1.3 Inequalities and Identities

Define $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Lemma: Let $c \in \mathbb{R}$, and $d \geq 0$. Then

$$|c| \leq d \quad \text{if and only if} \quad -d \leq c \leq d.$$

Proof: For any $x \in \mathbb{R}$, $x \leq |x|$
because of $x \geq 0$, $x = |x|$ and if $x < 0$, then $-x > 0$
and $x < 0 < -x = |x|$.

($\Rightarrow$) Assume $|c| \leq d$.
Then $c \leq |c| \leq d$ so $c \leq d$
and $-c \leq |-c| = |c| \leq d$ so $c \geq -d$. $-c \leq d$
Therefore, $-d \leq c \leq d$.

($\Leftarrow$) Assume $-d \leq c \leq d$. Since $|c| = c$ or $|c| = -c$,
Then $-d \leq -c \leq d$.
$-d \leq |c| \leq d$ so $|c| \leq d$.

Also, since $x \leq |x|$ with $c = x$ and $d = |x|$ we get

$$-|x| \leq x \leq |x|.$$

Thm 1.11 The Triangle Inequality

For any $a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$.

Proof: Since $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$
we get $-(|a|+|b|) = -|a| - |b|$
$\leq a + b$ by the inequalities above
$\leq |a| + |b|$ by the inequalities above

So by the lemma with $d = |a| + |b|$ and $c = a + b$ we get

$$|a+b| \leq |a| + |b|.$$

Proposition: For any $n \geq 2, n \in \mathbb{N}$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

Proof: (1) Using algebraic manipulation.

$$(a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) = (a-b)\sum_{j=0}^{n-1} a^{n-1-j}b^j$$

$$= a\sum_{j=0}^{n-1} a^{n-1-j}b^j - b\sum_{j=0}^{n-1} a^{n-1-j}b^j$$

$$= \sum_{j=0}^{n-1} a^{n-j}b^j - \sum_{j=0}^{n-1} a^{n-1-j}b^{j+1}$$

$$= \sum_{j=0}^{n-1} a^{n-j}b^j - \sum_{i=1}^{n} a^{n-i}b^i \quad , \text{ with } i = j+1$$

$$= a^n + \sum_{j=1}^{n-1} a^{n-j}b^j - \sum_{i=1}^{n-1} a^{n-i}b^i - b^n$$

$$= a^n - b^n.$$

The Geometric Sum Formula

For any $r \neq 0$

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$$

Proof: 1) Let $a = 1$, $b = r$ in the proposition above + then divide both sides by $1-r$.

2) Let $S = 1 + r + r^2 + \cdots + r^{n-1}$ then $rS = r + r^2 + \cdots + r^{n-1} + r^n$ so $rS - S = r^n - 1$ then $S = \dfrac{r^n - 1}{r-1} = \dfrac{1-r^n}{1-r}$

3) By induction on n.

when $n=1$ $\quad \frac{1-r}{1-r}=1$

Assume $1+r+\cdots+r^{n-1}=\frac{1-r^n}{1-r}$

Then, $1+r+\cdots+r^n = 1+r+\cdots+r^{n-1}+r^n$

$= \frac{1-r^n}{1-r}+r^n$ by the induction hyp.

$= \frac{1-r^n+r^n-r^{n+1}}{1-r}$

$= \frac{1-r^{n+1}}{1-r}.$

Therefore, $1+r+\cdots+r^{n-1}=\frac{1-r^n}{1-r}$ for all $n\in\mathbb{N}$.

or $\quad \frac{1-r^n}{1-r}=\sum_{j=0}^{n-1} r^j.$

The Binomial Formula

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \quad \text{where} \quad \binom{n}{k}=\frac{n!}{k!(n-k)!}$$

Proof: Exercise (by induction on n)

First, prove that $\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}$.

