Seiberg-Witten

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This is an oversimplified version of the lecture notes *Mark Hamilton, Dieter Kotschick, Mathematical Gauge Theory II.* which I followed as I tried to learn about Seiberg-Witten equations. Many important things in the lecture notes were omitted here, including the discussion of Donaldson's theory for 4-manifolds and computations of Seiberg-Witten invariants for several examples. I initially intended to make my own notes as self-contained as possible, but I realized it is not quite possible. So this note is perhaps only useful to me, and for anyone else, the lecture notes should be more helpful.

1 $Spin$ and $Spin^c$ Structures

1.1 Clifford Modules

Let H be a n-dimensional real vector space, equipped with a Euclidean scalar product g. A Clifford Module for H is a finite dimensional complex vector space with a Hermitian product together with a linear map $\gamma : H \mapsto End(V)$:

- $\gamma(v)^t = -\gamma(v)$
- $\gamma(w)\gamma(v) + \gamma(v)\gamma(w) = -2g(v,w)Id_V$

The elements of V are called spinors.

A Clifford module corresponds to a representation of the Clifford algebra $Cl(H,g)$. Such a representation is called irreducible if there are no nontrivial submodules. We have:

Proposition 1.1. *If $dimH = 2m$ then there is a unique irreducible Clifford module (V,γ) with $dim\ V = 2^m$. If $dimH = 2m+1$, then there exist two irreducible Clifford modules (V,γ) and $(V,-\gamma)$ with $dim\ V = 2^m$*

Example 1.1. *Consider $\mathbb{R}^4$ with the Euclidean metric. For $V = \mathbb{C}^4$, we define γ as follows. Choose an ONB $\{e_0, e_1, e_2, e_3\}$ of $\mathbb{R}^4$. We set*

$$\gamma(e_j) = A_j := \begin{pmatrix} 0 & -B_j^t \\ B_j & 0 \end{pmatrix}$$

where $B_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $B_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
It is easy to check that this defines a Clifford module.

The Clifford multiplication $\mathbb{R}^4 \times \mathbb{C}^4 \mapsto \mathbb{C}^4$ extends to the exterior algebra: for $i_1 < i_2 < ... < i_l$, we define

$$\gamma(e_{i_1} \wedge ... \wedge e_{i_l}) = \gamma(e_{i_1})...\gamma(e_{i_l})$$

For the standard Clifford module for $\mathbb{R}^4$, we have

$$\gamma(e_0 \wedge e_1 \wedge e_2 \wedge e_3) = \begin{pmatrix} -Id_2 & 0 \\ 0 & Id_2 \end{pmatrix}$$

Definition 1.1. *An isomorphism of Clifford modules (V, γ) and (V', γ') for H is a linear isometry $f : V \mapsto V'$ such that $f \circ \gamma(v) = \gamma'(v) \circ f$ for all $v \in H$*

Lemma 1.1. *(Schur) Let (V, γ) be an irreducible Clifford module of H. Then every automorphism of V is of the form $f = \lambda Id_V$ for a constant $\lambda \in S^1$*

Proof. f is an isometry, hence unitary. So it can be diagonalized, each eigenvalue has unit norm. f commute with γ, which means that γ preserves the eigenspaces of f, so each eigenspace is a submodule. By irreducibility, V is an eigenspace of f. $f = \lambda Id$ for some $\lambda \in S^1$. □

1.2 Principal Bundle Point of View

1.2.1 $Spin^c$ Structures

We assume that n is even, $\gamma : \mathbb{R}^n \mapsto End\mathbb{C}^N$ denotes the standard Clifford module.

Definition 1.2. *The Lie group $Spin^c(n)$ is defined as the set of pairs $(\tau, \sigma) \in SO(n) \times U(N)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\tau} & \mathbb{R}^n \\ \downarrow{\scriptstyle \gamma_0} & & \downarrow{\scriptstyle \gamma_0} \\ End\mathbb{C}^N & \xrightarrow{Ad\sigma} & End\mathbb{C}^N \end{array}$$

where $Ad\sigma(\mu) = \sigma\mu\sigma^{-1}$ is the adjoint action.

Lemma 1.2. *The homomorphism $Spin^c(n) \mapsto SO(n)$, $(\tau, \sigma) \mapsto \tau$ is surjective with kernel $\{Id_n, S^1\} \cong S^1$*

Definition 1.3. *A $Spin^c$ structure $\mathcal{S}$ on a real, oriented vector bundle equipped with a Euclidean bundle metric $H \to X$ is a pair (V, γ) where $V \to X$ is a complex vector bundle with a Hermitian bundle metric, and $\gamma : H \mapsto End(V)$ is a bundle homomorphism which is fiberwise a standard (irreducible) Clifford module.*

Lemma 1.3. *Specifying a $Spin^c$ structure $\mathcal{S} = (V, \gamma)$ for an oriented Euclidean bundle $H \to X$ is equivalent to specifying a principal $Spin^c(n)$ bundle $Q \to X$ together with an isomprphism $Q/S^1 \cong Fr(H)$, where $Fr(H)$ is the oriented orthonormal frame bundle.*

Proof. Given a $Spin^c$ structure $\mathcal{S} = (V, \gamma)$, we can define a $Spin^c(n)$-bundle $Q \to X$. Consider the pairs of orientation preserving linear isometries

$$(t, s) \in Isom(\mathbb{R}^n, H_x) \times Isom(\mathbb{C}^N, V_x) \cong SO(n) \times U(N)$$

that make the following diagram commute:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "t"] \arrow[d, "\gamma_0"] & H_x \arrow[d, "\gamma_x"] \\
End(\mathbb{C}^N) \arrow[r, "Ad(s)"] & End(V_x)
\end{tikzcd}$$

