Chapter 3

3.1 Continuity

Let $D \subseteq \mathbb{R}$ (domain)

$f : D \to \mathbb{R}$ - a real valued function

Definition A function $f : D \to \mathbb{R}$ is continuous at x_0 in D if

$(x_n)_{n \in \mathbb{N}}$ in D, $x_n \to x_0$

implies

$f(x_n) \to f(x_0)$ in $\mathbb{R}$.

example ① $D = \mathbb{R}$, $f(x) = 2x+3$

Then, f is continuous at $5 \in D$.

Assume $x_n \to 5$.

Then $2x_n \to 2\cdot 5 = 10$, by Thm. on convergence (i)

$2x_n + 3 \to 2\cdot 5 + 3 = 13$ by Thm on convergence (ii)

So $f(x_n) \to f(5)$.

f is continuous at 5.

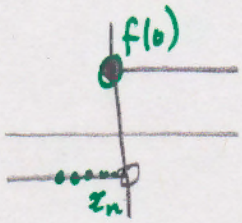
Generalize: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax+b$ is continuous at any $x_0 \in \mathbb{R}$.

Generalize further: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a_n x^n + \cdots + a_1 x + a_0$ is continuous at any $x_0 \in \mathbb{R}$.

example ② $f(x): \mathbb{R} \to \mathbb{R}$

$$x \to 1 \quad x \geq 0$$
$$x \to -1 \quad x < 0$$

is not continuous at 0 because



$-\frac{1}{n} \to 0$

but $f(-\frac{1}{n}) \to -1 \neq f(0)$.

f is continuous at any $x_0 \neq 0$.

example ③ $f(x): \mathbb{R} \to \mathbb{R}$

$$x \to 1 \quad x \in \mathbb{Q}$$
$$x \to -1 \quad x \notin \mathbb{Q}$$

is not continuous at <u>any</u> $x_0 \in \mathbb{R}$.

If $x_0 \in \mathbb{Q}$, find $x_n \to x_0$ with $x_n \notin \mathbb{Q}$
(the irrationals are dense in $\mathbb{R}$)
then $f(x_n) = -1 \to -1$ but $f(x_0) = 1$

If $x_0 \notin \mathbb{Q}$, find $x_n \to x_0$ with $x_n \in \mathbb{Q}$
(the rationals are dense in $\mathbb{R}$)
then $f(x_n) = 1 \to 1$ but $f(x_0) = -1$.

example ④ $f(x): \mathbb{R} \to \mathbb{R}$.

$$x \to x \quad x \in \mathbb{Q}$$
$$x \to 0 \quad x \notin \mathbb{Q}$$

is continuous at 0 but not anywhere else.

If $x_n \to 0$, then $f(x_n) = \begin{cases} x_n & x_n \in \mathbb{Q} \\ 0 & x_n \notin \mathbb{Q} \end{cases} \to 0$

If $a \in \mathbb{Q}$, find $x_n \notin \mathbb{Q}, x_n \to a$ then $f(x_n) = 0 \not\to a$.

If $a \notin \mathbb{Q}$, find $x_n \in \mathbb{Q}, x_n \to a$ then $f(x_n) = x_n \to a \neq f(a)$

Theorem 3.4 (Sums, Products, Quotients of Cts. Functions)

Assume $f, g : D \to \mathbb{R}$ are continuous at $x_0 \in D$.

Then

(i) $\alpha f : D \to \mathbb{R}$, $\alpha \in \mathbb{R}$, is continuous at x_0.
$x \to \alpha f(x)$

(ii) $f+g : D \to \mathbb{R}$ is continuous at x_0.
$x \to f(x)+g(x)$

(iii) $fg : D \to \mathbb{R}$ is continuous at x_0.
$x \to f(x)g(x)$

(iv) If $g(x) \neq 0$ for any $x \in D$,
$\frac{f}{g} : D \to \mathbb{R}$ is continuous at x_0.
$x \to \frac{f(x)}{g(x)}$

Proof. All will follow from the definition of continuity and Thm. on convergence of sequences.

