MFE 409: Financial Risk Management
Problem Set 1

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1 VaR for an exponential distribution

1. **Assume an asset value in 10 days follows an exponential distribution with mean W0. Derive a formula for the value-at-risk for confidence c and reference level W0 for the asset. Apply with W0 = 200 and c = 99.9%.**

$$\int_0^{W_0-VaR} \lambda e^{\lambda W} dW = 1-c$$
$$\left[-e^{-\lambda W}\right]_0^{W_0-VaR} = 1-c$$
$$-e^{-\lambda(W_0-VaR)} + e^{-\lambda\times 0} = 1-c$$
$$\ln(c) = -\lambda(W_0 - VaR)$$
$$VaR = \frac{\ln(c)}{\lambda} + W_0$$

$For\ \mu = W_0 = 200\ and\ c = 99.9\%$:

$$\frac{1}{\lambda} = \mu = 200 \Rightarrow \lambda = \frac{1}{200}$$
$$VaR = \frac{\ln(99.9\%)}{\frac{1}{200}} + 200 = 199.79$$

2. **Now assume you are short this asset. Compute your value-at-risk for confidence c and apply with the same numerical values.**

$For\ shorting\ this\ asset, we\ want\ W \geq W_0 + VaR, then\ we\ can\ derive\ formula\ for\ VaR$:

$$\int_{W_0+VaR}^{+\infty} \lambda e^{-\lambda W} dW = 1-c$$
$$\left[-We^{-\lambda W} - \frac{1}{\lambda}e^{-\lambda W}\right]_{W_0+VaR}^{+\infty} = 1-c$$
$$\left[-e^{-\lambda W}\right]_{W_0+VaR}^{+\infty} = 1-c$$
$$e^{-\lambda(W_0+VaR)} = 1-c$$
$$VaR = -\frac{\ln(1-c)}{\lambda} - W_0$$

$Then\ for\ W_0 = 200\ and\ c = 99.9\%, VaR = -\frac{\ln(1-99.9\%)}{\frac{1}{200}} - 200 = 1181.551.$

3. **Comment on the similarity or difference between the results in the two questions.**

$The\ above\ 2\ VaRs\ are\ both\ from\ an\ exponential\ distribution. Becasue\ the\ exponential\ distribution$

$is\ not\ symmetric, tails\ on\ different\ sides\ of\ the\ distribution\ is\ also\ different, which\ gives\ us\ 2$

$different\ VaRs\ from\ the\ same\ distribution.$

4. **Repeat the previous questions with expected shortfall.**

$For\ longing\ asset:$

$$\begin{aligned}
ES &= W_0 - E(W|W \le W_0 - VaR) \\
&= W_0 - \frac{\int_0^{W_0-VaR} W\lambda e^{-\lambda W} dW}{\int_0^{W_0-VaR} \lambda e^{-\lambda W} dW} \\
&= W_0 - \frac{-(W_0 - VaR)e^{-\lambda(W_0-VaR)} - \frac{1}{\lambda}e^{-\lambda(W_0-VaR)} + \frac{1}{\lambda}}{1 - e^{-\lambda(W_0-VaR)}} \\
&= \frac{(W_0 - VaR)e^{-\lambda(W_0-VaR)}}{1 - e^{-\lambda(W_0-VaR)}}
\end{aligned}$$

$$ES_{long} = \frac{(200 - 199.79)e^{-\frac{1}{200}(200-199.79)}}{1 - e^{-\frac{1}{200}(200-199.79)}} = 199.895\ where\ VaR = 199.79$$

