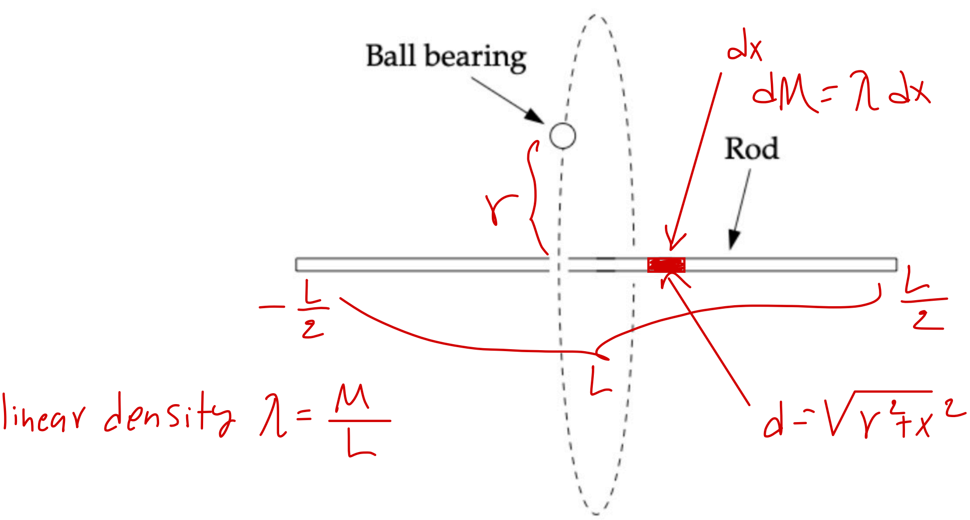
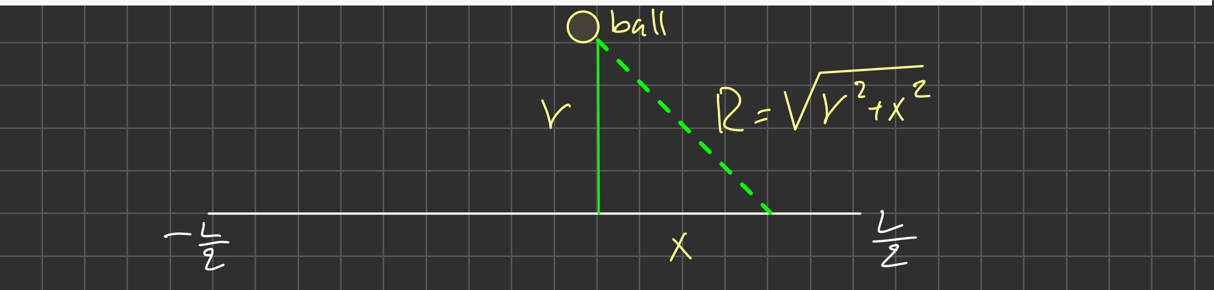
HW6 – Exercise 8.8: Space Garbage

A heavy steel rod and a spherical ball-bearing, discarded by a passing spaceship, are floating in zero gravity and the ball bearing is orbiting around the rod under the effect of its gravitational pull.



For simplicity we will assume that the rod is of negligible cross-section and heavy enough that it doesn't move significantly, and that the ball bearing is orbiting around the rod's mid-point in a plane perpendicular to the rod.



a) Treating the rod as a line of mass M and length L and the ball bearing as a point of mass m, convince yourself that the attractive force F felt by the ball bearing in the direction toward the center of the rod is give by of

$$F = \frac{GMm}{\sqrt{(x^2+y^2)(x^2+y^2+\frac{1}{4}L^2)}}$$

Hence one finds that the equations of motion for the position (x,y) of the ball bearing in the xy-plane are

$$\frac{d^2x}{dt^2} = -GM\frac{x}{r^2\sqrt{r^2+\frac{1}{4}L^2}} \quad \frac{d^2y}{dt^2} = -GM\frac{y}{r^2\sqrt{r^2+\frac{1}{4}L^2}}$$

where $r = \sqrt{x^2+y^2}$.

