Preliminaries + Appendix A

The Number System

$\mathbb{N}$: natural numbers $\qquad \mathbb{N} = \{1,2,3,4,\dots\}$

$\mathbb{Z}$: integers $\qquad \mathbb{Z} = \{\dots,-2,-1,0,1,2,3,\dots\} = \{0,1,-1,2,-2,3,-3,\dots\}$

$\mathbb{Q}$: rational numbers $\qquad \mathbb{Q} = \{\frac{m}{n} : m,n \in \mathbb{Z}, n \neq 0\} = \{0,1,-1,\frac{1}{2},\frac{-1}{2},2,-2,\frac{1}{3},\frac{-1}{3},\frac{2}{3},\frac{-2}{3},\frac{3}{2},\frac{-3}{2},3,-3,\dots\}$

$\mathbb{R}$: real numbers

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \ (\subseteq \mathbb{C})$$

$\mathbb{C}$: complex numbers

$(\mathbb{Z},+)$ is a group (abelian/commutative) it is closed under the operation $+$, with an identity element 0 and an inverse $-a$ for each $a \in \mathbb{Z}$ under $+$.

$(\mathbb{Z},+,\cdot)$ is a ring (look it up)

$\mathbb{Q}$ and $\mathbb{R}$ are fields, besides an additive inverse ($-a$ for every a) there is a multiplicative inverse for each $a \neq 0$. Here is a list of all the field axioms: There are operations $+$ and $\cdot$ such that for any a,b,c in the set

Commutativity	of addition	$a+b=b+a$
	of multiplication	$a\cdot b = b\cdot a$
Identity	of addition	$0+a = a+0 = a$
	of multiplication	$1\cdot a = a\cdot 1 = a$
Inverses	of addition	$(-a)+a = a+(-a) = 0$
	of multiplication	$\frac{1}{a}\cdot a = a\cdot\frac{1}{a} = 1$, $a \neq 0$

Associativity of addition $a+(b+c)=(a+b)+c$

of multiplication $a\cdot(b\cdot c)=(a\cdot b)\cdot c$

Distributive Property $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$

Nontriviality $1\neq 0$ (true of $\mathbb{Q}$ + $\mathbb{R}$, of course)

You can also order the elements of $\mathbb{Q}$ or $\mathbb{R}$, i.e. compare them $a<b$ or $b<a$ or $a=b$.

this follows from the

Positivity Axioms

There is a set $P\subseteq\mathbb{R}$ such that

P1 If $a,b\in P$, then $ab, a+b\in P$.

P2 For any $a\in\mathbb{R}$, exactly one of

$a\in P$, $-a\in P$, $a=0$

hold.

Now, we define $a>b$ by $a-b\in P$

$a<b$ by $b-a\in P$

$a\geq b$ by $a>b$ or $a=b$

$a\leq b$ by $a<b$ or $a=b$

Proposition A7. If $a<b$ and $c<0$, then $ac>bc$.
If $a<b$ and $c>0$, then $ac<ab$.

Proof: Assume $a<b$ and $c<0$.

Then $b-a \in P$, by definition of $a<b$
And $-c \in P$, by definition of $c<0$
So $(b-a)(-c) \in P$, by P1
Then $b(-c)+(-a)(-c) \in P$, by distributive property
So $-bc+ac \in P$, from HW question 1
Therefore $ac>bc$, definition of $ac>bc$

The proof of the second part is similar.

<u>Interval Notation</u>

For $a<b$, we have the

<u>open intervals</u>
$(a,b)=\{x \in \mathbb{R}: a<x<b\}$
$(a,\infty)=\{x \in \mathbb{R}: a<x\}$
$(-\infty,a)=\{x \in \mathbb{R}, x<a\}$

<u>closed intervals</u>
$[a,b]=\{x \in \mathbb{R}: a\leq x\leq b\}$
(if $a=b$, $[a,b]=\{a\}$)
$[a,\infty)=\{x \in \mathbb{R}: a\leq x\}$
$(-\infty,a]=\{x \in \mathbb{R}: x\leq a\}$

