# Project 2 Jiaqi Li

January 24, 2019

**The following are packages and user-defined functions with random number generators**

```
In [101]: import numpy as np
          import math
          import matplotlib.pyplot as plt
          import scipy.stats as bs

          a = 7**5
          b = 0
          m = 2**31-1

          #This function generates uniform distribution
          def f_unif(n,x_0):
              U = [None] * (n+1)
              U[0] = x_0
              i = 1
              while i < (n+1):
                  U[i] = np.mod(a*U[i-1]+b,m)
                  i = i+1
              del U[0]
              U = [x/m for x in U]
              return U

          #This function generates normal distribution
          def f_norm(n,U):
              Z_1 = [None]*n
              Z_2 = [None]*n
              for i in range(n):
                  Z_1[i] = np.sqrt(-2*np.log(U[2*i]))*np.cos(2*math.pi*U[2*i+1])
                  Z_2[i] = np.sqrt(-2*np.log(U[2*i]))*np.sin(2*math.pi*U[2*i+1])
                  i = i + 1
              return Z_1+Z_2

          def f_w(T,Z):
              W = [np.sqrt(T)*x for x in Z]
              return W
```

## 1 Problem 1.

In [102]:
```
#1----------------------------------------
cov = -0.7
n = 1000

#cholesky decomposition
covM = np.array([[1,cov],[cov,1]])
L = np.linalg.cholesky(covM)

#set seed (x0)
np.random.seed(5)
x0 = np.random.randint(100,size = 2)

#generate 2 uniform distribution with 1000 observations in each
U1 = f_unif(n,x0[0])
U2 = f_unif(n,x0[1])
#generate 2 normal distribution with 1000 observations in each
x = 500
Z1 = f_norm(x,U1)
Z2 = f_norm(x,U2)

#create X and Y that used to generate 2 correlated normal distributions
X = [x*L[0,0] for x in Z1]
Y = [a+b for a,b in zip([y*L[1,0] for y in Z1], [y*L[1,1] for y in Z2])]

#calculate sampling mean of X and Y
Xe = np.mean(X); Ye = np.mean(Y)

#simulate rho
numerator = 0
VX = VY = 0
for i in range(n):
    numerator = numerator + (X[i]-Xe)*(Y[i]-Ye)
    VX = VX + (X[i]-Xe)**2
    VY = VY + (Y[i]-Ye)**2
rho = numerator/(n-1) / (np.sqrt(VX/(n-1))*np.sqrt(VY/(n-1)))
print(rho)
```

```
-0.6913662593058951
```

```
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  app.launch_new_instance()
```

$\rho$ is approximately -0.7.

## 2 Problem 2.

In [103]:
```
#2---------------------------------------
cov = 0.6
covM = np.array([[1,cov],[cov,1]])
L = np.linalg.cholesky(covM)

#set seed (x0)
np.random.seed(1)
x0 = np.random.randint(100,size = 2)

#generate 2 uniform distribution with 1000 observations in each
U1 = f_unif(n,x0[0])
U2 = f_unif(n,x0[1])
#generate 2 normal distribution with 1000 observations in each
x = 500
Z1 = f_norm(x,U1)
Z2 = f_norm(x,U2)

#create X and Y that used to generate 2 correlated normal distributions
X = [x*L[0,0] for x in Z1]
Y = [a+b for a,b in zip([y*L[1,0] for y in Z1], [y*L[1,1] for y in Z2])]

E = [None]*n
for i in range(n):
    if (X[i]**3+np.sin(Y[i])+X[i]**2*Y[i]) > 0:
        E[i] = (X[i]**3+np.sin(Y[i])+X[i]**2*Y[i])
    else:
        E[i] = 0
E_mean = np.mean(E)
print(E_mean)
```

```
1.5289656039193773
```

```
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  app.launch_new_instance()
```

Estimated value of $E[max(0, X^3 + sin(Y) + X^2Y]$ is 1.53.

