Math 300D Final Exam Solutions Winter 2016

1. a) There is a prime p and numbers a, b such that $p|ab$ and p does not divide a and does not divide b.

b) If p doesn't divide a and doesn't divide b, then p is not prime or p does not divide ab.

2. a) No. If $y \in Im\, f$ then $y+4$ has a rational 5th root, hence $y+4 = n^5$ for some $n \in \mathbb{Z}$ by a theorem from the notes. So for example, $0 \notin Im\, f$ since 4 is not the fifth power of an integer.

b) Yes. The domain has cardinality 100 and the codomain has cardinality 144. Since $100 < 144$, an injection exists.

3. (10) I'll use proof by contradiction. Suppose S^1 is countable. Define a function $f : S^1 \longrightarrow [-1, 1]$ by $f(x, y) = x$. Then f is surjective, so by one of our theorems $[-1, 1]$ is countable. Since any subset of a countable set is countable, $[0, 1)$ is countable, contradicting Cantor's Theorem. QED.

4. a) Base case $n = 1$: This says $f(1) = 1 \cdot f(1)$, which I'd say is pretty darn clear.

Now suppose (inductive hypothesis) that $f(n) = n \cdot f(1)$. We must show $f(n+1) = (n+1)f(1)$. For this we simply compute

$$f(n+1) = f(n) + f(1) = n \cdot f(1) + f(1) = (n+1)f(1),$$

where the first equality is by the original hypothesis on f, and the second is by the inductive hypothesis.

b) $f(0) = f(0+0) = f(0) + f(0)$, so subtracting $f(0)$ from both sides we get $f(0) = 0$.

c) $0 = f(0) = f(x - x) = f(x + -x) = f(x) + f(-x)$; now solve for $f(-x)$.

d) Suppose f is injective. Then every fiber $f^{-1}x$ has at most one element, so $f^{-1}\{0\} = \{0\}$.

Conversely, suppose $f^{-1}\{0\} = \{0\}$. Then suppose $f(x_1) = f(x_2)$. Then using part (b) we get

$$0 = f(x_1) - f(x_2) = f(x_1 - x_2).$$

Hence $x_1 - x_2 \in f^{-1}\{0\}$, so $x_1 - x_2 = 0$ by hypothesis. Then $x_1 = x_2$, QED.

5.(20) Let X denote the set of pairs A, B of subsets of $[n]$ such that $|A| = 3$, $|B| = 8$, and $A \subset B$. (We assume that $n \geq 8$ so that the problem makes sense.)

a) First we choose the set B; there are $\binom{n}{8}$ subsets B of cardinality 8. Then choose three elements from B, yielding $|X| = \binom{n}{8}\binom{8}{3}$.

Alternatively, one could choose A first, then complete it to a set of 8 elements by choosing 5 elements from the complement of A. This yields $|X| = \binom{n}{3}\binom{n-3}{5}$. Both formulas are correct.

b)

$$\binom{9}{8}\binom{8}{3} = 9 \cdot \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 504.$$

6. I'll first prove a handy lemma. It's intuitively pretty obvious if you just look at examples, but we need to prove it!

Lemma: Suppose $(a_1, a_2, ...)$ has period d. For any $n \in \mathbb{N}$, let $n = sd + r$ as in the division theorem. Then $a_n = a_r$ if $r \neq 0$, and $a_n = a_d$ if $r = 0$. (This funny shift could have been avoided if I'd numbered my sequences starting with a_0.)

Proof of lemma: By induction on s. The base case $s = 0$ just says $a_r = a_r$ (note $r = 0$ doesn't occur in the base case.) Now suppose (inductive hypothesis) that the result is true for s; we must prove it for $s + 1$. If $r \neq 0$, we have

$$a_{(s+1)d+r} = a_{sd+r} = a_r,$$

where the first equality is by definition of "periodic" and the second by inductive hypothesis. In the case $r = 0$ we similarly obtain

$$a_{(s+1)d} = a_{sd} = a_d.$$

This completes the proof of the lemma.

Now let X denote the set of all periodic sequences, and let X_d denote the periodic sequences that have period d. Define a function $f : X_d \longrightarrow \mathbb{Z}^d$ by $f(a_1, a_2, ...) = (a_1, a_2, ..., a_d)$. (In other words, just pick off the first d entries of the sequence.) Then f is injective by the lemma (which shows that the sequence is uniquely determined by its first d entries). Since $\mathbb{Z}$ is countable, and any finite product of countable sets is countable, $\mathbb{Z}^d$ is countable. Then since f is injective, by a theorem from the notes we conclude that X_d is countable. Finally, X is the union of the X_d's, where $d \in \mathbb{N}$. Hence X is the union of a countable number of countable sets, hence is countable.