MATA 32
TT2 review part 1

Lecture title	
Review Part 1 多伦多时间: 周一-11/07 19:00-22:00	1. Derivative 公式表总结 2. implicit differentiation 3. logarithmic differentiation 4. higher-order derivatives 5. Newton's method 6. The marginal concept and elasticity of demand
Review Part 2 多伦多时间: 周三-11/09 19:00-22:00 **Review Part 3** 多伦多时间: 周四-11/10 20:00-22:00	1. First and second derivative test 2. extreme value theorem 3. applied max-min problems 4. the indefinite integral 5. integration with initial conditions 6. Substitution rule

- 考试时间：
  - 多伦多时间: **Saturday November 12 from 1pm - 3pm** 总共两小时
- 考试地点：in-person
- **Midterm test 2 占分比例：** 20%

1. 关于 DERIVATIVE 求导

Derivative 公式表

原方程	*导数 derivative*
$cf(x)$	$cf'(x)$
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$ "前导后不导 + 后导前不导"
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ $\frac{\text{“上导下不导} - \text{下导上不导”}}{\text{分母的平方}}$
x^n	nx^{n-1}
数字	0
$ln(x)$	$\frac{1}{x}(1)$
e^x	$e^x(1)$
$\log_b x$	$\frac{1}{xln(b)}(1)$
数字 b^x	$b^x \ln(b)(1)$
Chain rule: $f(g(x))$	$f'(g(x)) \cdot g'(x)$ "从外往里导，先处理外围的，再处理里面的"

✧ 转换公式 $\log_b x = \frac{\log_a x}{\log_a b} = \frac{lnx}{lnb}$ ------记住：$\log_b x = \frac{lnx}{lnb}$ 然后可以以这个形式求导

【知识点】关于求导 chain rule

1. 什么是复合函数(composition function)?

- the composition of f and g

$$(f \circ g)(x) = f(g(x))$$

2. 关于求导 chain rule

两种说法:

(1) If g is a differentiable at x and f is differentiable at g(x), then the composite function $f \circ g$ is differentiable at x and $(f \circ g)'(x)$ *is given by*:

$$(f \circ g)'(x) = [f(g(x))]' = f'(g(x)) \cdot g'(x)$$

(2) If y is a differentiable function of u and u is a differentiable function of x, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

【practice question】

【2021 past midterm】

1. If $y = e^x x^3 + 8\sqrt{x} + \frac{6}{x-2}$, then find $y'(4)$

2. If $h(x) = \sqrt{9x^2 + \ln(x)} + \sqrt[3]{x^2}$, then find $h'(1)$

3. If $k > 1$ is a constant and $g(x) = kx^2 + kx + \log_5(2x+3)$, then find $g'(1)$

4. If $w = x^2 + 2$ and $x = \sqrt{u} + e^u$, and $u = 6t^2 + t + 1$, then fine the value of $\frac{dw}{dt}$ when $t = 0$

【2020 past midterm】

1. If $y = 4u^4 - 5u^3 + 9$ and $u = 4x^2 - 2\sqrt{x}$, then $\frac{dy}{dx}|_{x=1}$ equals:

A.7 B.408 C.28 D.476 E.496 F. none of the above

2. If $y = 3^x x^3 + 3e^{x-1}$, then $y'(1)$ equals
A.12 B. $6 + 2\ln(3)$ C. $12 + 3\ln(3)$ D. $10\ln(3)$ E. $6 + 3\ln(3)$

3. If $a > 1$ is a constant and $f(x) = 5x^3 + \ln(ax) - \frac{a}{x}$, then find $f'(1)$

4. Let f and g be MATA32 functions that are differentiable for all real x. Assume we have the values:
$g'(0) = 6,\ g(0) = 1,\ g'(2) = 6,\ f(0) = 3,\ f'(0) = 2,\ g(3) = 4, and\ g'(3) = 5.$
It is then the case that the derivative of the function $g \circ f$ evaluated at x = 0 is equal to ________

5. Suppose $f(x) = h\big(g(x)k(x)\big)$.
if $g(1) = 3, k(1) = 0, h(1) = -5, g'(1) = 2, k'(1) = -6, h'(1) = 3, and\ h'(0) = 2$
$find\ \ f'(1)$

【2019 past midterm】

1. Find $f'(x)\ where\ f(x) = (\frac{3x+2}{x+5})^2$

2. Let f and g be differentiable functions such that $f\big(g(x)\big) = x$ and $f'(x) = 1 + (f(x))^2$ for all real x. Find $g'(0)$

