



T-distribution and the One-sample t-test

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Inhoud

- t distribution
 - population distribution
 - t statistic
 - one or two sided
 - effect size
 - power
- one sample t test
 - hypothesis
 - type 1 error
 - p value one sided
 - p value two sided

T-distribution

Gosset



In probability and statistics, Student's t-distribution (or simply the t-distribution) is any member of a family of continuous probability distributions that arises when estimating the mean of a normally distributed population in situations where the sample size is small and population standard deviation is unknown.

In the English-language literature it takes its name from William Sealy Gosset's 1908 paper in *Biometrika* under the pseudonym "Student". Gosset worked at the Guinness Brewery in Dublin, Ireland, and was interested in the problems of small samples, for example the chemical properties of barley where sample sizes might be as low as 3.

Source: [Wikipedia](#)



Population distribution

```
layout(matrix(c(2:6,1,1,7:8,1,1,9:13), 4, 4))

n = 56      # Sample size
df = n - 1  # Degrees of freedom

mu = 100
sigma = 15

IQ = seq(mu-45, mu+45, 1)

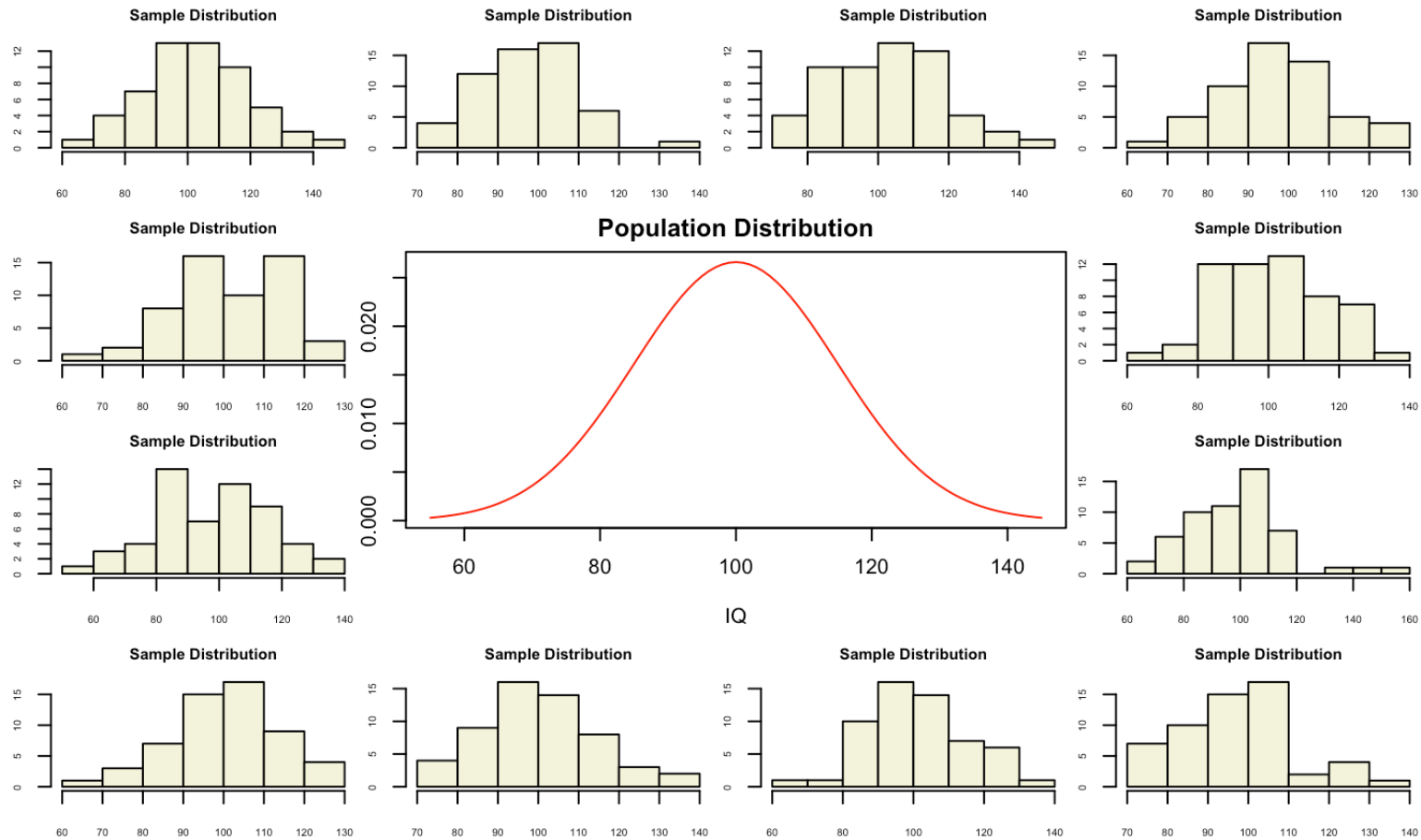
par(mar=c(4,2,2,0))
plot(IQ, dnorm(IQ, mean = mu, sd = sigma), type='l', col="red", main = "Population Distribution")

n.samples = 12

for(i in 1:n.samples) {

  par(mar=c(2,2,2,0))
  hist(rnorm(n, mu, sigma), main="Sample Distribution", cex.axis=.5, col="beige", cex.main = .7!)

}
```



T-statistic

$$T_{n-1} = \frac{\bar{x} - \mu}{SE_x} = \frac{\bar{x} - \mu}{s_x / \sqrt{n}}$$

So the t-statistic represents the deviation of the sample mean \bar{x} from the population mean μ , considering the sample size, expressed as the degrees of freedom $df = n - 1$

