Multi-objective Filtering for Discrete-time Systems in the Presence of Data Packet Drops[Feng, 2018]

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April 4, 2024

Overview

Introduction

- Lossy channel
- Discrete-time $\mathcal{H}_\infty$ Gaussian filtering (reflects the trade-off between the inherently conflicting $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances), in the presence of data packet dropout on infinite time horizon.
- The multi-objective filtering , which turns out to be much more involved, due to the complexity of characterization conditions, gives a good compromise between robustness and transient performance
- The prior work showed that the robust optimal filter can be obtained by solving a set of cross-coupled Riccati equations

Preliminaries

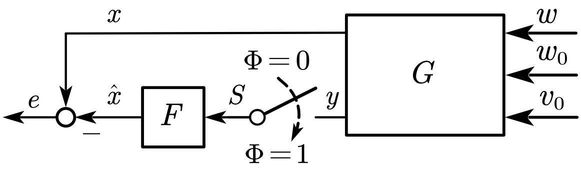

Figure: Multi-objective filtering over lossy network

$\Phi(k)$ is an i.i.d. Bernoulli process

$$\mathrm{Prob}\{\Phi(k)\} = \begin{cases} \mu, & \Phi(k) = 1; \\ 1 - \mu, & \Phi(k) = 0; \end{cases} \qquad \mu \in (0, 1]$$

then it follows

$$\mathbb{E}\{\Phi(k)\} = \mu, \quad \mathbb{E}\left\{[\Phi(k) - \mu]^2\right\} = \mu - \mu^2.$$

Preliminaries

The plant

$$G : \begin{cases} x(k+1) = Ax(k) + w(k) + w_0(k), \\ y(k) = C_2 x(k) + v_0(k), \quad x(0) = x_0, \end{cases}$$

$\mathcal{H}_\infty$ Gaussian filter

$$F : \begin{cases} \hat{x}(k+1) = A\hat{x}(k) + L\left[\Phi(k) C_2 \hat{x}(k) - s(k)\right], \\ s(k) = \Phi(k) y(k), \quad \hat{x}(0) = \hat{x}_0. \end{cases}$$

Estimation error is

$$e(k) = x(k) - \hat{x}(k).$$

Combine G and F, we have error system

$$e(k+1) = [A + L\Phi(k) C_2] e(k) + w(k) + w_0(k) + L\Phi(k) v_0(k).$$

Preliminaries

Bounded Power signal

$$\|u\|_{\mathcal{P}} = \sqrt{\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathrm{E}\left\{\|u(k)\|^2\right\}}$$

Cost functionals are

$$J_1(L, w(k), w_0(k), v_0(k)) = \gamma_1^2 \|w(k)\|_{\mathcal{P}}^2 - \|e(k)\|_{\mathcal{P}}^2$$
$$J_2(L, w(k), w_0(k), v_0(k)) = \|e(k)\|_{\mathcal{P}}^2$$

Problem Statement

Consider the plant and the filter. The problem of infinite time horizon $\mathcal{H}_\infty$ Gaussian filtering in the presence of data packet dropouts is to find a finite filter gain L_* and a bounded worst case disturbance signal $w_*(k)$ such that

$$J_1\left(L_*, w_*(k), w_0(k), v_0(k)\right) \leq J_1\left(L_*, w(k), w_0(k), v_0(k)\right)$$
$$J_2\left(L_*, w_*(k), w_0(k), v_0(k)\right) \leq J_2\left(L, w_*(k), w_0(k), v_0(k)\right)$$

Theorem

Theorem

Consider the previous plant G, the filter F, and the two cost functionals J_1 and J_2. Assume that there exist two matrices $P_1 > 0$ and $P_2 > 0$ satisfying the following cross-coupled MAREs

$$
\begin{aligned}
P_1 &= A_L' P_1 A_L + I_n + \left(\mu - \mu^2\right) C_2' L' P_1 L C_2 + \gamma_1^{-2} A_L' P_1 \Delta^{-1} P_1 A_L, \\
P_2 &= A_W P_2 A_W' + Q - \mu A_W P_2 C_2' \Lambda^{-1} C_2 P_2 A_W', \\
\text{where } A_L &= A + \mu L_* C_2, A_W = A + W, \\
\Delta &= I_n - \gamma_1^{-2} P_1, \\
\Lambda &= R + C_2 P_2 C_2', \\
L_* &= -A_W P_2 C_2' \left(R + C_2 P_2 C_2'\right)^{-1}, \\
W &= \left(\gamma_1^2 P_1^{-1} - I_n\right)^{-1} A_L.
\end{aligned}
$$

Then, the filter gain L_* and the worst case disturbance signal $w_*(k) = We(k)$ solve problem.