## 3 Problem 3.

**(a)**

In [104]:
```
#3---------------------------------------
#a-----------
#set seed and generates uniform and normal distribution
```

```
x0_3 = 2
x = 500
U_3 = f_unif(n,x0_3)
Z_3 = f_norm(x,U_3)

#The following results are estimated by Monte Carlo Simulation
W5 = f_w(5,Z_3)
E5 = np.array(W5)**2 + np.sin(W5)
E5_mean = np.mean(E5)
E5_v = np.sqrt(np.var(E5))

W05 = f_w(0.5,Z_3)
E05 = np.exp(0.5/2)*np.cos(W05)
E05_mean = np.mean(E05)
E05_v = np.sqrt(np.var(E05))

W32 = f_w(3.2,Z_3)
E32 = np.exp(3.2/2)*np.cos(W32)
E32_mean = np.mean(E32)
E32_v = np.sqrt(np.var(E32))

W65 = f_w(6.5,Z_3)
E65 = np.exp(6.5/2)*np.cos(W65)
E65_mean = np.mean(E65)
E65_v = np.sqrt(np.var(E65))

print(E5_mean, E05_mean, E32_mean, E65_mean)
print(E5_v, E05_v, E32_v, E65_v)
```

```
4.715685353777641 1.0194192307817886 1.1736742534509648 1.0528421655822626
7.571114466508359 0.34733827572240733 3.272860230200786 18.48038535552159


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  app.launch_new_instance()
```

| value | Ea1 | Ea2 | Ea3 | Ea4 |
|---|---|---|---|---|
| Expected | 4.71568 | 1.01941 | 1.17367 | 1.05284 |
| error | 7.57111 | 0.34733 | 3.27286 | 18.48038 |

**(b).**

Observing values of Ea2, Ea3, Ea4, we can conclude that they are all very close to 1. This may indicate that the exact value of $E[e^{\frac{t}{2}}cos(W_t)]$ is 1.

**(c)Use variance reduction techniques**

Here we use the Control Variate method:

```
In [105]: #c-----------
          #Set control variate Y = W^2 where W is the corresponding
          #brownian motion for different T
          ConV5 = np.array(W5)**2
          cov5 = np.cov(np.array(E5),ConV5)
          gamma5 = cov5[1,0]/np.var(ConV5)
          T5 = np.array(E5) - gamma5*(ConV5-5)
          T5_e = np.mean(T5)
          T5_v = np.sqrt(np.var(T5))

          ConV05 = np.array(W05)**2
          cov05 = np.cov(np.array(E05),ConV05)
          gamma05 = cov05[1,0]/np.var(ConV05)
          T05 = np.array(E05) - gamma05*(ConV05-0.5)
          T05_e = np.mean(T05)
          T05_v = np.sqrt(np.var(T05))

          ConV32 = np.array(W32)**2
          cov32 = np.cov(E32,ConV32)
          gamma32 = cov32[1,0]/np.var(ConV32)
          T32 = E32 - gamma32*(ConV32-3.2)
          T32_e = np.mean(T32)
          T32_v = np.sqrt(np.var(T32))

          ConV65 = np.array(W65)**2
          cov65 = np.cov(E65,ConV65)
          gamma65 = cov65[1,0]/np.var(ConV65)
          T65 = E65 - gamma65*(ConV65-6.5)
          T65_e = np.mean(T65)
          T65_v = np.sqrt(np.var(T65))

          print(T5_e, T05_e, T32_e, T65_e)
          print(T5_v, T05_v, T32_v, T65_v)
```

```
5.009442721021386 1.0064266518583218 1.098125650994369 0.8775578950787397
0.6996982107601377 0.09730750463381024 2.63678518983461 17.924554721519904
```

The expected values and corresponding error are shown as following:

| MC | Expected | error | VR | Expected | error |
|---|---|---|---|---|---|
| a1 | 4.71568 | 7.57111 | b1 | 5.00944 | 0.69969 |
| a2 | 1.01941 | 0.34733 | b2 | 1.00642 | 0.09730 |
| a3 | 1.17367 | 3.27286 | b3 | 1.09812 | 2.63678 |
| a4 | 1.05284 | 18.48038 | b4 | 0.87755 | 17.92455 |

By looking at the expected value, Eb1, Eb2, Eb3 are significantly improved. Only Eb4 does not have obvious improvement. By looking at the error, all results are improved, especially Eb1 and Eb2.

