Math 327 Homework 3

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1. Give examples of sequences with the following properties. If there is no such sequence, explain why not, citing a reference.

   (a) Non-increasing, convergent.

   $a_n = \sum 1 - \frac{1}{n^2}$

   (b) Not monotone, convergent.

   $a_n = 1$

   (c) Not monotone, divergent.

   $a_n = (-1)^n$

   (d) Unbounded, monotone.

   $a_n = n$

   (e) Bounded, increasing.

   $a_n = \sum \frac{1}{n^2}$

   (f) Unbounded, convergent.

   It is not possible. *Monotone Convergence Theorem* tells that a monotone sequence converges if and only if it is bounded. So if a sequence is unbounded, it could not be convergent.

   (g) Monotone, divergent.

   $a_n = n$

   (h) Decreasing, unbounded.

   $a_n = \sum -\frac{1}{n}$

2. Prove that the set of prime numbers is closed in $\mathbb{R}$. A prime number is a positive integer greater than 1 whose only positive integer factors are 1 and itself.

- Let $S = \{\text{all prime numbers}\}$. Assume $(a_n)_{n\in\mathbb{N}}$ in S, and $a_n \to a$, $a \in \mathbb{R}$. Let $\varepsilon = \frac{1}{2}$, then for all $n \geq N$, $|a_n - a| < \frac{1}{2}$. So

$$\begin{aligned}|a_{n+1} - a_n| =& |a_{n+1} - a + a - a_n| \\ &\leq |a_{n+1} - a| + |a_n - a| \text{ by Triangular inequality} \\ &< \frac{1}{2} + \frac{1}{2} = 1 \text{ for any } n \geq N\end{aligned}$$

  Since $a_{n+1} - a_n \in S$, then $a_{n+1} - a_n = 0$

  So $a_n = a_N$ for all $n \geq N$.

  Thus for any $\varepsilon > 0$, there holds $|a_n - a_N| = 0 < \varepsilon$

  Therefore, $a = a_N \in S$ Q.E.D.

3. Prove the Sandwich Theorem: Let (a_n), (b_n) and (c_n) be sequences such that $a_n \leq b_n \leq c_n$ for all n. Assume further that $lim_{n\to\infty} a_n = lim_{n\to\infty} c_n = m$. Then, $lim_{n\to\infty} b_n = m$.

   - Proof: Given $a_n \leq b_n \leq c_n$ for all n.

     Since a_n, c_n converges to m, thus for $n \geq N$, let $|a_n - m| < \varepsilon_1$, and $|c_n - m|, \varepsilon_2$ with $\varepsilon_1,\ \varepsilon_2 > 0$, then we have $-\varepsilon_1 < a_n - m < \varepsilon_1$ and $-\varepsilon_2 < c_n - m < \varepsilon_2$. Then,

$$\begin{aligned}-\varepsilon_1 < a_n - m \leq b_n - m \leq c_n - m < \varepsilon_2 \\ \Rightarrow -\varepsilon_1 < b_n - m < \varepsilon_2 \\ \Rightarrow |b_n - m| < min\{\varepsilon_1, \varepsilon_2\}\end{aligned}$$

     Since either ε_1 or $\varepsilon_2 > 0$, then b_n also converges to m. Q.E.D.

4. Suppose that the sequence (a_n) is monotone. Prove that (a_n) converges if and only if (a_n^2) converges. Show by an example that his result does not hold without the monotone assumption.

   - Example: $(a_n) = (-1)^n$. In this case, (a_n) is not monotone and divergent but (a_n^2) converges.

   - Proof: ($\Rightarrow$) Assume (a_n) is monotone and converges. Then *Monotone Convergence Theorem* tells that (a_n) is bounded. So $\left|(a_n)\right| \leq \sqrt{\varepsilon}\forall n$, with $\varepsilon > 0$. Then $(a_n^2) = \left|(a_n^2)\right| \leq \varepsilon\forall n$. Thus (a_n^2) converges.

     ($\Leftarrow$) Assume (a_n^2) converges and (a_n) is monotone. Then (a_n^2) is bounded since $\left|(a_n^2)\right| = (a_n^2) < \varepsilon$.

Let $\left|(a_n^2)\right| \leq M^2 \forall n$, with $M \in \mathbb{R}$. Then $-M \leq (a_n) \leq M$. So (a_n) is bounded. *Monotone Convergence Theorem* tells (a_n) converges.

Q.E.D.

5. If (a_n) and (b_n) are monotone sequences, is $(a_n + b_n)$ monotone? Is $(a_n b_n)$ monotone? Prove or give a counterexample.

   - $(a_n + b_n)$ can be non-monotone. Counterexample: $(a_n)_{n\in\mathbb{N}} = 1, 2, 3, 4, ... = n$, $(b_n)_{n\in\mathbb{N}} = -1, -1, -3, -3, -5, -5, ...$. Both (a_n) and (b_n) are monotone, but $(a_n + b_n) = 0, 1, 0, 1, 0, 1, ...$, which is not monotone.
   - $(a_n b_n)$ can be non-monotone. Counterexample: $(a_n)_{n\in\mathbb{N}} = 2, 4, 8, 16, 32, ... = 2^n$, $(b_n)_{n\in\mathbb{N}} = \frac{1}{4}, \frac{1}{16}, \frac{1}{16}, \frac{1}{64}, \frac{1}{64}, ...$. Both (a_n) and (b_n) are monotone, but $(a_n \cdot b_n) = \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, ...$, which is not monotone.

6. Suppose that $(a_n) \to a$. Use induction to prove that for any $m \in \mathbb{N}$, the sequence (a_n^m) converges to a^m.

   - Proof by induction on $m \in \mathbb{N}$. Assume $a_n \to a$
     - Base Case $m = 1$

       $$a_n \to a \text{ (by assumption)}$$

     - Inductive Step

       Assume $a_n^m \to a^m$. Prove $a_n^{m+1} \to a^{m+1}$.

       Let $\varepsilon > 0$ be given. Since (a_n) converges, it is bounded. Then there exists $M > 0$ such that $|a_n| \leq M \ \forall n \in \mathbb{N}$.

       Since $a_n \to a$ by assumption, there exists N_0 such that $|a_n - a| < \frac{\varepsilon}{2M}$ when $n \geq N_0$.

       Since $a_n^m \to a^m$ by inductive hypothesis, there exists N_m such that $|a_n^m - a^m| < \frac{\varepsilon}{2(2+|a^m|)}$ when $n \geq N_m$

Define $N = max\{N_0, N_m\}$. Then

$$\begin{aligned}
\left|a_n^{m+1} - a^{m+1}\right| &= |a_n^m \cdot a_n - a^m \cdot a| \\
&= |a_n^m a_n - a^m a_n + a^m a_n - a^m a| \\
&\leq |a_n^m a_n - a^m a_n| + |a^m a_n - a^m a| \text{ (Triangular Inequality)} \\
&= \left|a_n(a_n^m - a^m)\right| + \left|a^m(a_n - a)\right| \\
&= |a_n||a_n^m - a^m| + |a^m||a_n - a| \\
&< M\frac{\varepsilon}{2M} + |a^m|\frac{\varepsilon}{2(2 + |a^m|)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ since } \frac{|a^m|}{2 + |a^m|} < 1 \\
&= \varepsilon
\end{aligned}$$

So $\left|a_n^{m+1} - a^{m+1}\right| < \varepsilon$. Therefore, $a_n^{m+1} \to a^{m+1}$.