Deriving Internal Energy of an ideal monoatomic gas

internal energy per unit mass: $u = \frac{U}{M} = C_v T$

monoatomic gas: $C_v = \frac{3}{2} \frac{k_B}{\mu m_p}$

$$u = \frac{3}{2} \frac{k_B T}{\mu m_p}, \quad e = \frac{3}{2} n k_B T, \quad \text{where } n = \frac{\rho}{\mu m_p}$$

$$u = \frac{1}{\gamma - 1} \frac{k_B T}{\mu m_p}, \quad \gamma = \frac{3}{2}$$

T_{virial}: halo's Circular velocity: $k_B T_{vir} = \frac{1}{2} \mu m_p V_c^2$

$$T_{vir} = \frac{\mu m_p}{2 k_B} V_c^2, \quad V_c^2 = \frac{GM}{R} \text{ at } R_{vir}$$

$$T_{vir} = \frac{\mu m_p}{2 k_B} \frac{GM}{R_{vir}}$$

Gas reheating to T_{vir}

$$\Delta u = \frac{3}{2} \frac{k_B}{\mu m_p} (T_{vir} - T_i), \text{ if } T_i \ll T_{vir}, \text{ we assume:}$$

$$\Delta u \approx \frac{3}{2} \frac{k_B T_{vir}}{\mu m_p} \Rightarrow \Delta u \approx \frac{3}{2} \frac{k_B}{\mu m_p} \left(\frac{\mu m_p}{2 k_B} V_c^2 \right) = \frac{3}{4} V_c^2$$

$$T_{vir} \sim \frac{3}{4} V_c^2$$

$$\boxed{\Delta U \approx M_g \Delta u \approx \frac{3}{2} \frac{M_g k_B T_{vir}}{\mu m_p}}$$