# 4 Problem 4

**(a)Using Monte Carlo Simulation to estimate the European Call Option**

```
In [106]: #4---------------------------------------
          #a---------------
          n = 1000
          Call = [None]*n
          S0 = 88
          sigma = 0.2
          r = 0.04
          K = 100
          T = 5
          u = f_unif(n,1)
          z = f_norm(500,u)
          w = f_w(T,z)

          #simulate the stock price at time T
          ST = S0*np.exp(sigma*np.array(w)+(r-(sigma**2/2))*T)

          #generate call option price
          for i in range(n):
              if (ST[i]-K) > 0:
                  Call[i] = ST[i]-K
              else:
                  Call[i] = 0
          c = np.mean(np.array(Call)*np.exp(-r*T))

          print(c)
```

```
17.494260392677816
```

```
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  app.launch_new_instance()
```

Estimated price of the call option is 17.49

**(b)Use variance reduction techniques**

Here we use the Antithetic Variates method:

```
In [107]: #b---------------
          z1 = f_norm(500,u)
          z2 = [-x for x in z1]
          w1 = f_w(T,z1)
```

```
    w2 = f_w(T,z2)
    x1 = S0*np.exp(sigma*np.array(w1)+(r-(sigma**2/2))*T)
    x2 = S0*np.exp(sigma*np.array(w2)+(r-(sigma**2/2))*T)
    ST_red1 = [None]*n
    ST_red2 = [None]*n
    for i in range(n):
        if (x1[i]-K) > 0:
            ST_red1[i] = x1[i]-K
        else:
            ST_red1[i] = 0
        if (x2[i]-K) > 0:
            ST_red2[i] = x2[i]-K
        else:
            ST_red2[i] = 0
    c_red1 = np.array(ST_red1)*np.exp(-r*T)
    c_red2 = np.array(ST_red2)*np.exp(-r*T)
    c_red = np.mean((np.array(c_red1)+np.array(c_red2))/2)
    print(c_red)

    #Using Black-Scholes formula
    d1 = (np.log(S0/K) + (r+(sigma**2)/2)*T)/(sigma*np.sqrt(T))
    d2 = d1 - sigma*np.sqrt(T)
    F = S0*bs.norm.cdf(d1) - np.exp(-r * T)*K*bs.norm.cdf(d2)
    print(F)
```

```
18.172097162958003
18.28376570485581
```

Based on the Black-Scholes formula, the exact value this option is 18.28. Using the Antithetic Variates method, the call option price is 18.17; Monte Carlo simulation gives a price of 17.49. Thus, there is a great improvement by using the variance reduction technique.

# 5 Problem 5.

**(a)**

In [108]:
```
#5----------------------------------------
#a-------------
n = 1000
S0 = 88
sigma = 0.18
r = 0.04
U10 = f_unif(n,2870)
Z10 = f_norm(500,U10)
W10 = np.zeros((10,n))
S10 = np.zeros((10,n))
E10 = np.zeros(11)
```

```python
#simulate Sn for n ranging from 1 to 10
for i in range(10):
    W10[i,:] = np.sqrt(i+1)*np.array(Z10)
    S10[i,:] = S0*np.exp(sigma*W10[i,:]+(r-(sigma**2/2))*(i+1))
    E10[i+1] = np.mean(S10[i,:])

E10[0] = S0
plt.figure()
ax1 = plt.plot(E10)
plt.xlabel("rank")
plt.ylabel("price")
plt.title("ESn with sigma = 0.18")
print("when sigma = 0.18, ESn are",E10)
```

```
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  app.launch_new_instance()
```
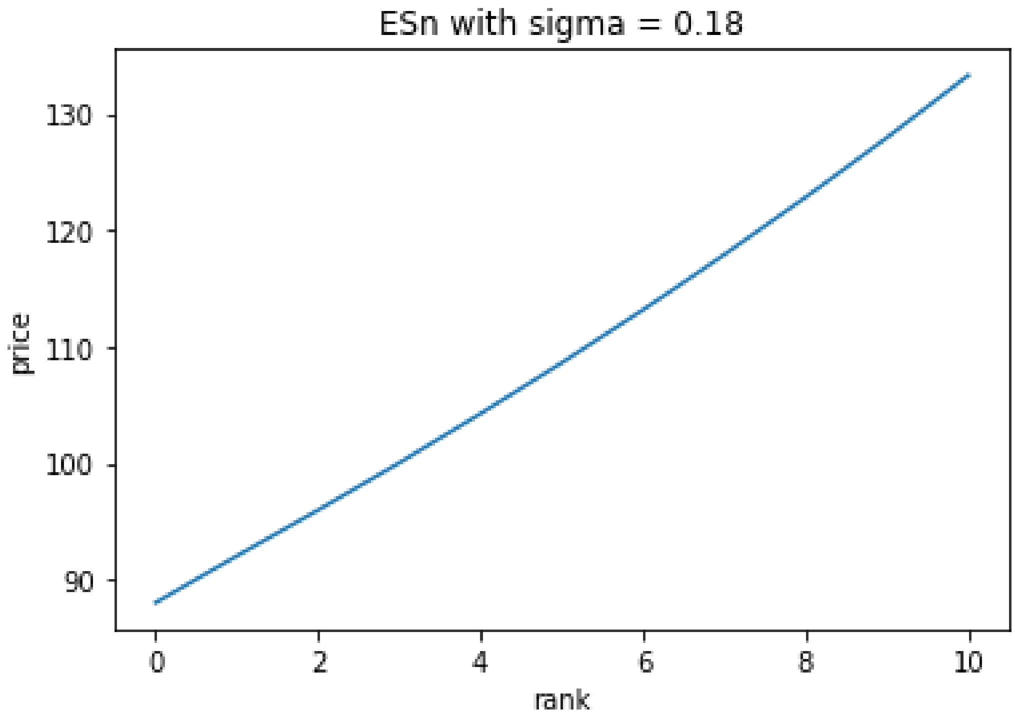

```
when sigma = 0.18, ESn are [ 88.          92.02321065  95.96074736 100.02376747 104.23668356
 108.6128875  113.16269903 117.89534309 122.81966241 127.94443486
 133.27853758]
```




