Research statement

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1 Overview

My research centers on both theoretical and applied probability. On the theoretical side, I focus on stochastic partial differential equations (SPDEs) on general metric measure spaces, where I have extended various comparison principles that hold for SPDEs on $\mathbb{R}$ to equations on more general spaces. I also computed the extinction probability and proved a universality-type result for the stochastic FKPP equations. Additionally, I have developed analytic tools for handling singular SPDEs on general spaces. On the applied side, I concentrate on spatial stochastic modeling of biological systems, where I have devised modeling schemes for the spatial distributions of viruses with mutations.

The following is a list that summarize my researches:

1. **Project: SPDEs on metric measure spaces**
   - I proved compact support and strong comparison principles for SPDEs on general spaces.
2. **Project: Extinction probability of stochastic FKPP and its dual particle system**
   - I derived a formula for the extinction probability of solutions to stochastic FKPP and identified the invariant distribution of Branching-Coalescing Brownian motions.
3. **Project: Universality of stochastic FKPP**
   - I established an SPDE equivalent to the Kingman coalescent universality.
4. **Project: Singular SPDEs: the sub-Gaussian case**
   - I proved Schauder and Besov-type estimates on general spaces with sub-Gaussian heat kernels.
5. **Project: Patch formation due to stochastic effects in biological modeling**
   - Developed a hybrid simulation scheme for 2D stochastic modeling in biological systems.

2 SPDEs: from theories to applications

Consider the stochastic heat equation in $\mathbb{R}^n$ with a trivial initial condition and space-time white noise $\dot{W}$:

$$\partial_t u_t = \frac{1}{2}\Delta u_t + \dot{W}, \quad u_0 \equiv 0.$$

The (mild) solution is given by $u_t(x) = \int_0^t \int_{\mathbb{R}^n} p_{t-s}(x,y)\dot{W}(dy,ds)$, where $p_t(x,y)$ is the n-dimensional heat kernel. By computing its second moment, one finds $\mathbb{E}[u_t(x)^2] < \infty$ if and only if $n < 2$. This implies that for dimensions $n \geq 2$, the solution to the above equation is not function-valued. Consequently, for $n \geq 2$, we cannot expect to solve a nonlinear heat equation in the classical sense. However, I demonstrated that function-valued solutions may still exist when $n \in (1,2)$ is non-integer and proved various comparison principles for such solutions. Additionally, I established key estimates on general metric measure spaces that carry a sub-Gaussian heat kernel, enabling us to solve certain singular SPDEs for dimensions $n \in (2,3)$.

2.1 Strong comparison principle of SPDEs on metric measure spaces[5]

I focus on spaces $(\mathbb{X}, m, d)$ that carry a heat kernel and consider non-linear equations with multiplicative Gaussian noises that is white in time and possibly colored in space:

$$\partial_t u = \mathcal{L}u + b(u) + \sigma(u)\dot{W}, \tag{1}$$

where $\mathcal{L}$ is the generator to an $\mathbb{X}$-valued Hunt process. An important question is: when does the strong comparison principle (**SCP**) hold? i.e. when is it true that if u_1, u_2 solves (1) with $u_1(0,\cdot) - u_2(0,\cdot)$ being a nontrivial nonnegative function on $\mathbb{X}$, then $\mathbb{P}(u_1(t,x) > u_2(t,x), \forall t > 0, x \in \mathbb{X}) = 1$? Prior to my results, not even the weak comparison principle was shown for SPDEs in general settings. However, I resolved this question in the following theorem:

Theorem 1 (Strong Comparison Principle)**.** *Suppose $(\mathbb{X}, d, m)$ is a locally compact, doubling length space, and $\mathcal{L}$ generates a Hölder continuous heat kernel that satisfies, for some $1 \le \alpha \le \beta$ and a positive, non-decreasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$,*

$$p_t(x,y) \asymp t^{-\alpha/\beta}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right), \quad \textit{and} \quad \Phi(a) + \Phi(b) \lesssim \Phi(a+b).$$

Then, if $\dot{W}$ is space-time white noise, the equation (1) *is well-posed, and the SCP holds provided b and σ are globally Lipschitz on $\mathbb{R}$.*

Remark 1. *Theorem 1 opens up the possibility of solving the KPZ equation on fractals, such as the Sierpiński Gasket, via a Cole-Hopf type transform.*

Example 1 ((Fractional) Parabolic Anderson Model)**.** *Let $n \ge 2$ be an integer, let $(\mathbb{X}, d, m)$ denote the n-dimensional Sierpiński Gasket, and let $\mathcal{L}$ denote the natural Laplacian on $\mathbb{X}$. Suppose $\delta \ge 0$ is sufficiently small. For all initial values $u_0 \in \mathcal{C}_b^+(\mathbb{X})$, there exists a unique solution to the parabolic Anderson model given by:*

$$\partial_t u = -(-\mathcal{L})^{1-\delta}u + u\dot{W},$$

where $-(-\mathcal{L})^1 := \mathcal{L}$. If u_0 is not identically zero, then $\mathbb{P}(u_t(x) > 0, \forall t > 0, x \in \mathbb{X}) = 1$.