Proof

Target: if there exist matrices $P_1 > 0$ and $P_2 > 0$ satisfying the MAREs, then L_* and $w_*(k) = We_x(k)$ are solutions to the Problem.

Mean Square Stability

When $w(k) = 0$, $w_0(k) = 0$, and $v_0(k) = 0$, the error system is said to be MS stable, if $\Xi(k) := \mathbb{E}\{e(k)e'(k)\}$ is well defined for all k and converges to zero asymptotically.

1) First prove noise-free error system is MS stable with filter gain L_*

$$e(k+1) = [A + L_*\Phi(k)C_2]e(k).$$

Define Lyapunov function as $V_1(k) = \mathrm{Tr}\{\Xi(k)P_1\}$

$$\begin{aligned}
V_1(k+1) = \mathrm{Tr}\{\Xi(k+1)P_1\} &= \mathrm{Tr}\{\mathbb{E}\{[A + L_*\mu C_2 + L_*(\Phi(k) - \mu)C_2]e(k)\cdot^{\intercal}\}P_1\} \\
\text{(Trace: cyclic permutation invariance)} &= \mathrm{Tr}\{\Xi[A_L' P_1 A_L + (\mu - \mu^2)C_2' L_*' P_1 C_2 L_*]\} \\
(P_1 \text{ equation}) &= \mathrm{Tr}\left\{\Xi\left(P_1 - I_n - \gamma_1^{-2} A_L' P_1 \Delta^{-1} P_1 A_L\right)\right\} \\
&= V_1(k) - \mathrm{Tr}\{\Xi(I_n + \gamma_1^{-2} A_L' P_1 \Delta^{-1} P_1 A_L)\} < V_1(k).
\end{aligned}$$

Proof

2) Then, we show that the first Nash inequality $J_1\left(L_*, w_*(k), w_0(k), v_0(k)\right) \leq J_1\left(L_*, w(k), w_0(k), v_0(k)\right)$ holds. To this end, let us denote the Lyapunov function $V_2(k) = e'(k)P_1e(k)$, replacing L with L_*, recall

$$e(k+1) = [A + L_*\Phi(k)C_2]e(k) + w(k) + w_0(k) + L_*\Phi(k)v_0(k).$$

then,

$$\begin{aligned}
&\gamma_1^2\|w(k)\|^2 - \|e(k)\|^2 + V_2(k) - V_2(k+1) \\
=&\gamma_1^2 w'(k)w(k) - e'(k)e(k) + e'(k)P_1e(k) - e'(k+1)P_1e(k+1) \\
=&\gamma_1^2 w'(k)w(k) - e'(k)e(k) + e'(k)P_1e(k) \\
&- w'(k)P_1w(k) - e'(k)\left[A + L_*\Phi(k)C_2\right]' P_1 \left[A + L_*\Phi(k)C_2\right] e(k) \\
&- \left[w_0(k) + L_*\Phi(k)v_0(k)\right]' P_1 \left[w_0(k) + L_*\Phi(k)v_0(k)\right] \\
&- \mathbf{He}\left\{e'(k)\left[A + L_*\Phi(k)C_2\right]' P_1 w(k)\right\} - \mathbf{He}\left\{w'(k)P_1\left[w_0(k) + L_*\Phi(k)v_0(k)\right]\right\} \\
&- \mathbf{He}\left\{e'(k)\left[A + L_*\Phi(k)C_2\right]' P_1\left[w_0(k) + L_*\Phi(k)v_0(k)\right]\right\}
\end{aligned}$$

