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Modelling Population Growth via Laguerre-Type Exponentials

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Abstract—We use the Laguerre-type exponentials, i.e., eigenfunctions of the Laguerre-type derivatives, in order to construct new models for population growth. Relevant modifications of the classical exponential, logistic, and Volterra-Lotka models are investigated.

Keywords—Laguerre-type exponentials, Population growth models.

1. INTRODUCTION

In recent articles, Dattoli and Ricci [1–3] introduced the so-called Laguerre-type exponentials (shortly L-exponentials) satisfying, with respect to the Laguerre derivative $D_L := DxD$ (and its iterations), an eigenvalue property generalizing the classical property of the exponential, with respect to the ordinary derivative D.

In the same paper, the L-circular and L-hyperbolic functions were introduced, corresponding to the ordinary circular and hyperbolic ones under the action of a differential isomorphism, defined by means of the correspondence,

$$D \longrightarrow D_L = DxD, \qquad x \longrightarrow D_x^{-1}, \tag{1.1}$$

where

$$D_x^{-1} f(x) := \int_0^x f(t)\, dt \tag{1.2}$$

denotes the antiderivative, such that

$$D_x^{-n}(1) = \frac{x^n}{n!}. \tag{1.3}$$

According to such an isomorphism, the exponential function e^x is transformed into the Tricomi function,

$$C_0(-x) := \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2},$$

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the powers $(x+y)^n$, (which can be considered as Hermite polynomials $H_n^{(1)}(x,y)$ of order one, see e.g., [4–6], correspond to the two variable Laguerre polynomials,

$$\mathcal{L}_n(-x,y) := n! \sum_{r=0}^{n} \frac{y^{n-r}x^r}{(n-r)!(r!)^2},$$

and so on.

The consideration of such an isomorphism allowed the introduction of new classes of special functions, including higher-order Laguerre polynomials [2], Laguerre-type Bessel functions [7], generalized Appell polynomials [8], etc.

Applications to the solution of Laguerre-type integral or partial differential problems in the half-space were considered in [3,9,10].

In this article, we apply the Laguerre-type exponential in the study of the classical exponential, logistic, and Lotka-Volterra models [11–13].

2. RECALLING L-EXPONENTIALS

For every positive integer n, the nL-exponential function is defined by

$$e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}.$$

This function reduces to the classical exponential when $n=0$, so that we can put $e_0(x) := e^x$.

Consider the differential operator (containing $n+1$ derivatives),

$$\begin{aligned} D_{nL} :&= Dx \cdots DxDxD = D\left(xD + x^2D^2 + \dots x^nD^n\right) \\ &= S(n+1,1)D + S(n+1,2)xD^2 + \cdots + S(n+1,n+1)x^nD^{n+1}, \end{aligned} \tag{2.1}$$

where $S(n+1,1), S(n+1,2), \dots, S(n+1,n+1)$ denote Stirling numbers of the second kind (see [14, p. 835] for an extended table). In [1], the following theorem is proved.

THEOREM 2.1. *Let a be an arbitrary real or complex constant. The n^{th} Laguerre-type exponential $e_n(ax)$ is an eigenfunction of the operator D_{nL} i.e.,*

$$D_{nL}e_n(ax) = ae_n(ax). \tag{2.2}$$

For $n=0$, $D_{0L} := D$, and therefore, equation (2.2) gives back the classical property of the exponential function,

$$De^{ax} = ae^{ax}.$$

It is worth noting that $\forall\, n$, the nL-exponential function satisfies $e_n(0)=1$, and is an increasing convex function for $x \geq 0$; furthermore,

$$e^x = e_0(x) > e_1(x) > e_2(x) > \cdots > e_n(x) > \dots, \qquad \forall\, x > 0.$$

For this reason the L-exponential functions could be used in order to substitute the exponential functions in many frameworks, including the models for population growth.

This will be done in the following, by considering mainly the Laguerre-type derivative DxD, i.e., the function $e_1(x)$, instead of the ordinary exponential function.