A sample

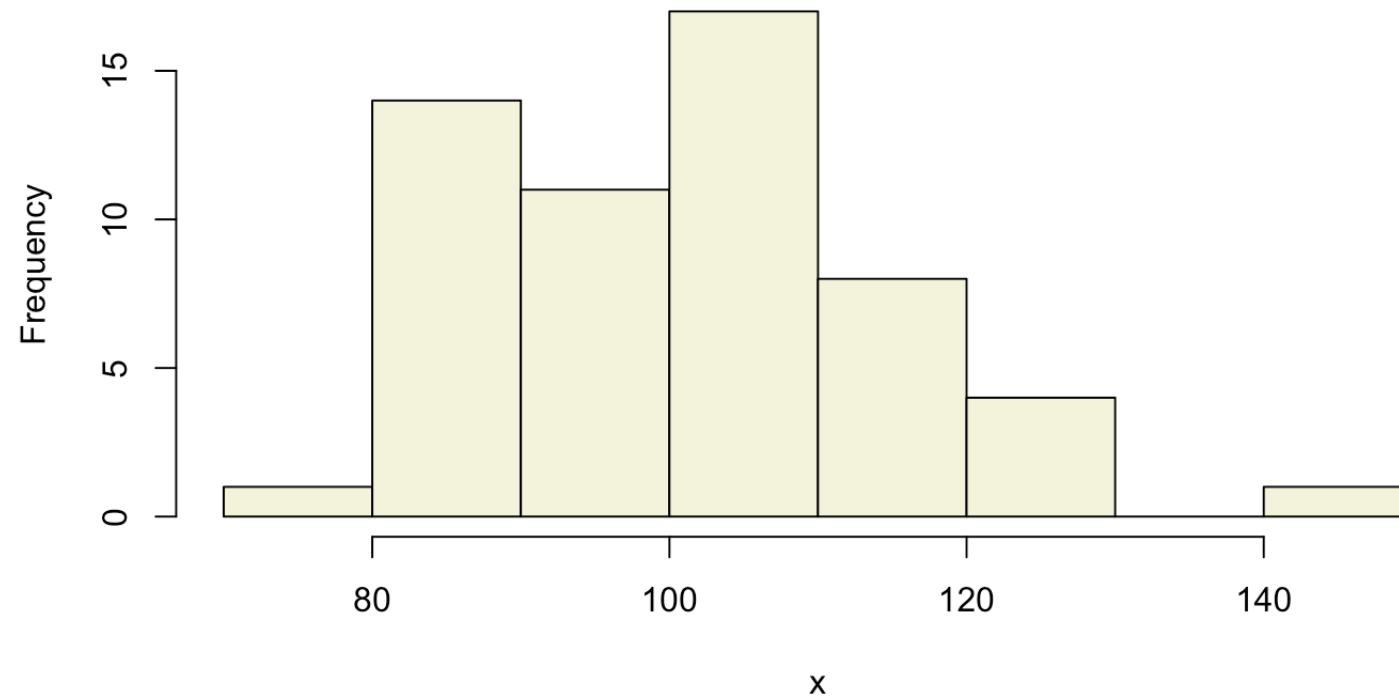
Let's take one sample from our normal population and calculate the t-value.

```
x = rnorm(n, mu, sigma); x
```

```
## [1] 83.89770 107.70912 89.72175 84.04451 107.91092 86.89489 101.89445  
## [8] 73.46541 88.68426 91.31640 129.11417 81.22425 100.64905 104.71157  
## [15] 122.65132 105.74678 86.50262 107.12507 81.71844 103.93729 90.08263  
## [22] 105.81273 97.43061 100.07212 97.62266 93.97758 91.46354 101.93830  
## [29] 90.90682 116.98224 108.18529 98.64524 117.22691 84.64899 81.28618  
## [36] 106.86087 114.33585 80.59898 88.11833 91.41706 81.20882 115.37149  
## [43] 111.30458 99.02562 97.05674 114.37217 120.61504 88.47761 123.67044  
## [50] 100.39810 105.96654 103.09806 140.79855 100.57394 110.62844 118.02008
```

```
hist(x, main = "Sample distribution", col = "beige")
```


Sample distribution



mean(x)

t-value

$$T_{n-1} = \frac{\bar{x} - \mu}{SE_x} = \frac{\bar{x} - \mu}{s_x / \sqrt{n}}$$

```
t = (mean(x) - mu) / (sd(x) / sqrt(n)); t
```

```
## [1] 0.2587839
```

More samples

let's take more samples.

```
n.samples      = 1000
mean.x.values = vector()
se.x.values    = vector()

for(i in 1:n.samples) {
  x = rnorm(n, mu, sigma)
  mean.x.values[i] = mean(x)
  se.x.values[i]   = (sd(x) / sqrt(n))
}
```

