Stat 462 HW3 Jiaqi Li

1.

$$y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i \qquad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \qquad \hat{\beta}_1 = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}$$

$$R^2 = 1 - \frac{RSS}{Total\,SS} = \frac{Total\,SS - RSS}{Total\,SS}$$

$$= \frac{ESS}{Total\,SS}$$

$$= \frac{\sum_i (\hat{y}_i - \bar{y})^2}{\sum_i (y_i - \bar{y})^2}$$

$$\sum_i (\hat{y}_i - \bar{y})^2 = \sum_i (\hat{\beta}_0 + \hat{\beta}_1 x_i - \bar{y})^2$$

$$= \sum_i (\bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x_i - \bar{y})^2$$

$$= \hat{\beta}_1^2 \sum_i (x_i - \bar{x})^2$$

$$= \frac{[\sum_i (x_i - \bar{x})(y_i - \bar{y})]^2 \sum_i (x_i - \bar{x})^2}{[\sum_i (x_i - \bar{x})^2]^2}$$

$$= \frac{[\sum_i (x_i - \bar{x})(y_i - \bar{y})]^2}{\sum_i (x_i - \bar{x})^2}$$

$$R^2 = \frac{[\sum_i (x_i - \bar{x})(y_i - \bar{y})]^2}{\sum_i (x_i - \bar{x})^2} \Big/ \sum_i (y_i - \bar{y})^2$$

$$= \frac{[\sum_i (x_i - \bar{x})(y_i - \bar{y})]^2}{\sum_i (x_i - \bar{x})^2 \sum_i (y_i - \bar{y})^2}$$

$$= \left(\frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2 \sum_i (y_i - \bar{y})^2}} \right)^2 = \left(\frac{\frac{1}{n} \sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\frac{1}{n} \sum_i (x_i - \bar{x})^2 \frac{1}{n} \sum_i (y_i - \bar{y})^2}} \right)^2$$

$$= \left(\frac{Cor(x, y)}{\sqrt{Var(x)\, Var(y)}} \right)^2$$

$$= [corr(x, y)]^2$$

Note: $Cor(x,y) = \frac{1}{n} \sum_i (x_i - E(x))(y_i - E(y)) = \frac{1}{n} \sum_i (x_i - \bar{x})(y_i - \bar{y})$

2. For $H_0: \beta_1 = 0$ $\quad H_1: \beta_1 \neq 0$, we have $p = 2$

$$F = \frac{SS_{reg}/(p-1)}{RSS/(n-p)} = \frac{SS_{reg}}{RSS/(n-2)}$$

$$= \frac{\sum_i (\hat{y}_i - \bar{y})^2}{\hat{\sigma}^2}$$

$$= \frac{\sum_i (\hat{\beta}_0 + \hat{\beta}_1 x_i - \bar{y})^2}{\hat{\sigma}^2}$$

$$= \frac{\sum_i (\bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x_i - \bar{y})^2}{\hat{\sigma}^2}$$

$$= \frac{\hat{\beta}_1^2 \sum_i (x_i - \bar{x})^2}{\sigma^2}$$

$$= \frac{\hat{\beta}_1^2}{\hat{\sigma}^2 / \sum_i (x_i - \bar{x})^2}$$

$$= \frac{\hat{\beta}_1^2}{se(\hat{\beta}_1)^2} = \left(\frac{\hat{\beta}_1}{se(\hat{\beta}_1)} \right)^2 = t^2$$

Note: $se(\hat{\beta}_1) = \sqrt{\dfrac{\hat{\sigma}^2}{\sum_i (x_i - \bar{x})^2}}$ based on $\hat{\beta}_1 \sim N\left(\beta_1, \dfrac{\sigma^2}{\sum_i (x_i - \bar{x})^2}\right)$.