3. EXPONENTIAL AND L-EXPONENTIAL MODELS

Consider the number $N(t)$ of population individuals at time t and let $N(0) = N_0$ the initial number at time $t = 0$. The Malthus model assumes a variation proportional to $N(t)$, i.e.,

$$\frac{d}{dt}N(t) = rN(t), \qquad N(0) = N_0, \tag{3.1}$$

where the *growth rate* r is a positive constant.

Therefore, the solution is given by the exponential function,

$$N(t) = N_0 e^{rt}. \tag{3.2}$$

Using the Laguerre derivative, we can consider the following Laguerre-Malthus (shortly L-Malthus) model,

$$\frac{d}{dt}t\frac{dN}{dt} = rN(t), \qquad \text{i.e.,}\ \frac{dN}{dt} + t\frac{d^2N}{dt^2} = rN(t), \tag{3.3}$$

where r is a positive constant. Assuming the initial conditions,

$$N(0) = N_0, \qquad N'(0) = N_1 = N_0 r, \tag{3.4}$$

we find the solution,

$$N(t) = N_0 e_1(rt) = N_0 \sum_{k=0}^{+\infty} r^k \frac{t^k}{(k!)^2}. \tag{3.5}$$

In this case, the population growth increases according to the Laguerre exponential function $e_1(x)$, so that the relevant increasing is slower with respect to the classical Malthus model.

More generally, considering the nL-derivative D_{nL} and the corresponding L-exponential function $e_n(x)$, we can write

$$\begin{aligned} (D_{nL})_t N &= rN(t), \\ N(0) &= N_0, \\ N'(0) &= N_1 = N_0 r, \\ &\vdots \\ N^{(n+1)}(0) &= N_{n+1} = N_0 \frac{r^{n+1}}{((n+1)!)^n}, \end{aligned} \tag{3.6}$$

and we find the solution,

$$N(t) = N_0 e_n(rt) = N_0 \sum_{k=0}^{+\infty} r^k \frac{t^k}{(k!)^{n+1}}. \tag{3.7}$$

REMARK 3.1. The possibility to choose different increasing behavior, fixing different values of the integer n could be useful, in order to approximate different behavior of population growth.

REMARK 3.2. Note that the initial conditions of the problem (3.6) have been chosen in such a way that the polynomial part in t of the solution vanishes. Of course, the initial conditions could be chosen more carefully by taking into account the observed initial values, and using discretization formulas for approximating derivatives.

3.1. Malthus vs. L-Malthus model

In Table 1, [13], we consider data relevant to world population growth in the period 1955–2005. The mean annual rate is computed according to the formula,

$$\text{annual rate} = \frac{N(t) - N(t-1)}{N(t-1)}.$$

Assuming that $t = 0$ corresponds to the year 1965, so that $N(0) = 3.346$, and $r = 0.022$ (relevant to the year 1970), the Malthus equation gives the solution,

$$N(t) = 3.346 \times 10^9 e^{0.022t}.$$

Accordingly, the forecasted value for the year 1970 is given by

$$N(5) = 3.346 \times 10^9 e^{0.022 \times 5} = 3.735 \text{ billions},$$

which is close to the real value of 3.708 billions.

Considering the $1L$-derivative $D_{1L} := D_L$, we should find the formula,

$$N(t) = 3.346 \times 10^9 e_1(0.022t)$$

and consequently,

$$N(5) = 3.346 \times 10^9 e_1(0.022 \times 5) = 3.724 \text{ billions}.$$

A table of estimated values using the $1L$-derivative is shown in Table 2.

A complete comparison between the estimated values using Malthus and $1L$-Laguerre-Malthus models is shown in Table 3.

Table 1. World population.

Year	$N(t)$ (Billions)	Annual Rate
1955	2.780	0.018
1960	3.040	0.019
1965	3.346	0.020
1970	3.708	0.022
1975	4.087	0.020
1980	4.454	0.018
1985	4.850	0.018
1990	5.276	0.018
1995	5.686	0.016
2000	6.079	0.014
2005	6.449	0.012

Table 2. Estimated values using $e_1(x)$.