$For\ shorting\ asset:$

$$\begin{aligned}
ES &= E(W|W \ge W_0 + VaR) - W_0 \\
&= \frac{\int_{W_0+VaR}^{+\infty} W\lambda e^{-\lambda W} dW}{\int_{W_0+VaR}^{+\infty} \lambda e^{-\lambda W} dW} - W_0 \\
&= \frac{(W_0 + VaR)e^{-\lambda(W_0+VaR)} + \frac{1}{\lambda}e^{-\lambda(W_0+VaR)}}{e^{-\lambda(W_0+VaR)}} - W_0 \\
&= \frac{1}{\lambda} + VaR
\end{aligned}$$

$$ES_{short} = 200 + 1181.551 = 1381.551\ where\ VaR = 1181.551$$

2 VaR for mixtures

You have to allocate \$1bn in the stock market. You are discussing with your partner regarding the volatility of returns. She has a view that, in line with historical averages, the volatility of returns will be of σ1=12% in the next year. However, you believe that volatility will be higher, in the orders of σ2=20% for the next year. After discussing with your partner, you agree in the following way: stocks returns follow a normal distribution with mean μ and σ1 with probability π and normal distribution with mean μ and σ2 with probability 1 − π . For now assume that μ = 8% and π = 0.7.

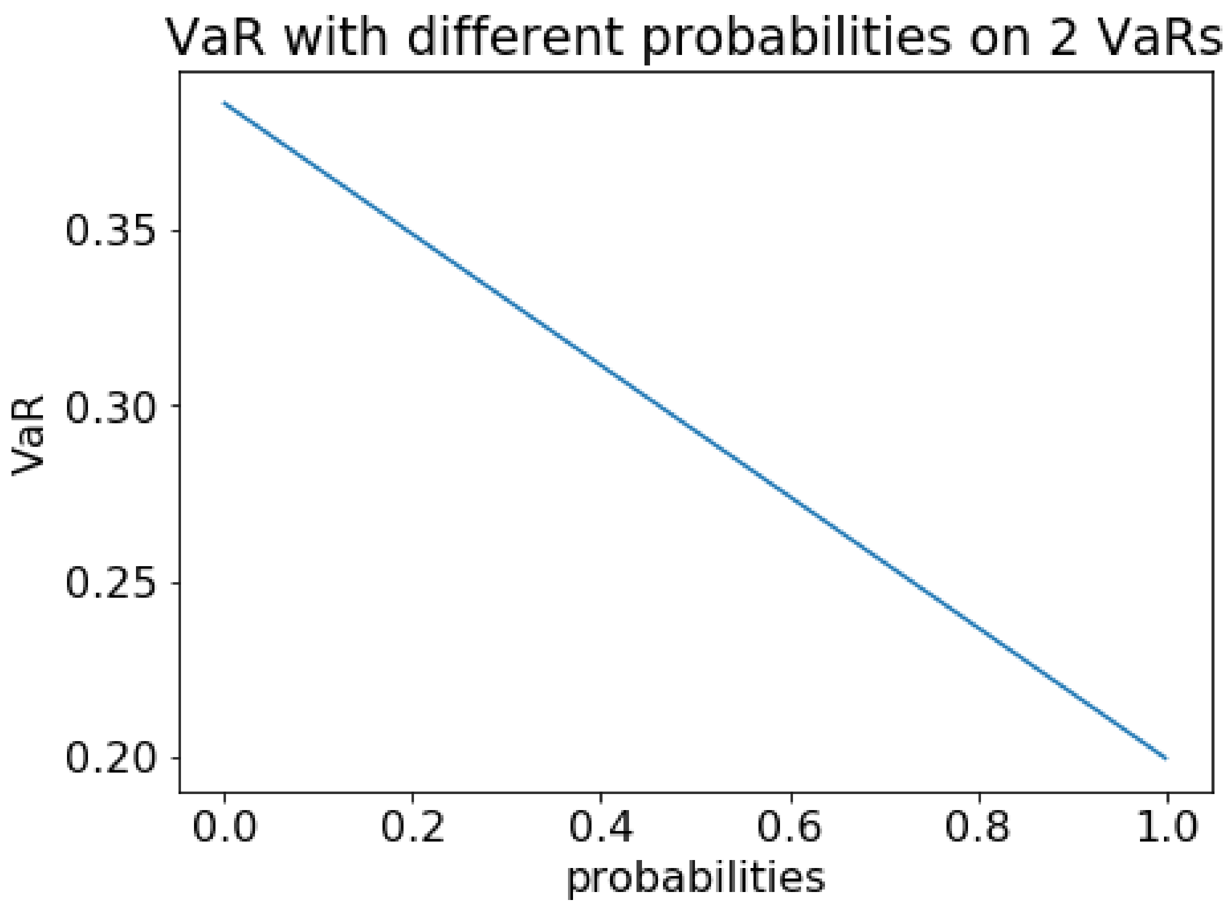
1. **Compute the 1-year 99% VaR with your view, with your partner view, and with the common view. Compare results and provide a very brief explanation as if you were presenting to your manager.**