2.2 Compact Support Property [5]

Previous research for SPDEs on $\mathbb{R}$, as outlined in [9], demonstrated that if the noise coefficients σ is degenerate in equation (1), then the compact support property (**CSP**), i.e. if $u_0 \in \mathcal{C}_c^+$, then $\mathbb{P}(u_t(\cdot) \in \mathcal{C}_c^+, \text{ for all } t > 0)$. In collaboration with Louis Fan and Zhenyao Sun, I have extended this result to general metric measure spaces.

Theorem 2. *Suppose $(\mathbb{X}, m, d)$ is a locally doubling space, and $\mathcal{L}$ generates an $\mathbb{X}$-valued diffusion process that admits a Hölder continuous heat kernel with a spectral dimension less than 2. Assume b and σ are continuous functions with linear growth on $\mathbb{R}$, satisfying, for some $C > 0$ and $\theta \in (0, 1/2]$,*

$$|b(u)| \le C|u|, \quad \sigma(0) = 0, \quad \textit{and} \quad \frac{1}{C}\sqrt{|u|} \le |\sigma(u)| \le C(|u| + |u|^\theta), \quad \textit{for all } u \in \mathbb{R}. \tag{2}$$

Then there exists a (probabilistic) weak solution to (1)*, and the CSP holds.*

Remark 2. *Theorem 2 enables us to discuss the limit shape of the compact support of the solution and its asymptotic speed on general spaces (e.g., regular graphs, fractals), particularly for the FKPP equation and super-Brownian motion density. On $\mathbb{R}$, the speed is detailed in [8].*

Example 2 (Super-Brownian Motion Density)**.** *Let $(\mathbb{X}, m, d)$ be the n-dimensional Sierpiński Gasket, and let $\mathcal{L}$ be as in Example 1. For any initial value $u_0 \in \mathcal{C}_c^+(\mathbb{X})$, there exists a non-negative solution u to the equation*

$$\partial_t u = \frac{\alpha}{2}\Delta u + \sqrt{u}\dot{W},$$

and the support of $u_t(\cdot)$ remains compact for all $t > 0$, almost surely.

2.3 Stochastic FKPP and interacting particle system [4]

I have computed the extinction probability of stochastic FKPP equations on metric graphs with space-time white noise: for $\beta, \gamma \geq 0$,

$$\partial_t u = \frac{\alpha}{2}\Delta u + \beta u(1-u) + \sqrt{\gamma u(1-u)}\dot{W}. \tag{3}$$

Theorem 3. *Suppose* $(\mathbb{X}, m, d)$ *is a metric graph, and* u *solves* (3) *with initial value* $u_0 \in \mathcal{B}(\mathbb{X};[0,1])$*. Then*

$$\mathbb{P}_{u_0}\left(\lim_{t\to\infty} u_t = 0\right) = \frac{\exp\left(-\frac{2\beta}{\gamma}m(u_0)\right) - \exp\left(-\frac{2\beta}{\gamma}m(\mathbb{X})\right)}{1-\exp\left(-\frac{2\beta}{\gamma}m(\mathbb{X})\right)},$$

where $m(\mathbb{X})$ *denotes the total mass of the space* $\mathbb{X}$*, and* $m(u_0) := \int_{\mathbb{X}} u_0(x)\, m(dx)$*. Here we assume* $e^{-\infty} := 0$ *in the case where* $\mathbb{X}$ *is not compact.*

Theorem 3 allows us to study the invariant measure of a system of branching Brownian motion with singular interactions. This model represents a system of independent Brownian motions that split into two at rate β. Additionally, each particle pair (i,j) coalesces into one at rate $\frac{\gamma}{2}L^{i,j}$, where $L^{i,j}$ is the intersection local time of the particle pair (i,j). This system of interacting particles is called branching coalescing Brownian motions (**BCBM**). It was shown in [1] that the discrete version of BCBM on combinatorial graphs, i.e., the branching coalescing random walk, has the Poisson random measure as its invariant distribution. The invariant distribution of BCBM is claimed to be a Poisson point process in the literature, though no proof was given.