3. Find $f'(0)$ where $f(x) = \sqrt{\frac{9+x^2}{4-x}} + \frac{2^x}{\ln(2)}$

【2018 Winter past midterm】

1. Let $f(u) = u^3 - 3u^2 + 2u + 1$ where $u = u(x) = 2x^2 + e^{x-1}$. Use the chain rule to find the value of $\frac{df}{dx}$ when $x = 1$

【2017 Fall past midterm/final】

1. Find $f(x) = xln(x) + 2\sqrt{x}$, then $\boldsymbol{f'(1)}$ **equals:**

A.0 B.2 C.1+e D.2+e E.3

2. If $w = \frac{5}{2-x^2}$ then $\frac{dw}{dx}$ equals

A. $2w^2x$ B. $\frac{2w^2x}{5}$ C. $\frac{4w^2x}{5}$ D. $-\frac{2w^2x}{5}$ E. $\frac{5w^2x}{2}$

3. Let $f(x) = \frac{e^{x^2}}{x}$ find $f'(2)$ and leave answer in terms of familiar mathematical constants, not decimals

4. Find $\frac{du}{dt}$ when $u = t^3 \log_3 t + 4\sqrt{t}$

5. Let $g(x)=\frac{10x}{x+1}$ and it's derivative is $g'(x)=\frac{10}{(x+1)^2}$
Calculate $\frac{du}{dx}|_{x=0}$ where $u=g(f(x))+f(0)\left(e^{g(x)}\right)$ and f is a differentiable function such that $f(0)=4$ and $f'(0)=5$

【2016 past midterm】

1. Let a and b be positive constants. If $u=\sqrt{3ax^2+b}$, then $\frac{du}{dx}$ equals

A. $\frac{2ax}{u}$ B. $2axu$ C. $\frac{3ax}{u}$ D. $\frac{3a}{u}$ E. $\frac{3ax}{\sqrt{u}}$ F. none of the above

2. If $y=x5^{(x^2+x)}$ then $\frac{dy}{dx}|_{x=1}$ equals :

A. 35 B. $25+30\ln(5)$ C. 50 D. $25+75\ln(5)$ E. none of the above

3. Suppose $y=h(x)$ and $x=g(t)$. Given that $g(3)=5$, $g'(3)=-4$, $h(-4)=1$, $h'(3)=3$, $h(3)=2$, and $h'(5)=-2$. evaluate $\frac{dy}{dt}$ at $t=3$

【2015 past midterm】

1. If $y = x^2\sqrt{3x^2+4}$, then $\frac{dy}{dx}$ when $x = 2$ is

A. $\frac{1}{2}$ B. 6 C. $\frac{33}{2}$ D.14 E.28 F. 22

2. Find $\frac{dy}{dx}$ in fully factored form where $y = (4x+3)^3(2x+5)^6$

3. Assumer $P(x)\ and\ Q(x)$ are differentiable functions and that for all real numbers x, $P\big(Q(x)\big) = x$ and $P'(x) = 4 + (\frac{P(x)}{2})^{\ 2}$. Find $Q'(0)$

【2014 past midterm】

1. If $f(x) = 4x^2\sqrt{4x+1} + 0.4x$ then $f'(6)$ is

A. 224.6 B.296 C.298 D.355.6 E.none of the above

2. Let $f(x)=\frac{3x+2}{7x+1}$ and $f'(x)=\frac{-11}{(7x+1)^2}$
   Assume g is a differentiable function such that $g(0)=g(2)=g'(2)=2$ and $g'(0)=4$
   Find $\frac{dA}{dx}$ when $x=0$ where $A(x)=g(x)g(f(x))$

【2014 past final exam】

1. Let S represent the future value of an ordinary annuity as a function of n , the number of compounding periods

$$\boldsymbol{S=R\cdot\frac{(1+r)^n-1}{r}}$$

Show that $\frac{dS}{dn}=\alpha(K+S)$ where $K=\frac{R}{r}$ $and\ \alpha=\ln(1+r)$

【2013 past midterm】

1. Let $f(x)=xe^{-x^2}$. Find $f'(x)$

2. Let $y=x^3\log_2 x$. Find $\frac{dy}{dx}$

2. IMPLICIT DIFFERENTIATION 隐函数

【知识点】对比：

(1) **explicit differentiation**(显函数求导)

- **形式：** $y =$ ____________

  单独的"y" 在等式左边，含有"x"的式子在等式右边

- **例如：** $y = 2x^3 + 1$

(2) **Implicit differentiation**(隐函数求导)