Year	Real Values	Estimated Values
1955	2.780×10^9	2.795×10^9
1960	3.040×10^9	3.057×10^9
1965	3.346×10^9	3.359×10^9
1970	3.708×10^9	3.735×10^9
1975	4.087×10^9	4.098×10^9
1980	4.454×10^9	4.472×10^9
1985	4.850×10^9	4.873×10^9
1990	5.276×10^9	5.306×10^9
1995	5.686×10^9	5.715×10^9
2000	6.079×10^9	6.098×10^9
2005	6.449×10^9	6.454×10^9

Table 3.

Year	Real Values	Estimated Values L-Malthus Model	Estimated Values Malthus Model
1955	2.780×10^9	2.790×10^9	2.795×10^9
1960	3.040×10^9	3.050×10^9	3.057×10^9
1965	3.346×10^9	3.351×10^9	3.359×10^9
1970	3.708×10^9	3.724×10^9	3.735×10^9
1975	4.087×10^9	4.088×10^9	4.098×10^9
1980	4.454×10^9	4.463×10^9	4.472×10^9
1985	4.850×10^9	4.863×10^9	4.873×10^9
1990	5.276×10^9	5.296×10^9	5.306×10^9
1995	5.686×10^9	5.706×10^9	5.715×10^9
2000	6.079×10^9	6.090×10^9	6.098×10^9
2005	6.449×10^9	6.449×10^9	6.454×10^9

4. LOGISTIC VS. L-LOGISTIC MODEL

Taking into account that the growth rate cannot be a constant, since it depends on the environmental resources, Verhulst considered the so called logistic model,

$$\begin{aligned} \frac{dN}{dt} &= r\left[1-\frac{1}{K}N(t)\right]N(t), \\ N(0) &= N_0, \end{aligned} \tag{4.1}$$

where r is called the *intrinsic growth rate*, and K denotes the *environmental capacity.*

The exact solution of problem (4.1) is given by

$$N(t) = \frac{N_0 K}{N_0 + (K - N_0)e^{-rt}}, \tag{4.2}$$

so that, if $N_0 < K$ the solution is a function monotonically increasing to K, whereas, if $N_0 > K$ the solution is monotonically decreasing to K.

In any case,

$$\lim_{t\to\infty} N(t) = K$$

and the value $N(t) = K$ is a stable equilibrium point for the logistic equation.

The Laguerre-logistic (shortly L-logistic) model is expressed by the equation,

$$(D_L)_t N = r\left[1-\frac{1}{K}N(t)\right]N(t), \tag{4.3}$$

i.e.,

$$\frac{d}{dt}t\frac{dN}{dt} = r\left[1-\frac{1}{K}N(t)\right]N(t). \tag{4.4}$$

Note that if in equation (4.4) N is small with respect to K, then N/K is close to 0 and consequently $\frac{d}{dt}t\frac{dN}{dt} \approx rN(t)$. If $N \to K$, then $N/K \to 1$, and $\frac{d}{dt}t\frac{dN}{dt} \to 0$.

Furthermore, if $N(0)$ is between 0 and K, then the right-hand side of equation (4.4) is positive, then $\frac{d}{dt}t\frac{dN}{dt} = \frac{dN}{dt} + t\frac{d^2N}{dt^2} > 0$, and consequently, the solution is initially increasing.

If $N(0) > K$, then $1 - N/K$ is negative, and $\frac{d}{dt}t\frac{dN}{dt} = \frac{dN}{dt} + t\frac{d^2N}{dt^2} < 0$, so that the solution is initially decreasing.

The L-logistic equation cannot be solved explicitly. Then, we will use a simple numerical approach to find approximate solutions.

