Math 327 Homework 1

Chongyi Xu

April 4, 2017

1. Prove that the product of a nonzero rational number and an irrational number is irrational. How about the product of two irrational numbers?
   (a) Let $a = \frac{m}{n}, b = \sqrt{2}$, and $m, n \in \mathbb{N}$, then $a \in \mathbb{Q}, b \in \mathbb{R}\backslash\mathbb{Q}$. $\Rightarrow a\dot{b} = \sqrt{2}\frac{m}{n}$, which is an irrational number. Q.E.D.
   (b) Let $a = \sqrt{\frac{m}{n}}, b = \sqrt{\frac{p}{q}}, m, n, p, q \in \mathbb{N}$, then $a \in \mathbb{R}\backslash\mathbb{Q}, b \in \mathbb{R}\backslash\mathbb{Q}$

$$\rightarrow if \sqrt{\frac{m}{n}} = \sqrt{\frac{p}{q}}, a \cdot b = \frac{m}{n} \in \mathbb{Q}$$
$$\rightarrow if \sqrt{\frac{m}{n}} \neq \sqrt{\frac{p}{q}}, a \cdot b = \sqrt{\frac{m \cdot p}{n \cdot q}} \in \mathbb{R}\backslash\mathbb{Q}$$

2. Find the inf (greatest lower bound), sup (least upper bound), max and min, if they exist, for the following sets. Prove your answers in (d) and (e). To prove m is the inf of a set, you have to show it is a lower bound and that no number bigger than m is a lower bound. To prove M is the sup of a set, you have to show it is an upper bound and that no number smaller than M is an upper bound.
   (a) $S = \{1, 3, 5, 7, 9\}$
   $infS = minS = 1,\ supS = maxS = 9$
   (b) $S = (3, \pi]$
   $infS = 3,\ supS = maxS = \pi$
   min does not exist.
   (c) $S = \{q \in \mathbb{Q} : 3 < q \leq \pi\}$
   $supS = \pi$
   min, max, and inf do not exist.
   (d) $S = \{\frac{1}{a} : a \in \mathbb{Z},\ a \neq 0\}$
   $infS = minS = -1,\ supS = maxS = 1$
   Proof:
   - infimum
     - $x \geq -1 \ \forall\ x \in S$. Thus -1 is a lower bound.
     - Assume -1 is not the infimum. Then $\exists r > 0,\ -1 + r = infS$

$$0 < \frac{r}{2} < r$$
$$-1 < -1 + \frac{r}{2} < -1 + r$$

since $-1 + \frac{r}{2} \in S, -1 + r$ is not the lower bound. Contradicts.
$\Rightarrow$ -1 is the infimum. And Since $-1 \in S$, minS = infS

- supremum
  - $x \leq 1 \; \forall \; x \in S$. Thus 1 is a upper bound.
  - Assume 1 is not the supremum. Then $\exists r > 0, \; 1 - r = supS$

$$0 < \frac{r}{2} < r$$
$$0 > -\frac{r}{2} > -r$$
$$1 - r < 1 - \frac{r}{2} < 1$$

  since $1 - \frac{r}{2} \in S, 1 - r$ is not the upper bound. Contradicts.
  $\Rightarrow$ 1 is the supremum. And Since $1 \in S$, maxS = supS

(e) $S = \{\frac{n+2}{2n+5} : n \in \mathbb{N}\}$
$infS = minS = \frac{3}{7}, supS = \frac{1}{2}$
max does not exist.
Proof:

- infimum
  - Since $\frac{d}{dn}\frac{n+2}{2n+5}$ is positive, the value of $\frac{n+2}{2n+5}$ will keep growing as n increases. Therefore, it has its lower bound when n is smallest, in the other word, when $\; n = 1$. When$n = 1, \; x \geq \frac{3}{7} \; for \forall x \in S$. Thus $\frac{3}{7}$ is the a lower bound.
  - Assume $\frac{3}{7}$ is not the infimum. Then $\exists r > 0, \; r + \frac{3}{7} = infS$

$$0 < \frac{r}{2} < r$$
$$\frac{3}{7} < \frac{3}{7} + \frac{r}{2} < \frac{3}{7} + r$$

  Since $\frac{3}{7} + \frac{r}{2} \in S, \; \frac{3}{7} + r$ is not a lower bound. Contradicts.
  $\Rightarrow \frac{3}{7}$ is the infimum. And since $\frac{3}{7} \in S, minS = infS$.
- supremum
  - Since L'Hospital's Rule tells $\lim_{x \to +\infty} \frac{n+2}{2n+5} = \frac{1}{2}$, it has its upper bound $\frac{1}{2}$ and has no maximum.
  - Assume $\frac{1}{2}$ is not the supremum. Then $\exists r > 0, \; \frac{1}{2} - r = supS$

$$0 < \frac{r}{2} < r$$
$$0 > -\frac{r}{2} > -r$$
$$\frac{1}{2} - r < \frac{1}{2} - \frac{r}{2} < \frac{1}{2}$$

  Since $\frac{1}{2} - \frac{r}{2} \in S, \; \frac{1}{2} - r$ is not the uppr bound. Contradicts.
  $\Rightarrow \frac{1}{2}$ is the supremum.

3. Suppose A and B are non-empty sets of real numbers that are both bounded above.

   (a) Prove that if $A \subset B$, then $supA \leq supB$.
   Proof by contradiction: Assume $supA > supB$ (in order to reach contradiction),

$$\exists C = \{k \in \mathbb{R} : k \in [supB, supA]\}$$

   And by the definition of supremum, $C \subset A \; but \; C \not\subset B$. Contradicts the condition $A \subset B$.
   Therefore, if $A \subset B$, then $supA \leq supB$. Q.E.D.

(b) Prove that $sup(A \cup B) = max\{supA, supB\}$
Proof:

- Show it is an upper bound.
  Assume $supA \geq supB$, then $max\{supA, supB\} = supA$(Otherwise, switch $supA$ with $supB$)
  Let $x \in A \cup B$

$$\rightarrow if\ \ x \in A,\ \ supA \geq x$$
$$\rightarrow if\ \ x \in B,\ \ supB \geq x,\ \ but\ supA \geq supB,\ \ supA \geq x$$

  Thus in any cases, $supA \geq x$. $supA$ is an upper bound.
- Show no smaller number works.
  Let $k < supA$, then $\exists x \in A$ with $k < x \leq supA$
  Since $A \subseteq A \cup B, x \in A \cup B$
  So k is not an upper bound for $A \cup B$
  Therefore, $sup(A \cup B) = max\{supA, supB\}$ Q.E.D

(c) Prove that if $A \cap B \neq emptyset$, then $sup(A \cap B) \leq min\{supA, supB\}$. Give an example to show that equality need not hold.

- Proof:
  - Show it is an upper bound.
    Assume $supA \leq supB$, then $mim\{supA, supB\} = supA$(Otherwise, switch $supA$ with $supB$)
    Assume $A \cap B \neq emptyset$. Let $x \in A \cap B$. Thus $x \in A$
    From the definition of supremum, $x \leq supA$. $supA$ is an upper bound. Q.E.D
- example:
  Two sets. $A = \{1, 3, 4, 5, 6\},\ B = \{2, 3, 5, 7\}$
  Then $A \cap B = \{3, 5\}$. $supA = 6,\ supB = 7 \Rightarrow\ min(supA, supB) = 6$. But $sup(A \cap B) = 5$. The equality does not hold.