Proof

Since $\Phi(k)$, $w_0(k)$, $v_0(k)$, $e(k)$, and $w(k)$ are mutually uncorrelated, taking expectation of the above equation leads to

$$
\begin{aligned}
&\mathrm{E}\left\{\gamma_1^2\|w(k)\|^2-\|e(k)\|^2+V_2(k)-V_2(k+1)\right\}, \\
=&\mathrm{E}\left\{\gamma_1^2 w'(k)\Delta w(k)-e'(k)e(k)+e'(k)P_1e(k)\right\} \\
-&\mathrm{E}\left\{e'(k)\left[A'P_1A+\mu\mathbf{He}\left\{A'P_1L_*C_2\right\}\right]e(k)\right\} \\
-&\mathrm{E}\left\{e'(k)\mu C_2'L_*'P_1L_*C_2e(k)\right\} \\
-&\mathrm{E}\left\{\mathbf{He}\left\{e'(k)\left[A+\mu L_*C_2\right]'P_1w(k)\right\}\right\} \\
-&\,\mathrm{Tr}\left\{P_1\mathrm{E}\left\{w_0(k)w_0'(k)\right\}+\mu L_*'P_1L_*\mathrm{E}\left\{v_0(k)v_0'(k)\right\}\right\} \\
=&\mathrm{E}\left\{\gamma_1^2 w'(k)\Delta w(k)-e'(k)e(k)+e'(k)P_1e(k)\right\} \\
-&\mathrm{E}\left\{e'(k)A_L'P_1A_Le(k)\right\}-\mathrm{E}\left\{\mathbf{He}\left\{e'(k)A_L'P_1w(k)\right\}\right\} \\
-&\mathrm{E}\left\{e'(k)\left(\mu-\mu^2\right)C_2'L_*'P_1L_*C_2e(k)\right\} \\
-&\,\mathrm{Tr}\left\{P_1Q+\mu L_*'P_1L_*R\right\},
\end{aligned}
$$

Proof

$$
\begin{aligned}
&\gamma_1^2\|w(k)\|_{\mathcal{P}}^2-\|e(k)\|_{\mathcal{P}}^2+\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\mathrm{E}\left\{V_2(k)-V_2(k+1)\right\}\\
=&\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\mathrm{E}\left\{\gamma_1^2w'(k)\Delta w(k)-\mathbf{He}\left\{e'(k)A_L'P_1w(k)\right\}\right\}\\
&+\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\mathrm{E}\left\{e'(k)\gamma_1^{-2}A_L'P_1\Delta^{-1}P_1A_Le(k)\right\}-\mathrm{Tr}\left\{P_1Q+\mu L_*'P_1L_*R\right\}\\
=&\gamma_1^2\Delta\|w(k)-We(k)\|_{\mathcal{P}}^2-\mathrm{Tr}\left\{P_1Q+\mu L_*'P_1L_*R\right\}.
\end{aligned}
$$

Note that

$$
\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\mathrm{E}\left\{V_2(k)-V_2(k+1)\right\}=\lim_{N\to\infty}\frac{1}{N}\mathrm{E}\left\{-V_2(N)\right\}.
$$

Proof

Since it has shown that the noise-free error system is MS stable, the estimation error $e(k)$ is bounded. Hence, we have $\lim_{N\to\infty} \mathrm{E}\left\{V_2(N)\right\} < \infty$. This fact indicates

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathrm{E}\left\{V_2(k) - V_2(k+1)\right\} = 0$$

Hence,

$$J_1\left(L_*, w(k), w_0(k), v_0(k)\right) = \gamma_1^2 \Delta \| w(k) - We(k) \|_{\mathcal{P}}^2 - \mathrm{Tr}\left\{P_1 Q + \mu L_*' P_1 L_* R\right\}.$$

Therefore, $w_*(k) = We(k)$ is the worse case signal under white noises and achieves

$$J_1\left(L_*, w_*(k), w_0(k), v_0(k)\right) \leq J_1\left(L_*, w(k), w_0(k), v_0(k)\right)$$

The second Nash inequality follows almost same techniques. More details are showed in the “Additional slides”.