$$VaR_{my} = -(0.08 - 2.33 \times 0.2) = 0.386$$

$$VaR_{partner} = -(0.08 - 2.33 \times 0.12) = 0.1996$$

$$VaR_{combined} = 0.386 \times 0.3 + 0.1996 \times 0.7 = 0.25552$$

2. **To understand the role of π, plot a chart with π between 0 and 1 on the horizontal axiswhich and the corresponding VaR on the vertical axis. Comment on your results.**



$Based\ on\ the\ plot, \pi\ decides\ how\ valatility\ of\ the\ return\ will\ be. We\ can\ explain\ \pi\ as\ a\ bargainning\ power,$

$which\ means\ who\ has\ the\ stronger\ power\ decides\ what\ the\ volatility\ should\ be.$

3. **More challenging. After presenting your common view to your manager, you are challenged with an alternative view about volatility: σ is timevarying. The volatility trader suggests that a sensible model for sigma is a gamma distribution. Explain in as many details as possible (either derive of formula or use a computer program) how to compute the VaR of your portfolio when returns have a normal distribution conditional on σ and σ is distributed according to a Gamma distribution.**

$Let\ \sigma \sim \Gamma(\alpha, \beta)\ and\ assume\ \alpha, \beta > 0$

$\Gamma(\alpha, \beta)\ has\ PDF\ of\ \ f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{1-\alpha} e^{-\beta x}$

$$E(\sigma) = \frac{\alpha}{\beta}$$

$$W - W_0 \sim N\big(\mu, \Gamma(\alpha, \beta)\big)$$

$$W = W_0 + N\big(\mu, \Gamma(\alpha, \beta)\big)$$

$$W \sim N\big(W_0 + \mu, \Gamma(\alpha, \beta)\big)$$

$$\begin{aligned} VaR &= W_0 - W \\ &= W_0 - [W_0 + \mu + z(c)\Gamma(\alpha, \beta)] \\ &= -[\mu + z(c)\Gamma(\alpha, \beta)] \\ &= -\left[\mu + z(c)\frac{\beta^\alpha}{\Gamma(\alpha)} W^{1-\alpha} e^{-\beta W}\right] \end{aligned}$$

$We\ can\ use\ this\ formula\ for\ to\ compute\ VaR\ with\ volatilities\ following\ a\ gamma\ distribution.$

$If\ we\ want\ to\ do\ the\ simulation, we\ can\ simulate\ volatility\ (\sigma)\ first, then\ plug\ \sigma\ into\ the\ formula$

$to\ compute\ VaR\ for\ each\ \sigma, then\ take\ mean\ of\ all\ VaRs\ to\ estimate\ the\ expected\ value\ of\ VaR.$

3 VaR for options

You will need to write code to solve this question. You are a hedge fund manager with \$100 million of capital, and you can take any long or short position as long as you keep the 99% 10-day VaR for your portfolio under your capital. You have limited liability. You have access to long and short positions on a stock, 3-month European calls and puts on the underlying, and risk free bonds. You choose your positions today and must hold them for the next 10 days. Assume the risk-free asset has annual rate of return r = 2%, and the stock price St has dynamics:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

with μ = 7% and σ = 16% (again, in annual units), and the current value S0 = 50. Assume there are 252 trading days in a year.

