Math 327 Homework 3

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1. Let c be a number with $|c| < 1$.

   (a) Show that there exists a $d > 0$ such that $|c| = \frac{1}{1+d}$. Then, use the binomial theorem to show that

   $$|c^n| \leq \frac{1}{1+nd} \leq \frac{1}{dn} \text{ for every integer } n \geq 1.$$

   - Assume $d > 0$, then $1 + d > 1$. So $0 < \frac{1}{1+d} < 1$. Therefore $\frac{1}{1+d} = |c|$ since $0 < |c| < 1$.
   - Obviously, $\frac{1}{1+nd} \leq \frac{1}{nd}$ for any $n \geq 1$ and $d > 0$. And since $|c^n| = \frac{1}{(1+d)^n}$, prove $(1+d)^n \geq 1+nd$ by induction on n.
     - Base Case($n = 1$).

       $(1 + d)^1 = 1 + d$
     - Inductive Step

       Binomial Theorem tells that $\binom{(1+d)^n=\sum_{k=0}^{n} n}{k(nd)^k}$. So

       $$\begin{aligned}(1 + d)^{n+1} &= (1 + d)^n \cdot (1 + d) \\ &\geq (1 + nd) \cdot (1 + d) \text{ Inductive Hypothesis} \\ &= 1 + d + nd + nd^2\end{aligned}$$

       Since $n \geq 1$, $nd^2 > 0$. Therefore, $(1 + d)^{n+1} \geq 1 + d + nd = 1 + (n + 1)d$

So we have

$$
\begin{aligned}
(1+d)^n &\geq 1+nd \geq nd \\
\frac{1}{(1+d)^n} &\geq \frac{1}{1+nd} \geq \frac{1}{nd} \\
|c^n| &\geq \frac{1}{1+nd} \geq \frac{1}{nd}
\end{aligned}
$$

Q.E.D.

(b) Use the Sandwich Theorem to give an alternative proof of $c^n \to 0$.

- $(c^n > 0)$. Let $a_n = 0$, then $a_n \to 0$.
- Let $b_n = \frac{1}{nd}$. In part(a), it has been proved that $c^n \leq \frac{1}{dn}$. And $\frac{1}{dn} \to 0$ since $d > 0$ and $n \geq 1$.

So $a_n \leq c^n \leq b_n$ and $a_n \to 0,\ b_n \to 0$. Sandwich Theorem tells c^n also converges to 0.

- $(c^n < 0)$. Let $\alpha_n = -a_n$, $\beta_n = -b_n$.

Then $\beta_n \leq c^n \leq \alpha_n$. Sandwich Theorem tells c^n converges to 0. Q.E.D.

(c) Prove that $\sqrt{n}c^n \to 0$.

$\sqrt{n}c^n$ is obviously monotonely decreasing. And c^n is bounded from part(b).

So there exists an M and m such that

$$
\begin{aligned}
m \leq &\quad c^n \leq M \\
\sqrt{n} \leq &\sqrt{n}c^n \leq \sqrt{n}M
\end{aligned}
$$

So $\sqrt{n}c^n$ is also bounded. Monotone Convergence Theorem tells if a monotone sequence is bounded, it converges. Claim $\sqrt{n}c^n$ converges to $inf\{\sqrt{n}c^n\} = 0$.

- (If $0 < c < 1$) Since $c^n > 0$, and $\sqrt{n} \geq 1$, 0 is a lower bound. Assume 0 is not the greatest lower bound for a contradiction. Let $r > 0$ be the greatest lower bound, then $\sqrt{n}c^n = \frac{\sqrt{n}c^n}{c} \geq \frac{r}{c}$. Then $\frac{r}{c} > r$ is also a greatest lower bound, contradicting. So r is not the greatest lower bound. So $\sqrt{n}c^n$ converges to 0.
- (If $-1 < c < 0$) Similarly, 0 is the least upper bound. So $\sqrt{n}c^n$ converges to 0.

In both cases, $\sqrt{n}c^n$ converges to 0. Q.E.D.

(d) Prove that if $0 < c < 1$, then $nc^n \to 0$.

2. For a pair of positive numbers α and β, $\frac{\alpha+\beta}{2}$ is called the arithmetic mean and $\sqrt{\alpha\beta}$ is called the geometric mean.

   (a) Prove that
   $$\frac{\alpha + \beta}{2} \geq \sqrt{\alpha\beta}$$

   (b) Let $a, b > 0$. Define sequences (a_n) and (b_n) recursively with $a_1 = 1,\ b_1 = b$,
   $$a_{n+1} = \frac{a_n + b_n}{2} \text{ and } b_{n+1} = \sqrt{a_n b_n}.$$

   Prove (a_n) and (b_n) are monotone and that they have the same limit. This limit is called the Gauss arithmetic-geometric mean on a and b.