 2.71828,$ $\qquad \gamma \approx 0.57721,$ $\qquad \phi = \frac{1+\sqrt{5}}{2} \approx 1.61803,$ $\qquad \hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1,024	29
11	2,048	31
12	4,096	37
13	8,192	41
14	16,384	43
15	32,768	47
16	65,536	53
17	131,072	59
18	262,144	61
19	524,288	67
20	1,048,576	71
21	2,097,152	73
22	4,194,304	79
23	8,388,608	83
24	16,777,216	89
25	33,554,432	97
26	67,108,864	101
27	134,217,728	103
28	268,435,456	107
29	536,870,912	109
30	1,073,741,824	113
31	2,147,483,648	127
32	4,294,967,296	131

Pascal's Triangle

```
                  1
                 1 1
                1 2 1
               1 3 3 1
              1 4 6 4 1
            1 5 10 10 5 1
          1 6 15 20 15 6 1
        1 7 21 35 35 21 7 1
      1 8 28 56 70 56 28 8 1
   1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
```

General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):
$$B_0 = 1, \; B_1 = -\tfrac{1}{2}, \; B_2 = \tfrac{1}{6}, \; B_4 = -\tfrac{1}{30},$$
$$B_6 = \tfrac{1}{42}, \; B_8 = -\tfrac{1}{30}, \; B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:
$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number e:
$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$
$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$
$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$
$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:
$$1, \; \tfrac{3}{2}, \; \tfrac{11}{6}, \; \tfrac{25}{12}, \; \tfrac{137}{60}, \; \tfrac{49}{20}, \; \tfrac{363}{140}, \; \tfrac{761}{280}, \; \tfrac{7129}{2520}, \ldots$$
$$\ln n < H_n < \ln n + 1,$$
$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:
$$1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \ldots$$
$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:
$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$
$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}.$$

Binomial distribution:
$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$
$$E[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:
$$\Pr[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad E[X] = \lambda.$$

Normal (Gaussian) distribution:
$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad E[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is
$$nH_n.$$

Probability

Continuous distributions: If
$$\Pr[a < X < b] = \int_a^b p(x)\, dx,$$
then p is the probability density function of X. If
$$\Pr[X < a] = P(a),$$
then P is the distribution function of X. If P and p both exist then
$$P(a) = \int_{-\infty}^a p(x)\, dx.$$

Expectation: If X is discrete
$$E[g(X)] = \sum_x g(x) \Pr[X = x].$$
If X continuous then
$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dP(x).$$

Variance, standard deviation:
$$\mathrm{VAR}[X] = E[X^2] - E[X]^2,$$
$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events A and B:
$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$
$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$
$$\text{iff } A \text{ and } B \text{ are independent.}$$
$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables X and Y:
$$E[X \cdot Y] = E[X] \cdot E[Y],$$
$$\text{if } X \text{ and } Y \text{ are independent.}$$
$$E[X + Y] = E[X] + E[Y],$$
$$E[cX] = c\,E[X].$$

Bayes' theorem:
$$\Pr[A_i|B] = \frac{\Pr[B|A_i] \Pr[A_i]}{\sum_{j=1}^n \Pr[A_j] \Pr[B|A_j]}.$$

Inclusion-exclusion:
$$\Pr\left[\bigvee_{i=1}^n X_i\right] = \sum_{i=1}^n \Pr[X_i] +$$
$$\sum_{k=2}^n (-1)^{k+1} \sum_{i_i < \cdots < i_k} \Pr\left[\bigwedge_{j=1}^k X_{i_j}\right].$$

Moment inequalities:
$$\Pr\left[|X| \geq \lambda\,E[X]\right] \leq \frac{1}{\lambda},$$
$$\Pr\left[|X - E[X]| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

Geometric distribution:
$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$
$$E[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$