Then each fiber is diffeomorphic to $Spin^c(n)$ by construction and $Q_x/S^1 \ni [t, s] \mapsto t$ is a fiberwise isomoprhism to $Fr(H)$.For the other direction, given a $Spin^c(n)$ bundle $Q \to X$, there is a representation of $Spin^c(n)$ on $\mathbb{C}^N$ given by $(\tau, \sigma) \mapsto \sigma \in U(N)$. This induces an associated vector bundle $V \to X$ with fiber $\mathbb{C}^N$, and it is easy to check that the standard Clifford module γ_0 induces a Clifford module γ for V. $\square$

1.2.2 Spin Structures and Stiefel-Whitney Classes

Definition 1.4. *If $V \to X$ is a complex vector bundle, then there exists a complex conjugate bundle $\bar{V} \to X$. Since $EndV \cong End\bar{V}$, we define charge conjugation as the map*

$$Spin^c(H) \longrightarrow Spin^c(H)$$

$$(V, \gamma) = \mathcal{S} \mapsto \bar{\mathcal{S}} = (\bar{V}, \gamma)$$

Definition 1.5. *A complex line bundle $L_{\mathcal{S}}$ such that $\mathcal{S} = \bar{\mathcal{S}} \otimes L_{\mathcal{S}}$ (unique up to isomorphism) is called the characteristic line bundle of $\mathcal{S}$.*

When the characteristic line bundle is trivial, we say that $\mathcal{S}$ arises from a *Spin* structure.

Definition 1.6. *A $Spin^c$ structure $\mathcal{S}$ together with an isomorphism $J : \bar{\mathcal{S}} \mapsto \mathcal{S}$ is called a Spin structure.*

Now consider $(\mathbb{C}^N, \gamma_0, J_0)$, the standard Cifford module for $\mathbb{R}^n$ with its charge conjugation map.

Definition 1.7. *$Spin(n)$ is defined as the subgroup of $Spin^c(n)$, consisting of elements (τ, σ) such that σ commutes with J_0.*

Lemma 1.4. *The homomorphism $Spin(n) \to SO(n)$, $(\tau, \sigma) \mapsto \tau$ is surjective with kernel $\{Id_n, \pm Id_N)\} \cong \mathbb{Z}_2$.*

Therefore, we have the following short exact sequence:

$$1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1$$

Comparing with

$$1 \to S^1 \to Spin^c(n) \to SO(n) \to 1$$

We get $Spin^c(n) \cong (Spin(n) \times S^1)/\mathbb{Z}_2$ with $\mathbb{Z}_2$ identifies (τ, σ, λ) with $(\tau, -\sigma, -\lambda)$. Similar to $Spin^c$-structures, which are lifts of an real, oriented vector bundle H to a $Spin^c(n)$-bundle Q such that $Q/S^1 \cong Fr(H)$, spin structures for H are lifts of $Fr(H)$ to a $Spin(n)$-bundle P such that $P/\mathbb{Z}_2 \cong Fr(H)$. We have a short exact sequence

$$0 \to S_{\mathbb{Z}_2} \to S_{Spin(n)} \to S_{SO(n)} \to 0$$

and hence a long exact sequence

$$... \to \check{H}^1(X; S_{Spin(n)}) \xrightarrow{q} \check{H}^1(X; S_{SO(n)}) \xrightarrow{\delta'} \check{H}^2(X; S_{\mathbb{Z}_2}) \cong H^2(X, \mathbb{Z}_2) \to ...$$

A lift $[P]$ for H exists if and only if $\delta'[H] = 0 \in H^2(X, \mathbb{Z}_2)$.

Definition 1.8. *Let H be a real,oriented vector bundle, and $Fr(H)$ its $SO(n)$-principal frame bundle. The second Stiefel-Whitney class $w_2(H)$ of H is defined as $w_2(H) = \delta'[H] \in H^2(X, \mathbb{Z}_2)$*

Proposition 1.2. *A manifold is Spin iff its tangent bundle has vanishing second Stiefel-Whitney class.*

For $Spin^c$-structure, the short exact sequence

$$0 \to S_{S^1} \to S_{Spin(n)} \to S_{SO(n)} \to 0$$

also gives rise to a LES:

$$... \to \check{H}^1(X; S_{Spin^c(n)}) \xrightarrow{p} \check{H}^1(X; S_{SO(n)}) \xrightarrow{\delta} \check{H}^2(X; S_{S^1}) \cong H^3(X, \mathbb{Z}) \to ...$$

And we have the obstuction class $o(H) = \delta[H] \in \check{H}^2(X; S_{S^1}) \cong H^3(X, \mathbb{Z})$.

Lemma 1.5. *The obstruction class $o(H)$ is the image of w_2 under $i^* : H^2(X, \mathbb{Z}_2) \to \check{H}^2(X; S_{S^1})$*

Lemma 1.6. *The map $i^* : H^2(X, \mathbb{Z}_2) \to \check{H}^2(X; S_{S^1}) \cong H^3(X; \mathbb{Z})$ is equal to the Bockstein homomorphism $\beta : H^2(X, \mathbb{Z}_2) \mapsto H^3(X, \mathbb{Z})$ induced by the SES:*

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{mod2} \mathbb{Z}_2 \to 0$$

To summarize, we have

Proposition 1.3. *The obsruction class $o(H) = \beta(w_2(H))$ vanishes iff $w_2(H)$ has a lift to $H^2(X; \mathbb{Z})$, i.e. if $w_2(H)$ is in the image of $H^2(X; \mathbb{Z}) \xrightarrow{mod2} H^2(X; \mathbb{Z}_2)$.*

1.3 $Spin^c$ connection and Dirac Operators

1.3.1 The Dirac Operator on $\mathbb{R}^n$

The Dirac operator comes from the idea of the "square root" of the Laplacian operator. Formally, we want to have an operator written as

$$D\Phi = \sum_{i=1}^{n} A_i \frac{\partial \Phi}{\partial x_i}$$

Where $\Phi : \mathbb{R}^n \longrightarrow \mathbb{C}^N$, and A_is are constant $N \times N$ matrices. The Laplace operator is given by

$$\Delta\Phi = -\sum_{i=1}^{n} Id_N \frac{\partial^2 \Phi}{\partial x_i^2}$$

Imposing the condition $D^2 = \Delta$, we obtain

$$A_i^2 = -Id_N \; ; A_i A_j + A_j A_i = 0 \; \forall i \neq j$$

Further, we want D to be formally self-adjoint with respect to the L^2 norm:

$$< D\Phi, \Psi > = < \Phi, D\Psi >$$

This is true iff A_is are skew-adjoint, i.e. $A_i^t = -A_i$, together, this implies that A_is are unitary.

