STAT 391 Homework 2

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1. Problem 1 - Estimation of small probabilities

a Estimate $\hat{\theta}_a, \hat{\theta}_b, \cdots \hat{\theta}_z$ from the given text I bascially use the same setup as I did for HW1 Question 4. The only difference is that I used $\theta^{M\ L} = 0$ for those letters that never appear.

```
import math

# Helper Methods
def langReader(file):
    '''
    langReader is used to read in files and compute for the
        probability
    for each letter

    Args:
        file: The name of the file containing the letter and
            probability.
    Returns:
        A dictionary containing letters as keys and probability
            as values
    '''
    pLang = {}
    for line in open(file):
        el = line.split(' ')
        letter = el[1].lower()
        pLang[letter] = float(el[2])/1000
```

```
    return pLang

def LetterCounter(testString):
    '''
    Counts for the number of each letter in the test string
        (sufficient statistics)

    Args:
        testString: The string to count
    Returns:
        A dictionary containing letters as keys and count
            number as values
    '''
    counter = {}
    for char in testString:
        try:
            counter[char] = counter[char] + 1
        except KeyError:
            counter[char] = 1
    return counter

# Initialize the probability map
path =
    r'C:\Users\johnn\Documents\UW\SchoolWorks\2018Spring\STAT391\HW1'

english = langReader(path + r'\english.dat')

# Read in lincoln_text.txt
txt = ''
for line in open(path + r'\lincoln_text.txt'):
    txt = txt + line
txt = ''.join(e for e in txt if e.isalnum()).lower()
counter = LetterCounter(txt)
length = len(txt)

# Since there might be some missing letters
lincolnEng = {}
for letter in english:
    if letter not in counter:
        counter[letter] = 0
```

```python
    lincolnEng[letter] = counter[letter] / length
print(lincolnEng)
```

As the result, I got my $\hat{\theta}$ to be

```
{'a': 0.07108239095315025, 'b': 0.015347334410339256,
'c': 0.020193861066235864, 'd': 0.029079159935379646,
'e': 0.11793214862681745, 'f': 0.02665589660743134,
'g': 0.021809369951534735, 'h': 0.05492730210016155,
'i': 0.07673667205169628, 'j': 0.0024232633279483036,
'k': 0.004038772213247173, 'l': 0.045234248788368334,
'm': 0.024232633279483037, 'n': 0.06704361873990307,
'o': 0.08239095315024232, 'p': 0.01615508885298869,
'q': 0.0008077544426494346, 'r': 0.08077544426494346,
's': 0.05977382875605816, 't': 0.0888529886914378,
'u': 0.03715670436187399, 'v': 0.01050080775444265,
'w': 0.018578352180936994, 'x': 0.0,
'y': 0.027463651050080775, 'z': 0.0008077544426494346}
```

b Let S be the sample space $\{a, b, c, \cdots, z\}$, with $m = |S| = 26$. Determine the sets $S_0, S_1, \cdots, S_n$, where $S_k = \{j \in S, \ n_j = k\}$.

```python
sample_count = LetterCounter(txt)
sk = {}
for letter in sample_count:
    if sample_count[letter] not in sk:
        sk[sample_count[letter]] = [letter]
    else:
        sk[sample_count[letter]].append(letter)
print(sk)
```

```
{23: ['w'], 68: ['h'], 88: ['a'], 110: ['t'], 25: ['c'],
102: ['o'], 83: ['n'], 74: ['s'], 95: ['i'], 46: ['u'],
146: ['e'], 19: ['b'], 56: ['l'], 100: ['r'], 5: ['k'],
33: ['f'], 34: ['y'], 36: ['d'], 20: ['p'], 27: ['g'],
30: ['m'], 13: ['v'], 1: ['z', 'q'], 3: ['j']}
```

c Let $r_k = |S_k|$ and r be the number of unique letters observed in the Lincoln-English corpus above. Verify that $r = \sum_{k=1}^{n} r_k$, $m = \sum k = 0^n r_k$,and $n = \sum_{k=1}^{n} kr_k$.

```
r = sum(len(sk[k]) for k in sk)
print(r)

n = sum(k*len(sk[k]) for k in sk)
print(n)

print(len(txt))
```

And the result I got is

$$
\begin{aligned}
r &= |S_1| + |S_3| + \cdots + |S_146| \\
&= 25 \\
m &= |S_0| + r \\
&= 26 \\
n &= 1 * |S_1| + 3 * |S_3| + \cdots + 146 * |S_146| \\
&= 1238 \\
length &= 1238
\end{aligned}
$$