**(b)Simulate 6 paths**

```
In [109]: #b-------------
          incre = np.sqrt(10/n)
          sim6 = np.zeros((6,n+1))
          Wincre = np.zeros((10,n))

          plt.figure()
          #simulate 6 paths of the change of the option price from time 0 to time T
          for i in range(6):
              Zincre = f_norm(500,f_unif(n,(i+1)))
              Wincre[i,0] = incre*Zincre[0]
              for j in range(1,n):
                  Wincre[i,j] = Wincre[i,j-1] + incre*np.array(Zincre[j])
              sim6[i,1:(n+1)] = S0*np.exp(sigma*Wincre[i,:]+(r-(sigma**2/2))*(i+1))
              sim6[i,0] = S0
              ax2 = plt.plot(sim6[i,:])
          plt.xlabel("path (timeline)")
          plt.ylabel("price")
          plt.title("6 paths with sigma = 0.18")
```

```
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  app.launch_new_instance()
```
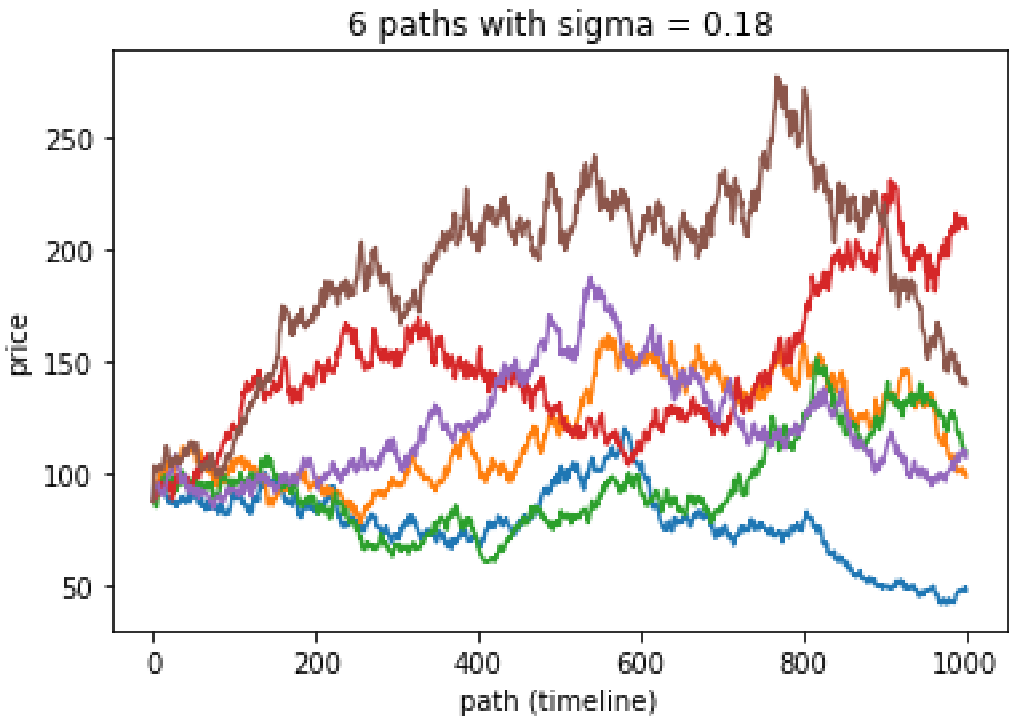
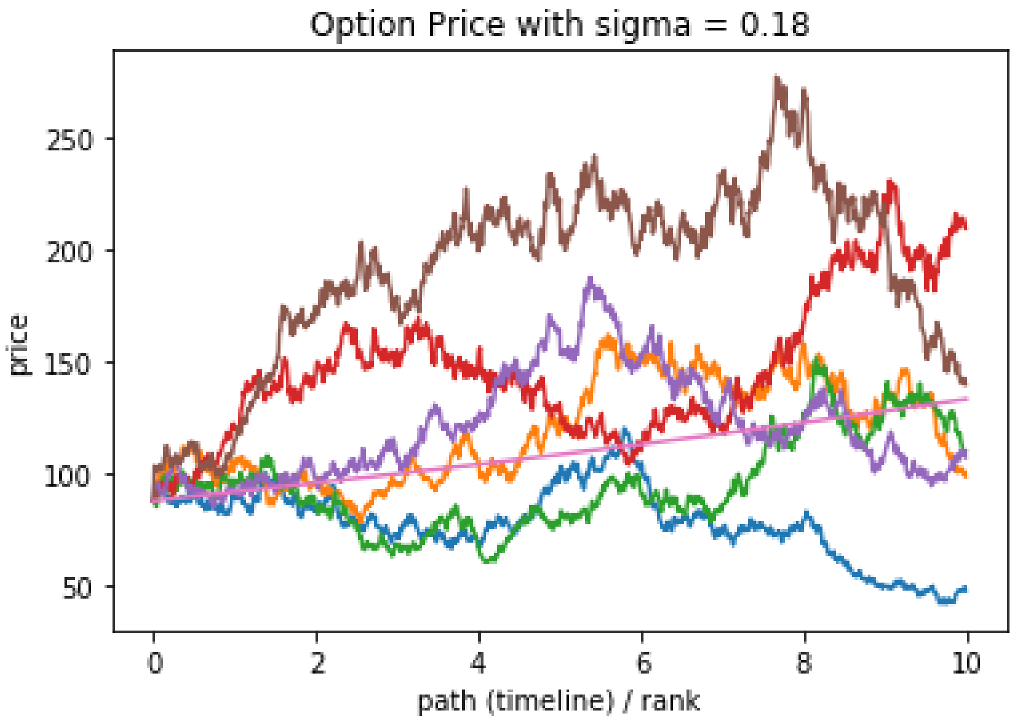

```
Out[109]: Text(0.5, 1.0, '6 paths with sigma = 0.18')
```