- 定义：defines y implicitly as a differentiable function of x and try to find the derivative $\frac{dy}{dx}$ $or\ y'$
- **形式：** x 和 y 都在等式同一边，并且无法将单独的一个"y" 放在等式一边
- **例如：** $xe^y + ye^x = 1$
- **做题方法/步骤：**

  (a) **涉及 function "y" 的求导，用到 chain rule**

  即：涉及到 function "y"的求导，求导公式和之前一样

  但是求完之后要再乘以一个 function y' 或者 $\frac{dy}{dx}$

  (b) **最后 isolate y' 在等式左边**

【例题】：find $\frac{dy}{dx}$ by implicit differentiation if $x^2 + 3y^2 = 2xy$

步骤：

(1) Differentiating both sides with respect to x

(2) Solving for $\frac{dy}{dx}$ side

【past exam questions】

1. (2020 winter final exam) assume the equation $x^3y + y^3x = 10$ defines y implicitly as a function of x, find the value of $\frac{dy}{dx}$ at (1,2) is

2. (2015 fall final exam) assume y is defined implicitly as a function of x by the equation $2\sqrt{y} + ln(xy^2) = 1$ Solve for x when $y = 1$ in this equation, and then evaluate $\frac{dy}{dx}$ at the point $(x, 1)$

3. (2014 Fall midterm) if y is defined implicitly by the equation $e^{xy} + y = 2 + (x+1)^2$, then find the value of $\frac{dy}{dx}$ evaluated at $(0,2)$

4. (2013 winter final exam)If y is defined implicitly as a function of x by the equation $xy^2 + y = 4x$ then find the value of $\frac{dy}{dx}$ When $x = 0$

【注意读题】

1. Differentiate the following equation with respect to t :

If $y = x^3 + 5x$ and $\frac{dx}{dt} = 7$, find $\frac{dy}{dt}$ where $x = 1$

2. If $Nu - 10u + N = 300$, find $\frac{dN}{du}$

3. LOGARITHMIC DIFFERENTIATION 对数微分

【知识点】

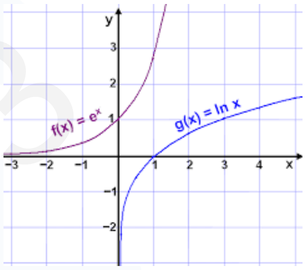
1. **properties of logarithmic**

转换形式：$lnx = y \Leftrightarrow e^y = x$	$ln(x^y) = ylnx$
$\ln(e^x) = x$	$ln\left(\frac{x}{y}\right) = ln(x) - ln\,(y)$
$e^{lnx} = x$	$ln(xy) = ln(x) + ln\,(y)$
$lne = 1$	
$\ln(1) = 0$	

对于e^x	
Domain：R	Range: (0 ,+∞)
对于$ln(x)$	
Domain：(0 ,+∞)	Range：R

2. **logarithmic differentiation** 对数求导法

let $y = f(x)$



- 做题步骤：
  - (1) **左右两边加 ln**
  - (2) 通过使用 properties of logarithmic 的公式去**简化式子**
  - (3) 左右两边**分别求导**
  - (4) **isolate y'** 在等式左边
  - (5) 等式右边的**y 替换成原方程**

- 涉及题型：
  - (1) 方程里面带有多个函数**相乘** 或者 **相除**
  - (2) **未知数**$^{\text{未知数}}$ 例如：x^x

- 【例题】：find y' using logarithmic differentiation

✧ **题型 1: 两个函数相乘**

$$y = (x+1)^2(x-2)(x^2+3)$$

✧ 题型 2: 两个函数相除

$$y = \frac{(2x-5)^3}{x^2\sqrt[4]{x^2-1}}$$

✧ 题型 3: 未知数^{未知数}

$$y = x^{x^2+1}$$

【past exam questions】

1. (2016 Fall final exam) find $y'(1)$ where $y = 8(ex)^{\sqrt{x}}$

2. (2014 past midterm) If $u = (e^2x)^{\sqrt{x}}$ then $u'(1)$ equals:

A. $2e^2$ B. $2e$ C. e^2 D. $2e^2 + 1$ E. none of the above

3. (2017 winter final exam) find $y'$where $y = \frac{x(1+x^2)^2}{\sqrt{2+x^2}}$

4. (2017 Fall final exam) find the exact value of $f'(1)$ where $f(x) = (4x^2+5)^x + \log_2 x$