1. **Find a formula for the 10-day VaR for one share of the stock as a function of μ and σ.** ***Hint: if X is a random variable with quantile c equal to x0, what is the quantile c of g(X) if g is a monotone function?***

$First\ derive\ the\ expression\ of\ the\ stock\ price\ as\ following:$

$$dS_t = S_t \mu dt + S_t \sigma dW_t$$

$Let\ X = \log(S_t)\ , then\ applies\ Ito's\ lemma\ on\ X\ we\ coud\ get:$

$$dX = 0 \times dt + \frac{1}{S_t} \times dS_t - \frac{1}{2}\frac{1}{S_t^2} \times dS_t^2$$

$Insert\ dS_t\ into\ dX\ and\ we\ could\ get:$

$$dX = (\mu dt + \sigma dW_t) - \left(\frac{1}{2}\sigma^2 dt\right)$$
$$dX = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$$
$$d(\log(S_t)) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$$
$$\int_t^T d(\log(S_t)) = \int_t^T \left(\mu - \frac{1}{2}\sigma^2\right) dt + \int_t^T \sigma dW_t$$
$$\log(S_T) - \log(S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)(T - t) + \sigma(W_T - W_t)$$
$$\log(S_T) = \log(S_t) + \left(\mu - \frac{1}{2}\sigma^2\right)(T - t) + \sigma(W_T - W_t)$$
$$S_T = S_t e^{\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W_T - W_t)}$$

$Thus, S_T\ is$ lognormal distrbution with mean $= \left(\mu - \frac{1}{2}\sigma^2\right)(T - t)\ and\ sd = \sigma\sqrt{T - t}$

$In\ this\ case, let\ Z = \left(\mu - \frac{1}{2}\sigma^2\right)(T - t) + \sigma(W_T - W_t), then\ Z \sim N(\left(\mu - \frac{1}{2}\sigma^2\right)(T - t), \sigma\sqrt{T - t})$

$Since\ Z\ is\ a\ random\ variable\ and\ g(Z) = e^Z\ is\ a\ monotone\ function, the\ quantile\ c\ of\ g(Z)\ is:$

$$e^{\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + z(c)\sigma\sqrt{T-t}} and\ we\ call\ this\ Z^*$$

$Since\ the\ stock\ price\ follows\ a\ lognormal\ distritbution\ and\ we\ want\ to\ get\ the\ VaR\ of\ the\ stock\ price\ at\ dat\ 10\ where\ T = \frac{10}{252}, \mu = 0.07, \sigma = 0.16, S_t = 50\ where\ t = 0, c = 0.99, r = 0.02, W_0 = 100,$
$Then,$

$$VaR = S_t - S_T^*$$
$$VaR = S_t - S_t \times e^{Z^*}$$
$$VaR = S_t - S_t e^{\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + z(c)\sigma\sqrt{T-t}}$$
$$VaR \approx 3.47$$

2. **If you can only invest in stocks and bonds and want to maximize the average return on equity (subject to the VaR constraint), which portfolio do you choose?**

$Let\ w\ be\ the\ weight\ of\ the\ stock\ and\ (1 - w) be\ the\ weight\ of\ the\ bond, then\ the\ return\ is:$

$$\mu_{portfolio} = R_{portfolio} = wR_{stock} + (1 - w)R_{rf}$$

$$\sigma_{potfolio} = w\sigma_{stock}\sqrt{T}$$

$with\ R_{stock} = \left(\mu - \frac{1}{2}\sigma^2\right)(T - t) = 0.0022698, R_{rf} = 0.02 \times (T - t), \sigma_{stock} = 0.16,$

$$VaR = -\left(\mu_{portfolio} + z(c)\sigma_{portfolio}\right) \times 100 millions$$

$$100\ millions = -\left(wR_{stock} + (1-w)R_{rf} + z(c)w\sigma_{stock}\sqrt{T}\right) \times 100\ millions$$

$$w = -\frac{1 + R_{rf}}{R_{stock} - R_{rf} + z(c)\sigma_{stock}\sqrt{T}}$$

$$w \approx 13.75$$

Then we should long $13.75 \times 100 = 1375$ *billions stocks and short* $-(1 - 13.75) \times 100 = 1275$ *billions bonds.*