1.3.2 The Dirac Operator on a Spinor bundle

Let $H \to X$ be a real,oriented vector bundle equipped with a Euclidean metric g and a compatible connection ∇^B. Assume H admits a $Spin^c$ structure $V \to X$ with Hermitian metric h.

Definition 1.9. *($Spin^c$ connection) A $Spin^c$ connection ∇^A on V is a covariant derivative which is*

- *Hermitian,i.e. $L_Y h(\Phi, \Psi) = h(\nabla_Y^A \Phi, \Psi) + h(\Phi, \nabla_Y^A \Psi)$ for every vector field Y, and sections $\Phi, \Psi \in \Gamma(V)$*
- *Compatible with ∇^B and the Clifford multiplication, i.e. for any vector field Y, $\Phi \in \Gamma(V)$ and $T \in \Gamma(H)$, we have*

$$\nabla_Y^A (T \cdot \Phi) = (\nabla_Y^B T) \cdot \Phi + T \cdot (\nabla_Y^A \Phi)$$

When $T \in \Gamma(H)$ is a parallel section with respect to ∇^B along the flow of a vector field Y, we have $\nabla_Y^A (T \cdot \Phi) = T \cdot (\nabla_Y^A \Phi)$.

Now we specialize to the setting $H = TX \to X$ with metric g, ∇ is the Levi-Civita connection, $\mathcal{S}$ be a $Spin^c$-structure on X.

Definition 1.10. *(Dirac Operator If ∇^A is a $Spin^c$-connection on V, the Dirac operator $D_A : \Gamma(V) \to \Gamma(V)$ is defined as*

$$D_A : \Gamma(V) \xrightarrow{\nabla^A} \Gamma(T^*X \otimes V) \xrightarrow{g} \Gamma(TX \otimes V) \xrightarrow{\gamma_{eval}} \Gamma(V)$$

where γ_{eval} is the composition of γ with the Clifford multiplication.

In coordinates, the Dirac operator can be expressed as following:

Lemma 1.7. *Let $\{e_1, ..., e_n\}$ be an orthonormal basis of (T_pX, g_p). Then*

$$D_A\Phi = \sum_{i=1}^{n} e_i \cdot \nabla^A_{e_i}\Phi$$

Suppose (X, g) is closed, then we can define a Hermitian product on the sapce of smooth sections $\Gamma(V)$:

$$< \Phi, \Psi >= \int_X h(\Phi, \Psi) vol_g$$

Proposition 1.4. *D_A is formally self-adjoint w.r.t. the Hermitian product.*

2 Dimension Four

We specialize to the setting that M is a closed,connected oriented smooth 4-manifold. Recall from example(1.1) the standard Clifford module for $\mathbb{R}^4$:

$$\gamma(e_j) = A_j := \begin{pmatrix} 0 & -B_j^t \\ B_j & 0 \end{pmatrix}$$

where $B_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $B_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
The Clifford multiplication $\mathbb{R}^4 \times \mathbb{C}^4 \to \mathbb{C}^4$ extends to the exterior algebra. We have

$$\gamma(e_0 \wedge e_1 \wedge e_2 \wedge e_3) = \begin{pmatrix} -Id_2 & 0 \\ 0 & Id_2 \end{pmatrix}$$

So we can split $\mathbb{C}^4 = \mathbb{C}^2_+ \otimes \mathbb{C}^2_-$ corresponding to the -1 and +1 eigenspace, i.e. $\gamma(vol)|_{\mathbb{C}^2_\pm} = \mp Id$. $\mathbb{C}^4$ is called the space of Dirac spinors, $\mathbb{C}^2_\pm$ is the space of positive/negative Weyl spinors.

Definition 2.1. *Hodge duality is a map $* : \Lambda^k(\mathbb{R}^n) \to \Lambda^{n-k}(\mathbb{R}^n)$, defined as*

$$*(e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}) = (-1)^\sigma (e_{j_1} \wedge ... \wedge e_{j_{n-k}})$$

where σ is the sign of the permutation which takes $(i_1 i_2 ... i_k j_1 j_2 ... j_{n-k})$ to $(12...n)$.

It is easy to see that $*$ fixes two forms and $*^2 = Id$ on two forms $\Lambda^2(\mathbb{R}^4)$. One can decompose two forms on $\mathbb{R}^4$ into a self-dual and anti self-dual part as

$$\omega = \omega_+ + \omega_- = \frac{1}{2}(\omega + *\omega) + \frac{1}{2}(\omega - *\omega)$$

This is a decompostion of $\Lambda^2(\mathbb{R}^4) = \Lambda^2_+(\mathbb{R}^4) \oplus \Lambda^2_-(\mathbb{R}^4)$. Taken the standard ONB $\{e_0, ...e_3\}$, we have the following basis for $\Lambda^2_\pm(\mathbb{R}^4)$:

$$e_0 \wedge e_1 \pm e_2 \wedge e_3$$

$$e_0 \wedge e_2 \mp e_1 \wedge e_3$$

$$e_0 \wedge e_3 \pm e_1 \wedge e_2$$

Under the standard Clifford module γ on $\mathbb{R}^4$, the self-dual 2-forms

$$e_0 \wedge e_1 + e_2 \wedge e_3$$

$$e_0 \wedge e_2 - e_1 \wedge e_3$$

$$e_0 \wedge e_3 + e_1 \wedge e_2$$

act on $\mathbb{C}^2_+$ nontrivially, as $2B_1, 2B_2, 2B_3$ respectively, and are zero on $\mathbb{C}^2_-$.A similar result holds for the basis set of $\Lambda^2_-(\mathbb{R}^4)$. As a consequence, we have the following:

Lemma 2.1. *γ induces isomorphisms*

$$(\Lambda^1(\mathbb{R}^4) \otimes \Lambda^3(\mathbb{R}^4)) \otimes \mathbb{C} \cong Hom(\mathbb{C}^2_+, \mathbb{C}^2_-) \otimes Hom(\mathbb{C}^2_-, \mathbb{C}^2_+)$$

$$\Lambda^2_\pm(\mathbb{R}^4) \otimes \mathbb{C} \cong End_0(\mathbb{C}^2_\pm)$$

$$\Lambda^4(\mathbb{R}^4) \otimes \mathbb{C} \cong \mathbb{C}Id_{\mathbb{C}^2_\pm}$$

2.1 $Spin(4)$ and $Spin^c(4)$

Recall that

$$SU(2) = \{\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} |a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\}$$

As a real vector space, we identify

$$\mathbb{R}^4 \cong \{\begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \in Mat(2 \times 2.\mathbb{C}) | u, v \in \mathbb{C}\} \cong \mathbb{C}^2$$

The action $SU(2) \times SU(2) \times \mathbb{R}^4 \to \mathbb{R}^4$, $(h_-, h_+, x) \mapsto h_- x h_+^{-1}$ defines a surjective homomorphism ϕ: $SU(2) \times SU(2) \to SO(4)$ with kernel $\{(Id, Id), (-Id, -Id)\} \cong \mathbb{Z}_2$. Hence,

$$Spin(4) \cong SU(2) \times SU(2) = \{\begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix} |B_+, B_- \in SU(2)\}$$

Remark 1. *ϕ is the universal covering of $SO(4)$.*

Recall $Spin^c(n) \cong (Spin(n) \times U(1))/\mathbb{Z}_2$ and find

$$\begin{aligned} Spin^c(4) &= (SU(2) \times SU(2) \times U(1))/((1,1,1) \backsim (-1,-1,-1) \\ &= \{ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} | A_+, A_- \in SU(2); \lambda \in U(1) \} \end{aligned} \tag{1}$$

Let $H \to X$ be a real rank-four Euclidean vector bundle equipped with a $Spin^c$-structure, which we see as a $Spin^c$-bundle Q. Then we can construct the associated Weyl spinor bundles $S_\pm$, with the positive and negative spinors correspond (fiberwise) to $\mathbb{C}^4 = \mathbb{C}^2_+ \oplus \mathbb{C}^2_-$, as vector bundles associated to Q. The representation used to associate $S_\pm$ to Q are

$$\rho_\pm : Spin^c(4) \to U(2)$$

$$\begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} \mapsto \lambda A_\pm$$

Since the map $(Spin(n) \times S^1)/\mathbb{Z}_2 \to Spin^c(n)$,$[\tau, \sigma, \lambda] \mapsto (\tau, \lambda\sigma)$ is an isomorphism, the following representation is well-defined:

$$\xi : Spin^c(4) \to U(1)$$

$$\begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} \mapsto \lambda^2 = det(\lambda A_+) = det(\lambda A_-)$$

The representation associates the determinant line bundle $L \cong det(S_+) = \Lambda^2(S_+) \cong det(S_-) = \Lambda^2(S_-)$ to Q. The conjugate representation is defined obviously as $\bar{S} = \bar{S}_+ \oplus \bar{S}_-$, associated to Q via

$$\bar{\rho_\pm} : Spin^c(4) \to U(2)$$

$$\begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} \mapsto \lambda \bar{A}_\pm$$

The next lemma shows that the fundamental representation of $SU(2)$ and its complex conjugate are isomorphic.

Lemma 2.2. *There exists a fixed matrix $M \in SU(2)$ such that $MAM^t = \bar{A}$ for all $A \in SU(2)$.*

Proof. Since the Pauli matrices form a basis of $SU(2)$, we can impose the required equations on Pauli matrices, and find

$$M = \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}$$

□

Also, we have $\lambda^2 \lambda \bar{A}_\pm = \lambda|\lambda|^2 \bar{A_\pm} = \lambda \bar{A_\pm} = M(\bar{\lambda} A_\pm)M^t$, which means $\rho_\pm \cong \xi \otimes \bar{\rho_\pm}$. Hence we have $S_\pm \cong \bar{S_\pm} \otimes L$, therefore L is the characteristic line bundle.

2.2 The Existence of $Spin^c$-connections

The construction in this section applies to all dimensions, not just in dimension 4. To understand the formulation of the Seiberg-Witten equations, it helps to understand the main ingredients in constructing $Spin^c$ connections.

Consider the $SO(n)$-principal bundle FrH, recall that there is a bijective correspondence between connection 1-forms on a principal bundle and covariant dereivatives on associated bundles. So the covariant derivative ∇^B corresponds to a principal connection B on $Fr(H)$, i.e. $B \in \Omega^1(Fr(H), so(n)) = \Gamma(T^*Fr(H) \otimes so(n))$,with:

- B being $SO(n)$ invariant, $Ad_g(r_g^*)B = B$
- $B(X_\xi) = \xi$ for any $\xi \in so(n)$, where X_ξ is the fundamental vector field associated to ξ, defined by differentiating the $SO(n)$ action on $Fr(H)$

Let $A \in \Omega^1(Q, spin^c(n))$ be a connection on the $Spin^c(n)$-bundle $Q \to X$, it defines a Hermitian covariant derivative ∇^A on $V \to X$, where $V \to X$ is the vector bundle associated to $Q \to X$ via the representation

$$\rho_1 : Spin^c(n) \to U(N)$$

$$(\tau, \sigma) \mapsto \sigma$$

Let $\rho_2 : Spin^c(n) \to SO(n)$,$(\tau, \sigma) \mapsto \tau$, denote the induced Lie algebra homomorphism by ρ_{1*} and ρ_{2*}.

