voltages are at frequencies within the pass band of the filter they can transfer power to the load and invalidate (18). In order to eliminate this effect, the following inequality must be satisfied:

$$\left| \omega_M - \sum_{m=1}^{M-1} \beta_{(m)}\omega_m \right| > \delta, \tag{28}$$

for all allowed $\beta_{(m)}$ except for the combination which corresponds to ω_M. It is seen that (27) is a special case of (28).

It is noted that there is no restriction on the impedance of the filter within the pass band. If this impedance Z has a real component R_L, the efficiency will be given by

$$\eta = \frac{R_L}{R_L + Re(Z)}. \tag{29}$$

In any practical case it is desirable to make Z as small as possible.

In the limit as N approaches infinity, condition (10) becomes the condition that all ω's be incommensurably related. However, in this case (28) can only be satisfied if δ is zero.

Conclusions

Power relationships, given in (18), have been derived. Although this equation differs from the result of Manley and Rowe, it nevertheless yields the same answers when applied to the specific cases which they consider. As in the case of Manley and Rowe's result, (18) does not depend on the detailed nature of the characteristic. Indeed (18) is satisfied by a linear characteristic in which case all the powers are zero. It is clear from this analysis that a nonlinearity of finite degree as given by (3) will not give rise to an infinite number of frequencies.

Requirements on the nature of the nonlinearity have been derived and are given in (23) and (25). The results derived here depend on restrictions on the generator frequencies, as given in (10) and (28), but these restrictions are less demanding than those of Manley and Rowe.

Solving Steady-State Nonlinear Networks of "Monotone" Elements*

G. J. MINTY†

Summary—**This paper treats the problem of finding the steady-state currents and voltage drops in an electrical network of two-terminal elements, each of which has the property that its current-vs-voltage-drop graph, or "characteristic," is a curve going upward and to the right. (Thus, "tunnel diodes" are excluded, but nonlinear resistances, current and voltage sources, rectifiers, etc. are permitted.)**

The construction methods are specifically designed for digital computation techniques (either automatic or manual). The principal tools are: 1) the application of theorems from graph theory ("network-topology"), and 2) quantization of the variables (permitting them to take on only a discrete set of values).

I. Introduction

We consider networks of two-terminal circuit elements, each element having the property that its current-vs-voltage-drop graph, or "characteristic," is a *curve going upward and to the right* (see Fig. 1). It is permitted to travel horizontally and/or vertically, and to have sharp corners. It will be called "passive" if the curve passes through the origin. (Note that it is *not* the graph of a monotone nondecreasing function if it contains vertical lines.)

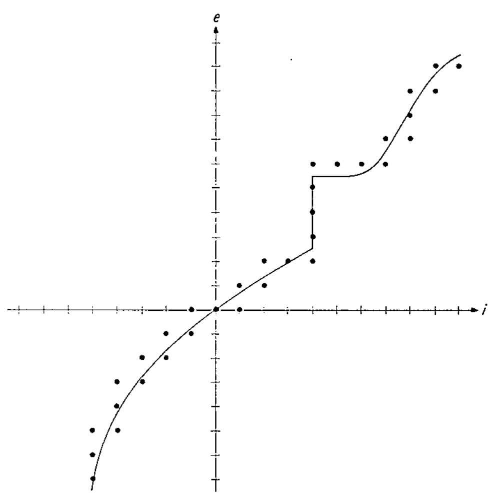

Fig. 1—Characteristic of a monotone circuit element, and quantized approximation thereto.

More formally: we impose a *partial order*[1] on the plane by saying that $(i_1, e_1) \leq (i_2, e_2)$ provided that $i_1 \leq i_2$ *and* $e_1 \leq e_2$, and insist that *a characteristic be a maximal*

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[1] G. Birkhoff and S. MacLane, "A Survey of Modern Algebra," The Macmillan Company, New York, N. Y., 1948.

simply-ordered subset of the plane. Thus, vertical and horizontal lines (current and voltage sources) are allowed, as are rectifiers, perfect conductors and nonconductors, Ohm's-law resistors with positive resistance, diodes (except "tunnel diodes"), and so on.