2. Problem 2 - Estimate letter probabilities from text

1. Get the sufficient statistics. Only print out $n_{a:j}$

```
import math

# Helper Methods
def LetterCounter(testString):
    '''
    Counts for the number of each letter in the test string
        (sufficient statistics)

    Args:
        testString: The string to count
    Returns:
```

```
        A dictionary containing letters as keys and count
            number as values
    '''
    counter = {}
    for char in testString:
        try:
            counter[char] = counter[char] + 1
        except KeyError:
            counter[char] = 1
    return counter

# Initialize the probability map
path =
    r'C:\Users\johnn\Documents\UW\SchoolWorks\2018Spring\STAT391\HW2'
english = langReader(path + r'\english.dat')

# Read in mlk-letter-estimation.txt
txt = ''
for line in open(path + r'\hw2-mlk-letter-estimation.txt'):
    txt = txt + line
txt = ''.join(e for e in txt if e.isalnum()).lower()
counter = LetterCounter(txt)
for letter in english: # Update missing letters
    if letter not in counter:
        counter[letter] = 0
length = len(txt)

for k in counter:
    if k <= 'j':
        print('n'+k+'='+str(counter[k]))
```

And the output is

```
na=24
ne=44
nf=17
nh=19
ng=3
nd=10
```

```
ni=29
nc=18
nb=2
nj=0
```

What is the fingerprint r_k of this dataset?

```python
sk = {}
for letter in counter:
    if counter[letter] not in sk:
        sk[counter[letter]] = [letter]
    else:
        sk[counter[letter]].append(letter)
rk = {}
for k in sk:
    rk[k] = len(sk[k])
    print('r'+str(k)+'='+str(rk[k]))
```

```
r39=1
r32=1
r16=1
r24=1
r4=1
r44=1
r11=1
r28=1
r17=1
r15=1
r19=1
r5=1
r3=2
r10=1
r29=1
r6=1
r18=1
r9=1
r2=2
r7=1
r0=4
```

2. Compute the ML estimates $\theta_{A:Z}^{ML}$

```
print('ML Estimation:')
ML = {}
for k in sk:
    letter = sk[k]
    for i in letter:
        ML[i] = k / length
        if letter.index(i) == 0:
            print('theta_'+i+'='+str(ML[i]))
        else:
            print('theta_'+i+'='+'theta_'+letter[0])
```

```
ML Estimation:
theta_t=0.11370262390670553
theta_o=0.09329446064139942
theta_s=0.04664723032069971
theta_a=0.06997084548104957
theta_v=0.011661807580174927
theta_e=0.1282798833819242
theta_m=0.03206997084548105
theta_n=0.08163265306122448
theta_f=0.04956268221574344
theta_r=0.043731778425655975
theta_h=0.05539358600583091
theta_p=0.014577259475218658
theta_g=0.008746355685131196
theta_w=theta_g
theta_d=0.029154518950437316
theta_i=0.08454810495626822
theta_y=0.01749271137026239
theta_c=0.052478134110787174
theta_u=0.026239067055393587
theta_b=0.0058309037900874635
theta_k=theta_b
theta_l=0.02040816326530612
theta_j=0.0
```

```
theta_q=theta_j
theta_x=theta_j
theta_z=theta_j
```

3. Compute Laplace $\theta_{A:Z}^{Lap}$ of the same probability

```
print('Laplace Estimation:')
lap = {}
m = 26
for k in sk:
    letter = sk[k]
    for i in letter:
        lap[i] = (k + 1) / (length + m)
        if letter.index(i) == 0:
            print('theta_'+i+'='+str(lap[i]))
        else:
            print('theta_'+i+'='+'theta_'+letter[0])
```

```
Laplace Estimation:
theta_t=0.10840108401084012
theta_o=0.08943089430894309
theta_s=0.04607046070460705
theta_a=0.06775067750677506
theta_v=0.013550135501355014
theta_e=0.12195121951219512
theta_m=0.032520325203252036
theta_n=0.07859078590785908
theta_f=0.04878048780487805
theta_r=0.04336043360433604
theta_h=0.05420054200542006
theta_p=0.016260162601626018
theta_g=0.01084010840108401
theta_w=theta_g
theta_d=0.02981029810298103
theta_i=0.08130081300813008
theta_y=0.018970189701897018
theta_c=0.051490514905149054
theta_u=0.02710027100271003
```

```
theta_b=0.008130081300813009
theta_k=theta_b
theta_l=0.02168021680216802
theta_j=0.0027100271002710027
theta_q=theta_j
theta_x=theta_j
theta_z=theta_j
```

