Johnny Basurto Rosales

Physics 31415

Professor Cottrell

Grover's Algorithm Project

**Introduction**

Any video on quantum computers will explain that quantum can  be more powerful that a classical computer. Famously, a quantum computer has $2^n$ states to work with while a classical computer only n. A misconception that I had about this is that all $2^n$ qubits can be observed, but that is not true –you'll notice this in the quantum circuits -only n measurements can be made. Therefore, the true power of a quantum computer lies in superposition, interference and entanglement.

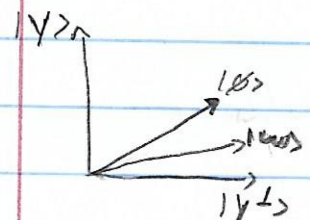
This paper will attempt to showcase an application of superposition, interference, and entanglement on the following problem: given records of data with assigned keys (0,1,2…N-1), find key y. This problem can be solved using the Grover's algorithm.

**Grover's Search Algorithm**

In short, Grover's algorithm initializes a superposition of equally likely states, |w>, and applies unitary operations so that the desired state, |y>, has a greater probability than the rest of the states. Since the total probability is constrained to 1, the probabilities of the undesired states in |w> are decreased.

This can be understood by first noting that the vector perpendicular to |y> can be derived from |w>. In the following demonstration, it is a given that |w> can be expressed in the |y>, |y,perpendicular> basis. Additionally, we look at an arbitrary state |ϕ> that can be interpreted as the |w> state to whom the unitary operations have been applied at least once.

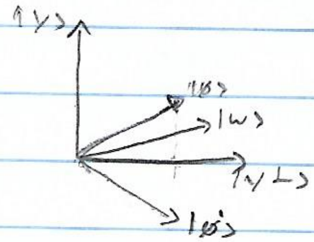
The first part shows a geometric understanding of what the unitary operations do to the state |ϕ>. The second part shows explicitly the effect the unitary operations have on |ϕ>. Finally, the third part shows the unitary operations applied n times to |ϕ> and ends with an example involving 2 qubits. All parts are numbered on the upper right corner.




$|w\rangle = \cos(\theta)|y^{\perp}\rangle + \sin(\theta)|y\rangle$

$|\phi\rangle = \cos(\phi)|y^{\perp}\rangle + \sin(\phi)|y\rangle$

$V_y$:



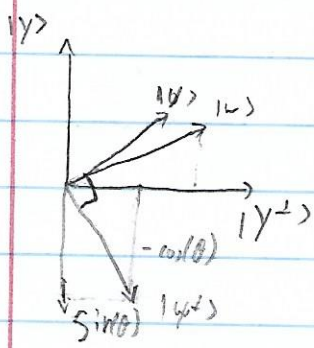

$|\phi'\rangle = \cos(\phi)|y^{\perp}\rangle - \sin(\phi)|y\rangle$

$U_0$ in three steps:

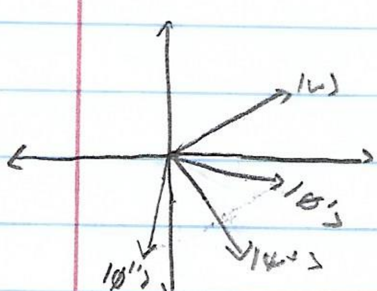
① We transform $|\phi'\rangle$ into the $|w\rangle, |w^{\perp}\rangle$ basis:

$|w\rangle = \cos(\theta)|y^{\perp}\rangle + \sin(\theta)|y\rangle$, $|w^{\perp}\rangle = \sin\theta(\theta)|y^{\perp}\rangle - \cos(\theta)|y\rangle$

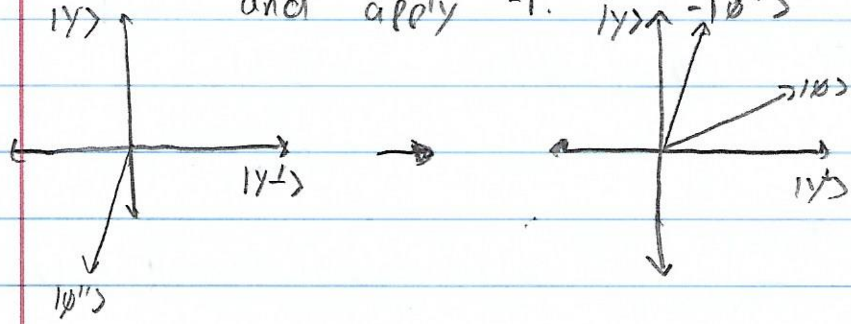
$\langle\psi^{\perp}|\psi\rangle = 0$




② In the $|\psi\rangle, |\psi^{\perp}\rangle$ basis, we reflect over $|\psi^{\perp}\rangle$!