<u>and the others</u>
$[a,b)=\{x \in \mathbb{R}: a\leq x<b\}$
$(a,b]=\{x \in \mathbb{R}: a<x\leq b\}$

Chapter 1 – Tools for Analysis

1.1 The Completeness Axiom

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ but none of these sets are equal:

$-1 \in \mathbb{Z}$ but $-1 \in \mathbb{N}$ ($\mathbb{N}$ does not have additive inverses)

$\frac{1}{2} \in \mathbb{Q}$ but $\frac{1}{2} \notin \mathbb{Z}$ ($\mathbb{Z}$ does not have (all) multiplicative inverses. Which ones does it have?)

$\sqrt{2} \in \mathbb{R}$ but $\sqrt{2} \notin \mathbb{Q}$

Proving $\sqrt{2} \notin \mathbb{Q}$ is a very common example of <u>proof by contradiction</u>: Assume $\sqrt{2} \in \mathbb{Q}$

⋮

(Oops!) Contradiction.

Most likely you have seen it before. If not, read it on page 7. (or look it up)

Proof of $\sqrt{2} \in \mathbb{R}$, i.e. there is a unique positive number $x \in \mathbb{R}$ ($x \in P$) such that $x^2 = 2$, is in your HW.

It follows from <u>The Completeness Axiom</u>.

First, we need:

$S \subseteq \mathbb{R}$ is <u>bounded above</u> if there is an $M \in \mathbb{R}$ such that $x \leq M$ for every $x \in S$.

Such an M is called <u>an upperbound for S</u>

Note that M is <u>not</u> unique. Any number larger than M is also an upper bound.

M is <u>the least upper bound</u> of S if

1) M is <u>an</u> upper bound
2) Any $k<M$ is <u>not</u> an upper bound i.e. there is $x \in S$ with $k<x$

(alternatively: For any $r>0$, $M-r$ is not an upperbound)

<u>The Completeness Axiom for $\mathbb{R}$</u>

Any set $S \subseteq \mathbb{R}$ which is bounded above has a (unique) least upper bound $M \in \mathbb{R}$.

We write $\sup S$ (supremum) for the least upper bound of S.

A set $S \subseteq \mathbb{R}$ is <u>bounded below</u> if there exists an $m \in \mathbb{R}$ such that $m \le x$ for all $x \in S$. Such an $\underline{m}$ is called a lower bound for S.

<u>Theorem 1.4</u> If $S \subseteq \mathbb{R}$ is bounded below, then it has a (unique) greatest lower bound.

Proof: Assume $S \subseteq \mathbb{R}$ is bounded below.
Then, there exists an $m \in \mathbb{R}$ such that $m \le x$ for all $x \in S$.
Let $T = \{-x : x \in S\} \subseteq \mathbb{R}$
Since $m \le x$, $-m \geq -x$ so $-m$ is an upper bound for T.

First we find a candidate for a greatest lower bound

By the Completeness Axiom, T has a least upper bound $M \in \mathbb{R}$. Since $M \geq -x$ for all $-x \in T$,
$\therefore -M \leq x$ for all $x \in S$ so $-M$ is a lower bound for S.

Showing no number bigger $-M+r$ works as a lower bound

For any $r > 0$, $M - r$ is not an upper bound for T.
So, there is $-x \in T$ with $-x > M - r$.
So $x < -M + r$, i.e. $-M + r$ is not a lower bound for S.

Therefore, $-M$ is the greatest lower bound of S.

We write $\inf S$ (infimum) for the greatest lower bound of S.

Note: Thm 1.4 + the completeness axiom are equivalent. (iff) We can alternatively make Thm 1.4 the axiom and use it to prove the CA

Proposition: Let A + B be subsets of $\mathbb{R}$ which are bounded below. Then,

(a) If $A \subseteq B$, then $\inf A \geq \inf B$.