5. (2013 Fall final exam) if $y = (\frac{9}{x^2})^x$ then $y'(3)$ equals

A. 3 B. -2 C. $\frac{2}{9}$ D. 2 E. None of the above

6. (2013 winter final exam) If $f(x) = (2x+3)^x$ then $f'(0)$ equals

A. 0 B. $\frac{2}{3} + \ln(3)$ C. $ln27$ D. $ln3$ E. None of the above

7. (2012 winter final exam) for $x > 0$, let $f(x) = \left(\frac{1}{x}\right)^x$. find the exact value of $f'(e)$ and simplify your answer. (“ exact value” mean no decimals)

4. HIGHER-ORDER DERIVATIVES

【知识点】：

1. Higher order derivatives:

- If $y = f(x)$, then we can write its nth derivative :

$$y^{(n)} \ \ or \ \ f^{(n)}(x) \ \ or \ \ \frac{d^n y}{dx^n}$$

- 例如：

(1) First derivative: $y' = f'(x) = \frac{dy}{dx}$

(2) Second derivative: $y'' = f''(x) = \frac{d^2y}{dx^2}$

(3) Third derivative: $y''' = f'''(x) = \frac{d^3y}{dx^3}$

(4) Fourth derivative: $y^{(4)} = f^{(4)}(x) = \frac{d^4y}{dx^4}$

【past exam questions】

1. (2016 Fall final exam) Find $g''(1)$ if $g(x) = e^{x^2} + x^3$

2. (2015 Fall final exam) let $f(x) = 2(5^x)$. Calculate $f^{(2)}(1)$

3. (2014 Fall final exam) if $y = 2(5^x)$ and n is a positive integer, then $y^{(n)}(1)$ equals

A. $10(ln5)^{-n}$ B. $(2ln5)5n$ C. $(10ln5)n$ D. $10\ln(5^n)$ E. $10(ln5)^n$

4. (2017 Fall final exam) if $y = 4(2^x)$ and n is a positive integer, then $y^{(n)}(1)$ equals

5. (2020 Fall final exam) if $f(x) = 4(3^x)$ and $n \geq 2$ is a positive integer, then $f^{(n)}(1)$ equals
A. $12(ln3)^n$ B. $12(\ln(3^n))$ C. $4n3^n$ D. $(12ln3)^n$ E. None of the above

6. (2014 Fall final exam)Find the cubic polynomial $y = Q(x) = ax^3 + bx^2 + cx + d$ having all these properties: $Q(1) = 31$, $Q'(1) = 7$, $Q''(1) = 24$, and $Q'''(1) = 24$

5. NEWTON'S METHOD

【知识点】

1. **Newton's method :** make an approximation to root of $f(x)$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- 有些式子我们通过解方程也解不出来的时候，我们就需要去估算"approximation"

做题步骤

(1) 找到函数表达式，求导 derivative :

(2) 写出公式表达式： $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

(3) 把题目给的初始估算值 initial estimate 带入

(4) 当看见答案相似的时候，就可以得出答案了 或者 根据题目要求算出答案

【past exam questions】

1. (2016 fall final exam) assume newton's method is used to approximate the number $\sqrt[4]{36}$. If $x_1 = 2$, calculate x_2 as a simplified function.

2. (2017 fall final exam) Let $x_1 = 2$ and select the appropriate function for Newton's method to approximate the number $\sqrt[3]{10}$, Calculate x_2.

3. (2015 fall final exam) assume newton's method is used to approximate the number $\sqrt{20}$. If $x_1 = 4$, calculate x_2

4. (2020 fall final exam) A number $c \in (3,4)$ has the special property that "its square is 7 more than the number". If we apply Newton's method to the most appropriate quadratic function that has c as a root and take $x_1 = 3.5$ as a first estimate of the number c, what would x_2 be (rounded down to 3 decimal places) ?
A. 2.974 B. 3.112 C.3.537 D. 3.208 E. none of the above

5. (2021 fall final exam) When you apply Newton's method to the equation $x^3 - 2x - 5 = 0$ with the starting value of $x_1 = 2$, find the value of x_3 rounded up to three decimal places

6.求导的应用公式 A RATE OF CHANGE

记住：(1) the rate of change = derivative $f'(x)$　　**(2) the relative rate of change = $\frac{f'(x)}{f(x)}$**

【知识点】关于 rate of change to economics

1. **Total cost function: $c = c(q)$**
   - c 代表：产生某种产品所产生的费用 total cost
   - q 代表：quantity（数量/产量）

2. **average cost（$\bar{c}$）:**

$$\bar{c} = \frac{c}{q} = \frac{total\ cost}{quantity}$$

3. **Marginal cost（边际成本）**

- 指的是 将要生产的下一 unit 单位产品 或者 额外生产 1 unit 单位的成本
- 两个公式:
  (1)　$c' = c'(q) = \frac{dc}{dq}$
  (2)　$c' = c'(q) \approx c(q+1) - c(q)$

