Recall

The Monotone Convergence Theorem

A monotone sequence converges if and only if it is bounded.

Cauchy Sequences

A sequence $(a_n)_{n\in\mathbb{N}}$ is a <u>Cauchy sequence</u> if for all $\varepsilon>0$ there is an N such that

$$n, m \geq N \quad \text{implies} \quad |a_n - a_m| < \varepsilon.$$

Proposition 9.12 Every convergent sequence is Cauchy.

Proof: Assume (a_n) converges to a.

Let $\varepsilon>0$ be given

there is N such that

$$n \geq N \quad \text{implies} \quad |a_n - a| < \frac{\varepsilon}{2}$$

since $a_n \to a$.

If $n, m \geq N$, then

$$\begin{aligned}
|a_n - a_m| &= |a_n - a + a - a_m| \\
&\leq |a_n - a| + |a - a_m| && \text{by triangle ineq} \\
&= |a_n - a| + |a_m - a| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{since } n, m \geq N \\
&= \varepsilon.
\end{aligned}$$

Therefore, (a_n) is Cauchy.

Lemma 9.3 Every Cauchy sequence is bounded.

Proof: Assume (a_n) is Cauchy.

There is N such that

$n,m \geq N$ implies $|a_n - a_m| < 1$

In particular, with $m = N$

$n \geq N$ implies $|a_n - a_N| < 1$

then $|a_n| - |a_N| \leq \big| |a_n| - |a_N| \big|$

$\leq |a_n - a_N|$, by HW #2

< 1

so $|a_n| < 1 + |a_N|$ for all $n \geq N$.

Let $M = \max\{|a_1|, \dots, |a_{N-1}|, 1 + |a_N|\}$

then $|a_n| \leq M$ for all n

so (a_n) is bounded.

Theorem 9.4 (The Cauchy Criterion)

A sequence converges if and only if it is Cauchy.

Proof: $(\Rightarrow)$ This is Proposition 9.2.

$(\Leftarrow)$ Assume (a_n) is Cauchy.

By Lemma 9.3 it is bounded.

By Thm 2.33 it has a convergent subsequence

$a_{n_k} \to a$.

We claim $a_n \to a$.

Let $\varepsilon > 0$ be given.

Since (a_n) is Cauchy, there is N such that

$m,n \geq N$ implies $|a_n - a_m| < \frac{\varepsilon}{2}$.

Also, $a_{n_k} \to a$ so there is $K \geq N$

$k \geq K$ implies $|a_{n_k} - a| < \frac{\varepsilon}{2}$.

Let $M = n_K \geq K \geq N$

If $n \geq M$

$$
\begin{aligned}
|a_n - a| &= |a_n - a_{n_K} + a_{n_K} - a| \\
&\leq |a_n - a_{n_K}| + |a_{n_K} - a| \\
&\leq \frac{\varepsilon}{2} + |a_{n_K} - a| \quad \text{since } \begin{array}{l} n \geq M \geq N \\ n_K \geq K \geq N \end{array} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{since } K \geq K \\
&= \varepsilon.
\end{aligned}
$$

Series + Sequences

From a sequence (a_n), we can make another one:

Sequence of partial sums:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n.$$

example: $a_n = \frac{1}{n}$ $\quad S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

$a_n \to 0$. we've seen $S_n \to \infty$
(increasing + unbounded)

If $S_n \to S$, we write

$$S = \sum_{k=1}^{\infty} a_k.$$

If (a_n) converges, (s_n) may or may not.

example: $a_n = \frac{1}{n^2} \to 0$, $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$ converges.

But if the converse holds:

Proposition 9.5 If $S_n = \sum_{k=1}^{n} a_k$ converges, then $a_n \to 0$.

Proof: Assum $\sum_{k=1}^{\infty} a_k = S$, i.e., $\sum_{k=1}^{n} a_k \to S$.

Then, $a_{n+1} = S_{n+1} - S_n$, $S_n \to S$, $S_{n+1} \to S$ (subsequence of S_n)

So the sequence $S_{n+1} - S_n \to S - S = 0$.

& so $a_{n+1} = S_{n+1} - S_n \to 0$.

Then, $a_n \to 0$.

