Math 327 HW8

Chongyi Xu

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1. Show that $f : (0,1) \to \mathbb{R}$ where $f(x) = \frac{1}{x^2}$ is not uniformly continuous.

   Let $(u_n)_{n\in\mathbb{N}} = \frac{1}{n+1}$, $(v_n)_{n\in\mathbb{N}} = \frac{1}{(n+1)^2}$, then $u_n - v_n \to 0$ but $f(u_n) - f(v_n) = (n+1)^2 - (n+1)^4 \to \infty$, which diverges. So f is not uniformly continuous

2. Let c be a number between 0 and 1. Without the use of Theorem 3.17, show that $f : [c,1] \to \mathbb{R}$ where $f(x) = \frac{1}{x^2}$ is uniformly continuous.

   Let $f(x) = \frac{1}{x^2} : [c,1] \to \mathbb{R}$ Let u_n, v_n be sequence in $[c,1]$. Assume $u_n - v_n \to 0$. Then

$$
\begin{aligned}
\left|f(u_n) - f(v_n)\right| &= \left|u_n^2 - v_n^2\right| \\
&= \left|(u_n + v_n)(u_n - v_n)\right| \\
&= |u_n - v_n| \cdot |u_n + v_n| \\
&\leq |u_n - v_n| \cdot (|u_n| + |v_n|)(\text{By Triangular Inequality}) \\
&\leq |u_n - v_n| \cdot (1+1)\text{since both } u_n \text{ and } v_n \in [c,1]) \\
&= 2|u_n - v_n| \to 0\text{since } u_n - v_n \to 0
\end{aligned}
$$

   So $\left|f(u_n) - f(v_n)\right| \to 0$. Therefore, $f(x)$ is uniformly continuous.

3. Assume that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are uniformly continuous.

   (a) Show by example that $fg : D \to \mathbb{R}$ does not have to be uniformly continuous.

   Let $f(x) = x$, $g(x) = x$, $D = [0,\infty]$. Then both $f(x)$ and $g(x)$ are uniformly continuous but $fg = x^2$ does not.

(b) Show that if f and g are also bounded, then $fg : D \to \mathbb{R}$ will be uniformly continuous.

Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be uniformly continuous and f, g are bounded. Let u_n, v_n be sequence in D, Assume $u_n - v_n \to 0$. Since f and g are bounded, assume $a_f < f(x) < b_f$ for all $x \in D$, $a_g < g(x) < b_g$ for all $x \in D$. Then

$$\begin{aligned}\left|f(u_n)g(u_n) - f(v_n)g(v_n)\right| &= f(u_n)[g(u_n) - g(v_n)] + g(v_n)[f(u_n) - f(v_n)] \\ &< b_f[g(u_n - g(v_n))] + b_g[f(u_n) - f(v_n)] \text{(By construction)} \\ &\to 0 + 0 = 0 \text{since } f \text{ and } g \text{ are uniformly continuous}\end{aligned}$$

So with f and g bounded, fg will also be uniformly continuous.

(c) Show that if D is compact, then $fg : D \to \mathbb{R}$ will be uniformly continuous.

By Extreme Value Theorem, if D is compact, f and g are continuous(uniformly continuous in this case), then f and g have max and min(bounded). So by part(b), it is proved that if f and g are bounded, fg will be uniformly continuous.

4. Determine wheter the following are true are false. If true, explain. If false, give a counter-example.

(a) A monotone function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one.

It is false. Let f be a constant funciton $f(x) = a, a \in \mathbb{R} \forall x \in \mathbb{R}$. In this case, $f(x_1) = f(x_2) = a$ does not necessary implies $x_1 = x_2$. So it is not one-to-one.

(b) A strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one.

It is true. Because f strictly increasing implies if $x_1 < x_2$, then $f(x_1) < f(x_2)$ by the definition. If $f(x_1) = f(x_2)$, x_1 and x_2 has to be equal.

(c) A strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

It is false. Consider a step function $f(x) = \begin{cases} x, & \text{for } x < 1 \\ x+1 & \text{for } x \geq 1 \end{cases}$.

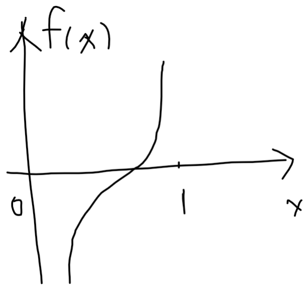
In this case, $f(x)$ is strictly increasing but not continuous.

(d) A one-to-one function $f : \mathbb{R} \to \mathbb{R}$ is continuous. It is false. Consider a step function $f(x) = \begin{cases} x, & \text{for } x < 1 \\ x+1 & \text{for } x \geq 1 \end{cases}$.

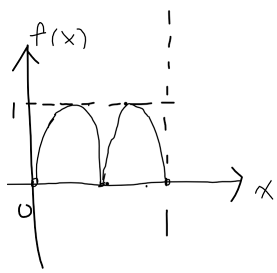
In this case, $f(x)$ is one-to-one but not continuous.

5. For the following, a picture is ok, a formula is better, proving continuity of your function using its formula is the best.

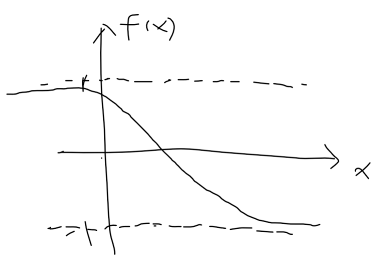
   (a) Find a continuous function $f : (0, 1) \to \mathbb{R}$ with an image equal to $\mathbb{R}$



   (b) Find a continuous function $f : (0, 1) \to \mathbb{R}$ with an image equal to $[0, 1]$.



   (c) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ with an image equal to $(-1, 1)$



6. Prove that there does not exist a strictly increaisng function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(\mathbb{Q}) = \mathbb{R}$.

   Suppose $f : \mathbb{Q} \to \mathbb{R}$ is strictly increasing. Since we have proved in preivous homeworks that $\mathbb{Q}$ is not compact, then there exists a convergent sequence $(a_n) \in \mathbb{Q}$ such that $a_n \to a$ but $a \notin \mathbb{Q}$. Since f is strictly increasing, $f(a_n) < f(a)$ for all n. So $f(a) \in \mathbb{R}$ is not in the image of $f(\mathbb{Q})$.