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Trigonometry

Pythagorean theorem:
$$C^2 = A^2 + B^2.$$

Definitions:
$$\sin a = A/C, \quad \cos a = B/C,$$
$$\csc a = C/A, \quad \sec a = C/B,$$
$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:
$$\tfrac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:
$$\sin x = \frac{1}{\csc x}, \qquad\qquad \cos x = \frac{1}{\sec x},$$
$$\tan x = \frac{1}{\cot x}, \qquad\qquad \sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$
$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$
$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$
$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$
$$\sin 2x = 2\sin x \cos x, \qquad \sin 2x = \frac{2\tan x}{1 + \tan^2 x},$$
$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$
$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2\cot x},$$
$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:
$$e^{ix} = \cos x + i\sin x, \qquad e^{i\pi} = -1.$$

Matrices

Multiplication:
$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.
$$\det A \cdot B = \det A \cdot \det B,$$
$$\det A = \sum_{\pi} \prod_{i=1}^{n} \text{sign}(\pi) a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g\begin{vmatrix} b & c \\ e & f \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} + i\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= \begin{array}{l} aei + bfg + cdh \\ -ceg - fha - ibd. \end{array}$$

Permanents:
$$\text{perm } A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Hyperbolic Functions

Definitions:
$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \text{csch } x = \frac{1}{\sinh x},$$
$$\text{sech } x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

Identities:
$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \text{sech}^2 x = 1,$$
$$\coth^2 x - \text{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$
$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$
$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh 2x = 2\sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$
$$2\sinh^2\tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2\tfrac{x}{2} = \cosh x + 1.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

... in mathematics you don't understand things, you just get used to them.
– J. von Neumann

More Trig.

Law of cosines:
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:
$$A = \tfrac{1}{2}hc,$$
$$= \tfrac{1}{2}ab\sin C,$$
$$= \frac{c^2 \sin A \sin B}{2\sin C}.$$

Heron's formula:
$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$
$$s = \tfrac{1}{2}(a + b + c),$$
$$s_a = s - a,$$
$$s_b = s - b,$$
$$s_c = s - c.$$

More identities:
$$\sin\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$
$$\cos\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$
$$\tan\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$
$$= \frac{1 - \cos x}{\sin x},$$
$$= \frac{\sin x}{1 + \cos x},$$
$$\cot\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$
$$= \frac{1 + \cos x}{\sin x},$$
$$= \frac{\sin x}{1 - \cos x},$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$
$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$
$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$
$$\sin x = \frac{\sinh ix}{i},$$
$$\cos x = \cosh ix,$$
$$\tan x = \frac{\tanh ix}{i}.$$

Theoretical Computer Science Cheat Sheet

Number Theory	Graph Theory

Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$C \equiv r_1 \bmod m_1$$
$$\vdots \quad \vdots \quad \vdots$$
$$C \equiv r_n \bmod m_n$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^{n} p_i^{e_i - 1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:
$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then
$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^{n} \frac{p_i^{e_i + 1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff
$$(n - 1)! \equiv -1 \bmod n.$$

Möbius inversion:
$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & \quad r \text{ distinct primes.} \end{cases}$$

If
$$G(a) = \sum_{d|a} F(d),$$

then
$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:
$$p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n}$$
$$+ O\left(\frac{n}{\ln n}\right),$$
$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2! n}{(\ln n)^3}$$
$$+ O\left(\frac{n}{(\ln n)^4}\right).$$

Graph Theory

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k - 1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \|S\|$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embeded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so
$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1, n_2}	Complete bipartite graph
$r(k, \ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.
$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

Distance formula, L_p and L_∞ metric:
$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$
$$\left[|x_1 - x_0|^p + |y_1 - y_0|^p\right]^{1/p},$$
$$\lim_{p \to \infty} \left[|x_1 - x_0|^p + |y_1 - y_0|^p\right]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):
$$\tfrac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:

$$\cos \theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):
$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:
$$A = \pi r^2, \qquad V = \tfrac{4}{3} \pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton

Theoretical Computer Science Cheat Sheet

π	Calculus

π

Wallis' identity:
$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:
$$\frac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

Gregrory's series:
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Newton's series:
$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:
$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$

Euler's series:
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$
$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$
$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining
$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$
where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:
$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$
where
$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}.$$
For a repeated factor:
$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$
where
$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Calculus

Derivatives:

1. $\dfrac{d(cu)}{dx} = c\dfrac{du}{dx},$ **2.** $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx},$ **3.** $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx},$

4. $\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx},$ **5.** $\dfrac{d(u/v)}{dx} = \dfrac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2},$ **6.** $\dfrac{d(e^{cu})}{dx} = ce^{cu}\dfrac{du}{dx},$

7. $\dfrac{d(c^u)}{dx} = (\ln c)c^u\dfrac{du}{dx},$ **8.** $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx},$

9. $\dfrac{d(\sin u)}{dx} = \cos u\dfrac{du}{dx},$ **10.** $\dfrac{d(\cos u)}{dx} = -\sin u\dfrac{du}{dx},$

11. $\dfrac{d(\tan u)}{dx} = \sec^2 u\dfrac{du}{dx},$ **12.** $\dfrac{d(\cot u)}{dx} = \csc^2 u\dfrac{du}{dx},$

13. $\dfrac{d(\sec u)}{dx} = \tan u \sec u\dfrac{du}{dx},$ **14.** $\dfrac{d(\csc u)}{dx} = -\cot u \csc u\dfrac{du}{dx},$

15. $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx},$ **16.** $\dfrac{d(\arccos u)}{dx} = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx},$

17. $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx},$ **18.** $\dfrac{d(\text{arccot}\, u)}{dx} = \dfrac{-1}{1+u^2}\dfrac{du}{dx},$

19. $\dfrac{d(\text{arcsec}\, u)}{dx} = \dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx},$ **20.** $\dfrac{d(\text{arccsc}\, u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx},$

21. $\dfrac{d(\sinh u)}{dx} = \cosh u\dfrac{du}{dx},$ **22.** $\dfrac{d(\cosh u)}{dx} = \sinh u\dfrac{du}{dx},$

23. $\dfrac{d(\tanh u)}{dx} = \text{sech}^2 u\dfrac{du}{dx},$ **24.** $\dfrac{d(\coth u)}{dx} = -\text{csch}^2 u\dfrac{du}{dx},$

25. $\dfrac{d(\text{sech}\, u)}{dx} = -\text{sech}\, u \tanh u\dfrac{du}{dx},$ **26.** $\dfrac{d(\text{csch}\, u)}{dx} = -\text{csch}\, u \coth u\dfrac{du}{dx},$

27. $\dfrac{d(\text{arcsinh}\, u)}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx},$ **28.** $\dfrac{d(\text{arccosh}\, u)}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx},$

29. $\dfrac{d(\text{arctanh}\, u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx},$ **30.** $\dfrac{d(\text{arccoth}\, u)}{dx} = \dfrac{1}{u^2-1}\dfrac{du}{dx},$

31. $\dfrac{d(\text{arcsech}\, u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx},$ **32.** $\dfrac{d(\text{arccsch}\, u)}{dx} = \dfrac{-1}{|u|\sqrt{1+u^2}}\dfrac{du}{dx}.$

Integrals:

1. $\displaystyle\int cu\,dx = c\int u\,dx,$ **2.** $\displaystyle\int (u+v)\,dx = \int u\,dx + \int v\,dx,$

3. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1,$ **4.** $\displaystyle\int \frac{1}{x}dx = \ln x,$ **5.** $\displaystyle\int e^x\,dx = e^x,$

6. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x,$ **7.** $\displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx,$

8. $\displaystyle\int \sin x\,dx = -\cos x,$ **9.** $\displaystyle\int \cos x\,dx = \sin x,$

10. $\displaystyle\int \tan x\,dx = -\ln|\cos x|,$ **11.** $\displaystyle\int \cot x\,dx = \ln|\cos x|,$

12. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|,$ **13.** $\displaystyle\int \csc x\,dx = \ln|\csc x + \cot x|,$

14. $\displaystyle\int \arcsin\frac{x}{a}dx = \arcsin\frac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0,$

Theoretical Computer Science Cheat Sheet

Calculus Cont.