Algorithm

Here, we use an iterative approach to solve the two mentioned coupled MAREs, as has been used in [chen, 2002] where the coupled AREs are numerically solved in the continuous-time setting. The procedures of this iterative approach are summarized below:

Algorithm

Consider the following iterative MDREs

$$
\begin{aligned}
P_1(k) &= A_L' P_1(k+1) A_L + I_n + \left(\mu - \mu^2\right) C_2' L'(k+1) P_1(k+1) L(k+1) C_2 \\
&\quad + \gamma_1^{-2} A_L'(k) P_1(k+1) \Delta^{-1}(k) P_1(k+1) A_L(k) \\
P_2(k) &= A_W(k) P_2(k+1) A_W'(k) + Q - \mu A_W(k) P_2(k+1) C_2' \Lambda^{-1}(k) C_2 P_2(k+1) A_W'(k)
\end{aligned}
$$

where

$$
\begin{aligned}
A_L(k) &= A + \mu L_*(k+1) C_2, \\
A_W(k) &= A + W(k+1), \\
\Delta(k) &= I_{m1} - \gamma_1^{-2} P_1(k+1), \\
\Lambda(k) &= R + C_2 P_2(k+1) C_2', \\
L_*(k) &= -A_W(k) P_2(k+1) C_2' \Lambda^{-1}(k), \\
W(k) &= \left[\gamma_1^2 P_1(k+1) - I_n\right]^{-1} A_L(k).
\end{aligned}
$$

Input: $M > 0$

Initialization;

for $k \leftarrow M$ **to** *1* **do**

  Update left column;

  Update above P_1, P_2;

end

Output: P_1, P_2, L_*, W

Numerical Example

$$A = \begin{bmatrix} 0.81 & 0.15 \\ 0.31 & 0.7 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 & 2.5 \end{bmatrix}.$$

The successful transmission rate for the lossy network is set to be 90%. The disturbance is given by $w(k) = |0.016\sin(0.1k)|$ with disturbance attenuation level $\gamma_1 = 2.5$. The covariances of process noise and measurement noise are given by $Q = \mathrm{diag}\{0.09, 0.09\}$ and $R = 0.05$, respectively. Here, we attempt to design an $\mathcal{H}_\infty$ Gaussian filter on the infinite time horizon with data loss.

Numerical Example

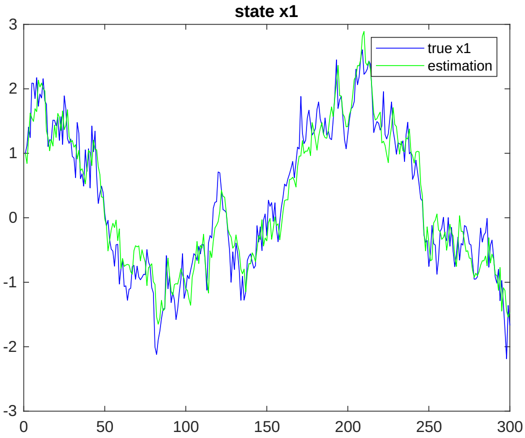

Figure: Estimation of $x_1(k)$ over lossy network

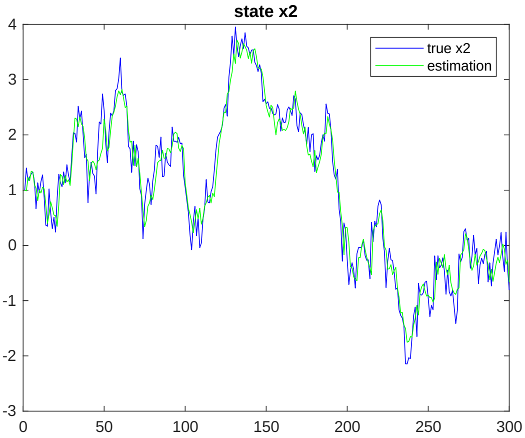

Figure: Estimation of $x_2(k)$ over lossy network

References

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The End

Additional Slides

3) Next, we prove that the second Nash inequality $J_2\left(L_*, w_*(k), w_0(k), v_0(k)\right) \leq J_2\left(L, w_*(k), w_0(k), v_0(k)\right)$ also holds. Substituting the worse case signal $w_*(k) = We(k)$ into the error system leads to

$$e(k+1) = \left[A_W + L\Phi(k)C_2\right]e(k) + w_0(k) + L\Phi(k)v_0(k). \tag{1}$$