The "standard" method of solving such networks consists in replacing all characteristics by straight lines and solving the resulting *linear* network by node or mesh analysis. Since the solution points usually do not lie exactly on the "original" characteristics, the straight lines are then replaced by others, using the first solution as a general guide, and then the new problem is solved. This process is repeated until the solution points come "satisfactorily close" to the original characteristics. This process is laborious even if a digital computer is used to solve the linear equations, and requires experience and judgment which are difficult to program for a computer; moreover, until now no good criterion for "satisfactorily close" has been available. The labor necessary to construct the "joint" characteristic of a two-terminal network by this method is great, since the process must be carried through for *each* point desired.

Another, somewhat easier, numerical method has been given by Birkhoff and Diaz,[2] but it suffers from the same kind of unprogrammability. Existence theorems (with rather nonconstructive proofs) have been given by Duffin[3] for a large class of these networks, and Dennis[4] has given an efficient method for solving networks of current sources, voltage sources, and rectifiers. The author, in a recent paper,[5] has shown that the fundamental theorems for these networks can be proved even if the currents and voltage drops are "quantized" (in a sense which will be clear later). The object of the present paper is to present an exposition of a programmable numerical method based on the proofs of the existence theorems for the quantized case. The methods are closely related to those of Dennis and of Ford and Fulkerson,[6] but "graphical" terminology has been preferred to the terminology of linear programming.

In the following sections, the algorithms ("loops" of the computer program) are presented in order of decreasing frequency of use. Thus, the first process given should be programmed for maximal efficiency, and succeeding processes do not require the same efficiency. The "basic loop" has been flow-charted; the reader may wish to arrange the other algorithms into flow-chart form himself.

Since, in the course of the constructions, large quantities of information are generated, but only a small amount of this information needs to be "carried forward" (*i.e.*, most of it can be erased from the computer memory), the author has employed the device of referring to quantities (or arrows, or other marks) which should sometimes be erased, by the phrase "in pencil," and to information which should be preserved through the whole course of the constructions by the phrase "in ink;" this convention is in keeping with the graphical terminology used throughout.

II. The Basic Loop

Consider a network of pipes filled with an incompressible fluid. In some of the pipes, there are one-way valves. Other pipes are "blocked," and still others are left "unblocked." Suppose a piston is placed in one of the pipes having a one-way valve, and that the piston is pushed in such a direction as to tend to force fluid through the valve in the valve's preferred direction. Then either 1) the piston will move, or 2) it will not move. Investigation of the combinatorial structure of the network reveals that in case 1) there is a *cycle* of "unblocked" pipes and one-way valves, including the pipe having a piston, in which all the one-way valves are similarly-directed; it is the presence of this cycle (which may not be unique) which permits the flow. In case 2), an increase in the force on the piston results in a pressure drop across the ends of the pipe. In the general case, it turns out that the pressures at the nodes of the network cannot be determined by statics (*i.e.*, there are several solutions consistent with the equations of static equilibrium). Combinatorial investigation shows that *a* solution of the system can be found in which the pressures at a certain set S of nodes all rise by the same amount, while the pressures at the other nodes remain "zero." The *coboundary* of this set S (defined as the set of branches having one node in S, and one node not in S) must consist of "blocked" pipes and one-way valves, must contain the pipe having a piston, and must have the one-way valves so directed that the pressure drop tends to *close* them rather than to open them.

Further investigation shows that the set of nodes can always be so chosen that its coboundary is a *co-cycle*. (A co-cycle of a connected network is a set of branches such that, if these branches are cut, the network falls into two pieces, and such that if one fails to cut any of these branches, it will still be in one piece.)