4. Compute the Witten-Bell estimates $\theta_{A:Z}^{W\ B}$

```
r = sum(len(sk[k]) for k in sk if k!=0)
print('Witten-Bell Estimation:')
wb = {}
for k in sk:
    letter = sk[k]
    for i in letter:
        if k != 0:
            wb[i] = k / (length + r)
        else:
            wb[i] = 1 / (m - r) * r / (length + r)
        if letter.index(i) == 0:
            print('theta_'+i+'='+str(wb[i]))
        else:
            print('theta_'+i+'='+'theta_'+letter[0])
```

```
Witten-Bell Estimation:
theta_t=0.10684931506849316
theta_o=0.08767123287671233
theta_s=0.043835616438356165
theta_a=0.06575342465753424
theta_v=0.010958904109589041
theta_e=0.12054794520547946
theta_m=0.030136986301369864
theta_n=0.07671232876712329
theta_f=0.04657534246575343
theta_r=0.0410958904109589
theta_h=0.052054794520547946
theta_p=0.0136986301369863
```

```
theta_g=0.00821917808219178
theta_w=theta_g
theta_d=0.0273972602739726
theta_i=0.07945205479452055
theta_y=0.01643835616438356
theta_c=0.049315068493150684
theta_u=0.024657534246575342
theta_b=0.005479452054794521
theta_k=theta_b
theta_l=0.019178082191780823
theta_j=0.015068493150684932
theta_q=theta_j
theta_x=theta_j
theta_z=theta_j
```

5. Compute the smoothed Good-Turing estimates $\theta_{A:Z}^{GT}$

```python
print('Good-Turing Estimation:')
gt = {}
for k in sk:
    letter = sk[k]
    for i in letter:
        try:
            gt[i] = ((k + 1) * rk[k+1] / rk[k]) / length
        except KeyError:
            gt[i] = 0
        if letter.index(i) == 0:
            print('theta_'+i+'='+str(gt[i]))
        else:
            print('theta_'+i+'='+'theta_'+letter[0])
```

```
Good-Turing Estimation:
theta_t=0
theta_o=0
theta_s=0.04956268221574344
theta_a=0
theta_v=0.014577259475218658
theta_e=0
```

```
theta_m=0
theta_n=0.08454810495626822
theta_f=0.052478134110787174
theta_r=0.04664723032069971
theta_h=0
theta_p=0.01749271137026239
theta_g=0.0058309037900874635
theta_w=theta_g
theta_d=0.03206997084548105
theta_i=0
theta_y=0.02040816326530612
theta_c=0.05539358600583091
theta_u=0.029154518950437316
theta_b=0.008746355685131196
theta_k=theta_b
theta_l=0
theta_j=0
theta_q=theta_j
theta_x=theta_j
theta_z=theta_j
```

6. Compute the Ney-Essen estimates $\theta^{N\ E}_{A:Z}$, taking $\delta = 1$

```
print('\nNey-Essen Estimation:')
D = 0
delta = 1
for letter in counter:
    D = D + min(counter[letter], delta)
ne = {}
for k in sk:
    letter = sk[k]
    for i in letter:
        ne[i] = (k - min(k, delta) + D / m) / length
        if letter.index(i) == 0:
            print('theta_'+i+'='+str(ne[i]))
        else:
            print('theta_'+i+'='+'theta_'+letter[0])
```

```
Ney-Essen Estimation:
theta_t=0.11325409284592958
theta_o=0.09284592958062346
theta_s=0.04619869925992375
theta_a=0.06952231442027361
theta_v=0.01121327651939897
theta_e=0.12783135232114823
theta_m=0.031621439784705094
theta_n=0.08118412200044853
theta_f=0.049114151154967485
theta_r=0.04328324736488002
theta_h=0.054945054945054944
theta_p=0.014128728414442699
theta_g=0.008297824624355236
theta_w=theta_g
theta_d=0.02870598788966136
theta_i=0.08409957389549226
theta_y=0.017044180309486432
theta_c=0.05202960305001122
theta_u=0.025790535994617628
theta_b=0.005382372729311505
theta_k=theta_b
theta_l=0.01995963220453016
theta_j=0.002466920834267773
theta_q=theta_j
theta_x=theta_j
theta_z=theta_j
```

