Stat 462 HW2 Jiaqi Li

1. By using $\hat{\beta} = (X^T X)^{-1} X^T y$,

$$\underbrace{\begin{bmatrix} 0.5 & 0 & -0.25 \\ 0 & 0.01 & 0 \\ -0.25 & 0 & 0.5 \end{bmatrix}}_{(X^T X)^{-1}} \underbrace{\begin{bmatrix} 12 \\ -50 \\ 20 \end{bmatrix}}_{X^T y} = \begin{bmatrix} 1 \\ -0.5 \\ 2 \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}$$

Then, $\hat{\beta}_0 = 1$, $\hat{\beta}_1 = -0.5$, $\hat{\beta}_2 = 2$

Thus, $\hat{y} = 1 - 0.5x_1 + 2x_2$

$$\text{Then } \hat{y} = \begin{bmatrix} 1 - 0.5 \times 5 + 2 \times 2 \\ 1 - 0.5 \times 5 + 2 \times 0 \\ 1 - 0.5 \times (-5) + 2 \times 2 \\ 1 - 0.5 \times (-5) + 2 \times 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -1.5 \\ 7.5 \\ 3.5 \end{bmatrix}$$

Thus, $\hat{y}_1 = 2.5$, $\hat{y}_2 = -1.5$, $\hat{y}_3 = 7.5$, $\hat{y}_4 = 3.5$.

2.

① For $\hat{\beta}_1$, we have:

$$\hat{\beta}_1 = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2} = \frac{\sum_i (x_i - \bar{x}) y_i}{\sum_i (x_i - \bar{x})^2}.$$

Since $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$.

$$\hat{\beta}_1 = \frac{\sum_i (x_i - \bar{x})(\beta_0 + \beta_1 x_i + \varepsilon_i)}{\sum_i (x_i - \bar{x})^2}$$

$$= \frac{\beta_0 \sum_i (x_i - \bar{x}) + \beta_1 \sum_i (x_i - \bar{x}) x_i + \sum_i (x_i - \bar{x}) \varepsilon_i}{\sum_i (x_i - \bar{x})^2}$$

$$= \frac{\beta_1 \sum_i (x_i - \bar{x}) x_i + \sum_i (x_i - \bar{x}) \varepsilon_i}{\sum_i (x_i - \bar{x})^2}$$

$$= \frac{\beta_1 \left[\sum_i (x_i - \bar{x})(x_i - \bar{x}) + \sum_i (x_i - \bar{x}) \bar{x} \right] + \sum_i (x_i - \bar{x}) \varepsilon_i}{\sum_i (x_i - \bar{x})^2}$$

1. $$\hat{\beta}_1 = \frac{\beta_1 \sum_i (x_i-\bar{x})^2 + \sum_i (x_i-\bar{x})\varepsilon_i}{\sum_i (x_i-\bar{x})^2}$$

$$= \beta_1 + \frac{\sum_i (x_i-\bar{x})\varepsilon_i}{\sum_i (x_i-\bar{x})^2}$$

$$Var(\hat{\beta}_1) = Var\left(\beta_1 + \frac{\sum_i (x_i-\bar{x})\varepsilon_i}{\sum_i (x_i-\bar{x})^2}\right)$$

$$= \frac{\sum_i (x_i-\bar{x})^2}{\left[\sum_i (x_i-\bar{x})^2\right]^2} \cdot \sigma^2 \quad \text{since } \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$$

$$= \frac{\sigma^2}{\sum_i (x_i-\bar{x})^2}$$

Since we assumed $\hat{\beta}_1$ is an unbiased estimator, $E(\hat{\beta}_1) = \beta_1$

Thus, $\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma^2}{\sum_i (x_i-\bar{x})^2}\right)$