It is easy to formulate the above considerations as a theorem on hydraulic networks. However, such a formulation tends to hide its applicability to electrical networks of perfect conductors, perfect nonconductors, and rectifiers, and to street networks of "open" streets, "blocked" streets, and one-way streets. The best language for revealing the full range of possible applications seems to be that of graph theory.

Theorem 1: In a directed (oriented) graph whose branches are colored red, green, and blue, and in which precisely one branch is colored dark green, there exists one, but not both, of the following:

[2] G. Birkhoff and J. B. Diaz, "Nonlinear network problems," *Quart. Appl. Math.*, vol. 13, pp. 431–443; January, 1956.

[3] R. J. Duffin, "Nonlinear networks. II," *Bull Amer. Math. Soc.*, vol. 53, pp. 963–971; October, 1947.

[4] J. B. Dennis, "Mathematical Programming and Electrical Networks," John Wiley and Sons, Inc., New York, N. Y. Technology Press, Cambridge, Mass.; 1959.

[5] G. J. Minty, "Monotone networks," *Proc. Roy. Soc. (London) A*, vol. 257, pp. 194–212; September, 1960.

[6] L. R. Ford, Jr. and D. R. Fulkerson, "A simple algorithm for finding maximal network flows and an application to the Hitchcock problem," *Canad. J. Math.*, vol. 9, pp. 210–218; April, 1957.

1) A cycle of red and green branches, containing the dark-green branch, and in which all the green branches are similarly-directed.

2) A co-cycle of blue and green branches, containing the dark-green branch, and in which all the green branches are similarly-directed.[7]

(*N. B.*, the dark-green branch may be regarded as merely a device for distinguishing the terminals of a two-terminal network; the cycle is essentially a *tie-set* and the co-cycle is a *cut-set*. Rephrasing of the theorem in this terminology is left to the reader.)

The algorithm for discovering the cycle or co-cycle is given in Fig. 2 (next page). In following through this process, the reader should have before him a picture of the network, drawn in ink; the branches are labelled R, G, or B in pencil, and one branch is labelled dg, in pencil; the arrows on the branches are drawn in pencil. For many applications, it is sufficient to find a *co-boundary* rather than a *co-cycle;* in this case, the algorithm should be interrupted at point A.

III. Constructing the Characteristic of a Two-Terminal Quantized Network of Passive Elements

It is assumed that the reader has before him a picture of the network, drawn in ink; the branches are represented merely by lines, and are numbered $1, \cdots, n$, in ink. Draw an "extra branch" in ink, connecting the two terminals, and label it dg (for "dark green") in pencil. On each branch, draw an arrowhead, in pencil; the current will be considered as positive if it goes in the direction of the arrow, and the voltage drop is the tail-node voltage minus the head-node voltage.

Obtain as many sheets of square-ruled graph paper as there are branches. Number the sheets, in ink, to show which branches they correspond to. Draw, in ink, on each sheet *except* the one for the dark-green branch, a "quantized" approximation to the characteristic of the circuit element of that branch. Leave the sheet belonging to the dark-green branch blank.

The "quantized approximation" should be a *maximal simply-ordered subset of the lattice points of the plane* (see Fig. 1), or at least, as much of such a subset as it is possible to get onto a sheet of graph paper; since we are assuming passive circuit elements, the quantized approximation *should include the origin.*

(It will help the visualization if the graph papers are spread out on the table, each being placed in close proximity to the branch it represents, in the network drawing.)

Place a mark (X) in pencil at (0, 0) of each graph paper, including the sheet for the dark-green branch. Note that these X's are a *solution* of the two-terminal network—all X's lie on their respective dot patterns, and "all currents and voltage-drops zero" satisfy the two Kirchhoff's Laws.

