Solutions to some practice problems

I. *The induction problems.*

1. Base case $n = 1$: This case says $x^0 = \frac{x-1}{x-1}$, which is certainly true since both sides $= 1$.
Inductive step. Suppose (inductive hypothesis) that the result is true for n. We must show it is true for $n+1$. We have

$$\sum_{i=0}^{n} x^i = (\sum_{i=0}^{n-1} x^i) + x^n = \frac{x^n - 1}{x-1} + x^n = \frac{x^{n+1} - 1}{x-1},$$

where the second equality is by the inductive hypothesis. QED.

2. Base case $n = 1$: We have $3^2 - 1 = 8$, which is certainly divisible by 8.
Inductive step: Suppose (inductive hypothesis) $8|(3^{2n} - 1$. We must show $8|3^{2(n+1)} - 1 = 3^{2n+2} - 1$. By inductive hypothesis $3^{2n} - 1 = 8k$ for some $k \in \mathbb{N}$. So

$$3^{2n+2} - 1 = 3^{2n} \cdot 9 - 1 = (8k+1)9 - 1 = 8k + 8 = 8(k+1).$$

QED.

3. Base case $n = 1$. This says $\frac{1}{1\cdot 2} = \frac{1}{2}$, which is certainly true.
Inductive step: Suppose (inductive hypothesis that the result is true for n. We must show it is true for $n+1$. We have

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = (\sum_{i=1}^{n} \frac{1}{i(i+1)}) + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2},$$

where the second equality is by inductive hypothesis.

The triplet primes problem: Let $n = 3k + r$, where $r = 0, 1, 2$.
If $r = 0$ then $3|n$. Since n is prime, we must have $n = 3$, and our triplet is $3, 5, 7$.
It remains to show the other two cases can't occur.
If $r = 1$ then $n + 2 = 3k + 3$ is divisible by 3, so not prime. Hence this case can't occur.
If $r = 2$ then $n + 4 = 3k + 6$, which is again divisible by 3, so not prime. Hence this case can't occur either. QED!.

The prime gaps problem. Consider the number $n! + k$, where $2 \le k \le n$. Since k divides $n!$ (by definition of $n!$), it divides $n! + k$. So $n! + k$ is not prime. Therefore the $n - 1$ consecutive numbers $n! + 2, n! + 3, ..., n! + n$ are all non-primes. Since n is arbitrary, this produces arbitrarily long prime gaps.

Proof by contradiction example. Start by negating the conclusion, i.e. suppose n is not prime, so $n = ab$ with $a, b < n$. Then $n|n$ but n does not divide a or b, contradiction.

6.6: 1. a) $\{x \in \mathbb{R} : 0 < x \le 1\}$.
b) $\{n \in \mathbb{N} : n \ge 2\}$.

c) $\mathbb{Q}$

2. a) $\{(1,6),(6,1),(2,3),(3,2)\}$

b) $(-\infty, 0]$.

c) If $c > 0$ we get a hyperbola lying in the first and third quadrants. If $c < 0$ we get a hyperbola lying in the second and fourth quadrants. If $c = 0$ we get the union of the x-axis and the y-axis.