Theorem 4. *Let* $(X_t)_{t\geq 0}$ *be a system of BCBM on a metric graph* $(\mathbb{X}, d, m)$ *with branching rate* β *and coalescing rate* $\frac{\gamma}{2}L^{i,j}$ *for each particle pair* (i,j)*. Then the unique stationary distribution of* X *is the Poisson random measure on* $(\mathbb{X}, m)$ *with intensity measure* $\frac{2\beta}{\gamma}m(dx)$*.*

The duality formula from [2] is a critical analytical tool in this research. For BCBM with index $\mathcal{I}(t)$, $(\{X_t^i\}_{i\in\mathcal{I}(t)})_{t\geq 0}$ with branching and coalescing rates β and $\frac{\gamma}{2}$ respectively, and where u is the solution to (3). If $X_0 = (x_1, \ldots, x_n) \in \mathbb{X}^n$, then we have the duality formula

$$\mathbb{E}\left[\prod_{i=1}^n u_t(x_i)\right] = \mathbb{E}_{X_0}\left[\prod_{i\in\mathcal{I}(t)} u_0(X_t^i)\right].$$

Besides the duality, I also employed techniques from functional analysis and the theory of Dirichlet forms.

2.4 Universality of stochastic FKPP [4]

I considered the universality of the FKPP equation (3) and showed that a class of Coordinated Particle systems, introduced in [3], converges to solutions of the stochastic FKPP equation (3). More precisely, let $\{\Lambda_i\}_i$ be a family of measures on $[0,1]$ such that $\int_0^1 \Lambda_i(dy) = c$. For each $n \in \mathbb{N}$, we let $\{V_i = V_i^{(n)}\}_{i=0}^{n-1}$ be a system of SDEs with jumps on a discrete circle, given by

$$\begin{aligned} \mathrm{d}V_i(t) = &\left[\alpha\frac{n^2}{2}\left(V_{i+1} - 2V_i + V_{i-1}\right) + \beta V_i(1-V_i)\right]\mathrm{d}s + \sqrt{\frac{\gamma V_i(1-V_i)}{h}}\,\mathrm{d}B_i^n(s) \\ &+ \int_{[0,t]\times[0,1]} y\left(1_{[0,V_i(s-)]}(u) - V_i(s-)\right)\mathcal{N}_i^n(\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}s), \end{aligned} \tag{4}$$

where $\{B_i^n\}$ is a system of independent Brownian motions, and $\{\mathcal{N}_i^n\}$ is a family of independent Poisson random measures on $(0,1] \times [0,1] \times \mathbb{R}_+$ with intensity measures $n\frac{\Lambda_i(\mathrm{d}y)}{y^2} \times \mathrm{d}u \times \mathrm{d}s$.

Observe that in equation (4), the jump size remains large as $n \to \infty$, so the standard method for proving tightness breaks down. However, using techniques from hydrodynamic limits, I demonstrated the universality of stochastic FKPP via the following convergence theorem:

Theorem 5. *Let $\{V^{(n)}\}_{n\in\mathbb{N}}$ be solutions to (4) on the discrete circles with fixed initial conditions bounded between 0 and 1. Then the sequence of processes is tight, and any limiting point is a solution to the stochastic FKPP equation:*

$$\partial_t u = \frac{\alpha}{2}\Delta u + \beta u(1-u) + \sqrt{(\gamma+c)u(1-u)}\,\dot{W}.$$

Remark 3. *Theorem 5 is the SPDE equivalent of Kingman coalescent universality.*

2.5 Singular SPDEs on Metric Measure Spaces

One of my ongoing projects is to consider the Φ^4 model on metric measure spaces $(\mathcal{X}, d, m)$ with sub-Gaussian heat kernel estimates. Specifically, there exist $d_w > 2$ and $d_h \geq 1$ such that the heat kernel generated by $\mathcal{L}$ satisfies, for $t \in (0,1]$ and $x, y \in \mathcal{X}$,

$$p_t(x,y) \lesssim t^{-\frac{d_h}{d_w}}\,\Phi\left(\frac{d(x,y)}{t^{\frac{1}{d_w}}}\right),$$

where $\Phi(r) = \exp(r^{\frac{d_w}{d_w-1}})$ for $r \geq 0$. It turns out there is a natural definition of Besov spaces $B^\alpha_{p,q}$ for any $p, q \in [1,\infty]$ and $\alpha \in \mathbb{R}$, in terms of the heat kernels. For $\alpha < 0$, $B^\alpha_{p,q}$ contains generalized functions that are not pointwise defined, such as space-time white noise.

