Math 383 Homework 1

Chongyi Xu

Oct, 17, 2017

1. Let $y(t)$ be the number of ^{210}Pb atoms per gram of ordinary lead at time t. Let t_0 be the time the pigment was manufactured and r the number of disintegrations of ^{226}Ra per gram of ordinary lead per unit time.

   (a) Explain why the following equations should govern the change in the amout of ^{210}Pb :

   $$\frac{dy}{dt} = -\lambda y + r \textit{ while in the ore}, \frac{dy}{dt} = -\lambda y \textit{ after manufacture}$$

   λ is the decay constant for ^{210}Pb.

   Because during the period in the ore, ^{210}Pb atoms and ^{226}Ra atoms are in "radioactive equilibrium", in the other words, ^{210}Pb atoms are decaying, but ^{226}Ra atoms are changing to ^{210}Pb atoms. After manufacture, all ^{226}Ra are eliminated, thus the only atoms that are decaying will be ^{210}Pb.

   (b) Measurements from a variety of ores over the earth's surface gave a range of values for the rate of disintegration of ^{226}Ra per gram of ordinary lead as

$$r = 0 - 200 \textit{ per miniute.}$$

Show that it is reasonable to assume that

$$\lambda_y(t_0) = r = 0 - 200 \textit{per miniute.}$$

Since at $t = t_0$, it can be assumed that the atoms inside of the ore are ${}^{210}Pb$ that are changing from ${}^{226}Ra$ atoms. And the ${}^{210}Pb$ atoms that was in the ore have already decayed. So at very beginning of the decay session after manufacture, the intial rate of change in the number of ${}^{210}Pb$ atoms should be directly the rate of disintegration of ${}^{226}Ra$ atoms.

(c) Solve subject to the intial condition

$$y(t_0) = r/\lambda$$

Solution:

$$\frac{dy}{dt} = -\lambda y$$
$$\int \frac{1}{y} = \int -\lambda dt$$
$$lny = -\lambda t + C,\ \ C \in \mathbb{R}$$
$$y(t) = y_0 e^{-\lambda t}$$
$$Let\ t = t_0,\ and\ given\ y(t_0) = r/\lambda$$
$$\frac{r}{\lambda} = y_0 e^{-\lambda t_0}$$
$$y_0 = \frac{r}{\lambda} * e^{-\lambda t_0}$$
$$Therefore,$$
$$y(t) = \frac{r}{\lambda} e^{-\lambda(t-t_0)}$$

(d) For the *Disciples at Emmaus* painting, it was measured that

$$-\frac{dy}{dt}(t) \simeq 8.5 per minute.$$

Estimate $t - t_0$ to decide if the painting can be 300 years old.

From part(c), we have

$$y(t) = \frac{r}{\lambda} e^{-\lambda(t-t_0)}$$

And from (4.10), with given value of $\frac{dy}{dt}$

$$\frac{dy}{dt} = -\lambda y \text{ after manufacture}$$

$$8.5 = -\lambda \frac{r}{\lambda} e^{-\lambda(t-t_0)}$$

With $\lambda = ln2/\tau$

$$e^{-\frac{ln2}{\tau}(t-t_0)} = \frac{8.5}{r}$$

$$-\frac{ln2}{\tau}(t-t_0) = ln\frac{8.5}{r}$$

$$t - t_0 = \frac{-ln8.5/r}{ln2}\tau$$

Taking $\tau = 300, r = 0\ 200$

$$t - t_0 = 1366.92 (r = 200)$$

$$t - t_0 = 0 (r = 8.5 \text{ Halflife could not be less than 0})$$

2. ... A drug therapy using RT (reverse transcriptase) inhibitors blocks infection, leading to $k \simeq 0$. Setting $k = 0$ in (4.11), solve for $T^*(t)$, Substitue it into (4.12) and solve for $V(t)$. Show that the solution is

$$V(t) = \frac{V(0)}{c - \delta}[ce^{\delta t} - \delta e^{-ct}] \tag{1}$$

Solution:

$$\begin{aligned}
&With\ k = 0, \\
&\frac{dT^*}{dt} = 0 - \delta T^* \\
&T^*(t) = T^*(0)e^{-\delta t} \\
&Substitute\ T^*(t) = T^*(0)e^{-\delta t}\ into\ P(t) \\
&P(t) = N\delta T^*(0)e^{-\delta t} \\
&Since\ the\ general\ solution\ to\ V(t)\ is \\
&V(t) = V(0)e^{-ct}\int_0^t e^c\xi P\xi d\xi \\
&Substitute\ P(t)\ into\ V(t), \\
&V(t) = V(0)e^{-ct}\int_0^t e^c\xi N\delta T^*(0)e^{-\delta\xi}d\xi \\
&Then\ we\ have \\
&V(t) = V(0)e^{-ct} - \frac{N\delta T^*(0)}{c-\delta}(e^{-ct} - e^{-\delta t}) \\
&To\ find\ N\delta T^*(0),\ assume\ equal\ clearance\ of\ production \\
&\frac{dV}{dt} = 0 = N\delta T^*(0) \\
&cV(0) = N\delta T^*(0) \\
&Then\ we\ have \\
&V(t) = V(0)e^{-ct} - \frac{cV(0)}{c-\delta}(e^{-ct} - e^{-\delta t}) \\
&= \frac{V(0)}{c-\delta}[ce^{-\delta t} - \delta e^{-ct}]
\end{aligned}$$