**(c)Combine 2 plots**

```
In [110]: #c-------------
          t = np.arange(0,10,10/1001)
          plt.figure()
          for i in range(6):
              plt.plot(t,sim6[i,:])
          ax3 = plt.plot(E10)
          plt.xlabel("path (timeline) / rank")
          plt.ylabel("price")
          plt.title("Option Price with sigma = 0.18")
```

```
Out[110]: Text(0.5, 1.0, 'Option Price with sigma = 0.18')
```




**(d)Now increase $\sigma$ from 18% to 35%**

```
In [111]: #d-------------
          n = 1000
          S0 = 88
          sigma = 0.35
          r = 0.04
          U10 = f_unif(n,2870)
          Z10 = f_norm(500,U10)
          W10 = np.zeros((10,n))
          S10 = np.zeros((10,n))
          E10 = np.zeros(11)
```

```
for i in range(10):
    W10[i,:] = np.sqrt(i+1)*np.array(Z10)
    S10[i,:] = S0*np.exp(sigma*W10[i,:]+(r-(sigma**2/2))*(i+1))
    E10[i+1] = np.mean(S10[i,:])
E10[0] = S0
plt.figure()
ax4 = plt.plot(E10)
plt.xlabel("rank")
plt.ylabel("price")
plt.title("ESn with sigma = 0.35")
print("when sigma = 0.35, ESn are",E10)
```

```