We want to impose conditions on A such that ∇^A is a $Spin^c$-connection.

Proposition 2.1. *Let $\pi : Q \to Fr(H)$ be the bundle map that identifies $Q/S^1 \cong Fr(H)$. A Hermitian covariant derivative ∇^A is a $Spin^c$-connection iff the following diagram commutes:*

$$\begin{array}{ccc} TQ & \xrightarrow{A} & spin^c(n) \\ \downarrow{\scriptstyle D\pi} & & \downarrow{\scriptstyle \rho_{2*}} \\ TFr(H) & \xrightarrow{B} & so(n) \end{array}$$

Let $L := L_s$ denote the characteristic line bundle associated to Q via the homomorphism

$$\xi : Spin^c(n) \to U(1)$$

$$[\tau, \sigma, \lambda] \mapsto \lambda^2$$

ξ_* is the induced Lie algebra homomorphism. Further we define $\mathcal{L}$ as the $U(1)$ principal bundle corresponding to L.

Proposition 2.2. *A connection A on Q induces a connection $\mathcal{A}$ on $\mathcal{L}$ such that the following diagram, where $c : Q \to \mathcal{L}$ is a bundle map, commutes.*

$$\begin{array}{ccc} TQ & \xrightarrow{A} & spin^c(n) \\ \downarrow{\scriptstyle Dc} & & \downarrow{\scriptstyle \xi_*} \\ T\mathcal{L} & \xrightarrow{\mathcal{A}} & u(1) \end{array}$$

Since $Spin^c(n) \cong Spin(n) \times U(1)/\mathbb{Z}_2$, the Lie algebra is given by $spin^c(n) \cong spin(n) \oplus u(1) \cong so(n) \oplus u(1)$, the isomorphism is given by $(\rho_1 \times \xi)_* : spin^c(n) \to so(n) \oplus u(1)$. This allows us to construct a connection on Q from connections on FrH and $\mathcal{L}$,giving us the following result:

Theorem 1. *Let B be a pricipal $SO(n)$-connection on $Fr(H)$, and $\mathcal{A}$ a principal $U(1)$-connection on $\mathcal{L}$. Then there exists a unique pricipal $Spin^c(n)$-connection A on Q such that the following diagrams commute.*

$$\begin{tikzcd} TQ \arrow[r, "A"] \arrow[d, "D\pi"] & spin^c(n) \arrow[d, "\rho_{2*}"] \\ TFr(H) \arrow[r, "B"] & so(n) \end{tikzcd} \qquad \begin{tikzcd} TQ \arrow[r, "A"] \arrow[d, "Dc"] & spin^c(n) \arrow[d, "\xi_*"] \\ T\mathcal{L} \arrow[r, "\mathcal{A}"] & u(1) \end{tikzcd}$$

Corollary 1. *The choice of a metric connection ∇^B on H and a Hermitian connection $\nabla^{\mathcal{A}}$ on $\mathcal{L}$ determines a unique $Spin^c$-connection on V.*

3 Seiberg-Witten Equations

3.1 Elliptic Operator

Let $E, F \to M$ be bundles, $P : \Gamma(E) \to \Gamma(F)$ be a first-order differential operator.

Definition 3.1. *The symbol of P is a bundle map $\sigma(P) : T^*M \to Hom(E, F)$, defined as follows. Take $\xi \in T^*M$, $e \in E_p$, choosing an extension $\tilde{e}$ of e to a section of E, and a smooth function $f \in C^\infty(M)$ with $f(p) = 0$ and $(df)_p = \xi$. Then*

$$(\sigma(P)(\xi))(e) := (P(f \cdot \tilde{e}))(p)$$

Definition 3.2. *P is called elliptic if $\sigma(P)(\xi)$ is an isomorphism for all $\xi \neq 0$.*

Example 3.1. *Let $P = D_A : \Gamma(V) \to \Gamma(V)$ be a Dirac operator of a $Spin^c$ stucture with a $Spin^c$ connection A. Pick $\phi \in V_p$ and extends to $\tilde{\phi} \in \Gamma(V)$, fix $f(p) = 0$ and $(df)(p) = \xi$ for a $\xi \in T_p^*V$. Then*

$$D_A(f \cdot \tilde{f})(p) = \gamma_{eval} \circ g(\nabla^A(f\tilde{\phi})(p)) = \gamma(\xi^*) \cdot \phi$$

where ξ^* is dual to ξ under g. Hence $\sigma(D_A)(\xi) = \gamma(\xi^*)$ and D_A is elliptic. Since D_A is self-adjoint, $ind(D_A) = 0$.

Now we specialize to the case of closed,smooth 4-manifold with a spin structure. In this case, the Clifford module V has a splitting $V = V_+ \oplus V_-$ which is preserved by a $Spin^c$ connection and interchanged by Clifford multiplication, which means the Dirac operator acts as

$$D_A\Phi = \begin{pmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{pmatrix} \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}$$

where $D_A^{\pm} : \Gamma(V_{\pm}) \to \Gamma(V_{\mp})$.

