Math 327 Homework 5

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1. Determine if the following series converge. Explain.

   (a) $\sum\limits_{n=1}^{\infty} \frac{1}{5n-2}$

   It diverges. Apply limit comparison test with $\sum\limits_{n=1}^{\infty} \frac{1}{n}$. $\lim\limits_{n\to\infty} \left|\frac{\frac{1}{5n-2}}{\frac{1}{n}}\right| = \lim\limits_{n\to\infty} \left|\frac{n}{5n-2}\right| = \frac{1}{5} \neq 0$. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges(harmonic series), then $\sum\limits_{n=1}^{\infty} \frac{1}{5n-2}$ also diverges.

   (b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

   It converges. In order to apply alternating series test, we need to prove $\frac{1}{\sqrt{n}}$ is non-increasing and converges to 0.

   - Claim: $\frac{1}{\sqrt{n}}$ is non-increasing.

     Proof: Let $\frac{1}{\sqrt{n}} = a_n$. Then $a_n - a_{n+1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}}$. Since $\sqrt{n(n+1)} > 0$ and $n+1 > n \iff \sqrt{n+1} > \sqrt{n}$ for $n \in \mathbb{N}$, then $a_n - a_{n+1} > 0$. So $\frac{1}{\sqrt{n}}$ is non-increasing(decreasing).

   - Claim: $\frac{1}{\sqrt{n}}$ converges to 0.

     Proof: Let $\varepsilon > 0$. By Archimedean Property, choose N with $\frac{1}{N} < \varepsilon^2$. Then for any $n \geq N$, $\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}}$ (since $n \geq N$) $< \varepsilon$ (by the choice of N). So $\frac{1}{\sqrt{n}}$ converges to 0.

   So by alternating sereis test, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.

   (c) $\sum\limits_{n=1}^{\infty} (\frac{n+1}{n^2+1})^3$

   It converges by applying comparison test.

   For $n = 1$, $(\frac{n+1}{n^2+1})^3 = 1$. So $\sum\limits_{n=1}^{\infty} (\frac{n+1}{n^2+1})^3 = 1 + \sum\limits_{n=2}^{\infty} (\frac{n+1}{n^2+1})^3$. Therefore, we only need to consider the $\sum\limits_{n=2}^{\infty} (\frac{n+1}{n^2+1})^3$ part for convergence.

Claim: $(\frac{n+1}{n^2+1})^3 < \frac{1}{n^3} \iff \frac{n+1}{n^2+1} < \frac{1}{n}$

$$
\begin{aligned}
\text{Since } n \geq 2,\ \frac{n+1}{n^2+1} = \frac{(n+1)(n-1)}{(n^2+1)(n-1)} &= \frac{n^2-1}{n^3-n^2+n-1} \\
&< \frac{n^2}{n^3-n^2+n-1} \\
\text{Since for } n \geq 2,\ -n^2+n-1 = n(1-n)-1 < 0 \\
&< \frac{n^2}{n^3} \\
&= \frac{1}{n}
\end{aligned}
$$

So we have $(\frac{n+1}{n^2+1})^3 < \frac{1}{n^3} < \frac{1}{n^2}$. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges, by comparison test, $\sum\limits_{n=1}^{\infty} (\frac{n+1}{n^2+1})^3$ converges

(d) $\sum\limits_{n=1}^{\infty} \frac{n!}{n^n}$

It converges by Ratio Test.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} &= \lim_{n\to\infty} \frac{(n+1)!n^n}{(n+1)^{n+1}n!} \\
&= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} (n+1) \cdot (\frac{n}{n+1})^n \cdot \frac{1}{n+1} \\
&= \lim_{n\to\infty} (\frac{n}{n+1})^n \\
&= \lim_{n\to\infty} (\frac{1}{1+\frac{1}{n}})^n \\
&= \frac{1}{e} < 1 \text{ since } \lim_{n\to\infty} (1+\frac{1}{n})^n = e
\end{aligned}
$$

Since $0 \leq \lim\limits_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} < 1$, by Ratio Test, it (absolutely) converges.

(e) $1 + \frac{1\cdot 2}{1\cdot 3} + \frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5} + ...$

It converges by Ratio Test.

Since $1 + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + ... = \sum\limits_{n=1}^{\infty} \frac{n!}{(2n-1)!}$

$$\begin{aligned}
\lim_{n \to \infty} \frac{\frac{(n+1)!}{[2(n+1)-1]!}}{\frac{n!}{(2n-1)!}} &= \lim_{n \to \infty} \frac{(n+1)!}{n!} \cdot \frac{(2n-1)!}{(2n+1)!} \\
&= \lim_{n \to \infty} (n+1) \cdot \frac{1}{2n} \cdot \frac{1}{2n+1} \\
&= \lim_{n \to \infty} \frac{n+1}{2n+1} \cdot \frac{1}{2n} \\
&< 1 \text{ since } n+1 < 2n+1 \text{ and } \frac{1}{2n} \leq 1
\end{aligned}$$

Since $0 \leq \lim\limits_{n \to \infty} \frac{\frac{(n+1)!}{[2(n+1)-1]!}}{\frac{n!}{(2n-1)!}} < 1$, by Ratio Test, it (absolutely) converges.

(f) $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$

It converges by Ratio Test.

Claim: $\frac{n+1}{n} \leq 2$

Proof: $\frac{n+1}{n} = \frac{1}{n} + 1$. Since $\frac{1}{n} \leq 1$ for $n \in \mathbb{N}$, $\frac{n+1}{n} \leq 2$.

$$\begin{aligned}
\lim_{n \to \infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} &= \lim_{n \to \infty} \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} \\
&= \lim_{n \to \infty} \frac{n+1}{n} \cdot \frac{1}{2} \\
&\leq 1 \text{ since as proved above, } \frac{n+1}{n} \leq 2
\end{aligned}$$

Since