Mean and SE for all samples

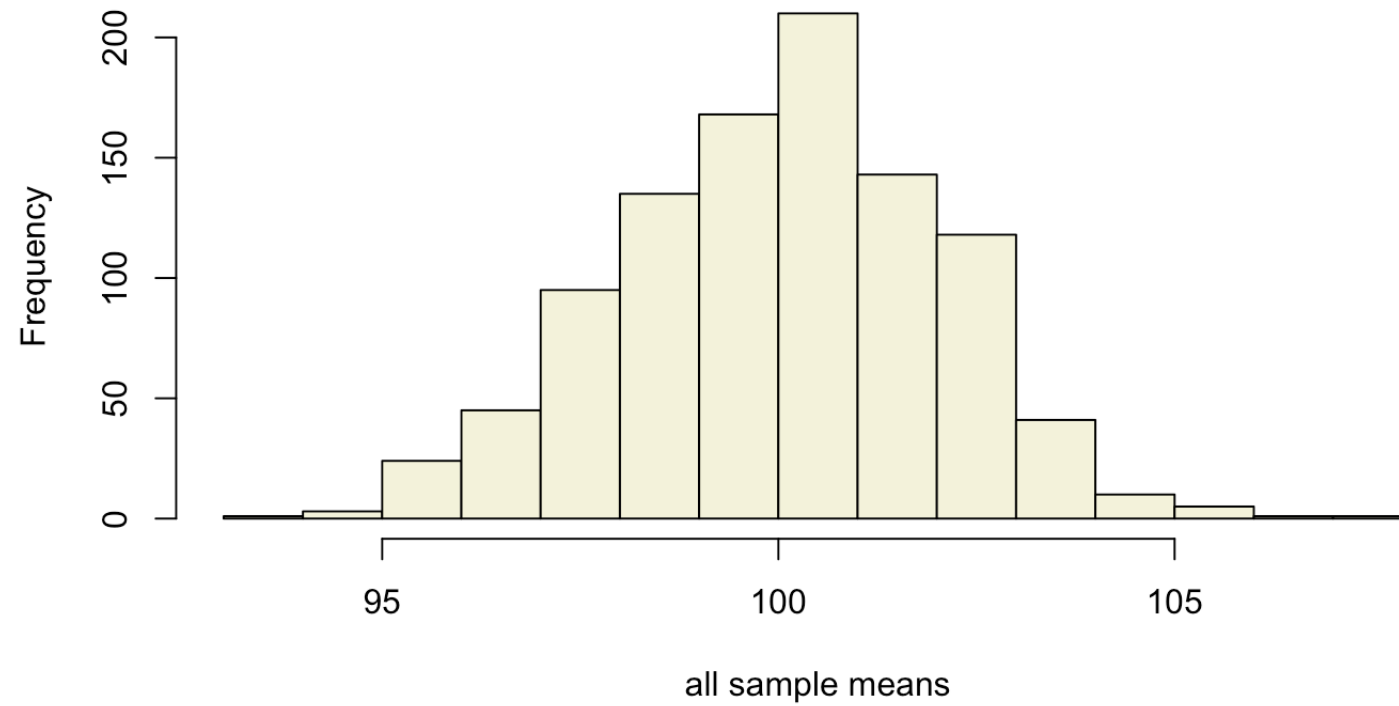
```
head(cbind(mean.x.values, se.x.values))
```

##	mean.x.values	se.x.values
## [1,]	97.30450	2.170377
## [2,]	98.54668	1.999864
## [3,]	98.78162	1.844834
## [4,]	99.52797	2.147194
## [5,]	99.36175	2.036269
## [6,]	96.92832	1.984855

Samples distribution

```
hist(mean.x.values,  
     col  = "beige",  
     main = "Samples distribution",  
     xlab = "all sample means")
```

Samples distribution



Calculate t-values

$$T_{n-1} = \frac{\bar{x} - \mu}{SE_x} = \frac{\bar{x} - \mu}{s_x / \sqrt{n}}$$

```
t.values = (mean.x.values - mu) / se.x.values
```

```
tail(cbind(mean.x.values, mu, se.x.values, t.values))
```

##		mean.x.values	mu	se.x.values	t.values
##	[995,]	95.39534	100	1.994954	-2.30815392
##	[996,]	98.87719	100	1.853329	-0.60583333
##	[997,]	98.62889	100	2.055854	-0.66693159
##	[998,]	100.06489	100	1.907915	0.03400932
##	[999,]	100.48619	100	1.757897	0.27657303
##	[1000,]	97.40763	100	2.149232	-1.20618354