The Atiya-Singer Index Theorem relates the index of D_A^+ to a topological quantity:

Theorem 2. *(Atiya-Singer Index Theorem)*

$$ind_{\mathbb{C}} D_A^+ = < \hat{A}(M), [M] >$$

where $\hat{A}(M) = 1 - (1/24)p_1(TM) + ...$ *On a 4-manifold, only the first Pontryagin class can be nonvanishing, so*

$$ind_{\mathbb{C}} D_A^+ = -\frac{1}{24} < p_1(TM), [M] >$$

Using the signature Formula

Theorem 3. *(Signature Formula, Thom-Hirzebruch) For a closed,connected,oriented,smooth 4 manifold, the signature is given by*

$$\sigma(M) = \frac{1}{3} < p_1(TM), [M] >$$

We then obtain

$$ind_{\mathbb{C}} D_A^+ = -\frac{1}{8}\sigma(M)$$

Corollary 2. *If M is spin, $\sigma(M)$ is divisible by 8.*

3.2 The Seiberg-Witten Equations

Let (X, g) be a COS Riemannian 4-manifold with a $Spin^c$-structure $\mathcal{S}$, equipped with a $Spin^c$-connection A. Let $\Gamma(V_+)$ denote the space of positive spinors, with $\Phi \in \Gamma(V_+)$ a positive spinor. The Seiberg-Witten equations are equations defined for a pair (A, Φ).
The first SW equation reads $D_A^+\Phi = 0$.
To write down the second equation, recall that γ induces and isomorphism $\Lambda_+^2 T^*X \otimes \mathbb{C} \cong End_0(V_+)$, where End_0 denotes the space of traceless endomorphisms. It maps real valued self-dual forms to traceless, skew-Hermitian endomorphisms and imaginary valued forms to traceless,Hermitian endomorphisms. In particular, $F_{\hat{A}}^+$ (where $\hat{A}$ is the $U(1)$-connection associated to A) is an element of $\Omega_+^2(X, i\mathbb{R})$, thus corresponds to a traceless Hermitian endomorphism under γ.

Let $\phi \in \Gamma(V_+)$ be a positive spinor. Define $\Phi \otimes \Phi^t \in End(V_+)$ by $(\Phi \otimes \Phi^t)(\phi) = \Phi h(\Phi, \phi)$, where h is the Hermitian metric on $\Gamma(V_+)$. The traceless part is denoted by $(\Phi \otimes \Phi^t)_0$. Consider $\Phi = \begin{pmatrix} a \\ b \end{pmatrix}$ in a basis for V_+, then

$$(\Phi \otimes \Phi^t) = \begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix}$$

so

$$(\Phi \otimes \Phi^t)_0 = \begin{pmatrix} \frac{1}{2}(|a|^2 - |b|^2) & a\bar{b} \\ \bar{a}b & \frac{1}{2}(|b|^2 - |a|^2) \end{pmatrix}$$

Definition 3.3. *We define $\sigma(\Phi, \Phi) \in \Omega^2_+(X, i\mathbb{R})$ through the relation*

$$\gamma(\sigma(\Phi, \Phi)) = (\Phi \otimes \Phi^t)_0$$

The second equation, called the curvature equation, reads $F^+_{\hat{A}} = \sigma(\Phi, \Phi)$. The SW equations for (A, Φ) are:

$$D^+_A \Phi = 0$$

$$F^+_{\hat{A}} = \sigma(\Phi, \Phi)$$

Sometimes it is necessary to perturb the SW equations by a self-dual imaginary valued form $\omega \in \Omega^2_+(X, i\mathbb{R})$. The perturbed equations read:

$$D^+_A \Phi = 0$$

$$F^+_{\hat{A}} = \sigma(\Phi, \Phi) + \omega$$

Definition 3.4. *(SW Parameter Space) The space of parameters for the SW equation on X is*

$$\mathcal{P} = \{(g, \omega) \in Met(X) \times \Omega^2_+(X, i\mathbb{R})\}$$

Definition 3.5. *The space*

$$\mathcal{C}_{\mathcal{S}} = \mathcal{A}_{\mathcal{S}} \times \Gamma(V_+)$$

where $\mathcal{A}_{\mathcal{S}}$ is the space of $Spin^c$-connections on V compatible with the Levi-Civita connection is called the Seiberg-Witten configuration space.

Since $\mathcal{A}_{\mathcal{S}}$ can be identified with the space of Hermitian connections on the characteristic line bundle $L_{\mathcal{S}}$, it is an affine space over the sapce $\Omega^1(X, i\mathbb{R})$ of imaginary valued 1-forms over X.

Consider the map

$$f_\omega : \mathcal{C}_{\mathcal{S}} \longrightarrow i\Omega^2_+(X) \times \Gamma(V_-)$$

$$(A, \Phi) \mapsto (F^+_{\hat{A}} - \sigma(\Phi, \Phi) - \omega, D^+_A \Phi)$$

then the solution space of the SW equations

$$\mathcal{Z}_\omega = f^{-1}_\omega(0)$$

3.3 Symmetries of the Seiberg-Witten Equations

3.3.1 Charge Conjugation

Recall the charge conjugation map $J : \bar{\mathcal{S}} \to \mathcal{S}$. Let A be a $Spin^c$-connection on $\bar{V}$: it then induces a $Spin^c$-connection A^* on V, defined by $\nabla_X^{A^*}(J\Phi) = J\nabla_X^A \Phi$. So we have a map

$$\tau : \mathcal{C}_{\bar{\mathcal{S}}} \to \mathcal{C}_{\mathcal{S}}$$

$$(A, \Phi) \mapsto (A^*, J\Phi)$$

$$(\tilde{A}^*, J^{-1}\Phi) \leftarrow\!\shortmid (\tilde{A}, \Phi)$$

It is easy to see that τ is an involution.

Lemma 3.1. *A pair $(A, \Phi) \in \mathcal{C}_{\bar{\mathcal{S}}}$ satisfies the SW equations fro (g, ω) iff $\tau(A, \Phi) \in \mathcal{C}_{\mathcal{S}}$ satisfies SW equations for $(g, -\omega)$.*

3.3.2 The action of the Gauge group

The gauge group of the SW equations, $\mathcal{G} = C^\infty(X, S^1)$, has actions defined as follows.