Chapter 2 – Convergent Sequences

2.1 The Convergence of Sequences

A <u>sequence</u> is a function $f: \mathbb{N} \longrightarrow \mathbb{R}$.
We usually write a_n, b_n, c_n, etc. instead of $f(n)$.
$\{a_n\}$ or $(a_n)_{n\in\mathbb{N}}$

example: 1) $a_n = n$ $\quad 1, 2, 3, 4, \cdots$

2) $b_n = \frac{1}{n}$ $\quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

3) $c_n = 1 - \frac{(-1)^n}{n}$ $\quad 2, \frac{1}{2}, \frac{4}{3}, \frac{3}{4}, \frac{6}{5}, \frac{5}{6}, \frac{8}{7}, \frac{7}{8}, \cdots$

For all the examples above, there is an <u>explicit formula</u> for the terms. We can also define a sequence <u>recursively</u>

<u>example 4</u> $a_1 = 1 \quad a_{n+1} = a_n + 3$ for every $n \geq 1$

$1, 4, 7, 10, \cdots$

Sometimes we can come up with an explicit formula for the sequence defined recursively:

guess: $a_n = 3n - 2$

check by induction (exercise)

<u>example 5</u> $a_1 = 1 \quad a_2 = 1 \quad a_{n+1} = a_n + a_{n-1}$ for all $n \geq 1$.

Sometimes guessing the formula is not that easy.

example 2.5: $a_n = \sum_{k=1}^{n} \frac{1}{k}$ (or recursively $a_1 = 1$, $a_{n+1} = a_n + \frac{1}{n+1}$)

$a_1 = 1$ $a_2 = 1 + \frac{1}{2} = \frac{3}{2}$ $a_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{10}{6}$

$a_4 = a_3 + \frac{1}{4} = \frac{23}{15}$. Formula for a_n in terms of n?

example 2.2 $a_1 = 1$ $a_{n+1} = \begin{cases} a_n + \frac{1}{n} & \text{if } a_n^2 \le 2 \\ a_n - \frac{1}{n} & \text{if } a_n^2 > 2 \end{cases}$

$a_1^2 < 2$ so $a_2 = a_1 + \frac{1}{1} = 2$

$a_2^2 > 2$ so $a_3 = a_2 - \frac{1}{2} = \frac{3}{2}$

$a_3^2 = \frac{9}{4} > 2$ so $a_4 = a_3 - \frac{1}{3} = \frac{3}{2} - \frac{1}{3} = \frac{7}{6}$

$a_4^2 = \frac{49}{36} < 2$ so $a_5 = a_4 + \frac{1}{4} = \frac{7}{6} + \frac{1}{4} = \frac{34}{24} = \frac{17}{12}$

Formula for a_n???

Observations Examples 1, 4, 5 seem to get larger and larger, exceeding any value.
In example 5, $a_n \geq n$ for any $n \geq 5$ (proof?)
Example 2.5 has increasing terms. Do they get very large? Or do they approach some large number?
In example 2, terms get closer + closer to 0
In example 3, terms get closer + closer to 1
Example 2.2 is hard to work with.

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A sequence $(a_n)_{n\in\mathbb{N}}$ <u>converges</u> to a if the terms a_n get arbitrarily close to a when n is sufficiently large, i.e.

For any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$\text{when } n \geq N, \quad |a_n - a| < \varepsilon.$$

If there is no such number a, the sequence $(a_n)_{n\in\mathbb{N}}$ <u>diverges</u>. The number a is called the <u>limit of the sequence</u>. We write <u>$a_n \to a$</u> or

$\lim_{n\to\infty} a_n = n$.

example: The sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ converges to 0.

Given $\varepsilon > 0$, by the Archimedean Property, there is an N with $\frac{1}{N} < \varepsilon$.

If $n \geq N$, then $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$ so $|\frac{1}{n} - 0| < \varepsilon$.

example: The sequence $(-1)^n$ does not converge:

For any a, let $\varepsilon = \frac{1}{2}$, for any $N \in \mathbb{N}$

If $|a_N - a| < 1/2$ and $|a_{N+1} - a| < 1/2$

then
$$\begin{aligned} |a_N - a_{N+1}| &= |a_N - a + a - a_{N+1}| \\ &\leq |a_N - a| + |a - a_{N+1}| \\ &< \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

but $|a_N - a_{N+1}| = |(-1)^N - (-1)^{N+1}| = |(-1)^N (1 - (-1))| = 2$

which is a contradiction.

(Thm 2.18)

Proposition: If $a_n \to a$ then (a_n) is bounded

i.e. there exists an M such that

$$|a_n| \leq M \text{ for all } n \in \mathbb{N}.$$

Proof: Assume $a_n \to a$.

