Deriving Jeans Dispersion Relation

a infinite, uniform, self-gravitating gas:

$\rho_0 = \text{const}$, $v_0 = 0$, $P_0 = \text{const}$

Small perturbations:

$\rho = \rho_0 + \delta\rho$, $P = P_0 + \delta P$, $v = \delta v$, $\Phi = \Phi_0 + \delta\Phi$

EOS: $\delta P = c_s^2 \delta\rho$

isothermal $c_s^2 = \dfrac{kT}{\mu m_p}$, for adiabatic $c_s^2 = \dfrac{\gamma P_0}{\rho_0}$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{1}{\rho} \nabla P - \nabla \Phi$$

$$\nabla^2 \Phi = 4\pi G \rho$$

$$\frac{\partial(\rho_0 + \delta\rho)}{\partial t} + \nabla \cdot \left[(\rho_0 + \delta\rho) \delta v \right] = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta v = 0$$

Euler linearized:

$$\frac{\partial \delta v}{\partial t} + (\delta v \cdot \nabla) \delta v = -\frac{1}{\rho_0 + \delta\rho} \nabla (P_0 + \delta P) - \nabla(\Phi_0 + \delta\Phi)$$

$$\frac{1}{\rho_0 + \delta\rho} \approx \frac{1}{\rho_0}$$

$$\frac{\partial \delta v}{\partial t} = -\frac{1}{\rho_0}\nabla \delta P - \nabla \delta \Phi \quad , \quad \delta P = c_s^2 \delta \rho$$

$$\frac{\partial \delta v}{\partial t} = -\frac{c_s^2}{\rho_0}\nabla \delta \rho - \nabla \delta \Phi$$

Poisson linearized: $\nabla^2(\Phi_0 + \delta\Phi) = 4\pi G(\rho_0 + \delta\rho)$

$\nabla^2 \delta\Phi = 4\pi G \delta\rho$ (only using perturbation eq)

Divergence:

$$\nabla \cdot \frac{\partial \delta v}{\partial t} = -\frac{c_s^2}{\rho_0}\nabla \cdot \nabla \delta\rho - \nabla \cdot \nabla \delta \Phi$$

$$\frac{\partial}{\partial t}(\nabla \cdot \delta v) = -\frac{c_s^2}{\rho_0}\nabla^2 \delta\rho - \nabla^2 \delta\Phi$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}(\nabla \cdot \delta v) = 0$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \rho_0 \left(-\frac{c_s^2}{\rho_0}\nabla^2 \delta\rho - \nabla^2 \delta\Phi\right) = 0$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c_s^2 \nabla^2 \delta\rho - \rho_0 \nabla^2 \delta\Phi = 0$$

$$\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \delta\rho - 4\pi G \rho_0 \delta\rho = 0$$

Plain wave sltn

Assuming Fourier Mode:

$$\delta\rho(x,t) = \delta\rho_k\, e^{i(k\cdot x - \omega t)}$$

$$\frac{\partial^2}{\partial t^2} \to -\omega^2;\quad \nabla^2 \to -k^2$$

$$-\omega^2 \delta\rho_k - c_s^2(-k^2)\delta\rho_k - 4\pi G\rho_0\,\delta\rho_k = 0$$

$$-\omega^2 + c_s^2 k^2 - 4\pi G\rho_0 = 0$$

$$\boxed{\omega^2 = c_s^2 k^2 - 4\pi G\rho_0}$$