3. If you can only invest in ATM calls and bonds, which portfolio do you choose?

In that case, we should long the call options as many as possible, we can compute the option weight by simulations as following:

```
###############################################################################
#                                 3.Question 3                                #
###############################################################################
S0 = 50
r = 0.02
sd = 0.16
T = 1/4 #3 months maturity
paths = 1000 #1000 simulations
steps = 21*3 #91 days in 3 months
K = 50 #ATM call option stick price

stock = StockPrices(S0,r,sd,T,paths,steps) #simulate 10000 stock prices
S10 = stock[:,10] #take stock prices at day 10
C0 = f_BS(S0,K,T,sd,r,"call")
#compute Call optoin price today using black-scholes
C10 = f_BS(S10,K,T-10/252,sd,r,"call")
#compute call option prices at day 10 for all simulations
Cc = np.quantile(C10,0.01)
#take the 0.01th option price at day 10 from the distribution
VaR_C = C0 - Cc #compute VaR

N_Coption = 100/VaR_C #number of options to buy
Cost_C = C0*N_Coption #total cost of all the options bought
Bond_C = 100-Cost_C #total bond to buy/short
w_option_C = Cost_C/(Cost_C+Bond_C) #weight on call option
w_bond_C = 1 - w_option_C #weight on bond
```

By doing so, we can get:

```
In [212]: w_option_C
Out[212]: 1.2466926925192754

In [213]: w_bond_C
Out[213]: -0.2466926925192754
```

This mean: long $1.2467 \times 100 = 124.67$ *millions of call options*

short $0.2467 \times 100 = 24.67$ *millions of bonds at risk free rate*

4. **If you can only invest in ATM puts and bonds, which portfolio do you choose?**

In that case, we should short the put options as many as possible, we can compute the option weight by simulations as following:

```
################################################################################
#                              3.Question 4                                    #
################################################################################
S0 = 50
r = 0.02
sd = 0.16
T = 1/4 #3 months maturity
paths = 1000 #1000 simulations
steps = 21*3 #91 days in 3 months
K = 50 #ATM put option stick price

stock = StockPrices(S0,r,sd,T,paths,steps) #simulate 10000 stock prices
S10 = stock[:,10] #take stock prices at day 10
P0 = f_BS(S0,K,T,sd,r,"put")
#compute Put optoin price today using black-scholes
P10 = f_BS(S10,K,T-10/252,sd,r,"put")
#compute Put option prices at day 10 for all simulations
Pc = np.quantile(P10,0.99)
#take the 0.99th option price at day 10 from the distribution
VaR_P = P0 - Pc #compute VaR

N_Poption = 100/VaR_P #number of options to buy
Cost_P = P0*N_Poption #total cost of all the options bought
Bond_P = 100-Cost_P #total bond to buy/short
w_option_P = Cost_P/(Cost_P+Bond_P) #weight on put option
w_bond_P = 1 - w_option_P #weight on bond
```

By doing so, we can get:

```
In [215]: w_option_P
Out[215]: -0.687336088490109

In [216]: w_bond_P
Out[216]: 1.687336088490109
```