② For $\hat{\beta}_0$, we have $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$
Since $\bar{y} = \beta_0 + \beta_1 \bar{x} + \frac{1}{n}\sum_i \varepsilon_i$, plug in and we have:

$$\hat{\beta}_0 = \left(\beta_0 + \beta_1\bar{x} + \frac{1}{n}\sum_i \varepsilon_i\right) - \hat{\beta}_1 \bar{x} = \left(\beta_0 + \beta_1\bar{x} + \frac{1}{n}\sum_i \varepsilon_i\right) - \left(\beta_1\bar{x} + \frac{\sum_i (x_i-\bar{x})\varepsilon_i}{\sum_i (x_i-\bar{x})^2}\bar{x}\right)$$

$$= \beta_0 + \frac{1}{n}\sum_i \varepsilon_i - \frac{\sum_i (x_i-\bar{x})\varepsilon_i}{\sum_i (x_i-\bar{x})^2}\bar{x} = \beta_0 + \sum_i \left(\varepsilon_i\left[\frac{1}{n} - \frac{(x_i-\bar{x})\bar{x}}{(x_i-\bar{x})^2}\right]\right)$$

$$Var(\hat{\beta}_0) = Var\left(\beta_0 + \sum_i \left(\varepsilon_i\left[\frac{1}{n} - \frac{(x_i-\bar{x})}{(x_i-\bar{x})^2}\right]\right)\right)$$

$$= \sum_i \left[\frac{1}{n} - \frac{(x_i-\bar{x})\bar{x}}{(x_i-\bar{x})^2}\right]^2 \sigma^2 = \left[\sum_i \frac{1}{n^2} - \frac{2\bar{x}}{n}\frac{\sum_i (x_i-\bar{x})}{\sum_i (x_i-\bar{x})^2} + \frac{\sum_i (x_i-\bar{x})^2 \bar{x}^2}{\left[\sum_i (x_i-\bar{x})^2\right]^2}\right]\sigma^2$$

$$= \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_i (x_i-\bar{x})^2}\right]\sigma^2.$$

Since we assumed $\hat{\beta}_0$ is an unbiased estimator, $E(\hat{\beta}_0) = \beta_0$

Thus, $\hat{\beta}_0 \sim N\left(\beta_0, \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_i (x_i-\bar{x})^2}\right]\sigma^2\right).$

Application

A.

<u>Hypothesis Test on Weight</u>:

```
Call:
lm(formula = bodyfat.percent ~ Weight, data = bodyfat)

Residuals:
     Min       1Q   Median       3Q      Max
-27.1676  -4.6126   0.0375   4.9613  20.9494

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -13.94208    2.83363   -4.92 1.58e-06 ***
Weight        0.18490    0.01573   11.75  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.712 on 248 degrees of freedom
Multiple R-squared:  0.3577,   Adjusted R-squared:  0.3551
F-statistic: 138.1 on 1 and 248 DF,  p-value: < 2.2e-16

> confint(lm.bodyfat.W, level=0.99)
                  0.5 %     99.5 %
(Intercept) -21.2976226 -6.5865326
Weight        0.1440604  0.2257362
```

H_0: $\beta_{weight} = 0$ H_1: $\beta_{weight} \neq 0$

Test statistic is t distribution that t = 11.75 with df=248

p-value for test statistic is smaller than 2×10^{-16}.

Since p-value is < $\alpha = 0.05$, we reject null hypothesis.

Thus, we can conclude that Weight has impact on the body fat percentage.

For every 1 pound increase in Weight, we are 99% confident that there will be an increase of 0.1440604 to 0.2257362 percent in body fat percentage.

<u>Hypothesis Test on Height</u>:

```
Call:
lm(formula = bodyfat.percent ~ Height, data = bodyfat)

Residuals:
     Min       1Q   Median       3Q      Max
-19.3423  -6.5537   0.2821   6.2142  27.5375

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  29.8863    14.2522   2.097    0.037 *
Height       -0.1551     0.2026  -0.765    0.445
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 8.365 on 248 degrees of freedom
Multiple R-squared:  0.002357, Adjusted R-squared:  -0.001666
F-statistic: 0.5858 on 1 and 248 DF,  p-value: 0.4448
```

```
> confint(lm.bodyfat.H, level=0.99)
                  0.5 %     99.5 %
(Intercept) -7.1095881 66.8822117
Height      -0.6809313  0.3708134
```

H_0: $\beta_{height} = 0$ H_1: $\beta_{height} \neq 0$

Test statistic is t distribution that t = -0.765 with df=248

p-value for test statistic is 0.445.