Now, each branch is behaving *locally, with respect to this temporary solution*, as a perfect conductor, perfect nonconductor, or rectifier, according to the positions of the two "neighboring" dots of the pattern. Thus, if the X has dots both to its left and to its right, we mark the branch with the letter R ("red"), in pencil; if it has dots both above and below, we mark it with B ("blue"). If there is a dot below and a dot to the right, we mark it with G ("green"), and if there is a dot to the left and a dot above, we mark the branch with G, and also *erase the pencilled arrow on the branch, put a new arrow in the opposite direction, and rotate the corresponding sheet of graph paper* 180° *without lifting it from the table.* In this way, we give each "temporary rectifier" its natural orientation.

The next step is to find the cycle or the coboundary of Theorem 1, using the flow chart of Fig. 2 and breaking off the process at point A. Wherever Fig. 2 specifies that a red or blue branch be redirected, the corresponding graph paper is, of course, simultaneously rotated 180°.

If the cycle is found: Erase, one-by-one, the pencilled X's of branches of the cycle, and draw new X's one unit to the right of their old positions; also, place a pencilled dot where the "old X" of the dark-green branch was. Note that the Kirchhoff's Laws are still satisfied, so we have found a *new solution* of the two-terminal network.

If the coboundary is found: Shift its pencilled X's, as above, *one unit down*, and also record the "old position" of the X of the dark-green branch. Again there is a new solution of the two-terminal network; Kirchhoff's voltage law is still satisfied because this shift is equivalent to subtracting one unit of voltage from each node of the set determining the coboundary.

(*N. B.*, In the above processes, it is essential that *all* the branches of the cycle or coboundary be similarly-directed. If Fig. 2 is used for the construction, this is automatically taken care of, but if the cycle or coboundary is found by inspection, it must be taken care of in a separate step.)

In either case, we now have a new point of the characteristic. The process given above can be used iteratively to construct as many points as desired in the "to the right and downward" direction. For greater efficiency, we can "save," in each step, some of the work (check-marks and stars) of the previous construction of Fig. 2; this is left to the reader to figure out.

To find the "other end" of the characteristic, in the "to the left and upward" direction: Erase the X's and replace them at the origins, reverse the arrow of the dark-green branch (simultaneously rotating its graph paper), and proceed as above.

The result of the processes described above is the train of pencilled dots on the graph paper of the dark-green branch. The mirror image of this pattern is the characteristic of the two-terminal (quantized) network. It has been shown[8] that the characteristic is a simply-ordered

[7] Minty, *op. cit.*, Theorem 3.1.

[8] *Ibid.*, Lemma 6.1.