For example,

(iii) Assume f, g are cts. at $x_0 \in D$

To show continuity of fg, let $x_n \to x_0$.

Then $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$

So by part (iii) of Thm. on convergence

$f(x_n)g(x_n) \to f(x_0)g(x_0)$

i.e. $(fg)(x_n) \to (fg)(x_0)$

So, fg is cts. at $x_0 \in D$.

Corollary: Any rational function $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials is continuous every where on its domain
$$D = \{x \in \mathbb{R} : q(x) \neq 0\}.$$

In particular, any polynomial is continuous on $\mathbb{R}$.

Theorem 3.6 (Composition of Cts. functions)

Assume $f: D \to \mathbb{R}$ and $g: U \to \mathbb{R}$ where $f(D) \subseteq U$,
f is cts. at $x_0 \in D$ and g is cts. at $f(x_0) \in U$.
Then, the composition
$$g \circ f : D \to \mathbb{R}$$
$$x \to g(f(x))$$
is continuous at x_0.

Proof. Assume $x_n \to x_0$ in D.
Since f is cts. at x_0, $f(x_n) \to f(x_0)$.
Since g is cts. at $f(x_0)$, $g(f(x_n)) \to g(f(x_0))$.
Therefore, $g \circ f$ is cts. at x_0.

3.2 The Extreme Value Theorem

Let $f : D \to \mathbb{R}$.

If $f(D)$ has a maximum ($\sup f(D) \in f(D)$),
then there is $x_0 \in D$ such that

$$f(x) \le f(x_0) \quad \text{for all } x \in D.$$

$f(x_0)$ is called <u>the</u> maximum value of f (which is necessarily unique) and x_0 is called a maximizer of f (which is not necessarily unique).

Similarly, if $f(D) \subseteq \mathbb{R}$ has a minimum ($\inf f(D) \in f(D)$)
then there is $x_0 \in D$ such that

$$f(x_0) \le f(x) \quad \text{for all } x \in D.$$

$f(x_0)$ is called <u>the</u> minimum value of f and x_0 is called <u>a</u> minimizer of f.

examples: 1) $f : \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \to x^2 \end{array}$ has the minimum value 0 with minimizer 0.

f has no maximum value.

2) $f : \begin{array}{l} [1,3] \to \mathbb{R} \\ x \to x^2 \end{array}$ has the minimum value 1 with minimizer 1 and maximum value 9 with maximizer 3.

3) $f : \begin{array}{l} (0,\infty) \to \mathbb{R} \\ x \to \frac{1}{x} \end{array}$ has no minimum or maximum value.

$f((0,\infty))$ is bounded below by 0 but has no minimum.

Thm 3.9 (Extreme Value Theorem)

Let S be a compact set and $f: S \to \mathbb{R}$ a continuous function. Then f attains a minimum and maximum value.

Proof: Let $f: S \to \mathbb{R}$.
First, we'll prove $f(S)$ is compact.
Let $(f(x_n))_{n \in \mathbb{N}}$ be a sequence in $f(S)$.
Then (x_n) is a sequence in S.
Since S is compact, there is $x_{n_k} \to x_0 \in S$.
But f is cts., so $f(x_{n_k}) \to f(x_0) \in f(S)$.
Hence, $(f(x_n))_{n \in \mathbb{N}}$ has a subsequence which converges to an element of $f(S)$.
Therefore, $f(S)$ is compact.
By Bolzano-Weierstrass $f(S)$ is closed and bounded. By the Completeness Axiom $M = \sup f(S)$ and $m = \inf f(S)$ exist. Since $f(S)$ is closed, $M, m \in f(S)$.
i.e. $M = \max f(S)$ and $m = \min f(S)$.
Let $x_M, x_m \in S$ with $f(x_M) = M$ and $f(x_m) = m$.

Therefore, f attains a maximum and minimum value.

Note: It is worth remembering separately that the image of a (sequentially) compact set under a continuous function is (sequentially) compact.