This mean: short $0.6873 \times 100 = 68.73$ *millions of put options*

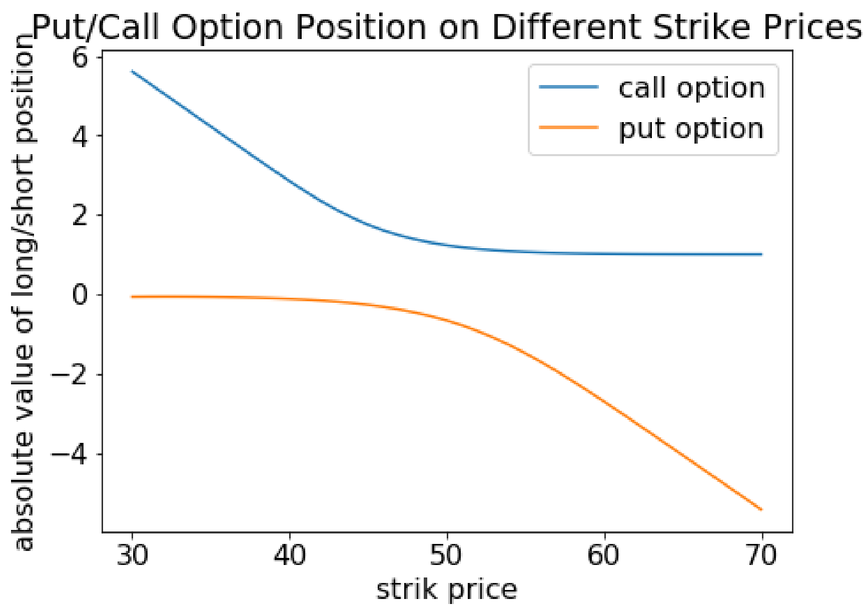
long $1.6873 \times 100 = 168.73$ *millions of bonds at risk free rate*