15. $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$ **16.** $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$

17. $\int \sin^2(ax) dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big),$ **18.** $\int \cos^2(ax) dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big),$

19. $\int \sec^2 x \, dx = \tan x,$ **20.** $\int \csc^2 x \, dx = -\cot x,$

21. $\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx,$ **22.** $\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx,$

23. $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx, \quad n \neq 1,$ **24.** $\int \cot^n x \, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx, \quad n \neq 1,$

25. $\int \sec^n x \, dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx, \quad n \neq 1,$

26. $\int \csc^n x \, dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x \, dx, \quad n \neq 1,$ **27.** $\int \sinh x \, dx = \cosh x,$ **28.** $\int \cosh x \, dx = \sinh x,$

29. $\int \tanh x \, dx = \ln|\cosh x|,$ **30.** $\int \coth x \, dx = \ln|\sinh x|,$ **31.** $\int \operatorname{sech} x \, dx = \arctan \sinh x,$ **32.** $\int \operatorname{csch} x \, dx = \ln\left|\tanh \frac{x}{2}\right|,$

33. $\int \sinh^2 x \, dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x,$ **34.** $\int \cosh^2 x \, dx = \frac{1}{4}\sinh(2x) + \frac{1}{2}x,$ **35.** $\int \operatorname{sech}^2 x \, dx = \tanh x,$

36. $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$ **37.** $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln|a^2 - x^2|,$

38. $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \dfrac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \dfrac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$

39. $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right), \quad a > 0,$

40. $\int \dfrac{dx}{a^2 + x^2} = \frac{1}{a}\arctan \frac{x}{a}, \quad a > 0,$ **41.** $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin \frac{x}{a}, \quad a > 0,$

42. $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8}\arcsin \frac{x}{a}, \quad a > 0,$

43. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$ **44.** $\int \dfrac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|,$ **45.** $\int \dfrac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}},$

46. $\int \sqrt{a^2 \pm x^2}\, dx = \frac{x}{2}\sqrt{a^2 \pm x^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{a^2 \pm x^2}\right|,$ **47.** $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right|, \quad a > 0,$

48. $\int \dfrac{dx}{ax^2 + bx} = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right|,$ **49.** $\int x\sqrt{a + bx}\, dx = \frac{2(3bx - 2a)(a+bx)^{3/2}}{15b^2},$

50. $\int \dfrac{\sqrt{a+bx}}{x}\, dx = 2\sqrt{a+bx} + a\int \frac{1}{x\sqrt{a+bx}}dx,$ **51.** $\int \dfrac{x}{\sqrt{a+bx}}\, dx = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}}\right|, \quad a > 0,$

52. $\int \dfrac{\sqrt{a^2 - x^2}}{x}\, dx = \sqrt{a^2 - x^2} - a\ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$ **53.** $\int x\sqrt{a^2 - x^2}\, dx = -\frac{1}{3}(a^2 - x^2)^{3/2},$

54. $\int x^2\sqrt{a^2 - x^2}\, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8}\arcsin \frac{x}{a}, \quad a > 0,$ **55.** $\int \dfrac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

56. $\int \dfrac{x\, dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$ **57.** $\int \dfrac{x^2\, dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin \frac{x}{a}, \quad a > 0,$

58. $\int \dfrac{\sqrt{a^2 + x^2}}{x}\, dx = \sqrt{a^2 + x^2} - a\ln\left|\frac{a + \sqrt{a^2 + x^2}}{x}\right|,$ **59.** $\int \dfrac{\sqrt{x^2 - a^2}}{x}\, dx = \sqrt{x^2 - a^2} - a\arccos \frac{a}{|x|}, \quad a > 0,$

60. $\int x\sqrt{x^2 \pm a^2}\, dx = \frac{1}{3}(x^2 \pm a^2)^{3/2},$ **61.** $\int \dfrac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{a}\ln\left|\frac{x}{a + \sqrt{a^2 + x^2}}\right|,$

Theoretical Computer Science Cheat Sheet

Calculus Cont.	Finite Calculus

Calculus Cont.