7. Now use the estimates you obtained to compute the log-probability of the text in hw2-test-letter-estimation-large.txt. Also compute the log-probability of the training data hw2-mlk-letter-estimation.txt

```python
from decimal import Decimal
# Helper Methods
def computeP(counter, probMap):
    '''
    Compute for the probability of with given letter counter
        and language probability map
```

```
    Args:
        counter: The letter counter of the test string.
        probMap: The probability map of the test language.
    Returns:
        A integer telling P(sentence)
    '''
    p = 1
    for letter in counter:
        try:
            p = p * Decimal(probMap[letter]**counter[letter])
        except KeyError:
            print("The letter ", letter, " is not in this
                language")
    return p

def MaxLogLikelihood(fileName, langDict):
    '''
    Find the maximum log-likelihood according to the fileName
        and output the guess.

    Args:
        fileName: The file name with txt to test
        langDict: The dictionary for all languages.
    Output:
        The log-likehood for each language and the best guess.
    '''
    print("\nConsidering the file: ", fileName)
    # Remove spaces and punctuation
    txt = ''
    for line in open(path + fileName):
        txt = txt + line
    testStr = ''.join(e for e in txt if e.isalnum()).lower()
    counter = LetterCounter(testStr)
    best = [-float('inf'), '']
    for lang in langDict:
        p = computeP(counter, langDict[lang])
        ll = p.ln() / Decimal(math.log(2))
        print("The log-likelihood for ", lang, " is ", ll)
        if best[0] < ll:
```

```
            best = [ll, lang]
    print("And as the result, the best guess is ", best[1], "
        with likelihood ", best[0], "\n")

langDict = {'ML Estimation':ML,\
            'Laplace Estimation':lap,\
            'Witten-Bell Estimation':wb,\
            'Smoothed Good-Turning Estimation':gt,\
            'Ney-Essen Estimation':ne}

MaxLogLikelihood(r'\hw2-mlk-letter-estimation.txt', langDict)
MaxLogLikelihood(r'\hw2-test-letter-estimation-large.txt',
    langDict)
```

```
Considering the file:  \hw2-mlk-letter-estimation.txt
The log-likelihood for  ML Estimation  is
-1380.9535216268537474873857811
The log-likelihood for  Laplace Estimation  is
-1387.341205133273515356164497
The log-likelihood for  Witten-Bell Estimation  is
-1411.716467071790303072722354
The log-likelihood for  Smoothed Good-Turning Estimation  is
-Infinity
The log-likelihood for  Ney-Essen Estimation  is
-1385.893096046358672696316814
And as the result, the best guess is  ML Estimation  with likelihood
-1380.9535216268537474873857811


Considering the file:  \hw2-test-letter-estimation-large.txt
The log-likelihood for  ML Estimation  is
-Infinity
The log-likelihood for  Laplace Estimation  is
-8814.732094912244450343732977
The log-likelihood for  Witten-Bell Estimation  is
-8969.144788769402325042405529
The log-likelihood for  Smoothed Good-Turning Estimation  is
-Infinity
```

```
The log-likelihood for  Ney-Essen Estimation  is
-8835.935483824767400464608135
And as the result, the best guess is  Laplace Estimation  with
likelihood
-8814.732094912244450343732977
```

From the result above, we can see that Laplace Estimation gives the highest likelihood for the new text data. And Max-Likelihood Method gives the highest likelihood for the training data.

3. Problem 3 - ML estimation

1. Write the expression of the probability $P(3,2,1,1,6)$

$$P(3,2,1,1,6) = \theta_o^3 \theta_e$$

2. Write the expression of $l(\theta_o, \theta_e)$ the log-likelihood of data set D as a function of θ_o, θ_e and the counts $n_{1:6}$

$$l(\theta_o, \theta_e) = n_1 log\theta_o + n_2 log\theta_e + n_3 log\theta_o + n_4 log\theta_e + n_5 log\theta_o + n_6 log\theta_e$$

3. Transform $l(\theta_o, \theta_e)$ into a function of one variable $l(\theta_e)$

$$l(\theta_o, \theta_e) = n_1 log(\frac{1}{3} - \theta_e) + n_2 log\theta_e + n_3 log(\frac{1}{3} - \theta_e) + n_4 log\theta_e + n_5 log(\frac{1}{3} - \theta_e) + n_6 log\theta_e$$

4. Now find the ML estimate of θ_e by equating the derivative of $l(\theta_e)$ with 0.

$$\begin{aligned}
l^{'}(\theta_e) = 0 &= -\frac{n_1}{\frac{1}{3} - \theta_e} + \frac{n_2}{\theta_e} - \frac{n_3}{\frac{1}{3} - \theta_e} + \frac{n_4}{\theta_e} - \frac{n_5}{\frac{1}{3} - \theta_e} + \frac{n_6}{\theta_e} \\
0 &= \frac{-3(n_1 + n_3 + n_5)}{1 - 3\theta_e} + \frac{n_2 + n_4 + n_6}{\theta_e} \\
0 &= -3(n_1 + n_3 + n_5)\theta_e + (n_2 + n_4 + n_6)(1 - 3\theta_e) \\
n_2 + n_4 + n_6 &= 3\theta_e(n_1 + \cdots + n_6) \\
\theta_e &= \frac{n_2 + n_4 + n_6}{3\sum_{i=1}^{6} n_i}
\end{aligned}$$

5. Explain why this result is natural
   This result is reasonable since the θ_e is found by the number of evens divided by 3, which is exactly the result I got from the computation above. Therefore, the ML I found is intuitive.