$F = \frac{GMm}{R^2}$ The Force

$F = \frac{GMm}{\left(\sqrt{r^2+x^2}\right)^2} \Rightarrow F = \frac{GMm}{r^2+x^2}$

$dF = \frac{GmdM}{r^2+x^2}$

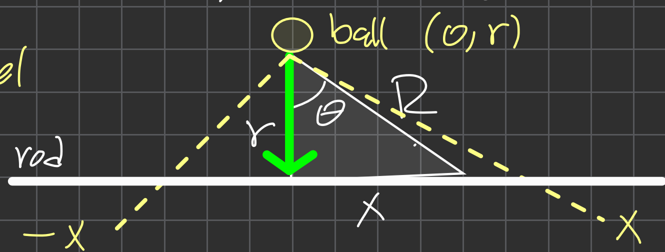
$dF = \frac{Gm\,dm}{r^2+x^2}$, force is angled, not 100% towards the center

due to symmetry, side components cancel

We only account for F straight toward the center

Pull from x & -x Cancel

$dF_{(inward)}$



$$\cos\theta = \frac{r}{R} = \frac{r}{\sqrt{r^2+x^2}}$$

$$dF_{(inward)} = (dF)(\cos\theta)$$

$$dF_{(inward)} = \left(\frac{GmdM}{r^2+x^2}\right)\left(\frac{r}{\sqrt{r^2+x^2}}\right)$$

$$dF(inward) = Gmr\frac{dM}{(r^2+x^2)^{3/2}}$$

$$\lambda = \frac{M}{L}$$

$$dm = \frac{M}{L}dx$$

$$dF(inward) = \frac{GmM}{L}\frac{dx}{(r^2+x^2)^{3/2}}$$

$$F = \int_{-\frac{L}{2}}^{\frac{L}{2}} dF_{(inward)}$$

$$F = \frac{GmMr}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dx}{(r^2+x^2)^{3/2}} \Rightarrow \text{even integrand}$$

$$F = \frac{GmMr}{L}\int_{0}^{\frac{L}{2}}\frac{dx}{(r^2+x^2)^{3/2}} \Rightarrow \text{Standard integral}$$

$$F = \frac{GmMr}{L}\left(\frac{x}{r^2\sqrt{r^2+x^2}}\right|_0^{\frac{L}{2}}$$

$$F = \frac{GmMr}{L}\left(\frac{\frac{L}{2}}{r^2\sqrt{r^2+\left(\frac{L}{2}\right)^2}}\right)$$

$$\boxed{F_{mag} = \frac{GMm}{r\sqrt{r^2+\left(\frac{L}{2}\right)^2}}}$$

Equations of motion

$$r = \sqrt{x^2+y^2} \Rightarrow r^2 = x^2+y^2$$

$$\hat{r} = \frac{r}{\|r\|} = \frac{(x,y)}{r}$$

$$F = -F_{mag}\,\hat{r} = -\frac{GMm}{r\sqrt{r^2+\left(\frac{L}{2}\right)^2}}\left(\frac{x,y}{r}\right) = -\frac{GMm}{r^2\sqrt{r^2+\left(\frac{L}{2}\right)^2}}(x,y)$$

$$a = \ddot{r} = -\frac{GM}{r^2\sqrt{r^2+\left(\frac{L}{2}\right)^2}}(x,y)$$

$$\frac{d^2x}{dt} = -\frac{GMx}{(x^2+y^2)\sqrt{x^2+y^2+\frac{L^2}{4}}}$$

$$\frac{d^2y}{dt} = -\frac{GMy}{(x^2+y^2)\sqrt{x^2+y^2+\frac{L^2}{4}}}$$

Converting 2^{nd} order To 1^{st} order

$$V_x = \frac{dx}{dt}, \quad V_y = \frac{dy}{dt}$$

$$\frac{dV_x}{dt} = -\frac{GMx}{(x^2+y^2)\sqrt{x^2+y^2+\frac{L^2}{4}}}$$

$$\frac{dV_y}{dt} = -\frac{GMy}{(x^2+y^2)\sqrt{x^2+y^2+\frac{L^2}{4}}}$$

$$\left.\begin{aligned} \frac{d^2x}{dt} &= -\frac{GMx}{r^2\sqrt{x^2+y^2+\frac{L^2}{4}}} \\ \frac{d^2y}{dt} &- \frac{GMy}{r^2\sqrt{x^2+y^2+\frac{L^2}{4}}} \end{aligned}\right\} \Rightarrow 4 \quad 1\text{st}$$

$$V_x = \frac{dx}{dt}, \quad V_y = \frac{dy}{dt}$$

$$\frac{dV_x}{dt} = -\frac{GMx}{r^2\sqrt{x^2+y^2+\frac{L^2}{4}}}$$

$$\frac{dV_y}{dt} - \frac{GMy}{r^2\sqrt{x^2+y^2+\frac{L^2}{4}}}$$