NOTES FOR JULY 8, 2021

1. Binomial Coefficients. Let n and k be positive integers. Then the binomial coefficient

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

has many combinatorial interpretations including:

(i) it is the number of binary strings of length n with exactly k appearances of 1;
(ii) it is the number of Dyck paths of length n with exactly k upward steps;
(iii) it is the number of size k subsets of a size n set.

One categorification of the binomial coefficient in the category of sets then might be:

$$\binom{-}{-} \colon \mathrm{Set} \times \mathrm{Set} \to \mathrm{Set}(N,K) \mapsto \{\, S \in \mathrm{Hom}(N,[2]) \mid S \text{ is isomorphic to } K \,\}.$$

In words, this is the functor that takes a pair of finite sets (N,K), produces the set $\mathrm{Hom}(N,[2])$ of all subsets of N—binary strings correspond to subsets of N!—and then picks out the subsets with the same size as K.[1] This is a categorification of the binomial coefficient in the sense that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Set} \times \mathrm{Set} \arrow[r, "\binom{-}{-}"] \arrow[d, "\#"'] & \mathrm{Set} \arrow[d, "\#"] \\
\mathbf{Z}_+ \times \mathbf{Z}_+ \arrow[r, "\binom{-}{-}"] & \mathbf{Z}_+
\end{tikzcd}$$

which means, explicitly, given finite sets N and K with sizes $n := \#N$ and $k := \#K$, there is the relation

$$\#\binom{N}{K} = \binom{\#N}{\#K} = \binom{n}{k}.$$

This might be carried out in other categories, too! The only thing that was special to the category of sets was the construction of the set of all subsets of N. But this amounted to saying that the object $[2]$ had the property that $\mathrm{Hom}(N,[2])$ classified subobjects of N: in other words, $[2]$ is a *subobject classifier* in the category of sets. So we might perform the same construction in a general category C by replacing $[2]$ in the above formula with a subobject classifier Ω in C.

2. Representations of Finite Groups. Let G be a group. A *representation of* G on a finite-dimensional complex vector space V is a linear action of G on V: that is, this is a binary operator $\cdot\colon G \times V \to V$ such that for every $g,h \in G$, $v,w \in V$, and $\lambda \in \mathbf{C}$,

(i) $g \cdot (v + \lambda w) = (g \cdot v) + \lambda(g \cdot w)$,
(ii) $h \cdot (g \cdot v) = (hg) \cdot v$, and
(iii) $e \cdot v = v$,

[1] Note, however, that in order to make this into an honest functor, one might have to be careful about what morphisms to take in the categories of sets in question.

where $e \in G$ is the identity element. Equivalently, let

$$\mathrm{GL}(V) := \{\, T : V \to V \mid T \text{ an invertible linear transformation} \,\},$$

then the data of a representation of G on V amounts to a homomorphism $\rho : G \to \mathrm{GL}(V)$.[2] A subspace $W \subseteq V$ is a *subrepresentation* if

$$g \cdot w \in W \quad \text{for all } g \in G \text{ and all } w \in W.$$

A representation V is called *irreducible* if whenever $W \subseteq V$ is a subrepresentation, then either $W = \{0\}$ or $W = V$.

The basic problem of representation theory is to classify irreducible representations of finite groups G, the point being that irreducible representations are the building blocks of all representations. Over the complex numbers, all representations are built out of irreducible ones in the simplest possible way:

3. Maschke's Theorem. — *Let G be a finite group. Then for every complex representation V, there is a decomposition*

$$V \cong \bigoplus\nolimits_{i \in I} V_i$$

where I is a finite index set, and the V_i are irreducible representations of G.

4. Moreover, the essential data of an irreducible representation turns out to be something quite intrinsic to G. Namely, let V be a representation of G, with associated homomorphism $\rho : G \to \mathrm{GL}(V)$. The *character* of this representation is the homomorphism

$$\chi := \mathrm{trace} \circ \rho : G \to \mathrm{GL}(V) \to \mathbf{C}$$

obtained by taking an element $g \in G$, considering the linear transformation $\rho(g) \in \mathrm{GL}(V)$, choosing any basis of V to make $\rho(g)$ into a complex matrix, and then taking the trace of the resulting matrix. The properties of the trace ensure that this is independent of the choice of basis chosen, and hence this is really an invariant of the representation ρ itself. Then the classification theorem for irreducible representations is

5. Character Theory of Finite Groups. — *Let G be a finite group. Let V and W be irreducible representations of G with characters χ and ψ, respectively. Then V and W are isomorphic representations if and only if χ and ψ are the same function on G.*