4. Problem4 - The ML estimate as a random variable

1. What is the set of possible values S_{θ_1} for θ_1^{ML}? Does the true θ_1 belong to S_{θ_1}?

   The set of possible values S_{θ_1} is

   $$S_{\theta_1} = \{\frac{100}{i} \; for \; i \in \mathbb{N}, 0 \leq i \leq 100\}$$

   The true θ_1 does not belong to this set since $i = 100 * 0.3141 = 31.41 \notin \mathbb{N}$

2. Write the expression of the probability of each outcome in S_{θ_1}

   $$P(\theta_1^{ML} = \frac{i}{100}) = \binom{i}{100} \theta_1^i (1 - \theta_1)^{100-i}$$

3. Make a plot of the probability distribution of θ_1^{ML}

```
import numpy as np
import math
import matplotlib.pyplot as plt

n = 100
p = 0.3141

log_P = [0]*101
for i in range(0, 101): # using ln-gamma to avoid overflow
    log_P[i] = math.log(math.gamma(n+1)) -
        math.log(math.gamma(i+1)) -\
            math.log(math.gamma(n-i+1)) + i*math.log(p) +
                (n-i)*math.log(1-p)
theta = [math.exp(log_P[i]) for i in range(len(log_P))]
```

```python
x = np.arange(0, 1.01, step=0.01)
plt.stem(x, theta)
plt.xlabel('theta_1')
plt.title('Probability of theta_1')
plt.show()

plt.stem(x, theta)
plt.xlabel('theta_1')
plt.title('Probability of theta_1')
plt.xlim([0.195, 0.405])
plt.show()
```

Figure 1 is the probability distribution of θ_1.

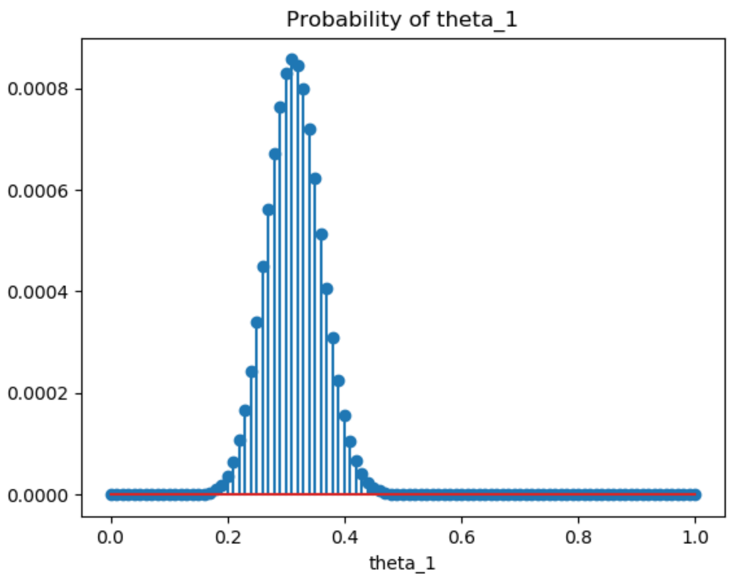
4. Let $\epsilon = 0.02$. Answer using the probability distribution computed previously

```python
e = 0.02
print('P{Absolute Error > 0.02} =',\
    sum(theta[i] for i in np.where(abs(x - p) > e)[0]))
print('P{Related Error > 0.02} =',\
    sum(theta[i] for i in np.where(abs(((x - p) / p)) > e)[0]))
```