Let $\varepsilon = 1$. There is an N such that for all $n \geq N$,

$$|a_n - a| < 1.$$

So,
$$\begin{aligned} |a_n| &= |a_n - a + a| \\ &\leq |a_n - a| + |a| \text{ by the } \Delta \text{ inequality} \\ &< 1 + |a| \end{aligned}$$

Let $M = \max\{|a_1|, \ldots, |a_{N-1}|, 1+|a|\}$

Then for any n,

If $n < N$, $|a_n| \leq M$ by def. of M

If $n \geq N$, $|a_n| < 1+|a|$ by above

$\leq M$ by def. of M.

In any case, $|a_n| \leq M$.

Theorem (Limit Properties) (Thms 2.11, 2.13, 2.15, Lemma 2.11)

Assume $a_n \to a$ and $b_n \to b$. Then

(i) $a_n + b_n \to a+b$

(ii) If $\alpha \in \mathbb{R}$, $\alpha a_n \to \alpha a$.

(iii) $a_n b_n \to ab$

(iv) If $b_n \neq 0$ for any n and $b \neq 0$, then

$$\frac{a_n}{b_n} \to \frac{a}{b}.$$

In particular, $\frac{1}{b_n} \to \frac{1}{b}$.

Proof: (i) Assume $a_n \to a$ and $b_n \to b$.

Let $\varepsilon > 0$ be given.

Since $a_n \to a$, there exists N_1 such that for all $n \geq N_1$ $|a_n - a| < \varepsilon/2$.

Similarly, $b_n \to b$ so there exists N_2 such that for all $n \geq N_2$, $|b_n - b| < \varepsilon/2$.

Let $N = \max\{N_1, N_2\}$

Assume $n \geq N$. Then

$$\begin{aligned} |(a_n+b_n)-(a+b)| &= |a_n - a + b_n - b| \\ &\leq |a_n - a| + |b_n - b| \quad \text{by the } \Delta \text{ inequality} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{since } n \geq N_1 \text{ and } n \geq N_2 \\ &= \varepsilon. \end{aligned}$$

Therefore, $a_n + b_n \to a + b$.

(ii) Let $\alpha \in \mathbb{R}$, assume $a_n \to a$.

If $\alpha = 0$, $\alpha a_n = 0$ for any n and $0 \to 0$
(for any $\varepsilon > 0$ take $N = 1$ since $|\alpha a_n - \alpha a| = |0 - 0| = 0 < \varepsilon$.

If $\alpha \neq 0$, given $\varepsilon > 0$, since $a_n \to a$
there exists N such that for any $n \geq N$ $|a_n - a| < \frac{\varepsilon}{|\alpha|}$.
Use the same N, let $n \geq N$

$$
\begin{aligned}
|\alpha a_n - \alpha a| &= |\alpha||a_n - a| \\
&< |\alpha| \frac{\varepsilon}{|\alpha|} \\
&= \varepsilon.
\end{aligned}
$$

So, $\alpha a_n \to \alpha a$.

(iii) This proof is different than the one in the text (Thm 2.13) using the Proposition instead of Lemma 2.12.

Assume $a_n \to a$ and $b_n \to b$.
Let $\varepsilon > 0$ be given
By the Proposition, there is an $M > 0$ such that $|a_n| \leq M$ for all n.
Since $a_n \to a$, there is an N_1, $n \geq N_1$ implies $|a_n - a| < \frac{\varepsilon}{(1+|b|)2}$
Since $b_n \to b$, there is an N_2, $n \geq N_2$ implies $|b_n - b| < \frac{\varepsilon}{2M}$
Let $N = \max\{N_1, N_2\}$. Assume $n \geq N$

$$
\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\
&= |a_n(b_n - b) + b(a_n - a)| \\
&\leq |a_n(b_n - b)| + |b(a_n - a)| \quad \text{by } \Delta \text{ inequality} \\
&= |a_n||b_n - b| + |b||a_n - a| \\
&\leq M|b_n - b| + |b||a_n - a| \quad \text{since } |a_n| \leq M. \\
&< M \cdot \frac{\varepsilon}{2M} + |b| \cdot \frac{\varepsilon}{(1+|b|)2} \quad \text{since } n \geq N \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{since } \frac{|b|}{1+|b|} < 1 \\
&= \varepsilon.
\end{aligned}
$$

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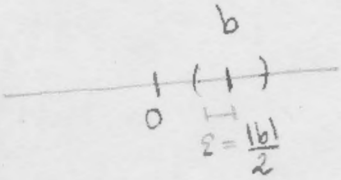
Therefore, $a_n b_n \to ab$.

