Homework 3

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2/4/2019

Problem 1: Year-on-year quarterly data and ARMA dynamics

A substantial amount of quantity data, such as earnings, exhibit seasonalities. These can be hard to model. It is therefore common to use so-called Year-on-Year data (e.g., Q1 earnings vs Q1 earnings a year ago, Q2 earnings vs Q2 earnings a year ago, etc). In this problem we will see that such a practice can induce MA-terms due to the overlap in the quarterly year-on-year observations.

Assume the true quarterly log market earnings follow:

$$e_t = e_{t-1} + x_t$$

$$x_t = \phi x_{t-1} + \epsilon_t$$

where $var(\epsilon_t) = \sigma_t^2 = 1$ and ϵ is i.i.d. over time t.

The earnings data you are given is year-on-year earnings growth, which in logs is:

$$y_t \equiv e_t - e_{t-4}$$

1. Assume $\phi = 0$. Derive autocovariances of order 0 through 5 for y_t. I.e., $cov(y_t, y_{t-1})$ for $j = 0, \dots, 5$.

Based on $x_t = \phi x_{t-1} + \epsilon_t$ where $\phi = 0$, we can get $e_t = e_{t-1} + \epsilon_t$ and :

$$\begin{aligned} y_t &= (e_{t-1} + \epsilon_t) - e_{t-4} \\ &= (e_{t-2} + \epsilon_{t-1}) + \epsilon_t - e_{t-4} \\ &= (e_{t-3} + \epsilon_{t-2}) + \epsilon_{t-1} + \epsilon_t - e_{t-4} \\ &= (e_{t-4} + \epsilon_{t-3}) + \epsilon_{t-2} + \epsilon_{t-1} + \epsilon_t - e_{t-4} \\ &= \epsilon_{t-3} + \epsilon_{t-2} + \epsilon_{t-1} + \epsilon_t \\ E(y_t) &= E(\epsilon_{t-3} + \epsilon_{t-2} + \epsilon_{t-1} + \epsilon_t) = 0 \end{aligned}$$

$$\begin{aligned} j = 0:\ cov(y_t, y_t) &= E(y_t^2) - E(y_t)E(y_t) \\ &= E(\epsilon_{t-3}^2 + \epsilon_{t-2}^2 + \epsilon_{t-1}^2 + \epsilon_t^2) - 0 \times 0 \\ &= 1 + 1 + 1 + 1 = 4 \\ j = 1:\ cov(y_t, y_{t-1}) &= E(y_t y_{t-1}) - E(y_t)E(y_{t-1}) \\ &= E(\epsilon_{t-3}^2 + \epsilon_{t-2}^2 + \epsilon_{t-1}^2) - 0 \times 0 \\ &= 1 + 1 + 1 = 3 \\ j = 2:\ cov(y_t, y_{t-2}) &= E(y_t y_{t-2}) - E(y_t)E(y_{t-2}) \\ &= E(\epsilon_{t-3}^2 + \epsilon_{t-2}^2) - 0 \times 0 \\ &= 1 + 1 = 2 \\ j = 3:\ cov(y_t, y_{t-3}) &= E(y_t y_{t-3}) - E(y_t)E(y_{t-3}) \\ &= E(\epsilon_{t-3}^2) - 0 \times 0 \\ &= 1 \\ j = 4:\ cov(y_t, y_{t-4}) &= E(y_t y_{t-4}) - E(y_t)E(y_{t-4}) \\ &= 0 - 0 \times 0 \\ &= 0 \end{aligned}$$

$$j = 4: \; cov(y_t, y_{t-4}) = E(y_t y_{t-4}) - E(y_t)E(y_{t-4})$$
$$= 0 - 0 \times 0$$
$$= 0$$

2. Assume $\phi = 0$. Determine the number of AR lags and MA lags you need in the ARMA(p,q) process for y_t. Give the associated AR and MA coefficients.

$$y_t = \epsilon_{t-3} + \epsilon_{t-2} + \epsilon_{t-1} + \epsilon_t$$
$$y_{t-1} = \epsilon_{t-4} + \epsilon_{t-3} + \epsilon_{t-2} + \epsilon_{t-1}$$

The red rectangle shows the overlapping between y_t and y_{t-1}. Thus, this demonstrates that y_t has a MA coefficient of $q = 3$ ($MA(3)$).
Since y_t only contains ϵ_t terms, which are all white noise, we can also say that y_t has a AR coefficient of $p = 0$ ($AR(0)$).
Therefore, y_t has an $ARMA(0,3)$ process.