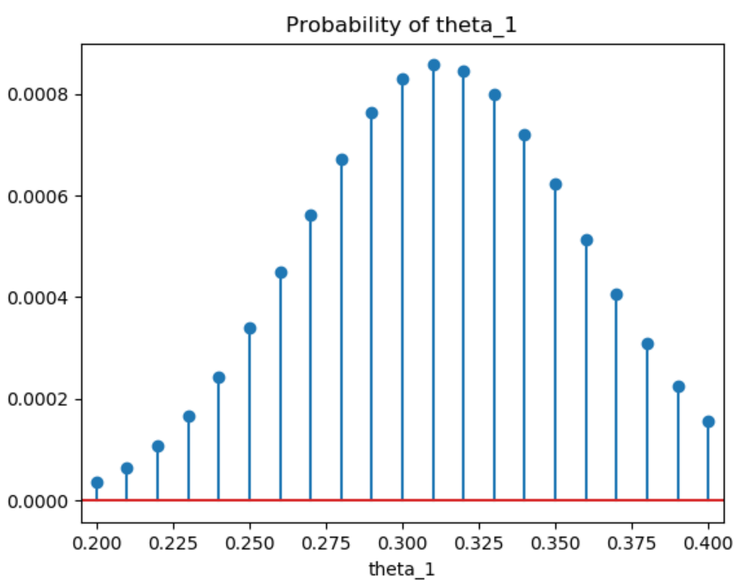
$$
\begin{aligned}
\delta_{abs} = P[|\theta_1^{ML} - \theta_1| > \epsilon] &\qquad = 0.6671038038886307 \\
\delta_{rel} = P[|\frac{\theta_1^{ML} - \theta_1}{\theta_1}| > \epsilon] &\qquad = 0.829757447824191
\end{aligned}
$$

5. For $\epsilon = 0, 0.005, 0.001, \cdots, 1$ plot the graph of $\delta_{abs} = P[|\theta_1^{ML} - \theta_1| > \epsilon]$ vs ϵ. Is the function $\delta(\epsilon)$ monotonically increasing, decreasing or neither?

```python
i = 0
epsilons = np.arange(0, 1, step=0.005)
delta = [0]*len(epsilons)
for e in epsilons:
    delta[i] = sum(theta[i] for i in np.where(abs(x - p) >
        e)[0])
    i = i + 1
```



(a) Plot of probability distribution of $theta_1$



(b) Plot of probability distribution of $theta_1$, enlarged from $\theta_1 = 20$ to $\theta_1 = 40$

Figure 1: Probability distribution of $theta_1$

```
plt.plot(epsilons, delta)
plt.xlabel('epsilon')
plt.title('Delta(epsilon) vs. epsilon')
plt.show()
```

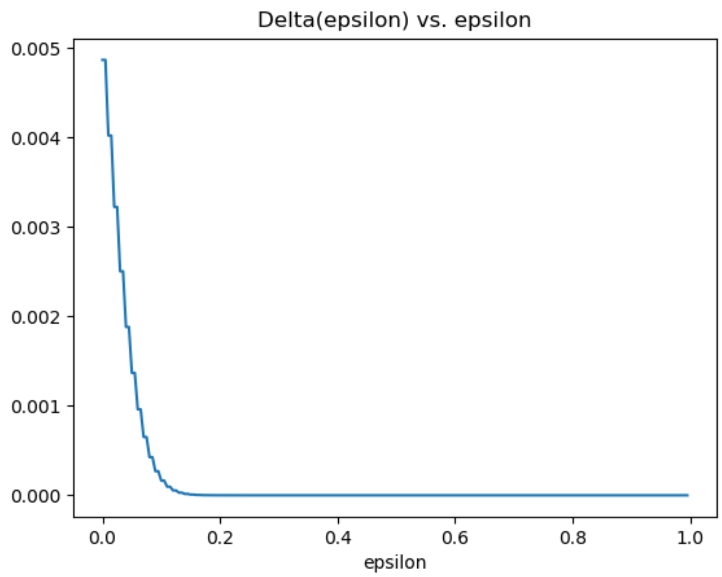


Figure 2: $\delta(\epsilon)$ vs. ϵ

From the figure 2, we can clearly see that the function $\delta(\epsilon)$ is monotonically decreasing(non-increasing).

6. Simulate tossing the coin with $\theta_1 = 0.3141$, $n = 100$ times and compute θ_1^{ML}. What is the value you θ_1^{ML} have obtained, and what are the absolute and relative error?

```
# Simulation
np.random.seed(999)
y = sum(np.random.binomial(1, p, n))
theta = y / n
print(theta - p)
print((theta - p) / p)
```

```
Absolute Error of Simulation =  0.08590000000000003
```

```
Relative Error of Simulation =  0.27347978350843694
```

Further, I have made 1000 observations to see the behavior of simulated θ_1

```
observations = 1000

yy = np.repeat(0, n+1)
for k in range(observations):
    yi = sum(np.random.binomial(1, p, n))
    yy[yi] = yy[yi] + 1

theta = [yy[i] / (n*observations) for i in range(0, n+1)]
x = np.arange(0, 1.01, step=0.01)
plt.stem(x, theta)
plt.xlabel('Simulated theta_1')
plt.title('Probability distribution of simulated theta_1')
plt.show()

e = 0.02
print('P{Absolute Error > 0.02}=',\
       sum(theta[i] for i in np.where(abs(x - p) > e)[0]))
print('P{Related Error > 0.02}=',\
       sum(theta[i] for i in np.where(abs((x - p) / p) >
           e)[0]))
```

```
P{Absolute Error > 0.02} = 0.6730000000000002
P{Related Error > 0.02} = 0.8120000000000003
```