Proposition 9.6 If $|r|<1$, $\sum_{k=1}^{\infty} r^k = \frac{1}{1-r}$,

i.e. the sequence $\sum_{k=1}^{n} r^k \to \frac{1}{1-r}$.

Proof: $\sum_{k=1}^{n} r^k = \frac{1-r^{n+1}}{1-r}$. Since $|r|<1$, by Prop. 2.28, $r^{n+1} \to 0$. So by Thm. on convergence $\frac{1-r^{n+1}}{1-r} \to \frac{1-0}{1-r}$.

Theorem 9.7 Assume $a_n \geq 0$ for all n.
Then $\sum_{k=1}^{\infty} a_k$ converges, i.e. $S_n = \sum_{k=1}^{n} a_k$ converges,
if and only if (S_n) is bounded.

Proof: Since $a_n \geq 0$, (S_n) is non-decreasing so the result follows from the Monotone Convergence Theorem.

Corollary 9.8 The Comparison Test for Series
Suppose that $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Then
(i) If $\sum_{k=1}^{\infty} b_k$ converges, then so does $\sum_{k=1}^{\infty} a_k$.
(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then so does $\sum_{k=1}^{\infty} b_k$.

Proof: Assume $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.
(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $S_n = \sum_{k=1}^{n} b_k$ is bounded. Then $t_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k = S_n$ is bounded so $\sum_{k=1}^{\infty} a_k$ converges.
(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $t_n = \sum_{k=1}^{n} a_k$ is unbounded. So is $S_n = \sum_{k=1}^{n} b_k \geq \sum_{k=1}^{n} a_k = t_n$. So $\sum_{k=1}^{n} b_k$ diverges.

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examples: (1) Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so does

any $\sum_{k=1}^{\infty} \frac{1}{k^p}$, $p<1$.

(2) Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so does

any $\sum_{k=1}^{\infty} \frac{1}{k^p}$ with $p \geq 2$

For $1<p<2$, we need another tool to decide.

(3) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}2^k}$, $\sum_{k=1}^{\infty} \frac{1}{k^2+k}$, $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ converge.

How about $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$? We can't use comparison with $\sum_{k=2}^{\infty} \frac{1}{k^2}$.

exercise: Use $\frac{1}{k^2-1} = \frac{1/2}{k-1} - \frac{1/2}{k+1}$ and then to compute $S_n = \sum_{k=2}^{n} \frac{1}{k^2-1}$

take the limit.

Theorem 9.15 The Alternating Series Test

If (a_k) is a sequence of non-increasing (monotone) positive terms with $a_k \to 0$, then the series

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \quad \text{converges.}$$

Proof: Let $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$

First, we claim the subsequence $S_{2n} \longrightarrow S$.

$$S_{2(n+1)} - S_{2n} = (-1)^{2n+2+1} a_{2n+2} + (-1)^{2n+1+1} a_{2n+1}$$

$$= -a_{2n+2} + a_{2n+1} \geq 0 \quad \text{since } a_{2n+2} \leq a_{2n+1}$$

So (S_{2n}) is non-decreasing.

Also,

$$S_{2n} = a_1 - a_2 + a_3 - \cdots - a_{2n}$$

$$= a_1 + (a_3 - a_2) + \cdots (a_{2n-1} - a_{2n-2}) - a_{2n}$$

$$\leq a_1 \text{ since } a_{2k+1} \leq a_{2k} \text{ and so } a_{2k+1} - a_{2k} \leq 0$$

So (S_{2n}) is bounded.

By Monotone Convergence Theorem, $S_{2n} \longrightarrow S$.

The subsequence $S_{2n+1} = S_{2n} + a_{2n+1} \longrightarrow S + 0 = S$.

Now, we claim $S_n \longrightarrow S$.

Let $\varepsilon > 0$ be given,

There are N_1, N_2 such that

$n \geq N_1$ implies $|S_{2n+1} - S| < \varepsilon$

$n \geq N_2$ imp $|S_{2n} - S| < \varepsilon$.

Let $N = 2 \cdot \max\{N_1, N_2\} + 1$.

