# Math 327 Homework 2

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1. Prove the following.

   (a) For any $a, b$ real numbers, $|a| - |b| \leq |a + b|$.

   Let $b = b + a - a$, then $|b| = |b + a - a| \leq |a + b| - |a|$(By Triangular Inequality)

   Since $-|b| \leq +|b|$, so $-|b| \leq |b| \leq |a + b| - |a|$

   Thus $|a| - |b| \leq |a| + |b| \leq |a + b|$ Q.E.D.

   (b) For any $a, b$ real numbers, $||a| - |b|| \leq |a + b|$.

   $$\rightarrow \text{if } |a| - |b| \geq 0,\ \text{part(a) tells it is true.}$$

   $$\rightarrow \text{if } |a| - |b| < 0,\ ||a| - |b|| = |b| - |a| \leq |b + a| \text{(By part(a))}.$$

   In both cases $||a| - |b|| \leq |a + b|$ Q.E.D.

   (c) For any $a, b$ real numbers, $||a| - |b|| \leq |a - b|$.

   Let $b = -c,\ c \in \mathbb{R}$. In part(b), it is proved that $||a| - |c|| \leq |a + c|$. So in this case, let $c = -b$, then we have $||a| - |b|| \leq |a - b|$ since $| - c| = c$. Q.E.D

2. Prove Bernoulli's Inequality

   $$(1 + b)^n \geq 1 + nb$$

   in two different ways:

   (a) For any $b \geq 0$, using the binomial formula.

   Binomial Formula tells that $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. So we have

Let $a = 1$,

$$(1+b)^n = \sum_{k=0}^{n} \binom{n}{k} b^k$$
$$\to \text{when } k=0,\ \binom{n}{0} \cdot b^0 = 1$$
$$\to \text{when } k=1,\ \binom{n}{1} \cdot b^1 = nb$$

So $(1+b)^n = 1 + nb + \sum_{k=2}^{n} \binom{n}{k} b^k$.

Since given $b \geq 0$, then $\sum_{k=2}^{n} \binom{n}{k} b^k \geq 0$.

Therefore $(1+b)^n \geq 1 + nb$. Q.E.D.

(b) For any $b > -1$, using mathematical induction.

Assume $b > -1$.

- Base Case($n = 0$)

  $(1+b)^0 = 1 \geq 1 + 0 \cdot b$

- Inductive Step

  Assume $(1+b)^n \geq 1 + nb$

  Prove $(1+b)^{n+1} \geq 1 + (n+1) \cdot b$

  - Proof:

$$(1+b)^{n+1} = (1+b)^n \cdot (1+b)$$
$$\text{Since } b > -1 \text{ by assumption}$$
$$\begin{aligned}(1+b)^n \cdot (1+b) &\geq (1+nb)(1+b)(\text{inductive hypothesis}) \\ &\geq 1 + b + nb + nb^2 \\ &\geq 1 + (n+1)b + nb^2\end{aligned}$$

  Since $b^2 \geq 0,\ nb^2 \geq 0$

  Therefore $(1+b)^{n+1} \geq 1 + (n+1)b$. Q.E.D.

3. Decide if the following are true or false. If true, give a short proof. If false, find a counter example.

(a) If the sequence $|a_n|$ converges, then so does $(a_n)$

*True.* Assume $|a_n|$ converges, then $\exists N \in \mathbb{N}$ such that

$||a_n| - a| < \varepsilon$ when $n \geq N$ for $\varepsilon > 0$

$\to$ if $a_n \geq 0$, then $|a_n - a| < \varepsilon$

Converges to $a\surd$

$\to$ if $a_b < 0$

$$\begin{aligned} |-a_n - a| &< \varepsilon \\ |a_n + a| &< \varepsilon \\ |a_n - (-a)| &< \varepsilon \\ \text{Let } b = -a&, \\ |a_n - b| &< \varepsilon \end{aligned}$$

Converges to b. Q.E.D.

$\Rightarrow$ Therefore, $(a_n)$ also converges if $|a_n|$ converges.

(b) If the sequence $(a_n + b_n)$ converges, then so do the sequences $(a_n)$ and $(b_n)$.

*False.* Let $a_n = 2n,\ b_n = -2n$, then $(a_n + b_n)$ converges to 0, but$a_n,\ b_n$ do not.

(c) If the sequences $(a_n + b_n)$ and $(a_n)$ converge, then so does the sequence $(b_n)$.

*True.* Assume $|a_n + b_n - c| < \frac{\varepsilon}{2},\ |a_n - a| < \frac{\varepsilon}{2}$ for any $n \geq N$, where $N \in \mathbb{N}$

Let $c = a + b$, then we have $|a_n + b_n - (a + b)| < \frac{\varepsilon}{2}$

$$\begin{aligned} |b_n - b| &= |a_n - a - a_n + a + b_n - b| \\ &= |a_n - a + b_n - b + (a - a_n)| \\ &\leq |a_n - a + b_n - b| + |a - a_n| \text{(Triangular Inequality)} \end{aligned}$$

Since $|a - a_n| = |a_n - a|$

$$\begin{aligned} |b_n - b| &\leq |a_n - a + b_n - b| + |a_n - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Therefore, $(b_n)$ converges. Q.E.D.