3. Protease inhibitors

   (a) Solve (4.15), substituting it into Eq. (4.13) to show that the solution for $T^*(t)$ is, assuming $T = T_0$ is a constant,

$$\begin{aligned} T^*(t) &= T^*(0)e^{-\delta t} + \frac{kT_0V_0(e^{-ct} - e^{-\delta t})}{\delta - c} \\ &= kV_0T_0[ce^{-\delta t} - \delta e^{-ct}]/[\delta(c-\delta)] \end{aligned}$$

Solution:

$$\begin{aligned} &\textit{Since } \frac{d}{dt}V_I = -cV_I, \\ V_I &= V_I(0)e^{-ct} \\ \textit{Substitute it into } &\frac{dT^*(t)}{dt} \\ \frac{dT^*(t)}{dt} &= kV_IT - \delta T^* \\ &= kV_I(0)e^{-ct}T(t) - \delta T^*(t) \\ \textit{Substitute } T = T_0,\ V_I(0) = V_0 & \\ \frac{dT^*(t)}{dt} &= kV_0e^{-ct}T_0 - \delta T^*(t) \\ T^*(t) &= T^*(0)e^{-\delta t} + e^{-\delta t}kV_0T_0\int_0^t e^{\delta\xi}\cdot e^{-c\xi} \\ &= T^*(0)e^{-\delta t} + \frac{kV_0T_0}{\delta - c}[(e^{-\delta t}\cdot e^{\delta t}\cdot e^{-ct}) - (e^{-\delta t})] \\ &= T^*(0)e^{-\delta t} + \frac{kT_0V_0(e^{-ct} - e^{-\delta t})}{\delta - c} \\ &= kV_0T_0[ce^{-\delta t} - \delta e^{-ct}]/[\delta(c-\delta)] \end{aligned}$$

(b) Substitue $T^*(t)$ found in (a) into (4.14) to show:

$$V_{NI}(t) = \frac{cV_0}{c-\delta}[\frac{c}{c-\delta}(e^{-\delta t} - e^{-ct}) - \delta te^{-ct}]$$

Solution:

$$
\begin{aligned}
\frac{dV_{NI}}{dt} &= N\delta T^*(t) - cV_{NI} \\
V_{NI}(t) &= V_{NI}(0)e^{-ct} + e^{-ct}\int_0^t e^{c\xi} N\delta T^*(\xi) d\xi \\
&= V_{NI}(0)e^{-ct} + e^{-ct}\int_0^t \frac{e^{c\xi} N\delta k V_0 T_0 [ce^{-\delta\xi} - \delta e^{-c\xi}]}{\delta(c-\delta)} d\xi \\
&= V_{NI}(0)e^{-ct} + e^{-ct}\frac{N\delta k V_0 T_0}{\delta(c-\delta)}\int_0^t ce^{\xi(c-\delta)} - e^{c\xi}(1-\delta) d\xi \\
&= V_{NI}(0)e^{-ct} + e^{-ct}\frac{N\delta k V_0 T_0}{\delta(c-\delta)}[\frac{ce^{t(c-\delta)} - c}{c-\delta} - \frac{(1-\delta)(e^{ct}-1)}{c}] \\
&= V_{NI}(0)e^{-ct} + e^{-ct}\frac{NkV_0T_0}{(c-\delta)}\frac{c^2e^{t(c-\delta)} - (c^2-\delta^2) - e^{ct}(c-\delta) - \delta e^{ct}(c-\delta) + c - \delta)}{c(c-\delta)} \\
&= \ldots \textit{ get lost, don't know where to go}
\end{aligned}
$$

(c) Adding $V_{NI}(t)$ and $V_I(t)$, show that the total virion concentration is given by

$$V(t) = V_0e^{-ct} + \frac{cV_0}{c-\delta}[\frac{c}{c-\delta}(e^{-\delta t} - e^{-ct}) - \delta te^{-ct}]$$

Since from part(a), we have

$$V_I = V_0e^{-ct}$$

from part(c), we have

$$V_{NI}(t) = \frac{cV_0}{c-\delta}[\frac{c}{c-\delta}(e^{-\delta t} - e^{-ct}) - \delta te^{-ct}]$$

Then put V_I and V_{NI} into the equation $V = V_I + V_{NI}$, we have

$$V(t) = V_0 e^{-ct} + \frac{cV_0}{c-\delta}[\frac{c}{c-\delta}(e^{-\delta t} - e^{-ct}) - \delta t e^{-ct}]$$