(3) Finally, we return back to the $|y^\perp\rangle, |y\rangle$ basis and apply $-1$:




I have shown that applying $-$Molly to an arbitrary state $|\phi\rangle$ rotates it closer to the target state $|y\rangle$.

Starting with $|\phi\rangle = \cos\phi|y^\perp\rangle + \sin\phi|y\rangle$ ②

$$U_y: \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \begin{matrix} |y^\perp\rangle \\ |y\rangle \end{matrix} = \begin{pmatrix} \cos\phi \\ -\sin\phi \end{pmatrix}$$

Now we're going to reflect over $|\omega\rangle$. This can be done by reflecting over $|\omega^\perp\rangle$ and negating. We first transform $|\phi\rangle$ into the $|\omega\rangle, |\omega^\perp\rangle$ basis.

① $$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi \\ -\sin\phi \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{pmatrix}$$

Now we reflect over $|\omega^\perp\rangle$, $U_0$:

② $$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{pmatrix} = \begin{pmatrix} -\cos\theta\cos\phi + \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{pmatrix}$$

Now we return to $\{y^\perp, y\}$ basis to see the effect!

③ $$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} -\cos\theta\cos\phi + \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{pmatrix}$$

$$= \begin{pmatrix} -\cos^2\theta\cos\phi + \cos\theta\sin\theta\sin\phi + \sin^2\theta\cos\phi + \sin\theta\cos\theta\sin\phi \\ -\sin\theta\cos\theta\cos\phi + \sin^2\theta\sin\phi - \cos\theta\sin\theta\cos\phi - \cos^2\theta\sin\phi \end{pmatrix}$$

$$= \begin{pmatrix} -(\cos^2\theta - \sin^2\theta)\cos\phi + 2\sin\theta\cos\theta\sin\phi \\ -(\cos^2\theta - \sin^2\theta)\sin\phi - 2\sin\theta\cos\theta\cos\phi \end{pmatrix}$$

$$\begin{pmatrix} -\cos(2\theta)\cos(\phi) + \sin(2\theta)\sin(\phi) \\ -\cos(2\theta)\sin(\phi) - \sin(2\theta)\cos(\phi) \end{pmatrix} = \begin{pmatrix} -\cos(\phi+2\theta) \\ -\sin(\phi+2\theta) \end{pmatrix}$$

$$\sim -\cos(\phi+2\theta)|y^{\perp}\rangle - \sin(\phi+2\theta)|y\rangle$$

Now we multiply by $-1$ to essentially reflect over $|w\rangle$.

$$\rightarrow \cos(\phi+2\theta)|y^{\perp}\rangle + \sin(\phi+2\theta)|y\rangle$$

In sum:

$$-M_\theta U_y|\phi\rangle = \cos(\phi+2\theta)|y^{\perp}\rangle + \sin(\phi+2\theta)|y\rangle$$

or

$$\underbrace{-\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}}_{-M_\theta \text{ diffusive operator}}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\substack{U_y \\ \text{blackbox}}}\underbrace{\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}}_{|\phi\rangle}$$

$$= \begin{pmatrix} \cos(\phi+2\theta) \\ \sin(\phi+2\theta) \end{pmatrix}$$

Applying $-M_0 U_y$ n times results in:

$$\cos(\phi + 2n\theta)\,|y^\perp\rangle + \sin(\phi + 2n\theta)\,|y\rangle$$

How many times was $-M_0 U_y$ applied?

We first let $\phi = \theta$. That is, we start off at $|w\rangle$ and apply $-M_0 U_y$ n times:

$$(-M_0 U_y)^n |w\rangle = \cos(\theta + 2n\theta)\,|y^\perp\rangle + \sin(\theta + 2n\theta)\,|y\rangle$$

And ideally $\pi/2 = \theta + 2n\theta \rightarrow \frac{\pi}{2}\left(\frac{1}{1+2n}\right) = \theta$

We can use this in another equation given by:

$$\langle y|w\rangle = \frac{1}{\sqrt{N}} = \sin\theta \approx \theta$$

Where the small-angle approximation was used because $\frac{1}{\sqrt{N}} \ll 1 \rightarrow N \gg 1$.

Finally: $\frac{1}{\sqrt{N}} \approx \frac{\pi}{2}\left(\frac{1}{1+2n}\right) \rightarrow \frac{1}{\sqrt{N}}(n) \approx \pi/2 \rightarrow \boxed{n \approx \frac{\sqrt{N}}{2}}$

where $n \gg 1$.