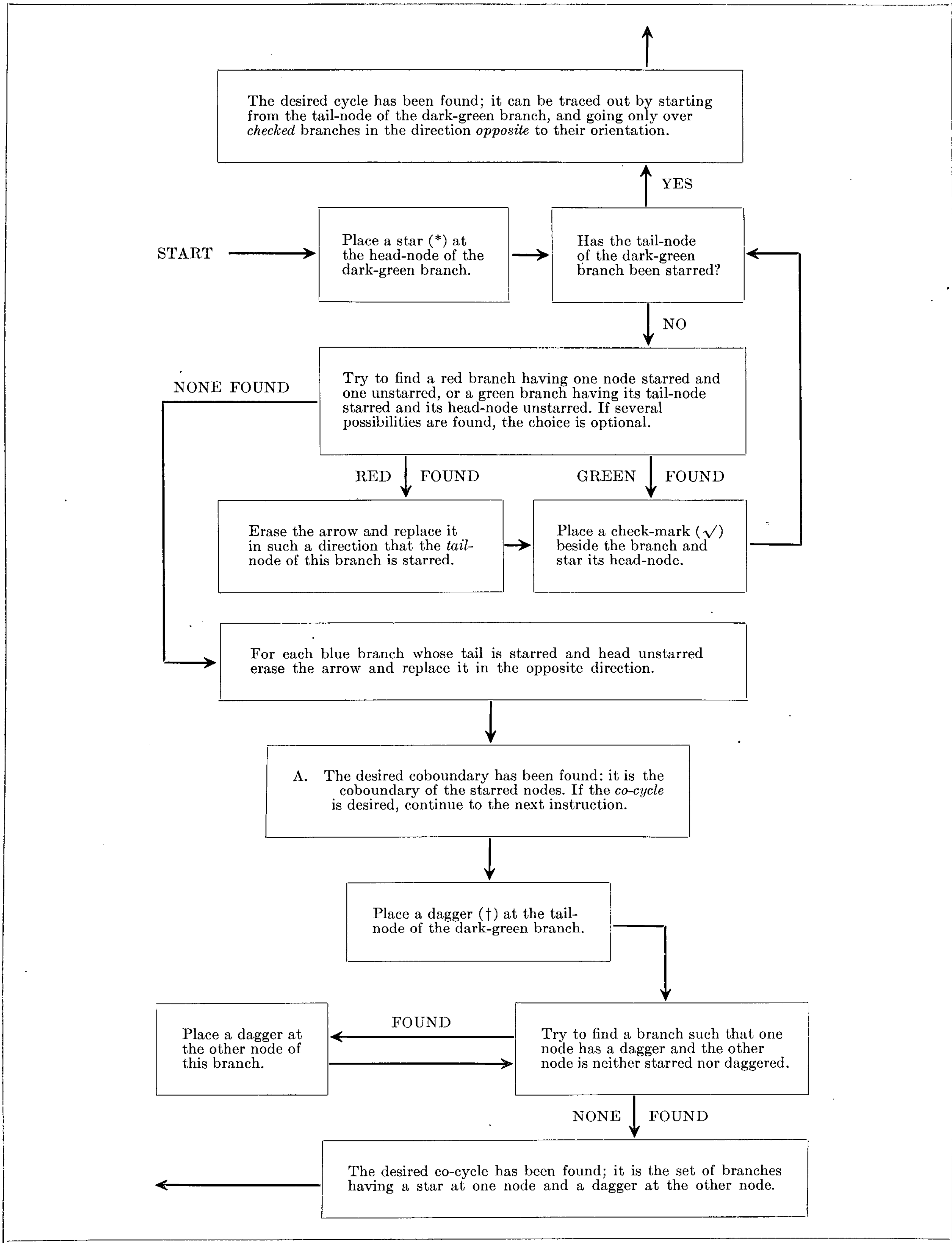

Fig. 2—Flow chart for performing the construction of Theorem 1.

subset of the lattice points of the plane, and since the construction process yields a *maximal* simply-ordered set of lattice points, it follows that *there is no point of the characteristic which is not constructed by the process.* (In particular: two people solving the same problem *may* find different "solutions" of the two-terminal network in the course of the construction, but they *will not* find different dot patterns for the characteristic.)

IV. Solving Networks of Nonpassive Elements

First, the *complete* (*i.e.*, not two-terminal) network is taken up.

It is assumed that there is a picture of the network, with the branches directed, and graph papers corresponding to the branches, with quantized approximations to the characteristics drawn on them. It is assumed also that X's have been placed, one on each graph paper, so that 1) they satisfy the two Kirchhoff's Laws, and 2) any vertical or horizontal line drawn through an X will strike a dot of the quantized characteristic. (The question of how to find these starting points routinely will be discussed in the next section.)

For each X which does not lie on a dot, draw (in pencil) vertical and horizontal rows of dots over to the inked patterns, so that the X lies at the corner of a pencilled L of dots. These are regarded as "temporary" dot patterns. Any inked dot which is not ordered with the X (by the partial ordering discussed earlier) *is regarded as temporarily deleted.*

Now erase the pencilled dots of one branch and regard it as the dark-green branch. Construct the characteristic of this two-terminal network, by the process already described, until the X lies on the inked dot pattern. Go to the next branch and repeat the process. Continue until all X's lie on the inked patterns.