- On $\mathcal{C}_{\mathcal{S}}$, for $u \in \mathcal{G}$,

  $$(A, \Phi) \mapsto (A, \Phi) \cdot u := ((u^{-1})^* A, u\Phi)$$

  and the $Spin^c$-connection transform as $\nabla^{(u^{-1})^* A} := u \nabla_A u^{-1}$
- On $i\Omega^2_+(X) \times \Gamma(V_-) \subseteq f_\omega(\mathcal{C}_{\mathcal{S}})$, the action is $(\iota, \phi) \mapsto (\iota, \phi) \cdot u := (\iota, u\phi)$

Lemma 3.2. *f_ω is equivivariant with respect to these actions of $\mathcal{G}$, i.e. $f_\omega((A, \Phi) \cdot v) = f_\omega(A, \Phi) \cdot v$*

Lemma 3.3. *If X^4 is connected, the stablizer $\mathcal{G}_{(A,\Phi)}$ of $(A, \Phi) \in \mathcal{C}_{\mathcal{S}}$ is given by*

$$\mathcal{G}_{(A,\Phi)} = \begin{cases} \{1\} & \text{if } \Phi \not\equiv 0 \\ U(1) & \text{if } \Phi \equiv 0 \end{cases} \tag{2}$$

Proof. $(A, \Phi) \cdot u = (A, \Phi)$ means $u\nabla^A u^{-1} = \nabla^A$ and $u\Phi = \Phi$. $u\nabla^A u^{-1} = \nabla^A + ud(u^{-1})$, so the first equation is satisfied iff $ud(u^{-1}) = 0$, hence $u \in S^1$ constant. □

Definition 3.6. *The space of irreducible configurations is*

$$\mathcal{G}^*_{(A,\Phi)} = \{(A, \Phi) \in \mathcal{G}_{(A,\Phi)} | \Phi \not\equiv 0\}$$

$\mathcal{G}_{(A,\Phi)} \setminus \mathcal{G}^*_{(A,\Phi)}$ is the set of reducible solutions.

4 Topology of the Moduli Space

Definition 4.1. *(SW Base and Moduli Spaces) We call $\mathcal{B} := \mathcal{C}_{\mathcal{S}}/\mathcal{G}$ the base space of the SW equations. The moduli space is $\mathcal{M}_\omega := \mathcal{Z}_\omega/\mathcal{G} \subset \mathcal{B}$.*

Roughly speaking,we want to show that the Moduli space is a compact smooth manifold of finite dimension.

Recall that the solutions of the SW equations is the zero set of the map

$$f_\omega : \mathcal{C}_{\mathcal{S}} \longrightarrow i\Omega^2_+(X) \times \Gamma(V_-)$$

This map is in fact a smooth map. Its differential gives the linearization of the SW equations.

$$\mathcal{T}_{(A,\Phi)} f_\omega : i\Omega^1(X) \times \Gamma(V_+) \to i\Omega^2_+(X) \times \Gamma(V_-)$$

Fixing $(A, \Phi) \in \mathcal{C}_{\mathcal{S}}$. The gauge group action induces a map

$$g = i\Omega^0(X) \xrightarrow{L_{(A,\Phi)}} T_{(A,\Phi)}\mathcal{C}_{\mathcal{S}} = i\Omega^1(X) \times \Gamma(V_+)$$

Now for a fixed (A, Φ), consider the composition of the two maps

$$i\Omega^0(X) \xrightarrow{L_{(A,\Phi)}} i\Omega^1(X) \times \Gamma(V_+) \xrightarrow{\mathcal{T}_{(A,\Phi)} f_\omega} i\Omega^2_+(X) \times \Gamma(V_-)$$

We have the following result

Proposition 4.1. *If $D^+_A\Phi = 0$, then for all $\omega \in i\Omega^2_+(X)$, the above is an elliptic complex with index $-\frac{1}{4}(c_1^2(L_{\mathcal{S}}) - (2\chi(X) + 3\sigma(X)))$*

Remark 2. *It is a general fact that*

$$\frac{1}{4}(c_1^2(L_{\mathcal{S}}) - (2\chi(X) + 3\sigma(X))) = c_2(V_+)$$

Definition 4.2. *For $(A, \Phi) \in \mathcal{Z}_\omega$, let $H^i_{(A,\Phi)}$ be the ith cohomology group of the elliptic complex. Its index is $dim H^0_{(A,\Phi)} - dim H^1_{(A,\Phi)} + dim H^2_{(A,\Phi)}$*

Lemma 4.1. *Let X be a CCOS. Then*

$$H^0_{(A,\Phi)} \cong \begin{cases} 0 & \text{if } \Phi \not\equiv 0 \\ \mathbb{R} & \text{if } \Phi \equiv 0 \end{cases} \tag{3}$$

Proof. $\xi \in H^0_{(A,\Phi)}$ iff $L_{(A,\Phi)}\xi = (0, 0)$, which means $d\xi = 0$ and $\xi\Phi = 0$.The statement follows. □

Now assume that (A, Φ) is an irreducible solution of the SW equations, i.e. $H^0_{(A,\Phi)} = 0$. S be a local slice for the $\mathcal{G}$-action on $\mathcal{C}_{\mathcal{S}}$ aound (A, Φ). This means a neighborhood of (A, Φ) admits a smoothly embedded Hilbert manifold S and the neighborhood is diffeomorphic to $S \times \mathcal{G}$.(S is a transverse submanifold to the $\mathcal{G}$ orbits.)

Consider $f_\omega|_S : S \to i\Omega^2_+(X) \times \Gamma(V_-)$. This map is Fredholm between Banach spaces, and a neighborhood of $[A,\Phi] \in \mathcal{M}_\omega$ is equal to $(f_\omega|_S)^{-1}(0)$.