Problem 2: Market-timing and Sharpe ratios

Much of this class is about prediction. In this problem you will derive how market timing can improve the unconditional Sharpe ratio of a fund. The market timing is based on "forecasting regressions" akin to those we undertake in a VAR. However, we are only forecasting one period ahead here.

Assume you have an estimate of expected annual excess market returns for each time t, called x_t. You estimate the regression

$$R^e_{t+1} = \alpha + \beta x_t + \epsilon_{t+1}$$

and obtain $\hat{\alpha} = 0, \hat{\beta} = 1$, and $\sigma(\hat{\epsilon}_{t+1}) = 15\%$. Further, the sample mean and standard deviation of x_t are both 5%.

1. Calculate the standard deviation of excess returns based on the information given.

$$Var_t[R^e_{t+1}] = V_t[\alpha + \beta x_t + \epsilon_{t+1}] = \hat{\beta}^2 V_t[x_t] + \sigma^2_{\epsilon_{t+1}} = 0.05^2 + 0.15^2 = 0.0025$$
$$\sigma_{R^e_{t+1}} = 0.1581$$

2. Calculate the R_2 of the regression based on the information given.

$$R^2 = \rho^2 \Rightarrow Cov(R^e_{t+1}, x_t) = E(R^e_{t+1} x_t) - E(R^e_{t+1})E(x_t)$$
$$= E\big((\alpha + \beta x_t + \epsilon_{t+1})x_t\big) - E(\alpha + \beta x_t + \epsilon_{t+1}) \times 0.05$$
$$= E(x_t^2) - E(x_t) \times 0.05$$

$$Var_t(x_t) = E(x_t^2) - E(x_t)^2$$
$$0.05^2 = E(x_t^2) - 0.05^2$$
$$E(x_t^2) = 0.005$$

$$Cov(R^e_{t+1}, x_t) = 0.005 - 0.05^2 = 0.0025$$

$$\rho = \frac{Cov(R^e_{t+1}, x_t)}{\sigma_{R^e_{t+1}} \sigma_{x_t}} = \frac{0.0025}{0.1581 \times 0.05} = 0.3163 \Rightarrow \rho^2 = \ 0.1 = R^2$$

3. Calculate the sample Sharpe ratio of excess market returns based on the information given.

$$Sharpe\ Ratio = \frac{excess\ return}{SD_{x_t}} = \frac{E(R^e_{t+1})}{\sigma_{R^e_{t+1}}} = \frac{0.05}{0.1581} = 0.3163$$

4. Recall from investments that myopic investors chooses a fraction of wealth

$$\alpha_t = \frac{E_t[R^e_{t+1}]}{\gamma \sigma_t^2 [R^e_{t+1}]}$$

in the risky asset (the market) at each time t, where we assume risk aversion coefficient, γ, equals 40/9. Further, assume that the residuals ϵ_{t+1} are i.i.d., so $\sigma_t(\epsilon_{t+1}) = 15\%$ for all t. Given this, calculate the weight the investor chooses to hold in the risky asset if $x_t = 0\%$ and if $x_t = 10\%$. What is conditional Sharpe ratio in each of these cases?