In this work, I have generalized two important analytic estimates used to study Φ^4 models on two-dimensional spaces $\mathbb{R}^2$: the paraproduct estimates and Schauder estimates.

Theorem 6 (Paraproduct Estimates)**.** *Suppose $(\mathcal{X}, d, m)$ is a metric measure space and $\mathcal{L}$ is a self-adjoint operator that generates a heat kernel satisfying the above sub-Gaussian estimates and is Hölder continuous in $(x,y) \in \mathcal{X}^2$. Then, for some $\theta > 0$ and $0 < \alpha + \beta < \theta$, if $f \in B^\alpha_{\infty,\infty}$ and $g \in B^\beta_{\infty,\infty}$, there exists a decomposition of $f \cdot g$ such that*

$$\|f\cdot g\|_{B^{\alpha\wedge\beta}} \lesssim \|f\|_{B^\alpha} \cdot \|g\|_{B^\beta},$$

where $\|\cdot\|_{B^\alpha}$ denotes the corresponding norm in the Besov space $B^\alpha_{\infty,\infty}$.

Let $\alpha \in \mathbb{R}$, and define for $v : \mathbb{R}_+ \to B^\alpha$ the operator $\mathcal{R}(v)_t := \int_0^t P_{t-s} v(s)\, ds$ for $t \geq 0$.

Theorem 7 (Schauder-Type Estimates)**.** *For any regularity index $\beta \in \mathbb{R}$, any positive time horizon T, and $v \in \mathcal{C}([0,T]; B^\beta)$, we have $\mathcal{R}(v) \in \mathcal{C}([0,T]; B^{\beta+d_w})$.*

Theorems 6 and 7 will enable us to study the Φ^4 equation on non-smooth spaces for which there exists an $\mathcal{L}$ generating a sub-Gaussian heat kernel and a space-time white noise W:

$$\partial_t u = \mathcal{L}u - u^3 + \dot{W}.$$

3 Spatial stochastic modeling in biological systems

My research on stochastic modeling mainly focuses on modeling the spatial phenomena of biological systems. This research direction is relatively new, and there are few models that consider the spatial behavior of biological systems in one- or two-dimensional domains.

Consider viruses and virus-like defective interfering particles (DIPs) growing on a petri dish. The spatial distribution of viruses and DIPs is never a perfect disk; instead, it appears rather patchy. However, conventional PDE modeling for spatial dynamics cannot capture this feature. On the other hand, modeling complex biological systems via Markov chains is computationally expensive. Motivated by this, in collaboration with Louis Fan, Wing-Cheong Lo, and Qiantong Liang in [7], we developed a mathematical model and a compartmental-based hybrid method that combines both deterministic and stochastic models. This approach is computationally efficient and effectively captures the patchiness of the spatial distributions of viruses and DIPs.

I am also working on a similar model to [7] with Louis Fan and John Yin. In this project, I consider viruses growing on a two-dimensional domain with mutations. It was shown experimentally in [10] that certain mutants with shorter genomes gained an advantage in both single-step growth and spatial spreading. I have developed a modeling scheme that allows us to sample the random emergence of mutations with growth advantages. I am also developing a mathematical model that captures the spreading speed of viruses and mutants.

4 Future Research Interests and Goals

In my future research endeavors, I aim to expand and diversify the scope of my work on SPDEs, stochastic modeling, and analysis as in [6]. The key projects I plan to undertake include:

1. In collaboration with Louis Fan and Adrián González-Casanova, I am focusing on applying the duality techniques discussed earlier to a broader array of SPDEs and studying the long-term behavior of both solutions to SPDEs and dual particle systems.
2. I plan to investigate wave propagation speed in solutions to the stochastic FKPP on d-regular metric graphs. This study will to provide valuable insights into wave dynamics in non-linear heat equations.
3. Utilizing the analytic estimates I have established, I intend to investigate singular SPDEs on general metric measure spaces, starting with the Φ^4 model.
4. For spatial modeling of biological systems, I plan to continue working with Louis Fan and John Yin to quantitatively capture the spatial features of interactions between viruses and mutants.

These research directions are not only a natural progression of my current work but also open avenues for exploring new and challenging problems in the realms of SPDEs and stochastic modeling in mathematical biology.

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