Since p-value is > α = 0.05, we do not reject null hypothesis.

Thus, we can conclude that Height does not have impact on the body fat percentage.

Since Height does not have any impact on body fat percentage, body fat percentage will not change with any increase of Height.

<u>Hypothesis Test on Abdomen Circumference</u>:

```
Call:
lm(formula = bodyfat.percent ~ AbdomenC, data = bodyfat)

Residuals:
     Min       1Q   Median       3Q      Max 
-23.1760  -3.5408   0.2143   3.1793  12.8435 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) -42.29941    2.82409  -14.98   <2e-16 ***
AbdomenC      0.66407    0.03042   21.83   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.899 on 248 degrees of freedom
Multiple R-squared:  0.6578,	Adjusted R-squared:  0.6564 
F-statistic: 476.7 on 1 and 248 DF,  p-value: < 2.2e-16

> confint(lm.bodyfat.A, level=0.99)
                   0.5 %      99.5 %
(Intercept) -49.6301857 -34.9686421
AbdomenC      0.5851118   0.7430221
```

H_0: $\beta_{abodmenc} = 0$ H_1: $\beta_{abdomenc} \neq 0$

Test statistic is t distribution that t = 21.83 with df=248

p-value for test statistic is smaller than 2×10^{-16}.

Since p-value is < α = 0.05, we reject null hypothesis.

Thus, we can conclude that Abdomen Circumference has impact on the body fat percentage.

For every 1 cm increase in Abdomen Circumference, we are 99% confident that there will be an increase of 0.5851118 to 0.7430221 percent in body fat percentage.

Comparing the 3 regressions, we can see that regression of Abdomen Circumference has that largest R^2 and regression of Height has the smallest R^2. Even though regression of Weight does not fit the data

as well as that of Abdomen Circumference, it still fits pretty well to the data. However, the regression of Height has very small R^2, which indicates that the regression of Height fits the data poorly. Also, from confident intervals, we can see that both interval of β of Abdomen Circumference and β of Weight do not contain 0 while interval of β of Height contain 0. Thus, Abdomen Circumference and Weight have significant impact on body fat percentage while Height does not have significant impact. This result meets the same conclusion with the hypothesis tests.

B.

```
> T=(summary.full$coefficients[2,1]-0.5)/summary.full$coefficients[2,2]
> T
[1] 6.351477
> pvalue=pt(T,df=n-p)
> pvalue
[1] 1
```

Computed from R, we have t=6.351477 with p-value ≈ 1

<u>**Hypothesis Test**</u>:

H_0: $\beta_{abodmenc} \geq 0.5$ H_1: $\beta_{abdomenc} < 0.5$

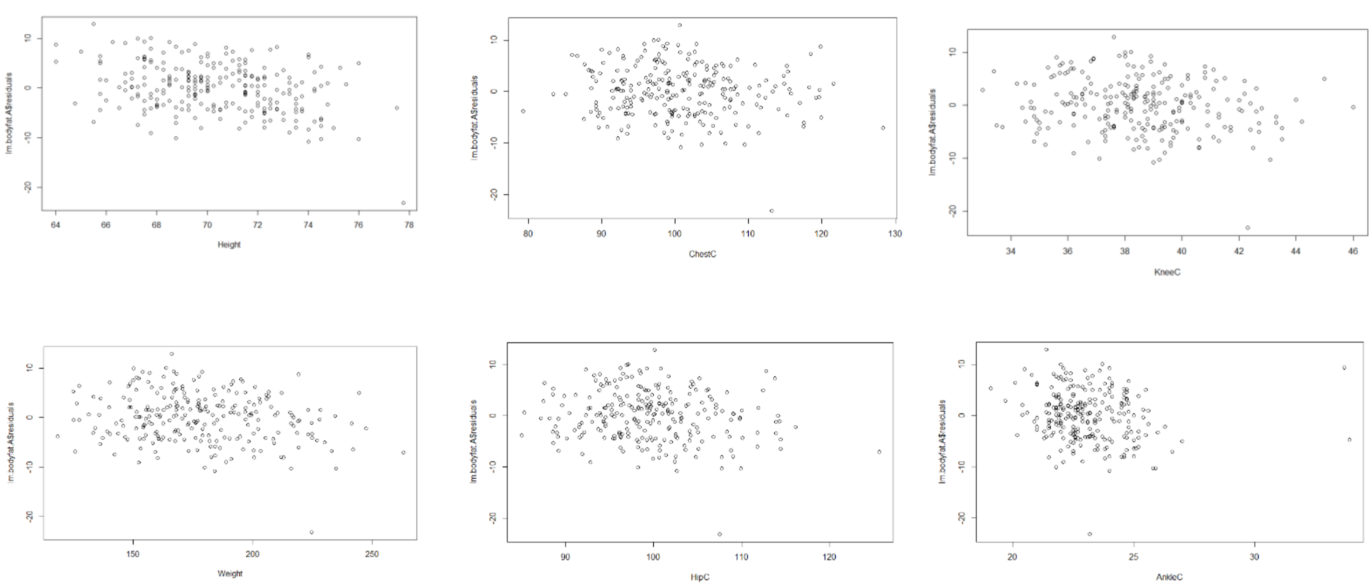
Test statistic is t distribution that t = 6.351477 with df=237

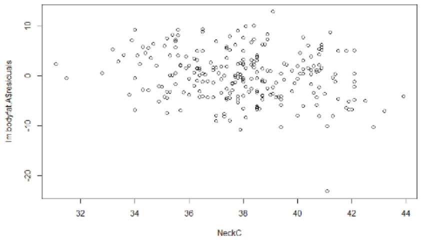
p-value for test statistic is approximately 1.

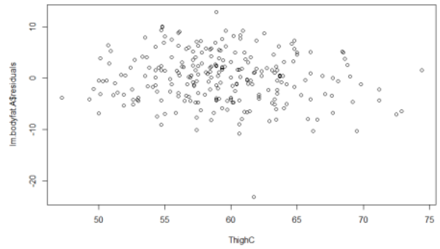
Since p-value is > α = 0.05, we accept null hypothesis.

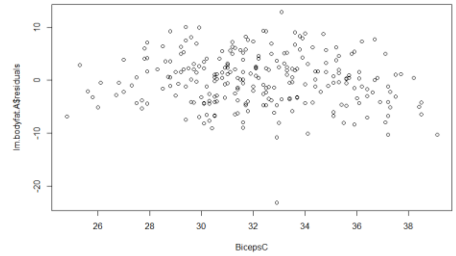
Thus, we can conclude that for each additional cm of abdomen circumference, the body fat percentage increases by more than 0.5 points

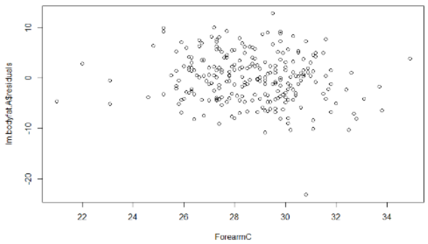
C.

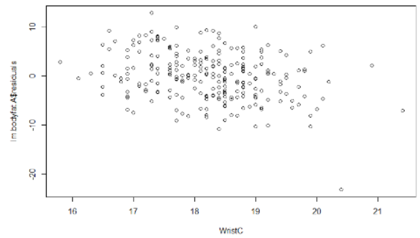












By observing the scatter plots, if we find some predictors that have correlation with the residuals of linear regression model for Abdomen Circumference, we can try to put those predictors in our AbdomenC model and test if those added predictors have significant impact on body fat percentage.

