State 462. Jiaqi Li

1. ① Prove $\bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$

for $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$.

we have $\varepsilon_i^2 = [y_i - (\beta_0 + \beta_1 x_i)]^2$

$$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} [y_i - (\beta_0 + \beta_1 x_i)]^2$$

$$= \sum_{i=1}^{n} [y_i^2 - 2y_i(\beta_0 + \beta_1 x_i) + (\beta_0 + \beta_1 x_i)^2]$$

$$= \sum_{i=1}^{n} [y_i^2 - 2y_i\beta_0 - 2y_i x_i \beta_1 + \beta_0^2 + 2\beta_0\beta_1 x_i + \beta_1^2 x_i^2].$$

Since we want to minimize $\sum_{i=1}^{n} \varepsilon_i^2$, differentiate $\sum_{i=1}^{n} \varepsilon_i^2$ & set to zero:

$$\frac{\partial \sum_{i=1}^{n} \varepsilon_i^2}{\partial \beta_0} = \sum_{i=1}^{n} [-2y_i + 2\hat{\beta}_0 + 2\hat{\beta}_1 x_i] = 0.$$

$$-2\sum_{i=1}^{n} y_i + 2\sum_{i=1}^{n} \hat{\beta}_0 + 2\hat{\beta}_1 \sum_{i=1}^{n} x_i = 0.$$

$$\Rightarrow \sum_{i=1}^{n} y_i = n\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{n} x_i \quad (*).$$

divide n on both sides $\Rightarrow$ $\bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$

② Prove $\sum_{i=1}^{n} \hat{y}_i = \sum_{i=1}^{n} y_i$

From (*), we have: $\sum_{i=1}^{n} y_i = n\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{n} x_i$

$$= \sum_{i=1}^{n} (\hat{\beta}_0 + \hat{\beta}_1 x_i). = \sum_{i=1}^{n} \hat{y}_i$$

$$\Rightarrow \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \hat{y}_i. \quad (*)'$$

③ Prove $\sum_{i=1}^{n} e_i = 0$:

Since $e_i = y_i - \hat{y}_i$, $\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - \hat{y}_i) = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \hat{y}_i = 0$ by (*)'.

Thus, $\sum_{i=1}^{n} e_i = 0$.

2. $Y = \beta_0 + \varepsilon$

$\Rightarrow \varepsilon_i = Y_i - \beta_0$.

$\Rightarrow \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} (Y_i - \beta_0)^2 = \sum_{i=1}^{n} (Y_i^2 - 2Y_i\beta_0 + \beta_0^2)$.

To minimize $\sum_{i=1}^{n} \varepsilon_i^2$, differentiate & set 0:

$$\frac{\partial \sum_{i=1}^{n} \varepsilon_i^2}{\partial \beta_0} = \sum_{i=1}^{n} (-2Y_i + 2\hat{\beta}_0) = 0.$$

$$n\hat{\beta}_0 = \sum_{i=1}^{n} Y_i$$

$$\hat{\beta}_0 = \overline{Y}$$

Since we can calculate $\bar{Y}$ by $\frac{1}{n}\sum_{i=1}^{n} Y_i$, $\hat{\beta}_0$ is a known quantity.