4. **Total Revenue (r)** 收入:

$$r = pq$$

   - p 代表：price per unit　　q 代表 quantity

(1) average revenue ($\bar{r}$) 平均收入:

$$\bar{r} = \frac{r}{q} = \frac{total\ revenue}{quantity}$$

(2) marginal revenue(r') 边际收益:

$$r' = \frac{dr}{dq}$$

5. **profit function** 利润方程:

$$profit = revenue - cost$$

- **marginal profit** = derivative of profit

6. **Demand function** 需求方程

$$p = f(q)$$

   - Where p is unit price(单价) and q is quantity(数量)

【知识点 2】

1. **Point elasticity of demand (η) 点弹性**

- 公式：if $p = f(q)$ is differentiable demand function, then the point elasticity of demand is :

$$\eta = \eta(q) = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{f(q)}{q}}{f'(q)}$$

- 注意：η 读做"eta", 它本身是**负数 negative**
- **三种 elasticity 种类**

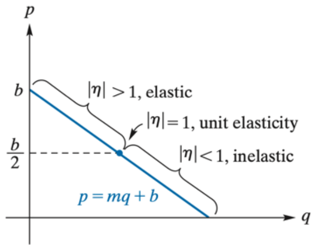


FIGURE 12.3 Elasticity for linear demand.

种类	特点
When $\|\eta\| > 1$	demand is **elastic** 需求富有弹性/高弹性
When $\|\eta\| = 1$	demand has **unit elasticity** 需求单一弹性
When $\|\eta\| < 1$	demand is **inelastic.** 需求缺乏弹性/低弹性

【公式小总结】

(1) If demand function is $p = f(q)$

$\eta = \frac{p}{q} \cdot \frac{1}{p'}$ where $p' = \frac{dp}{dq}$

- 注意：p'是 demand function $f'(q)$的导数。即：题目给了 p =________的方程
- 例如： $p = 10 - 0.04q$

(2) If demand function is $q = f(p)$

$\eta = \frac{p}{q} \cdot q'$ where $q' = \frac{dq}{dp}$

- 注意：q'是 demand function $f'(p)$的导数。即：题目给了q =______的方程
- 例如：$q = 1200 - 150p$

2. **关于 marginal revenue（r'）边际收益**

- 两种情况：

(1) If demand function is $p = f(q)$,

$$MR = r' = \frac{dr}{dq} = p'q + p$$

$$MR = r' = \frac{dr}{dq} = p(1 + \frac{1}{\eta})$$

(2) If demand function is $q = f(p)$,

$$MR = r' = \frac{dr}{dp} = q + pq'$$

$$MR = r' = \frac{dr}{dp} = q(1 + \eta)$$

【past exam questions】

1. (2017 winter final exam) let $p = \sqrt{1600 - q^2}$ be a demand function for $0 \leq q \leq 40$

(a) Show that the (point) elasticity of demand is $\eta = -\frac{p^2}{q^2}$

(b) For $q = 15$, determine if the (point) elasticity of demand is elastic or inelastic.

(c) Find the value(s) of q for which the (point) elasticity of demand is unit.

2. (2017 fall midterm) find the values of q for which the demand equation $p = 120 - 0.4q^2$ has unit elasticity

3. (2015 fall final exam) Assume $p = f(q) = (q+3)e^{-q}$ is a demand function where $q > 0$ is quantity. Find all values of q such that the (point) elasticity of demand is elastic.

4. (2014 fall final exam)if $p = -3q + 150$ is a demand function where the quantity q satisfies $0 < q < 50$, then we have unit elasticity at

A. $q = 48$ B. $q = 30$ C. $q = 25$ D. $q = 5$ E.none of the above

5. (2012 fall final exam) if $p = mq + b$ is a demand function where $m < 0 < b$, q is quantity and $0 < q < -\frac{b}{m}$ then we have unit elasticity when q equals

A. $-\frac{b}{3m}$ B. $-\frac{b}{2m}$ C. $\frac{b}{2m}$ D. no value of q E. none of the above

6. (2021 fall final exam)If a demand equation is $p = \frac{600}{3q+1}$ where $q > 0$ is quantity, For $q = 10$, determine if the (point) elasticity of demand is elastic or inelastic.

7. (2016 final exam) In all of this question let $p = 1200 - q^2$ be a demand function.
a) Find the point elasticity of demand.
b) Find the value of q for which the demand has unit elasticity.
c) Find the marginal revenue when $q = 10$.