If $n \geq N$ and n is even then $n = 2m \geq N$

so $m \geq N_2$ and $|S_n - S| = |S_{2m} - S| < \varepsilon$.

If $n \geq N$ and n is odd, then $n = 2m+1 \geq N$

so $m \geq N_1$ and $|S_n - S| = |S_{2m+1} - S| < \varepsilon$.

In any case, $|S_n - S| < \varepsilon$.

So $S_n \longrightarrow S$.

example: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges

Theorem 9.П The Cauchy Criterion for Series

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for each $\varepsilon > 0$ there is N such that

$$|a_{n+1} + \cdots + a_{n+k}| < \varepsilon \quad \text{for all } n \geq N \text{ and all } k.$$

Proof: Apply Cauchy Criterion to $S_n = \sum_{k=1}^{n} a_k$.

A series $\sum_{k=1}^{\infty} a_k$ <u>converges absolutely</u> if $\sum_{k=1}^{\infty} |a_k|$ converges.

examples: ① $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ does not converge absolutely although it converges.

② $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges absolutely and converges.

Absolute convergence is <u>stronger</u>, i.e.

The Absolute Convergence Test If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then it converges.

Proof: Follows from the Cauchy Criterion above since

$$|a_{n+1} + \cdots + a_{n+k}| \leq |a_1| + \cdots + |a_{n+k}|, \text{ by } \Delta \text{ inequa.}$$
$$= \big||a_1| + \cdots + |a_{n+k}|\big|.$$

example: $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ converges.

$\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$ converges from comparison to $\sum_{k=1}^{\infty} \frac{1}{k^2}$

so $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ converges absolutely.

Theorem 9.20 Suppose there is an r with $0 \le r < 1$ and $N \in \mathbb{N}$ such that $|a_{n+1}| \le r|a_n|$ for all $n \geq N$,

Then, $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

Proof: Assume $|a_{n+1}| \le r|a_n|$ for all $n \geq N$ with $0 \le r < 1$.

Then, $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{\infty} |a_k|$ converges if and only if $\sum_{k=N}^{\infty} |a_k| = \sum_{k=1}^{\infty} |a_{N+k}|$ does since $\sum_{k=1}^{N-1} |a_k|$ is a (finite) number.

$|a_{N+k}| \le r|a_{N+k-1}| \le r^2|a_{N+k-2}| \le \cdots \le r^k|a_N|$

But, since $\sum_{k=1}^{\infty} r^k|a_N|$ converges to $\frac{|a_N|}{1-r}$, by comparison so does $\sum_{k=1}^{\infty} |a_{N+k}|$.

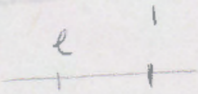
The Ratio Test: Suppose that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \ell$$

(i) If $\ell < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $\ell > 1$, then the series $\sum_{k=1}^{\infty} a_n$ diverges.

Proof: Assume $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \ell$.



(i) If $\ell < 1$, let $\varepsilon = \frac{1-\ell}{2}$, then

there is an $N \in \mathbb{N}$

$$\left| \frac{|a_{n+1}|}{|a_n|} - \ell \right| < \frac{1-\ell}{2}$$

so $$\frac{\ell-1}{2} < \frac{|a_{n+1}|}{|a_n|} - \ell < \frac{1-\ell}{2}$$

or $$\ell - \frac{1}{2} < \frac{|a_{n+1}|}{|a_n|} < \frac{1}{2} + \frac{\ell}{2} < 1$$

let $r = \frac{\ell+1}{2}$ and apply Thm 9.20.

(ii) Assume $\ell > 1$. Take $\varepsilon = \frac{\ell-1}{2}$ to find N with $\frac{\ell+1}{2} < \frac{|a_{n+1}|}{|a_n|}$ for all $n \geq N$. So with $r = \frac{\ell+1}{2}$,

$$|a_{n+1}| > r|a_n| \quad \text{for all } n \geq N.$$

Now mimic the proof of Thm 9.20, do comparison with the divergent $\sum_{k=1}^{\infty} r^k$.

note: The case $\ell=1$ in the ratio test is inconclusive. It may turn out both ways.

Try $\sum_{k=1}^{\infty} \frac{1}{k}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$.