62. $\int \dfrac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arccos\frac{a}{|x|}, \quad a>0,$ **63.** $\int \dfrac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\dfrac{\sqrt{x^2\pm a^2}}{a^2 x},$

64. $\int \dfrac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2},$ **65.** $\int \dfrac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\dfrac{(x^2+a^2)^{3/2}}{3a^2 x^3},$

66. $\int \dfrac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|, & \text{if } b^2>4ac, \\[2ex] \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}, & \text{if } b^2<4ac, \end{cases}$

67. $\int \dfrac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|, & \text{if } a>0, \\[2ex] \dfrac{1}{\sqrt{-a}}\arcsin\dfrac{-2ax-b}{\sqrt{b^2-4ac}}, & \text{if } a<0, \end{cases}$

68. $\int \sqrt{ax^2+bx+c}\,dx = \dfrac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \dfrac{4ac-b^2}{8a}\int \dfrac{dx}{\sqrt{ax^2+bx+c}},$

69. $\int \dfrac{x\,dx}{\sqrt{ax^2+bx+c}} = \dfrac{\sqrt{ax^2+bx+c}}{a} - \dfrac{b}{2a}\int \dfrac{dx}{\sqrt{ax^2+bx+c}},$

70. $\int \dfrac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right|, & \text{if } c>0, \\[2ex] \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}, & \text{if } c<0, \end{cases}$

71. $\int x^3\sqrt{x^2+a^2}\,dx = (\frac{1}{3}x^2-\frac{2}{15}a^2)(x^2+a^2)^{3/2},$

72. $\int x^n\sin(ax)\,dx = -\frac{1}{a}x^n\cos(ax) + \frac{n}{a}\int x^{n-1}\cos(ax)\,dx,$

73. $\int x^n\cos(ax)\,dx = \frac{1}{a}x^n\sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax)\,dx,$

74. $\int x^n e^{ax}\,dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}\int x^{n-1}e^{ax}\,dx,$

75. $\int x^n\ln(ax)\,dx = x^{n+1}\left(\dfrac{\ln(ax)}{n+1} - \dfrac{1}{(n+1)^2}\right),$

76. $\int x^n(\ln ax)^m\,dx = \dfrac{x^{n+1}}{n+1}(\ln ax)^m - \dfrac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx.$

$$
\begin{aligned}
x^1 &= & x^{\underline{1}} & & = & & x^{\overline{1}} \\
x^2 &= & x^{\underline{2}} + x^{\underline{1}} & & = & & x^{\overline{2}} - x^{\overline{1}} \\
x^3 &= & x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} & & = & & x^{\overline{3}} - 3x^{\overline{2}} + x^{\overline{1}} \\
x^4 &= & x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} & & = & & x^{\overline{4}} - 6x^{\overline{3}} + 7x^{\overline{2}} - x^{\overline{1}} \\
x^5 &= & x^{\underline{5}} + 15x^{\underline{4}} + 25x^{\underline{3}} + 10x^{\underline{2}} + x^{\underline{1}} & & = & & x^{\overline{5}} - 15x^{\overline{4}} + 25x^{\overline{3}} - 10x^{\overline{2}} + x^{\overline{1}}
\end{aligned}
$$

$$
\begin{aligned}
x^{\overline{1}} &= & x^1 & \qquad & x^{\underline{1}} &= & x^1 \\
x^{\overline{2}} &= & x^2 + x^1 & & x^{\underline{2}} &= & x^2 - x^1 \\
x^{\overline{3}} &= & x^3 + 3x^2 + 2x^1 & & x^{\underline{3}} &= & x^3 - 3x^2 + 2x^1 \\
x^{\overline{4}} &= & x^4 + 6x^3 + 11x^2 + 6x^1 & & x^{\underline{4}} &= & x^4 - 6x^3 + 11x^2 - 6x^1 \\
x^{\overline{5}} &= & x^5 + 10x^4 + 35x^3 + 50x^2 + 24x^1 & & x^{\underline{5}} &= & x^5 - 10x^4 + 35x^3 - 50x^2 + 24x^1
\end{aligned}
$$

Finite Calculus

Difference, shift operators:
$$\Delta f(x) = f(x+1) - f(x),$$
$$\mathrm{E}\, f(x) = f(x+1).$$