6. Thus the problem of classifying irreducible representations of G is greatly aided by the characters. Here are a few more facts:

(i) characters are *class functions*, that is, if $g, h \in G$ are in the same conjugacy class[3], then $\chi(g) = \chi(h)$;
(ii) the set of all complex-valued class functions out of G is a vector space with dimension equal to the number of conjugacy classes of G: take the indicator function for each conjugacy class;
(iii) there is an inner product on the space of class functions;
(iv) the irreducible characters form an orthogonal basis for the space of class functions;

[2] Note that when $V = \mathbf{C}^{\oplus n}$ is the standard n-dimensional vector space, then $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbf{C})$ is none other than the group of invertible square complex matrices of size n: this is the correspondence between linear transformations and matrices, upon picking a basis.

[3] Read: if up to a change of labels/coordinates, they are the same.

(v) each irreducible representations of G is a direct summand, appearing exactly once, of the *regular representation* of G:

$$\mathbf{C}G := \left\{ \sum_{g\in G} a_g g \;\middle|\; a_g \in \mathbf{G} \right\}$$

this is the vector space spanned by symbols indexed by elements of G and G acts on this by multiplication on the symbols.

7. Young Diagrams, Tableaux. A *Young diagram* is a set of boxes that is justified in some fashion. There are many conventions, but let's use the so-called English convention in which the boxes are left and upper aligned. The number of boxes in each row of a Young diagram Y form a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots)$, that is, a non-increasing sequence of nonnegative integers. The relevance of Young diagrams appears in their connection with the representation theory of the symmetric group: the set of irreducible representations of the symmetric group S_n on n elements are naturally indexed by a Young diagram with n boxes. In fact, given a Young diagram Y with n boxes, the corresponding irreducible representation can be made by making a vector space whose basis consists of certain fillings of Y.

Another related thing involves symmetric functions. There are certain symmetric polynomials called the Schur polynomials, and they can be specified by taking and counting certain Young diagrams. These are related to the irreducible polynomial representations of the general linear group $\mathrm{GL}_n(\mathbf{C})$, which in turn is related to the symmetric group situation via what is known as Schur–Weyl duality. Everything is connected!

8. Universal Invariants. Finally we discussed the notion of finding some sort of universal way of making an algebraic invariant for some type of object, satisfying some set of relations. The one that I know best is the notion of an *Euler characteristic*. The basic example would be: I am looking for an invariant χ on the set Vect of finite-dimensional vector spaces such that if I can write $V = V_1 \oplus V_2$, then

$$\chi(V) = \chi(V_1) + \chi(V_2).$$

Well, what sort of structure do I need to make sense of this? I just need to be able to add my invariants, so I am probably looking for a function

$$\chi \colon \mathrm{Vect} \to G$$

for some abelian group G. I want this to be *universal* in the sense that if I have any other map $\psi \colon \mathrm{Vect} \to H$ to some other abelian group H, then I would like there to be a unique homomorphism $\bar{\psi} \colon G \to H$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, "\chi"] \arrow[dr, "\psi"'] & G \arrow[d, "\bar{\psi}"] \\ & H \end{tikzcd} \qquad \text{that is, } \psi = \bar{\psi} \circ \chi.$$

To prove that this exists, I need to construct such an object. And the right thing to do is the least lossy thing possible: take G to be the free abelian group generated on isomorphism classes of objects on Vect, subject to the additive relations. Explicitly, the isomorphism classes of objects in Vect are represented by the standard vector spaces $\mathbf{C}^n$ of dimension n, for each integer $n \geq 0$. Then I would like to set G to be the abelian group generated by symbols $[\mathbf{C}^n]$. But I need to subject these to the direct sum relation. Note, however, that

$$\mathbf{C}^n = \mathbf{C}^{n-1} \oplus \mathbf{C} = \cdots \mathbf{C} \oplus \mathbf{C} \oplus \cdots \oplus \mathbf{C}$$

by peeling off coordinates. So this means that I need to impose the relations

$$[\mathbf{C}^n] = [\mathbf{C}] + [\mathbf{C}] + \cdots + [\mathbf{C}] = n[\mathbf{C}].$$

Hence G is the free abelian group generated on the single symbol $[\mathbf{C}]$, and hence is the group of integers $G \cong \mathbf{Z} \cdot [\mathbf{C}]$ and the function $\chi : \mathrm{Vect} \to G$ is none other than the dimension function!

9. This construction can be repeated when the category of vector spaces is replaced by any other category $\mathscr{A}$ in which some notion of "an object is the sum of two others" is represent. This is axiomitized in the notion of a abelian category and the resulting construction is known as the Grothendieck group of the associated category.