Math/Stat 394B **Solution to Final Exam** **December 10, 2007**

8:30-10:20 am.

Your Name ______________________________

Student Number ______________________________

Signature ______________________________

1. Print your name, student ID number and section number on this page. Do **NOT** separate the pages of the exam.
2. **WRITE LEGIBLLY AND SHOW ALL OF YOUR WORK.** Partial credit will only be given where you have made it clear that you understand part of the solution. Answers without justification may not receive full credit. **Place a box around your final answer to each equation.**
3. If you use a trial and error (or guess and check) method when an algebraic method is available, you will not receive full credits.
4. You may use two sheets (two-sided) of handwritten notes and a basic scientific calculator. **Do not share notes and no graphing calculator is allowed.** If you need more space to solve a problem, use the back of the page preceding that problem.
5. Read each question carefully. Work the problems in an order that will maximize your score. Good Luck!

Score

1.	(8)	
2.	(10)	
3.	(10)	
4.	(10)	
5.	(12)	
Total	(50)	

1. (8 points) If 8 rooks are randomly placed on a 8×8 chessboard, what is the probability that none of the rooks can capture any of the others. That is, find the probability that no row or file contains more than one rook.

Solution.

$$p = \frac{8^2 7^2 6^2 5^2 4^2 3^2 2^2 1^2}{64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{31 \cdot 61 \cdot 59 \cdot 29 \cdot 57}.$$

$\square$

2. (10 points) Assume that the customers arrive at a store at random times that follow a Poisson process with $\lambda = 2$ per minute. Given that the first arrival comes within the first minute after the store is open, find the conditional probability that there are exactly 6 arrivals during the first three minutes after the store opened.

Solution. Let $N(k)$ denote number of arrivals during the time interval $[0, k]$ (in minutes). Then

$$\begin{aligned}
p &= \mathbf{P}(N(3) = 6 | N(1) \geq 1) = \frac{\mathbf{P}(N(3) = 6 \text{ and } N(1) \geq 1)}{\mathbf{P}(N(1) \geq 1)} \\
&= \frac{\mathbf{P}(N(3) = 6) - \mathbf{P}(N(3) = 6 \text{ and } N(1) = 0)}{1 - \mathbf{P}(N(1) = 0)} \\
&= \frac{\frac{6^6}{6!} e^{-6} - e^{-2} \cdot \frac{4^6}{6!} e^{-4}}{1 - e^{-2}} \\
&= \frac{46}{9e^4(e^2 - 1)}.
\end{aligned}$$

$\square$

3. (10 points) 10 men went to a party, each wearing a hat. They put their hats on a table when they got there. After the party, each picks up a hat randomly from the hats that are on the table, one by one. Let X be the number of people who got his own hat back. Find $\mathbf{E}[X]$ and $\mathrm{Var}(X)$.
(*Hint*: Let $X_k = 1$ if the k^{th} man got his own hat back, and $X_k = 0$ otherwise. Then $X = \sum_{k=1}^{10} X_k$.)

Solution. Note that

$$\mathbf{E}[X_i] = \mathbf{P}(X_i = 1) = \frac{1}{10} \qquad \text{for every } 1 \leq i \leq 10.$$

For every $i \neq j$,

$$\mathbf{E}[X_i X_j] = \mathbf{P}(X_i = 1 \text{ and } X_j = 1) = \mathbf{P}(X_i = 1)\mathbf{P}(X_j = 1 | X_i = 1) = \frac{1}{10}\frac{1}{9} = \frac{1}{90}.$$

Therefore

$$\mathbf{E}[X] = \sum_{i=1}^{10} \mathbf{E}[X_i] = 10 \cdot \frac{1}{10} = 1,$$

and

$$\mathbf{E}[X^2] = \sum_{i=1}^{10} \mathbf{E}[X_i^2] + \sum_{i \neq j} \mathbf{E}[X_i X_j] = \sum_{i=1}^{10} \mathbf{E}[X_i] + (10^2 - 10)\frac{1}{90} = 1 + 1 = 2.$$

It follows then $\mathrm{Var}(X) = \mathbf{E}[X^2] - (E[X])^2 = 2 - 1 = 1$. $\square$

4. (10 points) Random variable Y is of standard normal distribution. What is the probability that the equation $x^2 + 2Yx + 2 - Y = 0$ has **no** real roots?

Solution. The probability that the equation $x^2 + 2Yx + 2 - Y = 0$ has **no** real roots is equal to

$$\begin{aligned} & \mathbf{P}((2Y)^2 - 4(2 - Y) < 0) = \mathbf{P}(Y^2 + Y - 2 < 0) = \mathbf{P}((Y + 2)(Y - 1) < 0) \\ = \; & \mathbf{P}(-2 < Y < 1) = \Phi(1) - \Phi(-2) = \Phi(1) - (1 - \Phi(2)) \\ = \; & \Phi(1) + \Phi(2) - 1 = 0.8413 + 0.9772 - 1 = 0.8185. \end{aligned}$$

$\square$

5. (12 points) Suppose that a continuous random variable X has density function $f_X(x) = c|x|\, e^{-x^2}$ for $-\infty < x < \infty$.
(a) Find the constant c;
(b) Find the density function of $Y = X^2$.

Solution. (a)

$$1 = \int_{-\infty}^{\infty} c|x|e^{-x^2} dx = 2c \int_0^{\infty} xe^{-x^2} dx = c(-e^{-x^2})\Big|_0^{\infty} = c.$$

So $c = 1$.
(b) For $y \le 0$, clearly $\mathbf{P}(Y \le y) = 0$. For $y > 0$,

$$\begin{aligned} \mathbf{P}(Y \le y) &= \mathbf{P}(X^2 \le y) = \mathbf{P}(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} |x|e^{-x^2} dx \\ &= 2\int_0^{\sqrt{y}} xe^{-x^2} dx = (-e^{-x^2})\Big|_0^{\sqrt{y}} = 1 - e^{-y}. \end{aligned}$$

The the probability density function of Y is given by

$$f_Y(y) = \frac{d}{dy}\mathbf{P}(Y \le y) = \begin{cases} 0 & \text{if } y \le 0 \\ e^{-y} & \text{if } y > 0 \end{cases}.$$

$\square$