Fundamental Theorem:
$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x) + C.$$
$$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:
$$\Delta(cu) = c\Delta u, \qquad \Delta(u+v) = \Delta u + \Delta v,$$
$$\Delta(uv) = u\Delta v + \mathrm{E}\,v\Delta u,$$
$$\Delta(x^{\underline{n}}) = nx^{\underline{n-1}},$$
$$\Delta(H_x) = x^{\underline{-1}}, \qquad\qquad \Delta(2^x) = 2^x,$$
$$\Delta(c^x) = (c-1)c^x, \qquad \Delta\binom{x}{m} = \binom{x}{m-1}.$$

Sums:
$$\sum cu\,\delta x = c\sum u\,\delta x,$$
$$\sum (u+v)\,\delta x = \sum u\,\delta x + \sum v\,\delta x,$$
$$\sum u\Delta v\,\delta x = uv - \sum \mathrm{E}\,v\Delta u\,\delta x,$$
$$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1}, \qquad \sum x^{\underline{-1}}\,\delta x = H_x,$$
$$\sum c^x\,\delta x = \frac{c^x}{c-1}, \qquad \sum\binom{x}{m}\,\delta x = \binom{x}{m+1}.$$

Falling Factorial Powers:
$$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n>0,$$
$$x^{\underline{0}} = 1,$$
$$x^{\underline{n}} = \frac{1}{(x+1)\cdots(x+|n|)}, \quad n<0,$$
$$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$$

Rising Factorial Powers:
$$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n>0,$$
$$x^{\overline{0}} = 1,$$
$$x^{\overline{n}} = \frac{1}{(x-1)\cdots(x-|n|)}, \quad n<0,$$
$$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$$

Conversion:
$$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$$
$$= 1/(x+1)^{\overline{-n}},$$
$$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$$
$$= 1/(x-1)^{\underline{-n}},$$
$$x^n = \sum_{k=1}^n \left\{n \atop k\right\} x^{\underline{k}} = \sum_{k=1}^n \left\{n \atop k\right\}(-1)^{n-k}x^{\overline{k}},$$
$$x^{\underline{n}} = \sum_{k=1}^n \left[n \atop k\right](-1)^{n-k}x^k,$$
$$x^{\overline{n}} = \sum_{k=1}^n \left[n \atop k\right] x^k.$$

Theoretical Computer Science Cheat Sheet

Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!}f^{(i)}(a).$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i,$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots = \sum_{i=0}^{\infty} x^{ni},$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{i=0}^{\infty} ix^i,$$

$$x^k \frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots = \sum_{i=0}^{\infty} i^n x^i,$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1}\frac{x^i}{i},$$

$$\ln\frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i},$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$$

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$$

$$(1+x)^n = 1 + nx + \tfrac{n(n-1)}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \binom{n+2}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$$

$$\frac{x}{e^x-1} = 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} = 1 + x + 2x^2 + 6x^3 + \cdots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = 1 + (2+n)x + \binom{4+n}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$$

$$\frac{1}{1-x}\ln\frac{1}{1-x} = x + \tfrac{3}{2}x^2 + \tfrac{11}{6}x^3 + \tfrac{25}{12}x^4 + \cdots = \sum_{i=1}^{\infty} H_i x^i,$$

$$\frac{1}{2}\left(\ln\frac{1}{1-x}\right)^2 = \tfrac{1}{2}x^2 + \tfrac{3}{4}x^3 + \tfrac{11}{24}x^4 + \cdots = \sum_{i=2}^{\infty} \frac{H_{i-1}x^i}{i},$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{i=0}^{\infty} F_i x^i,$$

$$\frac{F_n x}{1 - (F_{n-1}+F_{n+1})x - (-1)^n x^2} = F_n x + F_{2n}x^2 + F_{3n}x^3 + \cdots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^{n-1-k}y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i)x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1)a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} ia_i x^i,$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i}x^i,$$

$$\frac{A(x)+A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i}x^{2i},$$

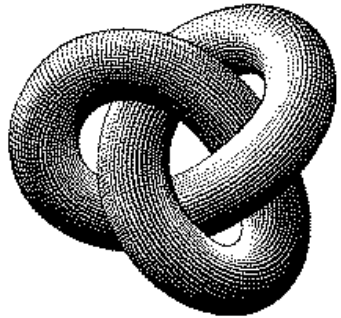$$\frac{A(x)-A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1}x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x}A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} a_j b_{i-j}\right)x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Theoretical Computer Science Cheat Sheet