For 2 qubits ($N=4$) we need $n \approx \frac{\sqrt{4}}{2} = 1$ applications of $-M_0 U_y$

$\frac{1}{\sqrt{4}} = \sin\theta \rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ $\quad \cos\left(\frac{\pi}{6} + 2\left(\frac{\pi}{6}\right)\right) = 0 \quad \sin\left(\frac{\pi}{6} + 2\left(\frac{\pi}{6}\right)\right) = 1$ ✓

## How to pick the gates for the quantum circuit?

For the black box, it easy to show the logic behind the gates I chose. The reflection over |y,perpendicular> is the same thing as applying a negative phase to the target state because |y,perp> is |w> that is not |y>. So for N qubits, we have to find any combination of gates that result in the target's phase to be flipped.

I was not able to find a similar idea for the diffusive operator. As part 2 of the calculations show, we can explicitly find the unitary operations that are to be applied on the space spanned by {|y>, |y,perp>}. But how that is to be translated to the quantum circuit and generalized to N qubits I failed to realize. At minimum, I can find a black box for 2 qubits and apply the quantum circuit given in lecture unto Qiskit and search for |11> and |10>.

## Grover's Algorithm Data

The Quantum Circuit for the search for 11:

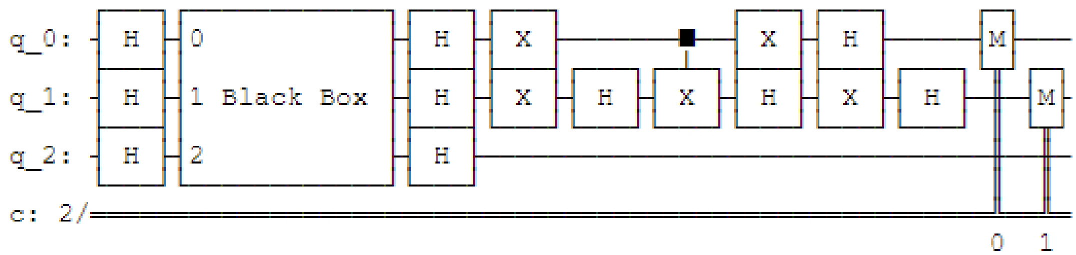



*Figure 1 Quantum circuit used to search for |11>.*

Results from IBM_Santiago:

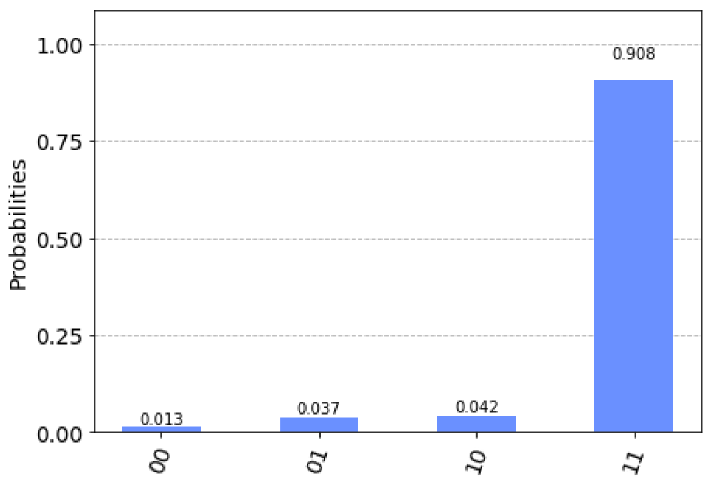



*Figure 2 Final states after running the quantum circuit.*

*Results from a classical simulation:*

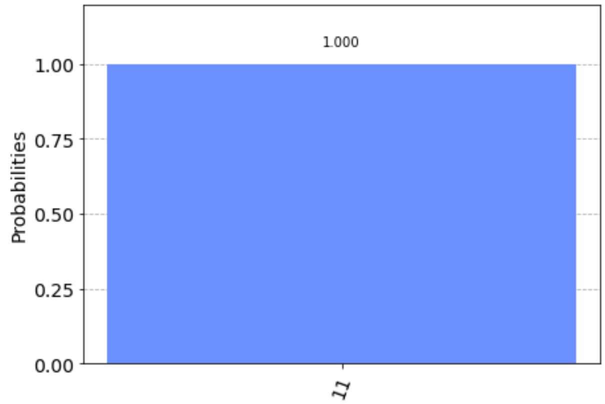



*Figure 3 States measured by a classical simulation where quantum errors cannot be simulated.*

The Quantum Circuit for the search of 10:

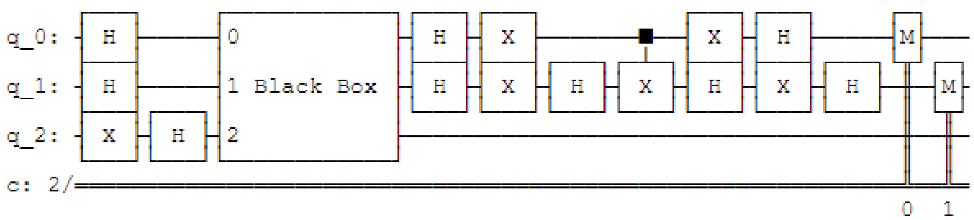



*Figure 4 Quantum circuit used to search for |10>.*

Results from IBM_Santiago:

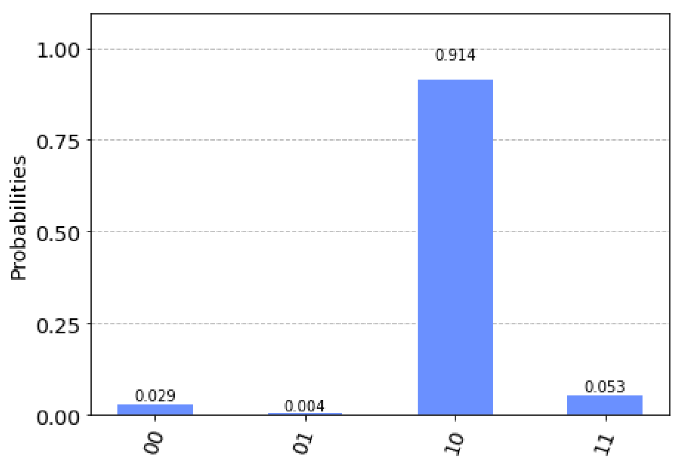
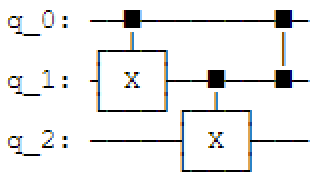



*Figure 5 States measured from a quantum computer calculation.*

Where the black box is defined as:




As both quantum searches show, there is a small probability of measuring undesired states. We have shown at the end of part 3 that this should theoretically be impossible so these results emphasize that quantum computers are still prone to phase errors. For the search of 10, there is a probability less than .1 of finding undesired states. In other words, for 1000 measurements in the 10 search you are likely to find 53 measurements resulting in the 11 state.

## Quantum Teleportation

The following is a classical simulation of quantum teleportation. I used it to learn about the relevant commands in python.[1] Qubit 0's state was transferred to qubit 2!

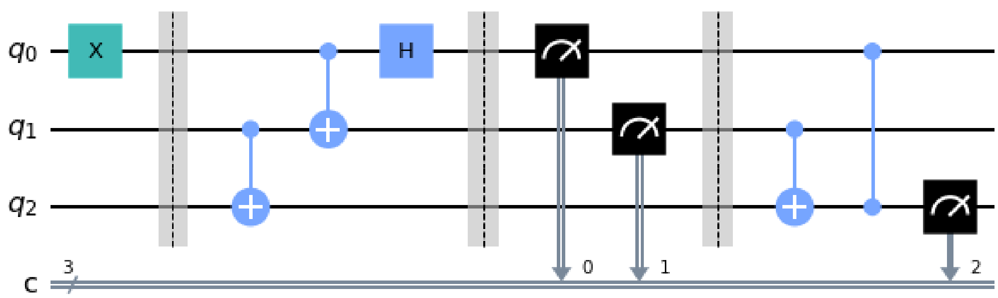



*Figure 6 Quantum circuit for Quantum Teleportation*

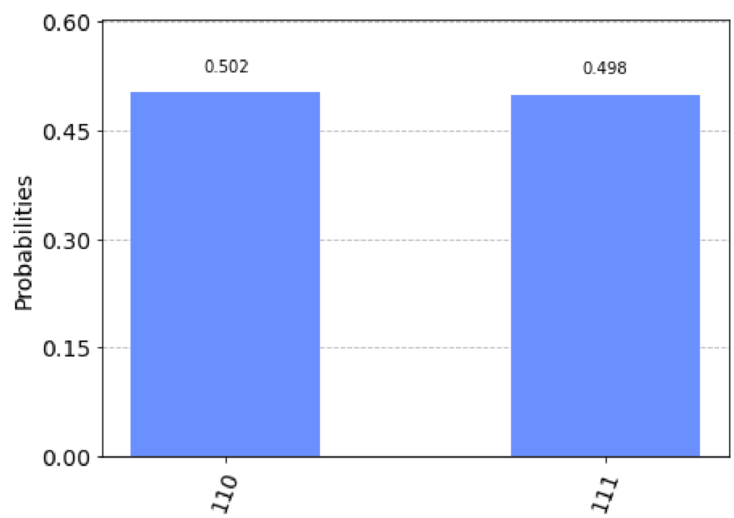



*Figure 7 Final states measured. The 3rd qubit is the bottom most number on the x-axis.*

1. Qiskit. “Quantum Teleportation Algorithm — Programming on Quantum Computers Season 1Ep 5.” YouTube, 30 Aug. 2019, www.youtube.com/watch?v=mMwovHK2NrE.