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  app.launch_new_instance()
```
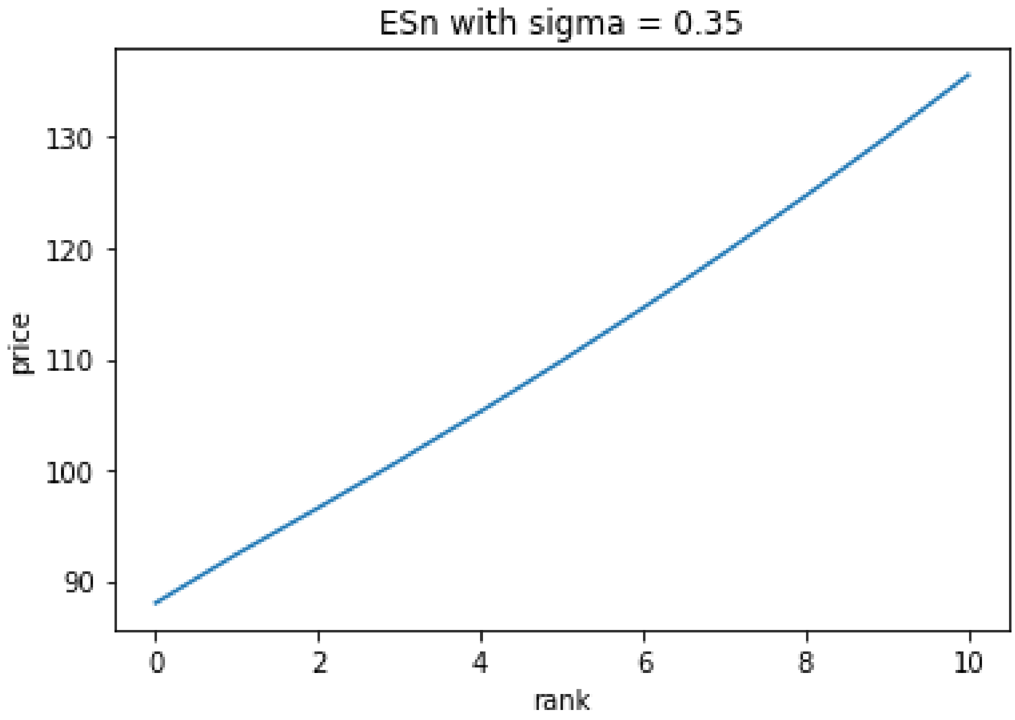

```
when sigma = 0.35, ESn are [ 88.          92.42540791  96.5749635  100.83963811 105.25548594
 109.8386506  114.59971034 119.54711969 124.68840022 130.03064428
 135.58075816]
```




```
In [112]: incre = np.sqrt(10/n)
          sim6 = np.zeros((6,n+1))
          Wincre = np.zeros((10,n))
```

```python
for i in range(6):
    Zincre = f_norm(500,f_unif(n,(i+1)))
    Wincre[i,0] = incre*Zincre[0]
    for j in range(1,n):
        Wincre[i,j] = Wincre[i,j-1] + incre*np.array(Zincre[j])
    sim6[i,1:(n+1)] = S0*np.exp(sigma*Wincre[i,:]+(r-(sigma**2/2))*(i+1))
    sim6[i,0] = S0

t = np.arange(0,10,10/1001)
plt.figure()
for i in range(6):
    plt.plot(t,sim6[i,:])
ax5 = plt.plot(E10)
plt.xlabel("path (timeline) / rank")
plt.ylabel("price")
plt.title("Option Price with sigma = 0.35")
```

```
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  app.launch_new_instance()
```
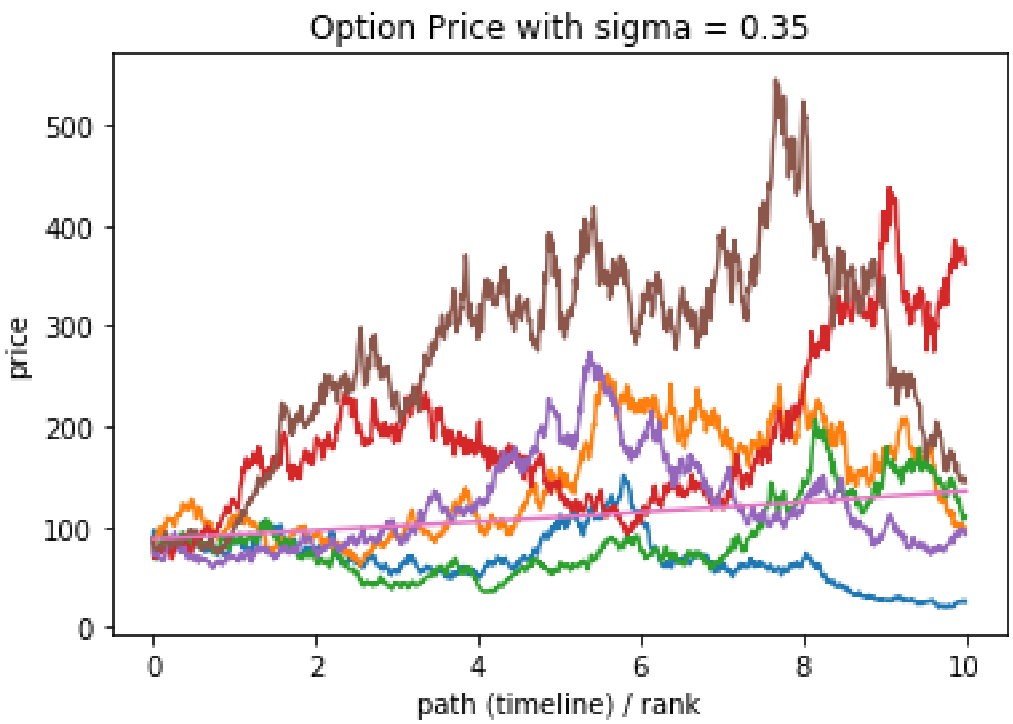

```
Out[112]: Text(0.5, 1.0, 'Option Price with sigma = 0.35')
```