(b) $\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$

(c) if $A \cap B \neq \emptyset$, $\inf(A \cap B) \geq \max\{\inf A, \inf B\}$

Proof: Let $m_A = \inf A$, $m_B = \inf B$
Assume $m_A \leq m_B$ so $\min\{m_A, m_B\} = m_A$
(otherwise switch names A + B)

Find the lower bound candidate m_A + show it is a lower bound

First, we show m_A is a lower bound for $A \cup B$.
Let $x \in A \cup B$.
If $x \in A$, then $m_A \leq x$ since m_A is a lower bound for A.
If $x \in B$, then $m_B \leq x$ since m_B is a lower bound for B
but $m_A \leq m_B$ so $m_A \leq x$ for any $x \in A \cup B$.

showing no $m > m_A$ works as a lower bound

Now, we show no larger number works.
If $m > m_A$, then since m_A is the greatest lower bound for A, m is not a lower bound.
So, there is an $x \in A$ with $x < m$.
Since $x \in A \cup B$ and $x < m$, m is not a lower bound for $A \cup B$.

Therefore, m_A is the least upper bound for $A \cup B$.
For part (a), you only need to show $\inf B$ is a lower bound for A. Why?
For part (c), you only need to show $\max\{\inf A, \inf B\}$ is a lower bound for $A \cap B$. Why? Equality does not necessarily hold.

ex: Let $A = \{1, 2, 3, 4, 5\}$ $B = \{-5, 5, 10\}$

Note: If $\inf S \in S$, we call it the <u>minimum</u> of S.
If $\sup S \in S$, we call it the <u>maximum</u> of S.

1.2 The Distribution of the Integers and the Rational Numbers

Thm 1.5 - The Archimedean Property ($\mathbb{N}$ is not bounded above)

(i) For any $c \in \mathbb{R}$ with $c > 0$, there is an $n \in \mathbb{N}$ such that $n > c$.

(ii) For any $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

You can see the two properties are equivalent by letting $c = \frac{1}{\varepsilon}$.

Proof: (i). By contradiction. Assume there is a $c > 0$ such that for all $n \in \mathbb{N}$ $n \leq c$.
Then $\mathbb{N}$ is bounded above. Let b be the least upper bound of $\mathbb{N}$.
Then $b - \frac{1}{2}$ is not an upper bound for $\mathbb{N}$
so there is an $n \in \mathbb{N}$ with $b - \frac{1}{2} < n \leq b$
Then $b - \frac{1}{2} + 1 < n+1$, so $b < b + \frac{1}{2} < n+1$
Since $n+1 \in \mathbb{N}$, this contradicts b being an upper bound
Therefore for all $c > 0$ there is an n with $n > c$.

<u>example</u>: Prove that 1 is the least upper bound of the set $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\}$.

Step 1 First, we prove 1 is an upper bound.
Let $1 - \frac{1}{n} \in S$
Since $n > 0$, $-\frac{1}{n} < 0$ so $1 - \frac{1}{n} < 1 + 0 = 0$, for any $n \in \mathbb{N}$.

Step 2 To show 1 is the least upper bound, we'll prove $1-r$ is not an upper bound for any $r>0$.

Find n_0 with $\frac{1}{n_0} < r$ by Thm 1.5

Then $-\frac{1}{n_0} > -r$ so $1-\frac{1}{n_0} > 1-r$

Therefore, $1-r$ is not an upper bound for S.

Hence, $1 = \sup S$.

Proposition 1.6 For any integer n, $(n, n+1) \cap \mathbb{Z} = \emptyset$

i.e. there is no integer in the open interval $(n, n+1)$.

Proof: By contradiction.

Assume there is an $n \in \mathbb{N}$ such that $(n, n+1) \cap \mathbb{Z} \neq \emptyset$

i.e. there is a $k \in \mathbb{Z}$ $n < k < n+1$, all integers

Then, adding $-n$

$0 < k-n < 1$

So there is a natural number $k-n < 1$

which is not possible.

Proposition 1.7 If $S \subseteq \mathbb{Z}$ is bounded above, then S has a maximum.

Proof: Assume $S \subseteq \mathbb{Z}$ is bounded above.

