Problem 2

Q1.

(a) $$Pr(X \le VaR(X)) = Pr(Z \le \frac{VaR(X)-\mu}{\sigma})$$

$$= \phi(\frac{VaR(X)-\mu}{\sigma})$$

$$= 1-c.$$

$$VaR = \phi^{-1}(1-c)\sigma + \mu,$$

$$ES = \frac{1}{1-c}\int_c^1 VaR_\alpha \, d\alpha$$

The we can get

$$ES = \frac{1}{1-c}\int_c^1 \phi^{-1}(1-\alpha)\sigma d\alpha + \mu.$$

$$\int_c^1 \phi^{-1}(1-\alpha)\sigma d\alpha = \int_{\phi^{-1}(c)}^{\phi^{-1}(1)} (\phi^{-1}(-y))\sigma \phi(y)\, dy.$$

$$= \int_{\phi^{-1}(c)}^{\infty} -\frac{y\sigma}{\sqrt{2\pi}} e^{(-\frac{y^2}{2})} dy$$

$$= \left[\frac{\sigma}{\sqrt{2\pi}} e^{(-\frac{y^2}{2})}\right]_{\phi^{-1}(c)}^{\infty}$$

$$= -\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\phi^{-1}(c)}{2}}$$

Thus, $ES = \mu - \frac{\sigma}{(1-c)\sqrt{2\pi}} e^{-\frac{\phi^{-1}(c)^2}{2}}$.

(b) $ES = W_0 - E(W \mid W \leq W_0 - VaR)$

$$= W_0 - \frac{\int_{-\infty}^{W_0 - VaR} w f(w) dw}{1-c}$$

$$VaR = W_0 - \phi^{-1}(1-c)$$

$$ES = \frac{1}{1-c} \int_c^1 VaR_\alpha \, d\alpha$$

$$= \frac{1}{1-c} \int_c^1 (W_0 - \phi^{-1}(1-\alpha)) d\alpha$$

$$= W_0 - \frac{1}{1-c} \int_c^1 \phi^{-1}(1-\alpha) d\alpha$$

Now, suppose that $1-\alpha = \phi(y)$, then:

$$y = \phi^{-1}(1-\alpha) \Rightarrow d\alpha = -\phi(y) dy$$

Then, $\int_c^1 \phi^{-1}(1-\alpha) d\alpha = \int_{\phi^{-1}(1-c)}^{\phi^{-1}(0)} y \phi(y) dy$

$$= -\int_{W_0 - VaR}^{\infty} y \phi(y) dy$$

Thus, $ES = W_0 - \frac{1}{1-c} \int_{-\infty}^{W_0 - VaR} y \phi(y) dy$.