Since the risk ($\sigma$) of the option increases significantly, the $ES_n$ grows faster and the price of the option is much more volatile.

# 6 Problem 6.

**(a)Using Euler's Method**

```
In [113]: #6----------------------------------------
          #a-------------
          #using Euler's discretization scheme
          n = 1000
          I = 1/n
          Euler = 0
          for i in range(n):
              Euler = Euler + np.sqrt(1-(I*(i+1))**2)*I
          Euler = Euler * 4
          print(Euler)
```

```
3.1395554669110277
```

Using Euler's method, the integral is approximately 3.14

**(b)Using the Monte Carlo Simulation**

```
In [114]: #b-------------
          Uint = f_unif(n,5)
          Monte = np.zeros(n)
          for i in range(n):
              Monte[i] = 4*np.sqrt(1-Uint[i]**2)
          MonteE = np.mean(Monte)
          Monte_error = np.sqrt(np.var(Monte))
          print(MonteE,Monte_error)
```

```
3.0964034571314283 0.9209680550482983


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  app.launch_new_instance()
```

Using the Monte Carlo Simulation, the integral is approximately 3.10 with an error of 0.92.

**(c)Using the Important Sampling method**

```
In [115]: #c-------------
          alpha = 0.74
          h = (1-alpha*np.array(Uint)**2)/(1-alpha/3)

          #using Acceptance-Rejection algorithm
          u_test = f_unif(n,3)
```

```python
U_select = np.zeros(n)
H = np.zeros(n)
for i in range(n):
    if u_test[i] <= h[i]/1.33:
        H[i] = h[i]
        U_select[i] = Uint[i]
H = [x for x in H if not x==0]
U_select = [x for x in U_select if not x==0]
Imp = (np.sqrt(1-np.array(U_select)**2))/np.array(H)
ImpE = 4*np.mean(Imp)
Imperror = np.sqrt(4*np.var(Imp))
print(ImpE, Imperror)
```

```
3.1465652971066445 0.11021834022421882
```

```
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  app.launch_new_instance()
```

Using the Important Sampling method, the integral is approximately 3.144 with an error of 0.11. Compared to the Monte Carlo simulation, there is a significant improvement both in expected value and error.