We first show that the above error system is MS stable when $L = L_*$, $w_0(k) = 0$, and $v_0(k) = 0$. Note that the MARE

$$P_2 = A_W P_2 A_W' + Q - \mu A_W P_2 C_2' \Lambda^{-1} C_2 P_2 A_W'$$

can be rewritten by

$$P_2 = (\tilde{A} + \tilde{B}\mu\tilde{C})P_2(\tilde{A} + \tilde{B}\mu\tilde{C})' + \tilde{B}H\tilde{B}' + \mathrm{Tr}\left\{Q + \mu L_*' L_* R\right\},$$

where $\tilde{A} = A_W, \tilde{B} = L_* C_2, \tilde{C} = I_n, H = \left(\mu - \mu^2\right)P_2$. It is easy to see that $\mathrm{Tr}\left\{\mu L_*' L_* R\right\} = \mathrm{Tr}\left\{\mu L_* R L_*'\right\} > 0$, for $\mu > 0$ and $R > 0$. Consequently,

$$P_2 > (\tilde{A} + \tilde{B}\mu\tilde{C})P_2(\tilde{A} + \tilde{B}\mu\tilde{C})' + \tilde{B}H\tilde{B}'$$

Additional Slides

Recall that $\mu \in (0,1]$. If $\mu \neq 1$, then $H > 0$. The error system (1) is MS stable with $L = L_*, w_0(k) = 0$, and $v_0(k) = 0$[xiao, 2012]. If $\mu = 1$, which means reliable transmission, i.e. $\Phi(k) = 1$, $\forall k$, then the error system (1) becomes

$$e(k+1) = (A_W + LC_2)\, e(k) + w_0(k) + Lv_0(k),$$

and above inequality is read to

$$P_2 > (A_W + L_* C_2)\, P_2\, (A_W + L_* C_2)'\,.$$

Hence, the error system is also stable with $L = L_*, w_0(k) = 0$, and $v_0(k) = 0$.

Additional Slides

Moreover, when white noises $w_0(k)$ and $v_0(k)$ are included, the error covariance at the time index k, defined as $\Sigma(k) = \mathbb{E}\left\{e(k)e'(k)\right\}$, follows the modified difference Riccati equation (MDRE)

$$
\begin{aligned}
\Sigma(k+1) =& \mathbb{E}\left\{\left[A_W + L\Phi(k)C_2\right]\Sigma(k)\left[A_W + L\Phi(k)C_2\right]'\right\} \\
& + \mathbb{E}\left\{\left[w_0(k) + L\Phi(k)v_0(k)\right]\left[w_0(k) + L\Phi(k)v_0(k)\right]'\right\} \\
=& A_W\Sigma(k)A_W' + Q + \mu L\left[R + C_2\Sigma(k)C_2'\right]L' + \mu\mathbf{He}\left\{A_W\Sigma(k)C_2'L'\right\} \\
=& A_W\Sigma(k)A_W' + Q + \mu\Theta(k)\left[R + C_2\Sigma(k)C_2'\right]\Theta'(k) \\
& - \mu A_W\Sigma(k)C_2'\left[R + C_2\Sigma(k)C_2'\right]^{-1}C_2\Sigma(k)A_W',
\end{aligned}
$$

where

$$
\Theta(k) = L + A_W\Sigma(k)C_2'(k)\left[R + C_2\Sigma(k)C_2'\right]^{-1}.
$$

Additional Slides

It is observed that $\Sigma(k+1)$ is minimized when

$$L = -A_W \Sigma(k) C_2' \left[R + C_2 \Sigma(k) C_2' \right]^{-1}.$$

Note that the error system is MS stable. Then, as $k \to \infty$, the MDRE converges to the MARE P_2 representation, $\Sigma(k) \to P_2$ and $L \to L_*$. Hence,

$$\mathrm{Tr}\left\{P_2\right\} = J_2\left(L_*, w_*(k), w_0(k), v_0(k)\right) \leq J_2\left(L, w_*(k), w_0(k), v_0(k)\right).$$

$\square$