$x_t = 0$: $\quad E_t(R^e_{t+1}) = \ E(\alpha + \beta x_t + \epsilon_{t+1}) = x_t = 0 \Rightarrow Sharpe\ Ratio = 0, \alpha = 0$

$x_t = 10\%$:

$$E_t(R^e_{t+1}) = \ E(\alpha + \beta x_t + \epsilon_{t+1}) = x_t = 0.1 \Rightarrow Sharpe\ Ratio = \frac{0.1}{0.15} = 0.67$$

$$\alpha_t = \frac{0.1}{\frac{40}{9} \times 0.15^2} = \frac{0.1}{0.1} = 1$$

5. (a)
$x = 0\% \Rightarrow E_t(R^e_{t+1}) = 0$
$x_t = 10\% \Rightarrow E_t(R^e_{t+1}) = 0.1$
Average Excess Return = 0.5$\times$ 0+0.5$\times$ $1 \times 0.1 = 0.05$

(b)
$$\begin{aligned} Var(\alpha_t R^e_{t+1}) &= E\left[\alpha_t^2\left(x_t^2 + \sigma_t^2(\epsilon_{t+1})\right)\right] - E[\alpha_t x_T]^2 \\ &= 0.5 \times 1^2 \times E[x_t^2 + \sigma_t^2(\epsilon_{t+1})] - 0.5^2 \times 1^2 \times x_t^2 \\ &= 0.5 \times [0.1^2 + 0.15^2] - 0.5^2 \times 0.1^2 \\ &= 0.01375 \end{aligned}$$
$$\sqrt{Var(\alpha_t R^e_{t+1})} = \sqrt{0.01375} = 0.1173$$

(c)

$$SR = \frac{excess\ return}{\sigma_{excess\ return}} = \frac{0.05}{0.1173} = 0.4264$$

(d)

$x_t = -5\%$: $\quad E_t(R_{t+1}^e) = x_t = -0.05$

$$\alpha_t = -\frac{0.05}{\frac{40}{9} \times 0.15^2} = -0.5$$

$$SR = -\frac{0.05}{0.15} = -0.33$$

$x_t = 15\%$: $\quad E_t(R_{t+1}^e) = x_t = 0.15$

$$\alpha_t = -\frac{0.15}{\frac{40}{9} \times 0.15^2} = 1.5$$

$$SR = \frac{0.15}{0.15} = 1$$

$E(\alpha_t R_{t+1}^e) = \frac{1}{2}(-0.5 \times -5\% + 1.5 \times 15\%) = 12.5\%$

$E(R_{t+1}^e) = \frac{1}{2}(-5\% + 15\%) = 5\%$

i. $R^2 = \rho^2 \Rightarrow \rho = \frac{Cov(R_{t+1}^e, x_t)}{\sigma_{R_{t+1}^e} \sigma_{x_t}} = \frac{0.0025}{0.1953 \times 0.05} = 0.2561 \Rightarrow \rho^2 = 0.0656$

$$\begin{aligned} where\ Cov(R_{t+1}^e, x_t) &= E(R_{t+1}^e x_t) - E(R_{t+1}^e)E(x_t) \\ &= E(x_t^2) - 0.05 \times E(R_{t+1}^e) \\ &= 0.005 - 0.05 \times 0.05 \\ &= 0.0025 \end{aligned}$$

$$\begin{aligned} Var(\alpha_t R_{t+1}^e) &= E\left[\alpha_t^2\left(x_t^2 + \sigma_t^2(\epsilon_{t+1})\right)\right] - E[\alpha_t x_T]^2 \\ &= 0.5 \times (-0.5^2) \times [(-0.05)^2 + 0.15^2] + 0.5 \times 1.5^2 \times (0.15^2 + 0.15^2) \\ &\quad - (0.5 \times -0.5 \times -0.05 + 0.5 \times 1.5 \times 0.15)^2 \\ &= 0.5^3 \times [(-0.05)^2 + 0.15^2] + 0.5 \times 1.5^2 \times (0.15^2 + 0.15^2) - 0.125^2 \\ &= 0.038125 \end{aligned}$$

$\sigma_{R_{t+1}^e} = 0.1953$

ii. Sharpe Ratio$= \frac{E(\alpha_t R_{t+1}^e)}{\sigma_{R_{t+1}^e}} = \frac{0.0125}{0.1953} = 0.6402$