Math 327 Homework 6

1. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series of positive numbers such that

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = l$$

   Prove that

   (a) If $l > 0$, the $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

   (b) If $l = 0$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges. Show by an example that in this case $\sum_{n=1}^{\infty} a_n$ may converge when $\sum_{n=1}^{\infty} b_n$ does not.

2. Determine if the following are true or false. Justify you answer.

   (a) If $f + g : \mathbf{R} \to \mathbf{R}$ is continuous, then the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are continuous.

   (b) If $f^2 : \mathbf{R} \to \mathbf{R}$ is continuous, then $f : \mathbf{R} \to \mathbf{R}$ is continuous.

   (c) Let S be a finite subset of $\mathbf{R}$. Any function $f : S \to \mathbf{R}$ is continuous.

3. A function $f : R \to \mathbf{R}$ is a *Lipschitz function* if there exists $C \geq 0$ such that

$$|f(x) - f(y)| \leq C|x - y|, \qquad \text{for all } x.y \text{ in R.}$$

   Prove that a Lipschitz function is continuous.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a function with the property that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$.

   (a) Show that there is an m such that $f(x) = mx$ for all $x \in \mathbf{Q}$. First, you have to find the m which would work.

   (b) Prove that if f is continuous, then $f(x) = mx$ for all $x \in \mathbf{R}$.

5. Suppose that S is a subset of $\mathbf{R}$ which is not sequentially compact. Show that there is a continuous function $f : S \to \mathbf{R}$ which is unbounded.