Deriving Toomre Q Parameter

For a razor thin, axisymmetric fluid disc

$$\omega^2(k) = \kappa^2 + c_s^2 k^2 - 2\pi G \Sigma_0 |k|$$

k = letter k
κ = kappa

minimizing ω^2

$$\frac{d\omega^2}{dk} = 2c_s^2 k - 2\pi G \Sigma_0$$

$$2c_s^2 k - 2\pi G \Sigma_0 = 0, \quad k_{min} = \frac{\pi G \Sigma_0}{c_s^2}$$

$$\omega^2_{min} = \kappa^2 + c_s^2\left(\frac{\pi G \Sigma_0}{c_s^2}\right)^2 - 2\pi G \Sigma_0 \left(\frac{\pi G \Sigma_0}{c_s^2}\right)$$

$$c_s^2 \frac{\pi^2 G^2 \Sigma_0^2}{c_s^4} = \frac{\pi^2 G^2 \Sigma_0^2}{c_s^2}$$

$$\frac{2\pi^2 G^2 \Sigma_0^2}{c_s^2}$$

$$\omega^2_{min} = \kappa^2 - \frac{\pi^2 G^2 \Sigma_0^2}{c_s^2}$$

for $\omega^2_{min} > 0$, $\kappa^2 - \frac{\pi^2 G^2 \Sigma_0^2}{c_s^2} \geq 0 \Rightarrow \kappa^2 \geq \frac{\pi^2 G^2 \Sigma_0^2}{c_s^2}$

$$\frac{c_s \kappa}{\pi G \Sigma_0} \geq 1$$

$$\boxed{Q = \frac{c_s \kappa}{\pi G \Sigma_0}}$$

$Q \geq 1$ Stable to axisymmetric collapse