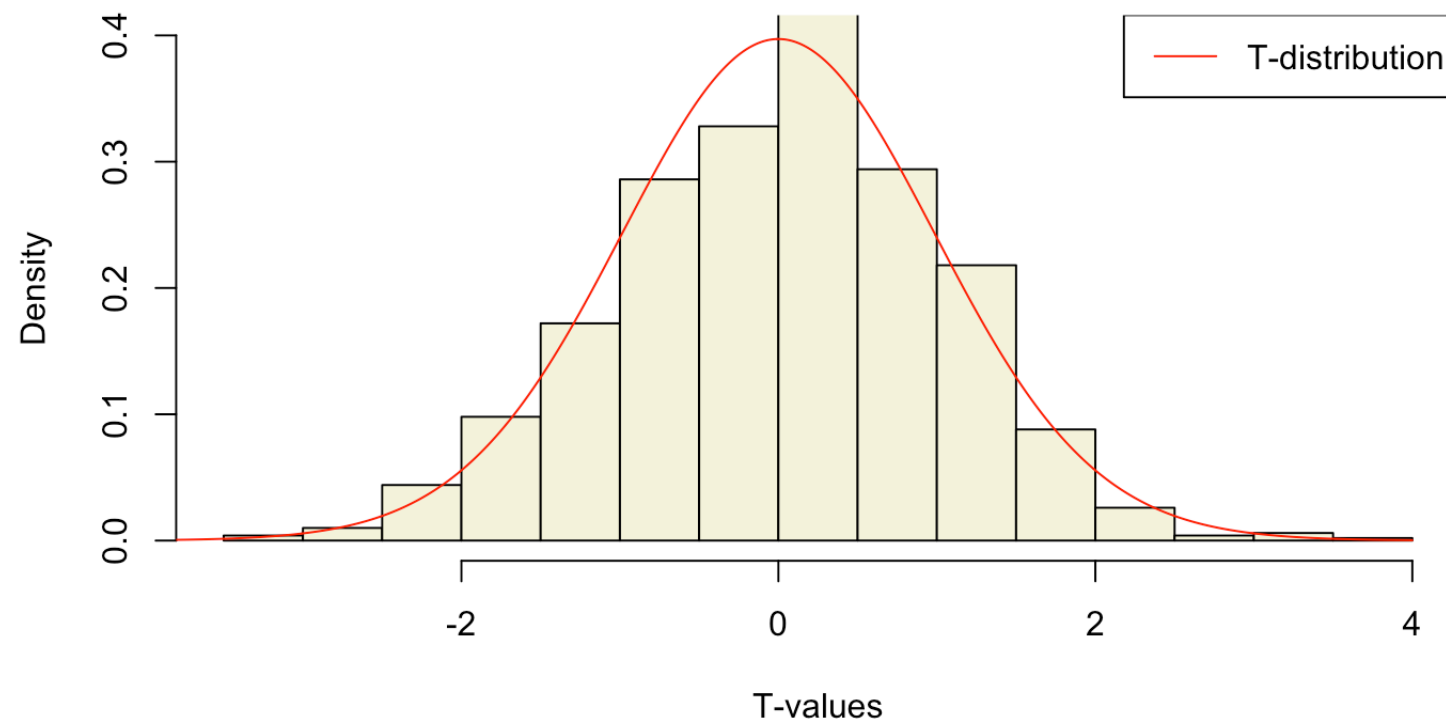


Sampled t-values

What is the distribution of all these t-values?

```
hist(t.values,  
     freq = F,  
     main = "Sampled T-values",  
     xlab = "T-values",  
     col  = "beige",  
     ylim = c(0, .4))  
T = seq(-4, 4, .01)  
lines(T, dt(T,df), col = "red")  
legend("topright", lty = 1, col="red", legend = "T-distribution")
```


Sampled T-values



T-distribution

So if the population is normally distributed (assumption of normality) the t-distribution represents the deviation of sample means from the population mean (μ), given a certain sample size ($df = n - 1$).

The t-distribution therefore is different for different sample sizes and converges to a standard normal distribution if sample size is large enough.

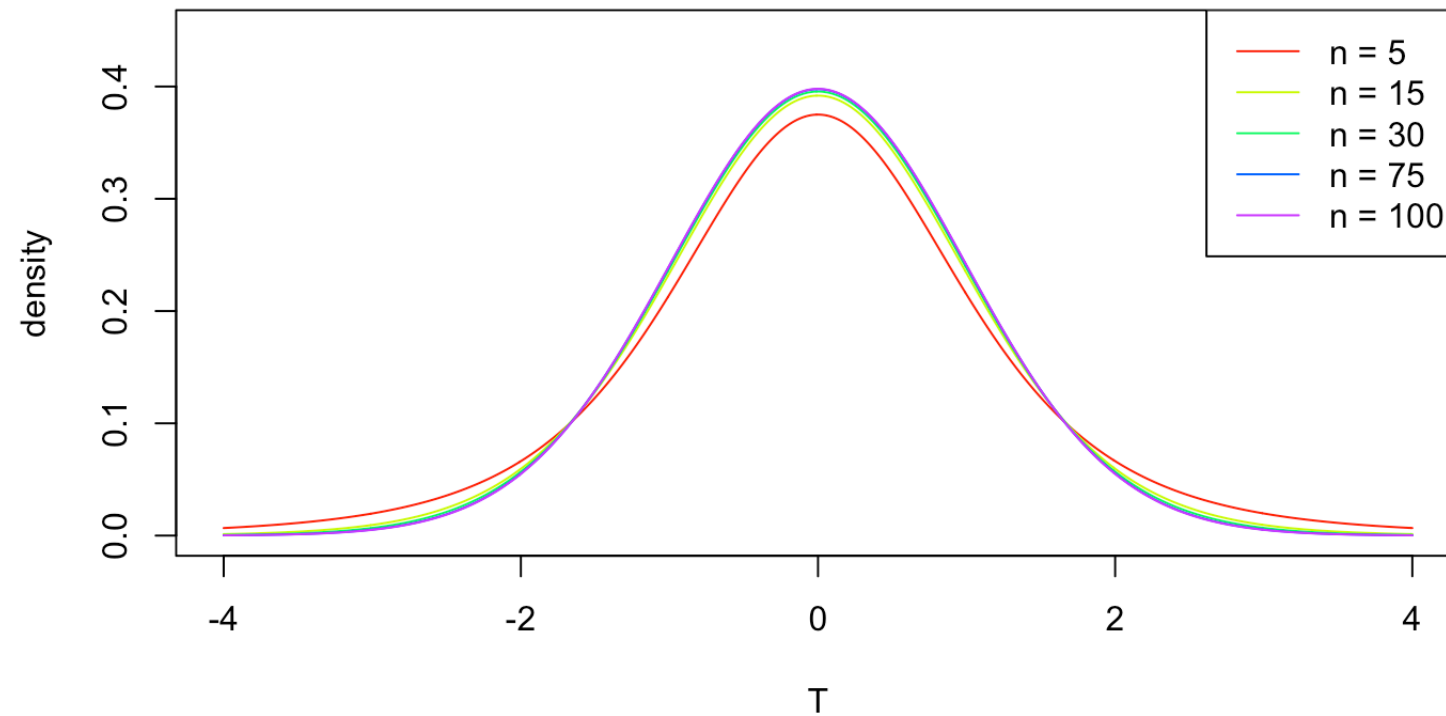
The t-distribution is defined by:

$$\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi} \Gamma(\frac{\nu}{2})} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

where ν is the number of degrees of freedom and Γ is the gamma function.