We consider the problem,

$$\begin{aligned} N'(t) + tN''(t) &= r\left[1-\frac{1}{K}N(t)\right]N(t), \\ N(0) &= N_0, \\ N'(0) &= N_1. \end{aligned} \tag{4.5}$$

Using the Euler explicit method, i.e., putting

$$\frac{N(t+\Delta t) - N(t-\Delta t)}{2\Delta t} + \frac{t\left[N(t+\Delta t) - 2N(t) + N(t-\Delta t)\right]}{(\Delta t)^2} = rN(t) - \frac{r}{K}N^2(t),$$

we have

$$\frac{2t\left[N(t+\Delta t) - 2N(t) + N(t-\Delta t)\right] + \Delta t\left[N(t+\Delta t) - N(t-\Delta t)\right]}{2(\Delta t)^2}$$
$$-\frac{2(\Delta t)^2 rN(t) - 2(\Delta t)^2 (r/K)N^2(t)}{2(\Delta t)^2} = 0,$$

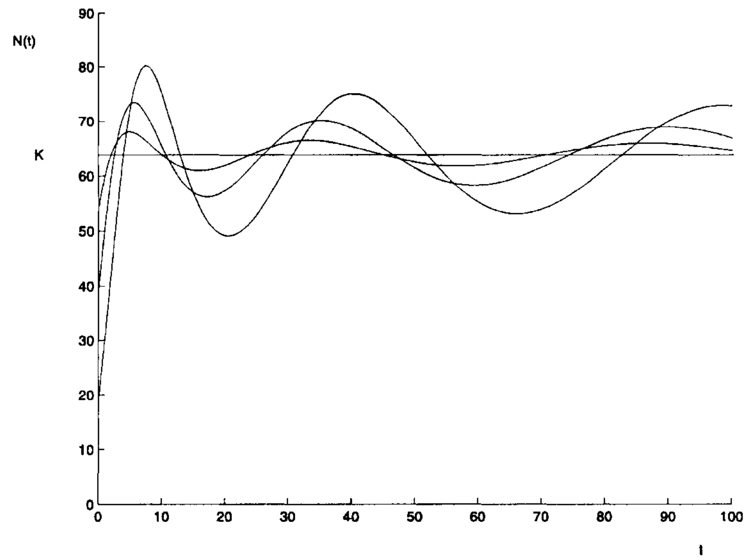

Figure 1. Solutions of problem (4.6) with $N(0) = N'(0) < K$.

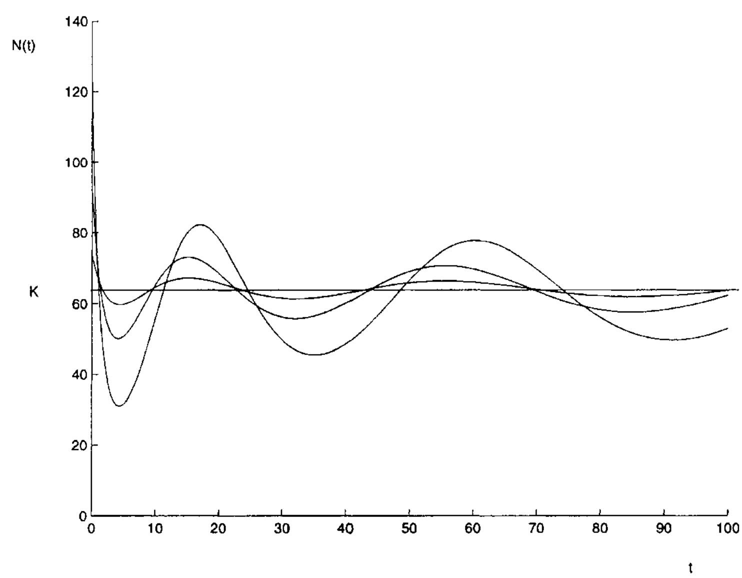

Figure 2. Solutions of problem (4.6) with $N(0) = |N'(0)| > K$.

so that, putting

$$N(t + \Delta t) := N_{n+1},$$
$$N(t - \Delta t) := N_{n-1},$$
$$N(t) := N_n,$$

we find the recursion

$$\begin{aligned} N_{n+1} &= \frac{\left(4t + 2(\Delta t)^2 r\right) N_n - 2(\Delta t)^2 (r/K) N_n^2 + (\Delta t - 2t)\, N_{n-1}}{2t + \Delta t}, \\ N(0) &= N_0, \\ N'(0) &= N_1. \end{aligned} \tag{4.6}$$

Using a simple personal computer, it is possible to have an idea of the relevant solutions, which are reported in Figures 1 and 2.