Series

Expansions:

$$\frac{1}{(1-x)^{n+1}}\ln\frac{1}{1-x} = \sum_{i=0}^{\infty}(H_{n+i}-H_n)\binom{n+i}{i}x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty}\begin{bmatrix}n\\i\end{bmatrix}x^i,$$

$$\left(\ln\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty}\begin{bmatrix}i\\n\end{bmatrix}\frac{n!x^i}{i!},$$

$$\tan x = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\mu(i)}{i^x},$$

$$\zeta(x) = \prod_{p}\frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty}\frac{d(i)}{x^i} \quad \text{where } d(n)=\sum_{d|n}1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty}\frac{S(i)}{x^i} \quad \text{where } S(n)=\sum_{d|n}d,$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!}\pi^{2n}, \quad n\in\mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty}(-1)^{i-1}\frac{(4^i-2)B_{2i}x^{2i}}{(2i)!},$$

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty}\frac{n(2i+n-1)!}{i!(n+i)!}x^i,$$

$$e^x\sin x = \sum_{i=1}^{\infty}\frac{2^{i/2}\sin\frac{i\pi}{4}}{i!}x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty}\frac{(4i)!}{16^i\sqrt{2}(2i)!(2i+1)!}x^i,$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty}\frac{4^i i!^2}{(i+1)(2i+1)!}x^{2i}.$$

$$\left(\frac{1}{x}\right)^{\overline{-n}} = \sum_{i=0}^{\infty}\begin{Bmatrix}i\\n\end{Bmatrix}x^i,$$

$$(e^x-1)^n = \sum_{i=0}^{\infty}\begin{Bmatrix}i\\n\end{Bmatrix}\frac{n!x^i}{i!},$$

$$x\cot x = \sum_{i=0}^{\infty}\frac{(-4)^i B_{2i}x^{2i}}{(2i)!},$$

$$\zeta(x) = \sum_{i=1}^{\infty}\frac{1}{i^x},$$

$$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\phi(i)}{i^x},$$

Escher's Knot

Stieltjes Integration

If G is continuous in the interval $[a,b]$ and F is nondecreasing then

$$\int_a^b G(x)\,dF(x)$$

exists. If $a \le b \le c$ then

$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x).$$

If the integrals involved exist

$$\int_a^b \big(G(x)+H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x),$$

$$\int_a^b G(x)\,d\big(F(x)+H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x),$$

$$\int_a^b c\cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c\cdot F(x)\big) = c\int_a^b G(x)\,dF(x),$$

$$\int_a^b G(x)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x)\,dG(x).$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a,b]$ then

$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx.$$

Cramer's Rule

If we have equations:
$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad\qquad \vdots$$
$$a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n$$

Let $A = (a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A \ne 0$. Let A_i be A with column i replaced by B. Then

$$x_i = \frac{\det A_i}{\det A}.$$

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

The Fibonacci number system: Every integer n has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$

where $k_i \ge k_{i+1} + 2$ for all i, $1 \le i < m$ and $k_m \ge 2$.

Fibonacci Numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Definitions:

$$F_i = F_{i-1}+F_{i-2}, \quad F_0 = F_1 = 1,$$
$$F_{-i} = (-1)^{i-1}F_i,$$
$$F_i = \frac{1}{\sqrt{5}}\left(\phi^i - \hat{\phi}^i\right),$$

Cassini's identity: for $i > 0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_k F_{n+1} + F_{k-1}F_n,$$
$$F_{2n} = F_n F_{n+1} + F_{n-1}F_n.$$

Calculation by matrices:

$$\begin{pmatrix}F_{n-2} & F_{n-1}\\ F_{n-1} & F_n\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^n.$$