Source: [wikipedia](https://en.wikipedia.org/wiki/Student%27s_t_distribution)

T-distributions



One or two sided

Two sided

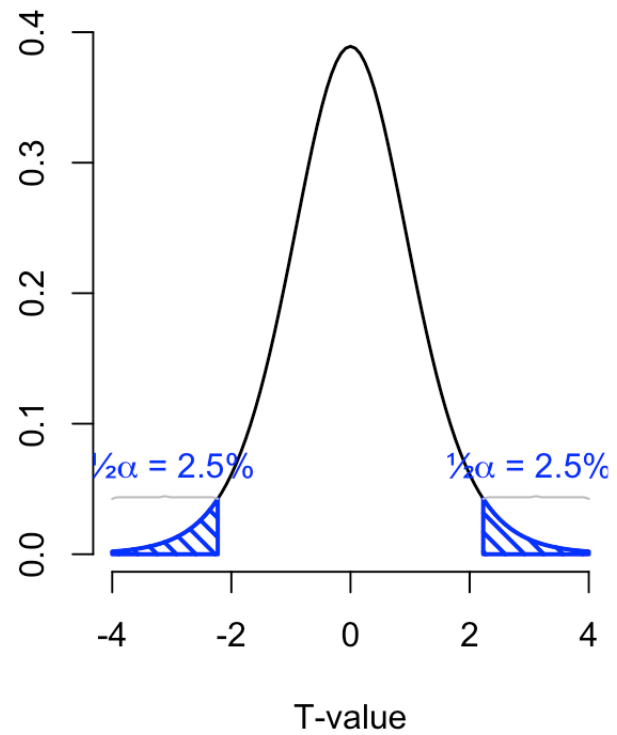
- $H_A : \bar{x} \neq \mu$

One sided

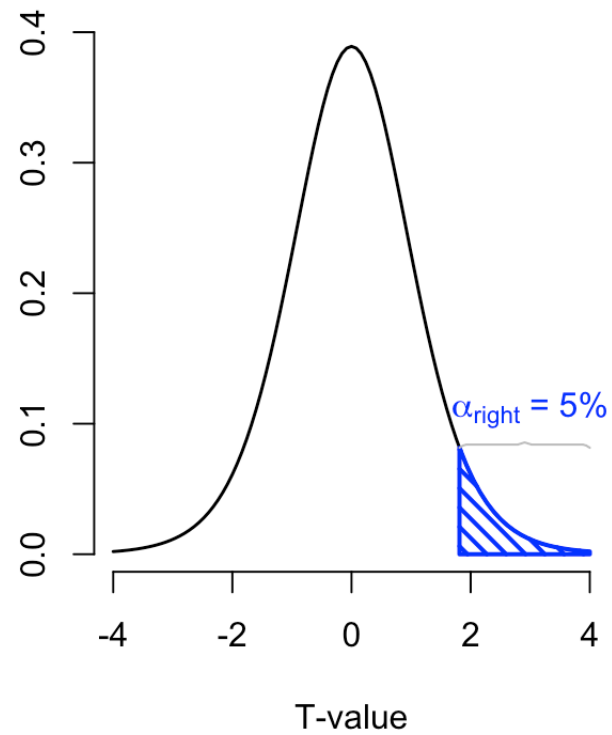
- $H_A : \bar{x} > \mu$

- $H_A : \bar{x} < \mu$

T distribution (df=10) with two sided alpha



T distribution (df=10) with one sided alpha



Effect-size

The effect-size is the standardised difference between the mean and the expected μ . In the t-test effect-size is expressed as r .

$$r = \sqrt{\frac{t^2}{t^2 + df}}$$

```
r = sqrt(t^2/(t^2 + df))
```

```
r
```

```
## [1] 0.2603778
```

Effect-size distribution

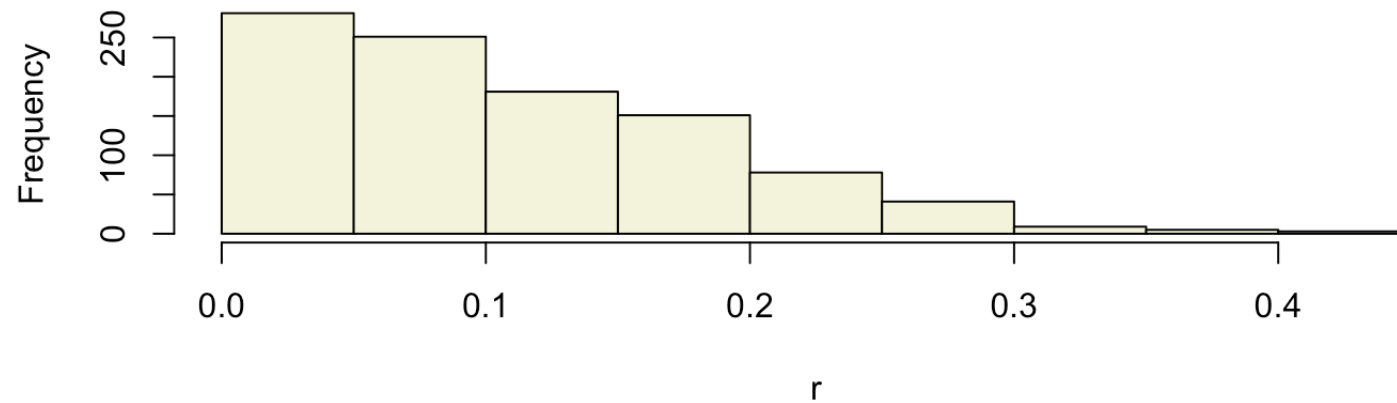
We can also calculate effect-sizes for all our calculated t-values. Under the assumption of H_0 the effect-size distribution looks like this.

```
r = sqrt(t.values^2/(t.values^2 + df))  
  
tail(cbind(mean.x.values, mu, se.x.values, t.values, r))
```