4. Use the definition of convergence to show the following limits.

(a) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

- Proof: Given $\varepsilon > 0$, Archimedean Property(2) tells that there exists a $M \in \mathbb{N}$, $\frac{1}{M} < \varepsilon$. Let $N = M^2$, then $\frac{1}{\sqrt{N}} < \varepsilon$

Assume $n \geq N$, then

$$\begin{aligned}\left|\frac{1}{\sqrt{n}} - 0\right| &= \left|\frac{1}{\sqrt{n}}\right| \\ &\leq \frac{1}{\sqrt{n}} \\ &\leq \frac{1}{\sqrt{N}}, \text{ since } n \geq N \\ &< \varepsilon\end{aligned}$$

Therefore, $\frac{1}{\sqrt{n}}$ converges to 0 $\iff \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

(b) $\lim_{n\to\infty} \frac{n^2}{n^2+n} = 1$

Prove $|\frac{n^2}{n^2+n} - 1| < \varepsilon$, for $n \geq N$

- Proof: Given $\varepsilon > 0$, Archimedean Property(2) tells that there exists a $N \in \mathbb{N}$, $\frac{1}{N} < \varepsilon$. And Archimedean Property(1) tells that there exists a $N+1 \in \mathbb{N}$. So $\frac{1}{N+1} < \frac{1}{N} < \varepsilon$.

Assume $n \geq N \iff n+1 \geq N+1$, then

$$\begin{aligned}\left|\frac{n^2}{n^2+n} - 1\right| &= \left|\frac{n^2 - n^2 - n}{n^2+n}\right| \\ &= \left|\frac{-n}{n^2+n}\right| \\ &= \left|\frac{-1}{n+1}\right| \\ &= \left|\frac{1}{n+1}\right| \\ &\leq \frac{1}{n+1} \\ &\leq \frac{1}{N+1} \\ &< \frac{1}{N} \\ &< \varepsilon\end{aligned}$$

Therefore, $\frac{n^2}{n^2+n}$ converges to 1 $\iff \lim_{n\to\infty} \frac{n^2}{n^2+n} = 1$

5. Discuss the convergence of the sequence $(\sqrt{n+1}-\sqrt{n})_{n\in\mathbb{N}}$

   *Claim* $(\sqrt{n+1}-\sqrt{n})_{n\in\mathbb{N}}$ converges.

   • Proof: Given $\varepsilon > 0$, Archimedean Property(2) tells that there exists a $M \in \mathbb{N}$, $\frac{1}{M} < 2\varepsilon$. Let $M = \sqrt{N}$, then $\frac{1}{\sqrt{N}} < 2\varepsilon \Rightarrow \frac{1}{2\sqrt{N}} < \varepsilon$.

   Assume $n \geq N$, then

$$\begin{aligned}
\left|\sqrt{n+1}-\sqrt{n}\right| &= \left|\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right| \\
&= \left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right| \\
&< \frac{1}{2\sqrt{n}} \\
&\leq \frac{1}{2\sqrt{N}} \\
&< \varepsilon
\end{aligned}$$

   Therefore, $(\sqrt{n+1}-\sqrt{n})_{n\in\mathbb{N}}$ converges to 0.

6. Let $a_1 = 1$ and for $n \geq 1$,

$$a_{n+1} = \begin{cases} a_n + \frac{1}{n} & \text{if } a_n^2 \leq 2 \\ a_n - \frac{1}{n} & \text{if } a_n^2 > 2 \end{cases}$$

   Show that for every $n$, $\left|a_n - \sqrt{2}\right| < \frac{2}{n}$ and prove that the sequence converges to $\sqrt{2}$.

   Since $a_1 = 1$ and $n \geq 1 \iff \frac{1}{n} \leq 1$, $a_n > 0$

7. For a sequence $(a_n)$ of positive numbers, prove that

$$a_n \to \infty \text{ if and only if } \frac{1}{a_n} \to 0.$$

   - ($\Rightarrow$) Assume $a_n \to \infty$, prove $\frac{1}{a_n} \to 0$.

     Proof: $a_n \to \infty$ implies that for every $\varepsilon > 0, |a_n - 0| > \varepsilon$ when $n \geq N$.

So we have

$$
\begin{aligned}
|a_n| &> \varepsilon \\
a_n &> \varepsilon, \text{ since } a_n \text{ are all positive numbers} \\
\frac{1}{a_n} &< \frac{1}{\varepsilon}
\end{aligned}
$$

Therefore, $\frac{1}{a_n} \to 0$ since for every $\frac{1}{\varepsilon} > 0$, $\left|\frac{1}{a_n}\right| < \frac{1}{\varepsilon}$ when $n \geq N$.

- ($\Leftarrow$) Assume $\frac{1}{a_n} \to 0$, prove $a_n \to \infty$.

  Proof: $\frac{1}{a_n} \to 0$ implies for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\left|\frac{1}{a_n} - 0\right| < \varepsilon$ when $n \geq N$.

  So we have

$$
\begin{aligned}
\left|\frac{1}{a_n}\right| &< \varepsilon \\
\frac{1}{a_n} &< \varepsilon, \text{ since } a_n \text{ are all positive numbers} \\
a_n &> \frac{1}{\varepsilon}
\end{aligned}
$$

  Therefore, $a_n \to \infty$ since $a_n$ will always be greater than $\frac{1}{\varepsilon}$, where $\varepsilon > 0$.

Q.E.D.