We further assume that $H^2_{(A,\Phi)} = 0$, which is going to be achieved by transversality. Using a version of implicit function theorem for Banach manifolds, we see that an open neighborhood of $[A,\Phi]$ looks like an open neighborhood of 0 in $H^1_{(A,\Phi)}$.Together with the index discussion, we have

Proposition 4.2. *If $H^0_{(A,\Phi)} = H^2_{(A,\Phi)} = 0$, a neighborhood of $[A,\Phi] \in \mathcal{M}_\omega$ is a smooth manifold of dimension*

$$dim^{exp}\mathcal{M}_\omega := \frac{1}{4}(c_1^2(L_\mathcal{S}) - (2\chi(X) + 3\sigma(X))) = c_2(V_+)$$

which is called the expected dimension of the moduli space.

An analogous statement holds for the reducible case:

Proposition 4.3. *If $H^0_{(A,\Phi)} = \mathbb{R}, H^2_{(A,\Phi)} = 0$, a neighborhood of $[A,\Phi] \in \mathcal{M}_\omega$ is the quotient of a smooth manifold of dimension $dim^{exp}\mathcal{M}_\omega + 1$ by a $U(1)$-action.*

Using transversality argument, we establish:

Theorem 4. *For any fixed Riemannian metric g on X, and a generic $\omega \in i\Omega^2_+(X)$, the irreducible part of the moduli space $\mathcal{M}^*_\omega = (\mathcal{Z}_\omega \cap \mathcal{C}^*_\mathcal{S})/\mathcal{G}$, is either a smooth manifold of dimension $c_2(V_+)$, or empty.*

Definition 4.3. *(Parametrized Moduli Space) The parametrized moduli space is defined as*

$$\mathcal{M} = \{([A,\Phi],\omega) \in \mathcal{B} \times i\Omega^2_+(X) | f_\omega(A,\Phi) = 0\}$$

The irreducible part is

$$\mathcal{M}^* = \mathcal{M} \cap (\mathcal{B}^* \times i\Omega^2_+(X))$$

Let $\pi : \mathcal{M}^* \to i\Omega^2_+(X)$ be the canonical projection, we observe that $\mathcal{M}^*_\omega = \pi^{-1}(\omega)$.

We have the following theorem

Theorem 5. *(Transversality)*

- *$\mathcal{M}^*$ is a Banach manifold*
- *The projection $\pi : \mathcal{M}^* \to i\Omega^2_+(X)$ is Fredholm with index $c_2(V_+)$*
- *The set of regular values of π is generic.For a regular value ω, $\mathcal{M}^*_\omega = \pi^{-1}(\omega)$ is either a smooth manifold of dimension $c_2(V_+)$, or empty.*

After assuming certain regularity conditions on the solutions of SW equations and the gauge transformations. There is the following compactness result:

Theorem 6. *Let (A_i, Φ_i) be a sequence of L_4^2 (Sobolev norm) solutions to the SW equations. Then there exists a sequence u_i of L_5^2 gauge transformations such that $u_i(A_i, \Phi_i)$ is a bounded sequence in L_k^2 for all k. Hence the solutions $u_i(A_i, \Phi_i)$ are C^∞ and there is a subsequence that converges in the C^∞ topology to a C^∞ solution (A, Φ) of the SW equations. In particular,the moduli space is sequentially compact in the C^∞ topology.*

The proof involves establishing certain bounds on the norms of the solutions of the equations, and then applying an elliptic regularity estimate. We omit the proof here.

5 Seiberg-Witten Invariants

We quote without proof the following result on the topology of the irreducible base space:

Theorem 7. *The quotient space $\mathcal{B}^* := \mathcal{C}_{\mathcal{S}}^*/\mathcal{G}$ is a smooth Hilbert manifold with the weak homotopy type of $\mathbb{C}P^\infty \times T^{b_1(X)}$, where $T^{b_1(X)}$ is the torus $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$.*

We assume X is closed,connected,oriented and smooth, with $b_2^+(X) \geq 2$. Given a *Spin*^{c} structure $\mathcal{S}$ and generic parameters $(g, \omega) \in \mathcal{P}$, $\mathcal{M}_\omega$ is a smooth,closed, oriented manifold of expected dimension. The fundamental class of the moduli space is an invariant of $\mathcal{S}$.

Theorem 8. *The map*

$$SW_X : Spin^c(X) \to H_*(\mathcal{B}^*; \mathbb{Z})$$

$$\mathcal{S} \mapsto [\mathcal{M}_\omega]$$

is an oriented diffeomorphism invariant of X. i.e. if $f : Y \to X$ is an orientation-preserving diffeomorphism, then $f^* \cdot SW_X = SW_Y \cdot f^*$.

5.1 Properties of Seiberg-Witten Invariants

- Let $\tau : Spin^c(X) \to Spin^c(X)$ be the charge conjugation map. Then $SW_X(\tau(\mathcal{S})) = \pm SW_X(\mathcal{S})$. This is because conjugation leads to an identification $(g, \omega) \leftrightarrow (g, -\omega)$, this yields a diffeomorphism of $\mathcal{M}_\omega$.
- SW_X has finite support
- If X admits a metric g_0 with the scalar curvature $s_{g_0} > 0$, then $SW_X(\mathcal{S}) \equiv 0$

  Remark 3. *This fails if $b_2^+(X) \leq 1$, exemplified by $\mathbb{C}P^2$.*

- In case X admits a scalar-flat metric, we have the following:

  Proposition 5.1. *Suppose X admits a metric g with $s_g \equiv 0$. If $2\chi(X) + 3\sigma(X) \geq 0$, then $SW_X(\mathcal{S}) = 0$ unless $c_1(L_{\mathcal{S}})_{\mathbb{R}} = 0$ and $2\chi(X) + 3\sigma(X) = 0$. In the latter case, $SW_X(\mathcal{S}) \in H_0(\mathcal{B}^*; \mathbb{Z})$.*