Let $a = \sup S$.

Since $a-1$ is not an upper bound for S, there exists $m \in S$ with $a-1 < m$ or $a < m+1$

Then, $S \subseteq (-\infty, a] \subseteq (-\infty, m+1)$

Since $(m, m+1) \cap S = \emptyset$ by Proposition 1.6

$S \subseteq (-\infty, m]$, with $m \in S$.

Then, m is the maximum of S.

Thm 1.8. For any $c \in \mathbb{R}$, there is exactly one $k \in \mathbb{Z}$ in $[c, c+1)$.

Proof: Existance: Let $S = (-\infty, c+1) \cap \mathbb{Z}$.

By Proposition 1.7, S has a maximum $k \in S \subseteq \mathbb{Z}$.

So $k < c+1$.

If $k < c$, then $k+1 < c+1$ so $k+1 \in S$

contradicting k was the maximum of S

so $k \geq c$.

Therefore $c \leq k < c+1$, i.e $k \in [c, c+1)$.

Uniqueness If $k, \ell \in [c, c+1)$, both integers

Assume $k \neq \ell$ and $k > \ell$ (otherwise switch names)

$c \leq k < c+1$ and $c \leq \ell < c+1$

so $-c-1 < -\ell \leq c$

so $-1 < k-\ell < 1$

since $k-\ell > 0$, $0 < k-\ell < 1$ which is a contradiction

Therefore, there is exactly one $k \in \mathbb{Z}$ in $[c, c+1)$.

We say $S \subseteq \mathbb{R}$ is <u>dense in $\mathbb{R}$</u> if
For every $a<b$, there is an $s\in S$ in (a,b).

Thm 1.9. $\mathbb{Q}$ is dense in $\mathbb{R}$.

do not include your thinking in your proofs

<u>coming up with the idea of the proof:</u>

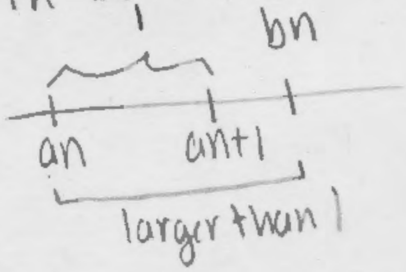
given $a<b$ real, want $\frac{m}{n}\in\mathbb{Q}$ $\quad a<\frac{m}{n}<b$, $m,n\in\mathbb{Z}$, $n>0$

integer

so $an < m < bn$

real

need: $bn-an$ to be larger than 1 so I guarantee an integer m is in between an and bn



want: $an+1<bn$

so $1<bn-an$

$1<(b-a)n$

$\frac{1}{n}<b-a$

we start here + "go up" as we write the proof.

Proof: Let $a,b\in\mathbb{R}$ with $a<b$.
By the Archimedean property, there is an $n\in\mathbb{N}$ with $\frac{1}{n}<b-a$.
Then $1<bn-an$ so $an+1<bn$
There is a unique integer m in $(an, an+1]$
by Theorem 1.8 $\quad (-m\in[-an-1,-an))$

So $an < m \leq an+1 < bn$

or $an < m < bn$

Dividing by n, we get

$$a < \frac{m}{n} < b \quad \text{with } \frac{m}{n} \in \mathbb{Q}$$

finishing the proof.

Corollary 1.10 The set of irrational numbers $\mathbb{R}\setminus\mathbb{Q}$ is also dense in $\mathbb{R}$.

Proof: Let $a, b \in \mathbb{R}$, $a < b$

Then $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$

By Thm 1.9 we can find $q \in \mathbb{Q}$

$$\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$$

then $a < \sqrt{2}q < b$ and $\sqrt{2}q \notin \mathbb{Q}$.

Exercise: Prove that if $q \in \mathbb{Q}$ and $r \notin \mathbb{Q}$ then $qr \notin \mathbb{Q}$.

Hint: Do a proof by contradiction:

assume $q \in \mathbb{Q}$, $r \notin \mathbb{Q}$ and $qr \in \mathbb{Q}$.