Math 300D Final Winter 2016

See other side for instructions!

1.(10) Consider the statement S: If p is prime and $p|ab$, then $p|a$ or $p|b$.
a) State the negation of S.
b) State the contrapositive of S.
No proof needed on this problem.

2.(10) (Short answer problem; give a brief explanation of your answer.)
a) Define $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ by $f(x) = x^5 - 4$. Is f surjective?
b) Is there an injection from $[5] \times [5] \times [4]$ to $[12] \times [12]$?

3. (10) Show that the unit circle in the plane is uncountable. (The unit circle is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Look for a short proof based on results we know.)

4.(20) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function with the property that $\forall x, y \in \mathbb{R}$, $f(x + y) = f(x) + f(y)$.
a) (10) Use induction on n to show that for all $n \in \mathbb{N}$, $f(n) = n \cdot f(1)$. The base case and inductive hypothesis must be clearly indicated.
b) (2) Show that $f(0) = 0$. (Hint: $0 + 0 = 0$.)
c) (2) Show that $f(-x) = -f(x)$ for all x.
d) (6) Show that f is injective $\Leftrightarrow f^{-1}\{0\} = \{0\}$.
Note: Part (a) is independent of the other parts.

5.(20) Let X denote the set of pairs A, B of subsets of $[n]$ such that $|A| = 3$, $|B| = 8$, and $A \subset B$. (We assume that $n \geq 8$ so that the problem makes sense.)
a) Find an explicit formula involving binomial coefficients for $|X|$. (Your formula should be valid for any $n \geq 8$.)
b) Use your formula to compute this number when $n = 9$. (Your answer to this second question should be a plain old number, not involving any factorials etc.)

6.(20) A sequence of integers $(a_1, a_2, ...)$ is called *periodic* if there exists $d \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, $a_{n+d} = a_n$. For example, the sequence $(1, 0, -3, 1, 0, -3, 1, 0, -3...)$ (where the pattern continues) is periodic with $d = 3$.
Show that the set of all periodic sequences of integers is countable.

Instructions. Remember that in this course it's not about merely getting the "correct" answer. Much of the grade goes to the logic used and how well the proof is written. Be sure to explain what you're doing and how you're drawing your conclusions; in particular, if you are applying a theorem from the notes, homework or class, say which theorem and why (you don't need to know the number of the theorem; just recall what the theorem says). All problems require proof, unless otherwise indicated. You never need to repeat proofs already done in the notes, homework or class, unless asked to do so.

Note: 1 sheet of notes (both sides) allowed. No electronic devices. Exam is 1 hr 50 min.

Finally, relax! It's not a big deal if you can't do every problem. If stuck on a problem, move on to another one and return to the one you were stuck on at the end, if time remains.

NOTE FOR WINTER 2017 STUDENTS: This final is for practice, but this doesn't mean that our final will be exactly like it. Some of the questions on our final could be quite different, since part of the point is to apply your proof skills to something new. The emphasis could be different too; for example, I might emphasize combinatorics more and countable/uncountable sets less. Be prepared for all topics!

Also, problem 3 turned out to be a bit too hard, so here's a hint: Try to find either (a) a surjection from the unit circle to a set already known to be uncountable, or (b) an injection to the unit circle from a set already known to be uncountable.