(iv) Assume $a_n \to a$, $b_n \to b$, $b_n \neq 0$, $b \neq 0$.

We'll prove $\frac{1}{b_n} \to \frac{1}{b}$.

Then the result will follow from (iii)

Since $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$.



First, we show that there is an $m > 0$ such that $|b_n| > m$ for all n.

Let $\varepsilon = \frac{|b|}{2}$. Then, there is an N such that

for all $n \geq N$, $|b_n - b| < \varepsilon = \frac{|b|}{2}$

Then, $|b| - |b_n| \leq \big||b| - |b_n|\big|$
$\leq |b - b_n|$ by HW2 problem
$< \frac{|b|}{2}$

so $|b| - \frac{|b|}{2} < |b_n|$

i.e. $\frac{|b|}{2} < |b_n|$ when $n \geq N$.

Let $m = \min\{|b_1|, \dots, |b_{N-1}|, \frac{|b|}{2}\} \neq 0$.

Then, for any n, $|b_n| \geq m$.

Now, given $\varepsilon > 0$, there is an N such that when $n \geq N$, $|b_n - b| < |b| m \varepsilon$

Then, for $n \geq N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b b_n}\right|$$

$$\leq \frac{|b - b_n|}{|b| m} \quad \text{since } |b_n| \geq m.$$

$$< \frac{\varepsilon |b| m}{|b| m} \quad \text{since } n \geq N$$

$$= \varepsilon.$$

Therefore $\frac{1}{b_n} \to \frac{1}{b}$ and using (iii)

$$\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} \to a \cdot \frac{1}{b} = \frac{a}{b},$$

finishing the proof of (IV) and hence of the theorem.

example: Let $a_n = \frac{n^2 + 2n - 1}{3n^2 + 2}$.

Then, $a_n \to \frac{1}{3}$.

(algebra trick from Math 124)

Proof: $a_n = \frac{n^2 + 2n - 1}{3n^2 + 2} = \frac{\frac{n^2 + 2n - 1}{n^2}}{\frac{3n^2 + 2}{n^2}}$

$$= \frac{1 + \frac{2}{n} - \frac{1}{n^2}}{3 + \frac{2}{n^2}}$$

Since $\frac{1}{n} \to 0$, $\frac{2}{n} \to 0$ from (ii)

$\frac{-1}{n^2} = (-1) \cdot \frac{1}{n} \cdot \frac{1}{n} \to -1 \cdot 0 \cdot 0$ from (ii) and (iii)

$1 \to 1$ (constant sequence)

So $1 + \frac{2}{n} - \frac{1}{n^2} \to 1 + 0 + 0$ from (i)

Similary, $\frac{2}{n^2} = -2\left(\frac{-1}{n^2}\right) \to -2 \cdot 0$ from (ii) + result above

$3 \to 3$ (constant sequence)

So $3 + \frac{2}{n^2} \to 3 + 0$ from (i)

Now, use (iv) since $3 \neq 0$ and $3 + \frac{2}{n^2} \neq 0$ for any n

$$\frac{1 + \frac{2}{n} - \frac{1}{n^2}}{3 + \frac{2}{n^2}} \to \frac{1}{3}.$$

We say $a_n \to \infty$ if
For every $c > 0$, there exists an N such that
for $n \geq N$, $a_n > c$.

examples: 1) The Archimedian Property says that $n \to \infty$.

2) $5n^2 - 3n + 1 \to \infty$.

Let $c > 0$ be given
By the Archimedian property, find N
with $N > c$.
Let $n \geq N$
Then,

$$\begin{aligned} 5n^2 - 3n + 1 &> 5n^2 - 3n \\ &= (5n-3)n \\ &> n \quad \text{since } 5n-3 \geq 5-3 \geq 2 > 1 \\ &\geq N \\ &> c \quad \text{by choice of } N. \end{aligned}$$

So, $5n^2 - 3n + 1 \to \infty$.