(It is easily seen that an equivalent, but more efficient, process is obtained by regarding any branch for which the X does not lie on the pattern, as a "green" branch, and orienting such branches so that the X always lies "above and to the left" of the dot pattern.)

Now let us try to find the characteristic of a two-terminal network of nonpassive elements. Adjoin a dark-green branch. On its graph paper, ink a dot pattern which represents an approximation to an Ohm's-Law resistor, with positive resistance. Proceed as above for a "complete" network. When all X's lie on the inked dot patterns, take a fresh sheet of graph paper for the dark-green branch, transfer the X to this sheet, and proceed exactly as in the case of a two-terminal network of passive elements.

V. Finding the Starting Point

A little reflection reveals that the requirements on a "starting point" for the above process are as follows: 1) the current components of the X's must satisfy Kirchhoff's current law, and must lie in the *projections* of the quantized characteristics on the current axes, and 2) the voltage-drop components of the X's must satisfy Kirchhoff's voltage-drop law, and must lie in the projections of the characteristics on the voltage-drop axes. Thus, the conditions on the current components and on the voltage-drop components are *separated*, and the problem splits naturally into two separate problems.

To find the current components: For each branch of the network, take a fresh sheet of graph paper. Place inked dots on the current axes which are the projections of the quantized characteristics onto these axes. If, on any sheet, the origin is not an inked dot, pencil in a row of dots on the current axis which connects the inked dots with the origin. Place X's at all origins. At each left-hand endpoint of an interval, *imagine* a row of dots proceeding straight down to $-\infty$; at each right-hand endpoint, imagine a row of dots proceeding straight up to $+\infty$. Now we see a complete network and a solution.

Erase the pencilled dots of one branch, and try to lead the X over to the "remaining" characteristic, in the manner already described. It should proceed horizontally along the current axis, as long as *cycles* continue to be found in the construction. If this process works, proceed to the next branch, and repeat; go through all branches in this way, so that finally all X's lie on inked dots.

If the process fails, it is because at some time of the construction, a *coboundary* is found instead of a cycle. In this case, the construction of Fig. 2 should be continued *beyond* point A; thus, an "unbalanced co-cycle" will be constructed, which renders the problem unsolvable—it is analogous to a co-cycle of current sources which tend to force current across the network in one direction, with no return path.

If we perform the analogous process with the voltage-drop projections, the X's must move vertically; if a *cycle* is found, causing an X to move horizontally, it is a *short circuit*, and the original (quantized) problem is unsolvable.

(The exact relationship between unsolvability of the quantized network and that of the original unquantized network has been investigated elsewhere;[5] for practical purposes, it amounts to the following: 1) If one wants the quantized problem to be *solvable* whenever the original problem is, one should take the quantized approximations to the characteristics so that their projections on the axes are *at least as wide as* those of the original problem. 2) If one wants it to be *unsolvable* whenever the original problem is, the projections should be at least as *narrow* as the original ones.)

The "currents" and "voltage-drops" found by the above two processes can now be used as a starting-point for the method of Section IV.

VI. Heuristics and Mechanical Analogs

There are two interesting mechanical analogs to electrical networks, each of which yields some insight into the heuristic nature of the above-described solution processes. The first is hydraulic in nature: Take a network of pipes, filled with hydraulic fluid, in the form of the underlying topological network of the electrical network. Assume the pipes have equal cross-sectional areas. Place a piston in each pipe, free to move in either direction, and place a scale (ruler) beside each pipe so that the piston position can be read off the scale. Couple each piston to an external mechanical agency so that the pressure-drop-vs-displacement graph is a step function approximating the characteristic of the electrical network. "Stops" are placed in the pipes where the characteristics have semi-infinite vertical lines; these "stops" are essentially infinite potential barriers. If all pistons are started out at "zero" on their scales, then the displacements at all future times will satisfy Kirchhoff's current law. If the electrical network has an unbalanced co-cycle, this model cannot even be built, and if it has a short circuit, the pistons will "move off to infinity" without ever coming to an equilibrium position. The analysis of the characteristic given in Section III is merely a step-by-step analysis of what happens in the network if one piston is gradually displaced unidirectionally. The fact that it can be done in "quantized" form seems to be a pure mathematical accident, and stems from the observation that only additions and subtractions are needed for the analysis.