It is worth to note that the solution tends to the environmental capacity K by an oscillating behavior.

This is the main difference with respect to the ordinary logistic model, since in that case the solution was monotonically increasing or decreasing to K.

Similar results could be obtained by using the nL-derivatives, introducing suitable initial conditions which can be easily derived from values of initial observations.

REMARK 4.1. It is worth noting that the oscillating asymptotic behavior of solutions has a real counterpart in practical situations. For example, the classical experiment of Gause [11] relevant to the proto-zoon paramecius shows such a typical behavior.

5. INVARIANCE OF THE VOLTERRA-LOTKA MODEL

Consider a predator species P and its prey N. The predator-prey system of Volterra and Lotka is given by

$$\frac{dN}{dt} = N(a - bP), \qquad \frac{dP}{dt} = P(cN - d), \tag{5.1}$$

where a, b, c, d are suitable positive constants.

The equilibrium points of the above system are

$$\begin{aligned} N(t) &= 0, \qquad & P(t) &= 0, \\ N(t) &= d/c, \qquad & P(t) &= a/b. \end{aligned}$$

The orbits in the phase plane are the solutions of the differential equation,

$$\frac{dP}{dN} = \frac{P(cN - d)}{N(a - bP)}, \tag{5.2}$$

so that they satisfy

$$a \log P - bP + d \log N - cN = K = \text{constant}.$$

It is well known that the orbits are closed curves around the equilibrium points, so that the solutions $N(t)$, $P(t)$ are periodic functions of time t.

Considering now a Laguerre-type predator-prey system in the form,

$$(D_L)_t N = N(a - bP), \qquad (D_L)_t P = P(cN - d). \tag{5.3}$$

Even in this case, the system reduces in the phase plane to the differential equation,

$$(D_L)_t N = N(a - bP) = 0, \qquad (D_L)_t P = P(cN - d) = 0, \tag{5.4}$$

and we find again the equilibrium point,

$$N(t) = d/c, \qquad P(t) = a/b.$$

Note that, in order to eliminate the t variable, it is possible to use the chain rule, which still holds for the Laguerre derivative,

$$\frac{d}{dt} t \frac{dP}{dt} = \frac{dP}{dN} \frac{d}{dt} t \frac{dN}{dt}, \qquad \text{i.e.,} \ (D_L)_t P = \frac{dP}{dN} (D_L)_t N,$$

so that

$$\frac{dP}{dN} = \frac{(D_L)_t P}{(D_L)_t N} = \frac{P(cN - d)}{N(a - bP)},$$

and we find again the differential equation,

$$\frac{dP}{dN} = \frac{P(cN - d)}{N(a - bP)}.$$

Consequently, the orbits of equation (5.3) are given again by the prime integral,

$$a \log P - bP + d \log N - cN = K = \text{constant}, \tag{5.5}$$

i.e.,

$$\frac{P^a}{e^{bP}} \frac{N^d}{e^{cN}} = K = \text{constant}. \tag{5.6}$$

In conclusion, the Volterra-Lotka model is invariant under the action of the above mentioned isomorphism transforming the ordinary into the Laguerre derivative. The solutions $N(t), P(t)$ are again periodic functions of time t, and the only difference is that we obtain the same constant K depending on different values of time t.

REMARK 5.1. Same results can be obtained by using the nL-derivative D_{nL} and the corresponding Laguerre exponential $e_n(x)$.

6. CONCLUSION

We take profit of new classes of exponential-like functions called L-exponentials in order to find new models for describing population growth. We can construct infinite many models depending on the choice of the integer n characterizing the relevant Laguerre derivative.

We introduced the L-Malthus model in the case of the $1L$-derivative, and checked its validity with respect to the classical model for human population growth, in comparison with the ordinary Malthus model.

We studied the $1L$-logistic model showing that the asymptotic behavior of its solutions exhibits an oscillating character which can be found e.g. in a classical experiment due to Gause.

We proved the orbit invariance property of the Volterra-Lotka model with respect to the introduction of the nL-derivative.

In a forthcoming article, we will extend our results to more sophisticated models and will analyze more carefully the relevant numerical results.

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