7. Let $\theta_1^{'}$ have the value θ_1^{ML} of the previous question. Repeat Question 3-6 using "the guess $\theta_1^{'}$ instead of "the truth" θ.

```
#============================================================#
# Use the guess theta
p = theta_1

log_P = [0]*101
for i in range(0, 101): # using ln-gamma to avoid overflow
```

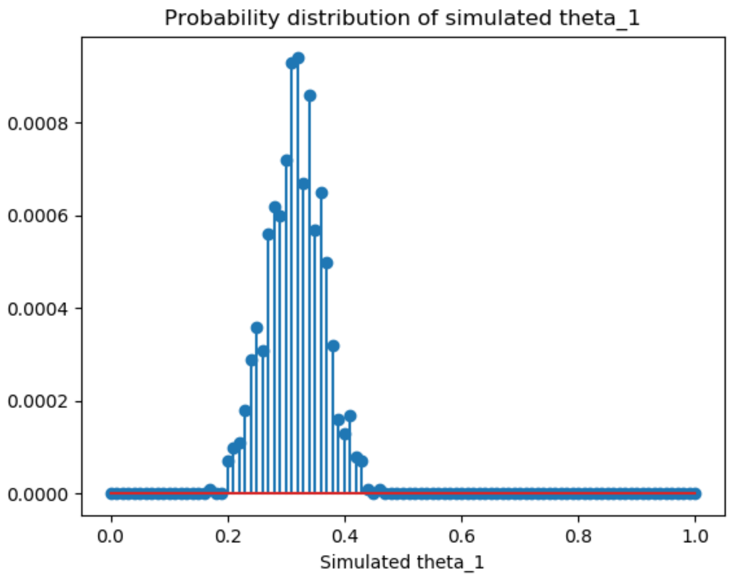


Figure 3: Probability distribution of simulated θ_1

```
    log_P[i] = math.log(math.gamma(n+1)) -
        math.log(math.gamma(i+1)) -\
            math.log(math.gamma(n-i+1)) + i*math.log(p) +
                (n-i)*math.log(1-p)
y = [math.exp(log_P[i]) for i in range(len(log_P))]
plt.stem(y)
plt.xlabel('k')
plt.title('Probability of observing outcome 1 with k times')
plt.show()

theta = [y[i] / n for i in range(0, n+1)]
x = np.arange(0, 1.01, step=0.01)
plt.stem(x, theta)
plt.xlabel('theta_1')
plt.title('Probability of theta_1')
plt.show()

e = 0.02
print('P{Absolute Error > 0.02}=',\
        sum(theta[i] for i in np.where(abs(x - p) > e)[0]))
```

```
print('P{Related Error > 0.02}=',\
       sum(theta[i] for i in np.where(abs((x - p) / p) >
           e)[0]))

i = 0
epsilons = np.arange(0, 1, step=0.005)
delta = [0]*len(epsilons)
for e in epsilons:
    delta[i] = sum(theta[i] for i in np.where(abs(x - p) >
        e)[0])
    i = i + 1
plt.plot(epsilons, delta)
plt.xlabel('epsilon')
plt.title('Delta(epsilon) vs. epsilon')
plt.show()

# Simulation
np.random.seed(999)
y = sum(np.random.binomial(1, p, n))
theta = y / n
print('Absolute Error of Simulation = ', abs(theta - p))
print('Relative Error of Simulation = ', abs((theta - p) / p))
```

From the figure 6, we can see that the function is still monotonically decreasing(non-increasing). And for the error part,

```
P{Absolute Error > 0.02}= 0.00685447787126666
P{Related Error > 0.02}= 0.009187808550038788
Absolute Error of Simulation =  0.06999999999999995
Relative Error of Simulation =  0.17499999999999988
```