##		mean.x.values	mu	se.x.values	t.values	r
##	[995,]	95.39534	100	1.994954	-2.30815392	0.297171280
##	[996,]	98.87719	100	1.853329	-0.60583333	0.081419332
##	[997,]	98.62889	100	2.055854	-0.66693159	0.089567590
##	[998,]	100.06489	100	1.907915	0.03400932	0.004585768
##	[999,]	100.48619	100	1.757897	0.27657303	0.037267194
##	[1000,]	97.40763	100	2.149232	-1.20618354	0.160532386

```
hist(r, main = "effect-size distribution", col = "beige")
```

effect-size distribution

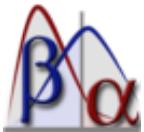


Cohen (1988)

- Small: $0 \leq .1$
- Medium: $.1 \leq .3$
- Large: $.3 \leq .5$

Power

- Strive for 80%
- Based on know effect size
- Calculate number of subjects needed
- Use [G*Power](#) to calculate



Alpha Power

```
T = seq(-3,6,.01)
N = 45
E = 2

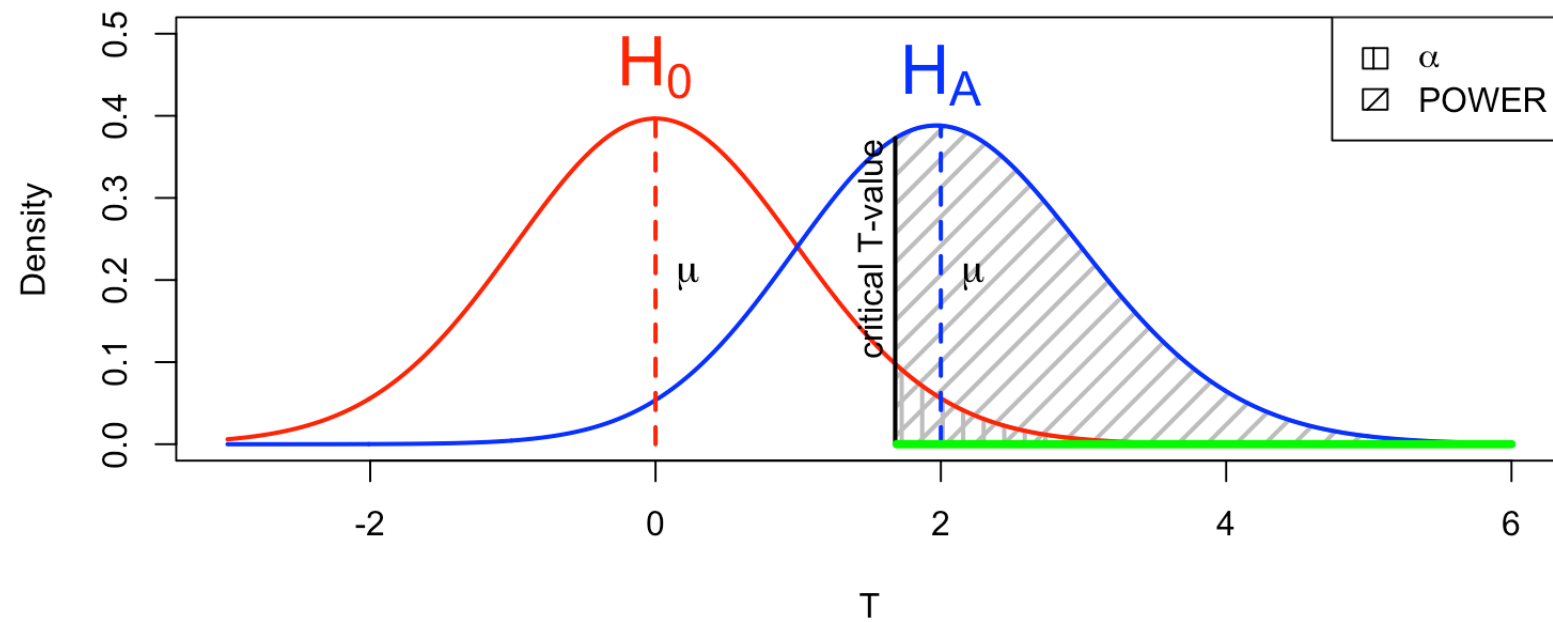
# Set plot
plot(0,0,
     type = "n",
     ylab = "Density",
     xlab = "T",
     ylim = c(0,.5),
     xlim = c(-3,6),
     main = "T-Distributions under H0 and HA")

critical_t = qt(.05,N-1,lower.tail=FALSE)

# Alpha
range_x = seq(critical_t,6,.01)
polygon(c(range_x,rev(range_x)),
        c(range_x*0,rev(dt(range_x,N-1,ncp=0))),
        col = "grey",
        density = 10,
```



T-Distributions under H_0 and H_A



– Bakan (1966)

“Statistical significance testing retards the growth of scientific knowledge; it never makes a positive contribution”

– Schmidt and Hunter (1997)

“The textbooks are wrong. The teaching is wrong. The seminar you just attended is wrong. The most prestigious journal in your scientific field is wrong.”