From the plots we have above, we can see that Ankle Circumference may have an correlation with residuals since we can see a trend in the plot of AnkleC, which means it may capture some factors that Abdomen Circumference may not where those factors can reduce the residuals of the fitted value. This makes sense because ankle circumference may have influence on the frequency and severity of body movement, which could influence body fat percentage.

D.

Equation of the population model employed:

$Y = \beta_0 + \beta_1X_1 + \beta_2X_2 + \beta_3X_3 + \beta_4X_4 + \beta_5X_5 + \beta_6X_6 + \beta_7X_7 + \beta_8X_8 + \beta_9X_9 + \beta_{10}X_{10} + \beta_{11}X_{11} + \beta_{12}X_{12} + \epsilon,\ \epsilon \sim N(0,1)$

```
Call:
lm(formula = bodyfat.percent ~ AbdomenC + Weight + Height + NeckC +
    ChestC + HipC + ThighC + KneeC + AnkleC + BicepsC + ForearmC +
    WristC, data = bodyfat)

Residuals:
     Min       1Q   Median       3Q      Max
-14.1320  -3.1309  -0.2276   3.2606   9.2764

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  4.68301   23.94904   0.196   0.8451
AbdomenC     1.03512    0.08425  12.286   <2e-16 ***
Weight      -0.05043    0.06748  -0.747   0.4556
Height      -0.30016    0.19614  -1.530   0.1273
NeckC       -0.36143    0.24045  -1.503   0.1341
ChestC      -0.14946    0.11089  -1.348   0.1790
HipC        -0.21225    0.14910  -1.423   0.1559
ThighC       0.07923    0.14196   0.558   0.5773
KneeC        0.07019    0.24391   0.288   0.7738
AnkleC       0.23758    0.22444   1.059   0.2909
BicepsC      0.27137    0.17448   1.555   0.1212
ForearmC     0.22654    0.21088   1.074   0.2838
WristC      -1.61083    0.51300  -3.140   0.0019 **
---
```

```
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.369 on 237 degrees of freedom
Multiple R-squared:  0.7399,   Adjusted R-squared:  0.7268
F-statistic: 56.19 on 12 and 237 DF,  p-value: < 2.2e-16
```

Least square estimates of the parameters:

β_0=4.68301, β_1=1.03512, β_2=- 1.03512, β_3=- 0.30016, β_4=- 0.36143, β_5=- 0.14946, β_6=- 0.21225, β_7=0.07923, β_8=0.07019, β_9=0.23758, β_{10}=0.27137, β_{11}=0.22654, β_{12}=- 1.61083, $\hat{\sigma}^2=4.369^2=19.088$

The equation of the estimated regression model:

$$Y = 4.68301 + 1.03512X_1 - 1.03512X_2 - 0.30016X_3 - 0.36143X_4 - 0.14946X_5 - 0.21225X_6 + 0.07923X_7 + 0.07019X_8 + 0.23758X_9 + 0.27137X_{10} + 0.22654X_{11} - 1.61083X_{12}$$

The value of the determination coefficient: $R^2 = 0.7399$

Hypothesis Test:

H_0: $\beta_0=\beta_1=\beta_2=\beta_3=\beta_4=\beta_5=\beta_6=\beta_7=\beta_8=\beta_9=\beta_{10}=\beta_{11}=\beta_{12}=0$

H_1: at least one of the predictors is significantly different from zero.

Test Statistic is F distribution with $F_{12,237} = 56.19$

p-value < 2.2×10^{-16}

Since p-value < $2.2 \times 10^{-16} < \alpha = 0.05$, we reject null hypothesis.

Thus, at least one of the predictors is significantly different from zero.

Is this model better?

Comparing with the model of Abdomen Circumference, this new model has $R^2 = 0.7399$, which is larger than $R^2 = 0.6578$ of the regression model of Abdomen Circumference.

Also, in this model, we can observe that 2 predictors, Abdomen Circumference and Wrist Circumference, have impact on body fat percentage since their p-value are both smaller than α =0.05.

Thus, this new model is a better model to predict body fat percentage.