5. Problem 5- Rare outcomes and data set size

1. What is the probability that the outcome sequence contains no 1′s?

$$P(n_1 = 0) = \theta_1^0 * (1 - \theta_1)^n$$

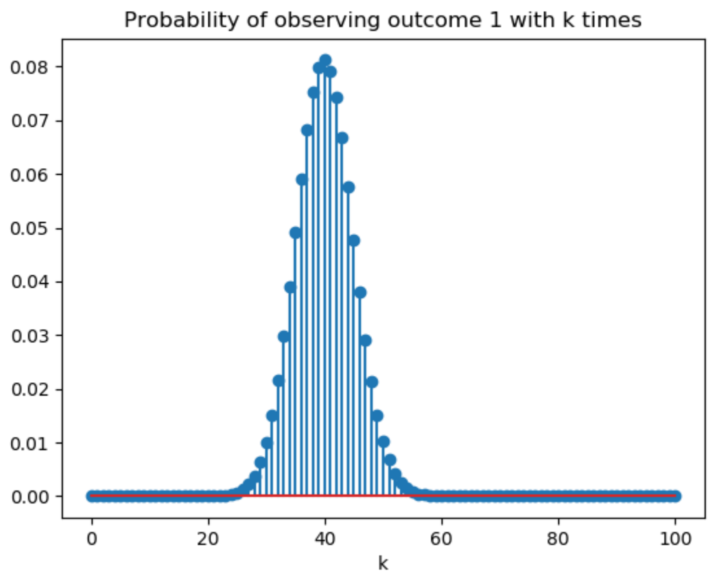


Figure 4: Probability distribution of outcome 1 using guess θ

2. What is the probability that the outcome sequence contains a single 1?

$$P(n_1 = 1) = \theta_1^1 * (1 - \theta_1)^{n-1}$$

3. Solve $p_0 = p_1$

$$\begin{aligned}
p_0 &\approx p_1 \\
P(n_1 = 0) &= P(n_1 = 0) + \epsilon \\
\theta_1^0 * (1 - \theta_1)^n &= \theta_1^1 * (1 - \theta_1)^{n-1} + \epsilon \\
(1 - \theta_1)^n &= \theta_1^1 * (1 - \theta_1)^{n-1} + \epsilon
\end{aligned}$$

   Say, the tolerance to be $\epsilon = 10^{-10}$

4. Compute the above n for $\theta_1 = 10^{-3}, 10^{-4}, 10^{-5}$.

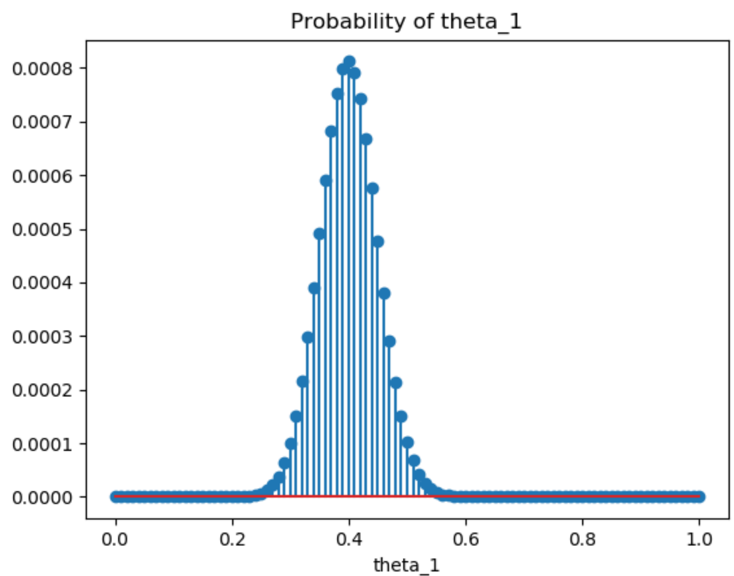


Figure 5: Probability distribution of θ_1 using guess θ

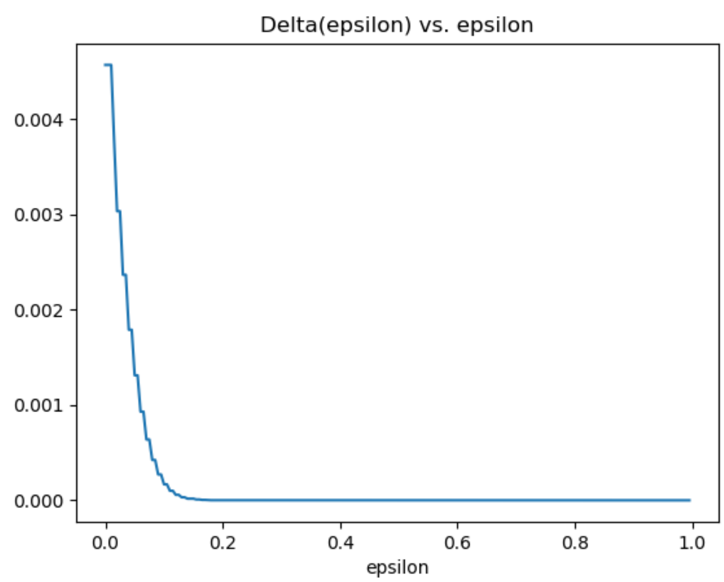


Figure 6: $\delta(\epsilon)$ vs. ϵ using guess θ