An alternative mechanical analog is the following one: Beads are strung on straight, parallel wires (one bead per wire). The beads are coupled pair-wise in such a way that the force exerted by one bead on another is a monotone step-function of their relative displacement, both force and displacement being measured parallel to the wires. Here, the beads represent nodes, the couplings represent branches, the relative displacements represent voltage drops, and the *forces*, in static equilibrium, satisfy Kirchhoff's current law! The characteristic-analysis traces out what happens as the relative displacement between two beads is slowly varied by some external agency.

The fact that the potential energies are minimized in static equilibrium of either system yields two minimization principles for electrical networks;[9] these are *not* energy- or power-minimization theorems in the electrical case.

VII. The Error of the Approximation; How to Add Decimal-Places

After finishing a quantized problem, we have X's on the graph papers which do not, in general, lie exactly on the original curves. To find out how good an approximation they are, proceed as follows: Measure the distances from the X's to the curves, parallel to the current-axes. Form the sum of these numbers, and call it δ'. Do similarly for the distances parallel to the voltage axes, and call the sum δ''. Draw, around each X as center, a rectangle with dimensions $2\delta'$, $2\delta''$. Now, it has been shown[10] that it is (theoretically) possible to draw X's *within these rectangles* so that they are an *exact* solution of the original problem. Thus, we have a good measure of how accurate our approximate solution is.

The method of proof of this theorem can be outlined as follows: first, we note that, with slight modifications, the characteristic-construction process can be made to work for "continuous" step functions. Next, we note that *any* characteristic can be approximated arbitrarily closely by such a continuous step function, so that one can construct arbitrarily close *approximate problems* to the original problem. Third, we note that in any process of constructing a characteristic, the maximum distance through which any X moves is bounded by the distance through which the X of the dark-green branch moves. Fourth, we find that, by a modification of the process for "increasing the number of decimal places" which we are about to give, each "approximate problem" can be solved, and that in the solution process, the maximum total distance through which it is necessary to move the X's of the successive dark-green branches is given by δ' in the current direction and δ'' in the voltage-drop direction. Thus, a sequence of points in $2n$-dimensional space can be constructed which remain inside a rectangular solid with dimensions $(2\delta', \cdots, 2\delta'; 2\delta'', \cdots, 2\delta'')$. By the Weierstrass-Bolzano theorem, this sequence has an accumulation point which lies inside that rectangular solid, and this accumulation point can be shown to be a solution of the original problem.

If these rectangles are uncomfortably large, proceed as follows: Take fresh sheets of graph paper, and draw on them the curves of the original problem, to larger scale. (Essentially, it is only necessary to draw that part of the curve which lies within the rectangle found above.) The scale must be larger by an *integer* factor. Transfer the X's to these sheets, and discard the old sheets. Draw inked dot patterns which are closer approximations to the curves (possible because of the finer quantization) and proceed as in Section IV. A "brute-force" procedure to make the X's satisfy the starting-point requirements is as follows: Project the X's, and the characteristics of the resistors, onto the axes; "lead" the projected X's into the projections of the characteristics by the method of Section V, and then compose them into a single X on each sheet.

[9] *Ibid.*, Corollary to Theorem 6.1; see also, W. Millar, "Some general theorems for nonlinear systems possessing resistance," *Phil. Mag.*, vol. 42, pp. 1150–1160; October, 1951.

[10] *Ibid.*, Theorem 8.1.