– Ziliak and McCloskey (2008)

These quotes were mostly taken from Nickerson’s (2000) excellent review “*Null Hypothesis Significance Testing: A Review of an Old and Continuing Controversy*”.

Suggestions

If you have any suggestions send me a message on **Twitter** (<https://twitter.com/krstoffr>) or use the **contact form** (<http://rpsychologist.com/about/>) on my site.

Created by **Kristoffer Magnusson** (<http://rpsychologist.com>), built with **D3.js** (<http://d3js.org/>), **jStat** (<http://jstat.org/>) and **Bootstrap** (<http://getbootstrap.com/>).



One-sample t-test

IQ next to you



<http://goo.gl/T6Lo2s>



Models

$$\text{outcome} = \text{model} + \text{error}$$

Compare sample mean

We use the one-sample t-test to compare the sample mean \bar{x} to the population mean μ .

Let's take a different sample of $n = 43$ and calculate the mean of this sample.

```
mu      = 120
n       = length(IQ.next.to.you)
x       = IQ.next.to.you
mean_x  = mean(x, na.rm = T)
sd_x    = sd(x, na.rm = T)
cbind(n, mean_x, sd_x)
```

```
##           n   mean_x    sd_x
## [1,] 47 122.7447 7.103178
```

Does this mean, differ significantly from the population mean $\mu = 120$?



Hypothesis

Null hypothesis

- $H_0 : \bar{x} = \mu$

Alternative hypothesis

- $H_A : \bar{x} \neq \mu$

- $H_A : \bar{x} > \mu$

- $H_A : \bar{x} < \mu$

Assumptions

- Normal samples distribution
- Measurement level
 - Interval
 - Ratio

T-statistic

$$T_{n-1} = \frac{\bar{x} - \mu}{SE_x} = \frac{\bar{x} - \mu}{s_x/\sqrt{n}} = \frac{122.74 - 120}{7.1/\sqrt{47}}$$

So the t-statistic represents the deviation of the sample mean \bar{x} from the population mean μ , considering the sample size.

```
t = (mean_x - mu) / (sd_x / sqrt(n)); t
```

```
## [1] 2.649037
```

Type 1 error

To determine if this t-value significantly differs from the population mean we have to specify a type I error that we are willing to make.

- Type I error / $\alpha = .05$

P-value one sided

Finally we have to calculate our p -value for which we need the degrees of freedom $df = n - 1$ to determine the shape of the t-distribution.

```
df = n - 1; df
```

```
## [1] 46
```

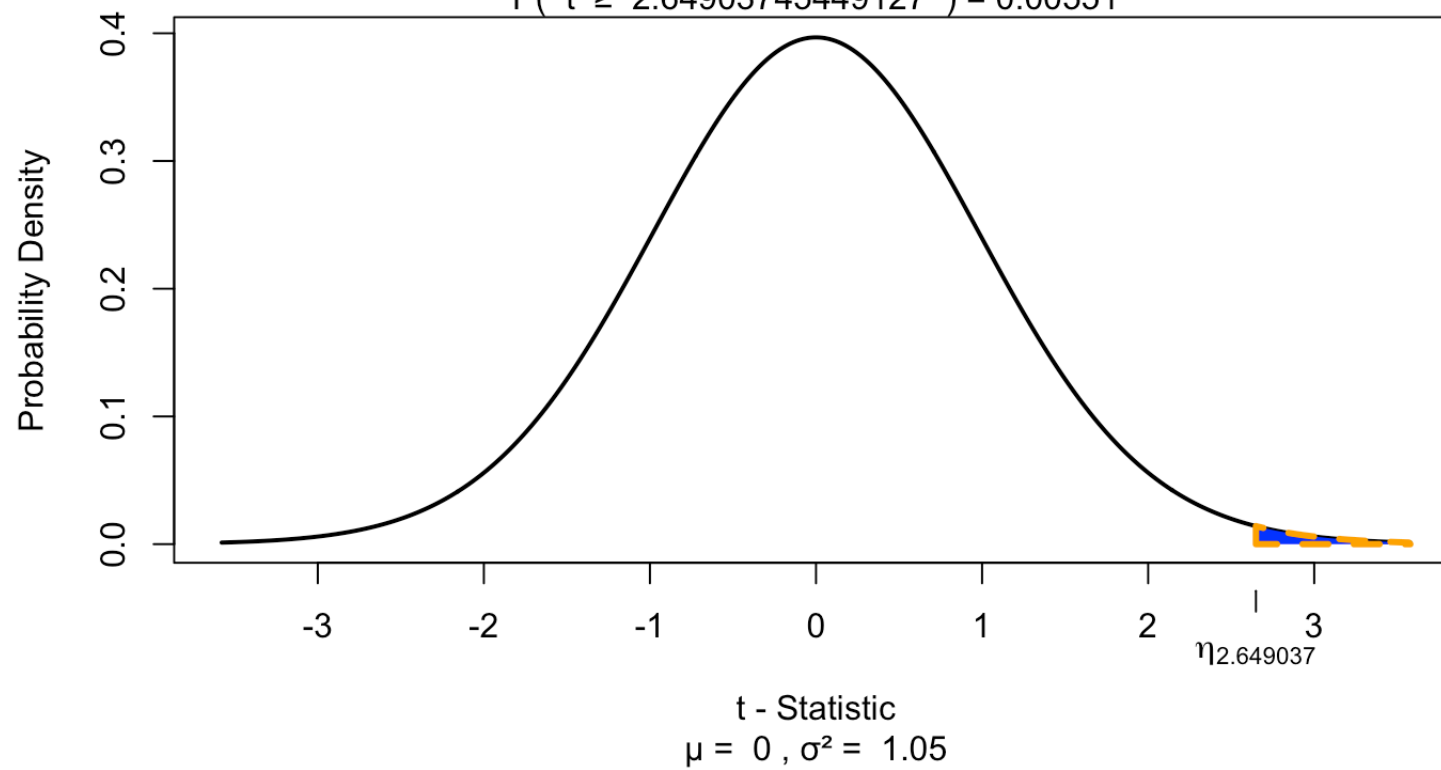
```
if(!"visualize" %in% installed.packages()) { install.packages("visualize") }  
library("visualize")
```

```
visualize.t(t, df, section = "upper")
```