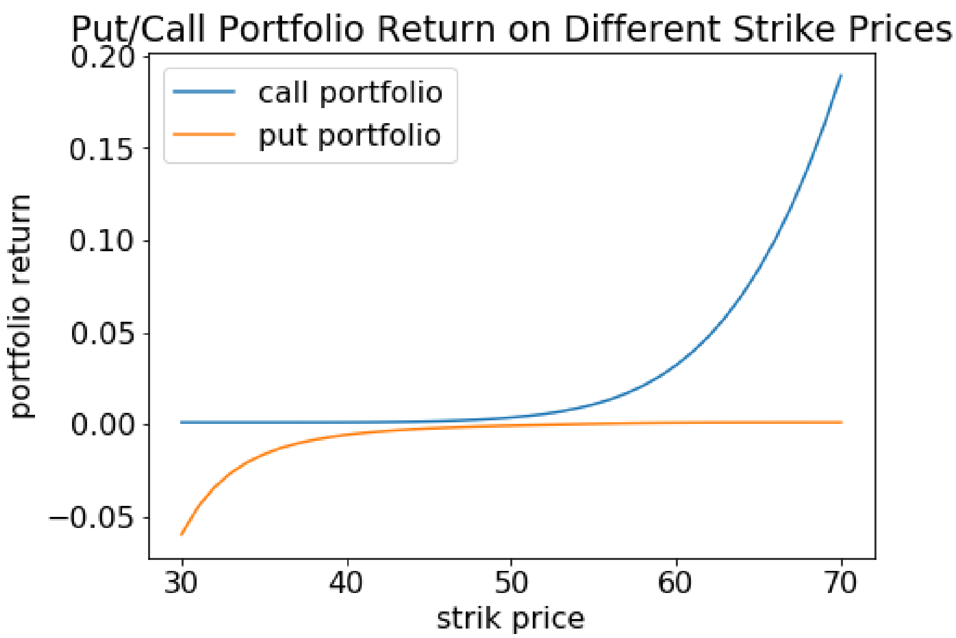
5. **Now you can choose one of the stock, call, or put (with arbitrary strike) to combine with bonds. Plot the optimal portfolio position as well as your expected return for each strike. Which strike and portfolio do you choose? Explain the intuition behind this result.**

```
###############################################################################
#                                3.Question 5                                 #
###############################################################################
S0 = 50
r = 0.02
sd = 0.16
T = 1/4 #3 months maturity
paths = 1000 #1000 simulations
steps = 21*3 #91 days in 3 months
K = np.arange(30,71) #put option stick price range
n = len(K) #length of all put option stick prices

#call option------------------------------------------------------------------
C0 = np.zeros(n)
for i in range(n):
    C0[i] = f_BS(S0,K[i],T,sd,r,"call")
#compute Put optoin price today using black-scholes
stock = StockPrices(S0,r,sd,T,paths,steps) #simulate 10000 stock prices
S10 = stock[:,10] #take stock prices at day 10
C10 = np.zeros((len(S10),n))
Cc = np.zeros(n)
for i in range(n):
    C10[:,i] = f_BS(S10,K[i],T-10/252,sd,r,"call")
    #compute Put option price at day 10 for each simulation
    Cc[i] = np.quantile(C10[:,i],0.01)
    #take the 0.99th option price at day 10 from the distribution
VaR_C = C0 - Cc #compute VaR

N_Coption = 100/VaR_C #number of options to buy
Cost_C = C0*N_Coption #total cost of all the options bought
Bond_C = 100-Cost_C #total bond to buy/short
w_option_C = Cost_C/(Cost_C+Bond_C) #weight on put option
w_bond_C = 1 - w_option_C #weight on bond
```

```
#put option-----------------------------------------------------------------
P0 = np.zeros(n)
for i in range(n):
    P0[i] = f_BS(S0,K[i],T,sd,r,"put")
#compute Put optoin price today using black-scholes
P10 = np.zeros((len(S10),n))
Pc = np.zeros(n)
for i in range(n):
    P10[:,i] = f_BS(S10,K[i],T-10/252,sd,r,"put")
    #compute Put option price at day 10 for each simulation
    Pc[i] = np.quantile(P10[:,i],0.99)
    #take the 0.99th option price at day 10 from the distribution
VaR_P = P0 - Pc #compute VaR

N_Poption = 100/VaR_P #number of options to buy
Cost_P = P0*N_Poption #total cost of all the options bought
Bond_P = 100-Cost_P #total bond to buy/short
w_option_P = Cost_P/(Cost_P+Bond_P) #weight on put option
w_bond_P = 1 - w_option_P #weight on bond

#plot the weights-----------------------------------------------------------
plt.figure(2,figsize=(7,5))
plt.rcParams.update({'font.size': 16})
ax1 = plt.plot(K,w_option_C)
ax2 = plt.plot(K,w_option_P)
plt.legend(["call option","put option"])
plt.xlabel("strik price")
plt.ylabel("absolute value of long/short position")
plt.title("Put/Call Option Position on Different Strike Prices")

```

```
#return---------------------------------------------------------------------
C_R = np.zeros(n);P_R = np.zeros(n)
for i in range(n):
    C_R[i] = w_option_C[i]*(np.mean(C10[:,i])-C0[i])/C0[i]+w_bond_C[i]*(r)*10/252
    P_R[i] = w_option_P[i]*(np.mean(P10[:,i])-P0[i])/P0[i]+w_bond_P[i]*(r)*10/252
plt.figure(3,figsize=(7,5))
plt.rcParams.update({'font.size': 16})
ax1 = plt.plot(K,C_R)
ax2 = plt.plot(K,P_R)
plt.legend(["call portfolio","put portfolio"])
plt.xlabel("strik price")
plt.ylabel("portfolio return")
plt.title("Put/Call Portfolio Return on Different Strike Prices")

```





Based on the plots, we should construct portfolios that short the put options if the put option is ITM, and long the call options if the call option is ITM. If both the options are at the money, we can either short put options or long call options. Thus, if the strike is larger than 50, *we should choose the portfolio that is shorting put; if the strike is smaller than* 50, *we should choose the portfolio that is longing the call.*

6. Extra question. What happens if you have to respect a constraint on expected shortfall instead of value-at-risk?