Deriving cumulative mass profile of an Navarro-Frenk-White Profile

Reference: universal density profile in Sims

MS-5-CGM-2025 Slides

$$\rho(r) \propto \frac{4}{\left(\frac{r}{cR_{vir}}\right)\left(1+\frac{r}{cR_{vir}}\right)^2}$$

$$x = \frac{r}{r_s} = \frac{r}{R_{vir}/c} = \frac{r}{cR_{vir}}$$

$$\rho(r) \propto \frac{1}{x(1+x)^2}$$

Enclosed mass $M(<R)$: for spherically symmetric density distribution

$$M(<r) = 4\pi \int_0^r \rho(r')r'^2 dr'$$

$$M(<r) = 4\pi \int_0^r \frac{\rho_0}{\left(\frac{r'}{cR_{vir}}\right)\left(1+\frac{r'}{cR_{vir}}\right)^2} r'^2 dr'$$

$$x' = \frac{r'}{cR_{vir}} \Rightarrow r' = (cR_{vir})x'$$

$$dr' = (cR_{vir})\,dx'$$

$$r'^2 = (cR_{vir})^2 x'^2, \quad \rho(r') = \frac{\rho_0}{x'(1+x')^2}$$

$$M(<r) = 4\pi \int_0^r \left(\frac{\rho_0}{x'(1+x')^2} \right) \left[(cR_{vir})^2 x'^2 (cR_{vir})\, dx' \right]$$

$$M(<r) = 4\pi \rho_0 (CR_{ivr})^3 \int_0^x \frac{x'^2}{x'(1+x')^2} dx', \quad x = \frac{r}{CR_{vir}}$$

$$M(<r) = 4\pi \rho_0 (cR_{vir})^3 \int_0^x \frac{x'}{(1+x')^2} dx'$$

let $u = 1+x \Rightarrow x = u-1, \; du = dx$

$$I = \int \frac{u-1}{u^2} du = \int \left(\frac{1}{u} - \frac{1}{u^2} \right) du$$

$$I = \ln u + \frac{1}{u} + C$$

$$= \ln(1+x) + \frac{1}{1+x} + C$$

$$\int_0^x \frac{x'}{(1+x')^2} dx' = \left(\ln(1+x') + \frac{1}{1+x'} \right|_0^x$$

$$= \ln(1+x) + \frac{1}{1+x} - 1$$

$$\frac{1}{1+x} - 1 = - \frac{x}{1+x}$$

$$\int_0^x \frac{x'}{(1+x')^2} dx' = \ln(1+x) - \frac{x}{1+x}$$

$$\boxed{M(<r) = 4\pi \rho_0 (CR_{vir})^3 \left(\ln(1+x) - \frac{x}{1+x} \right)}$$

where $x = \dfrac{r}{CR_{iv}}$