Student t Distribution

df = 46

$$P(t \geq 2.64903745449127) = 0.00551$$



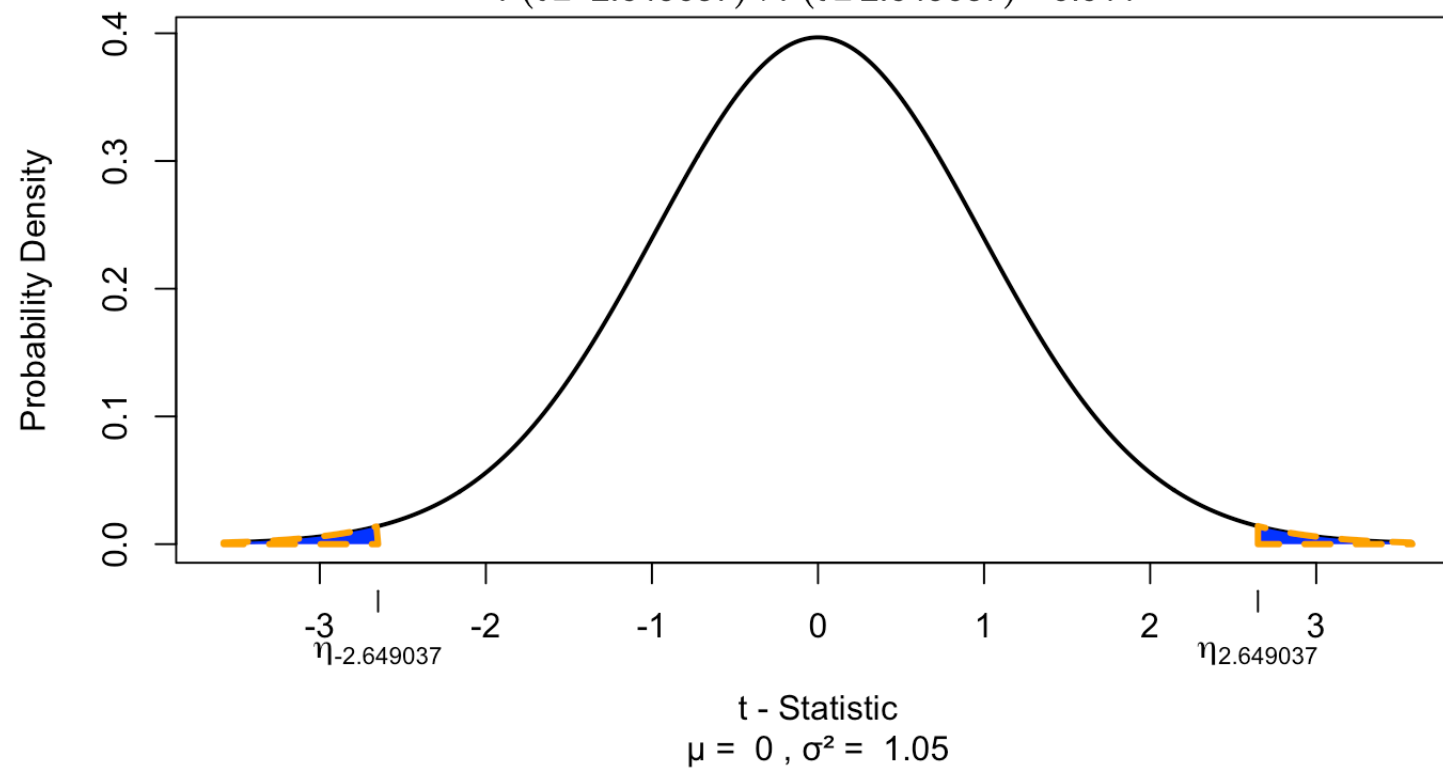
P-value two sided

```
visualize.t(c(-t, t), df, section = "tails")
```

Student t Distribution

df = 46

$$P(t \leq -2.649037) + P(t \geq 2.649037) = 0.011$$



Effect-size

$$r = \sqrt{\frac{t^2}{t^2 + df}}$$

```
r = sqrt(t^2/(t^2 + df))
```

```
r
```

```
## [1] 0.3638136
```

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END