Solutions for the 4th Morning

Exercise 0

1. $a(x, y) = \min\{1, \exp(-\frac{1}{2}(y^2 - x^2))\}$
2. Notice that the that the proposal is normally distributed and that $a(x, y) > 0$ for all x, y. Accordingly, for any subset $A \subset \mathbb{R}$ with $\Pi(A) > 0$ there is a positive probability that the proposal will be in A. Furthermore, there is a positive probability that this proposal will be accepted, i.e. $P(x, A) > 0$. Accordingly the Markov chain is irreducible.
   For all x there is a positive probability of proposing y so that $a(x, y) < 1$. I.e. there is positive probability that $x^{(t)} = x^{(t-1)}$ for all values of $x^{(t-1)}$. Hence, the Markov chain cannot be periodic.
3. The probability that $u < H(x, y)$ is exactly $\min\{1, H(x, y)\}$.
4. Example of `R` implementation

```
sigma = 2
x = 0 ## initial value
for(i in 2:10000)
    y = rnorm(1, x[i-1], sigma)
    u = runif(1)
    H = dnorm(y)/dnorm(x[i-1])
    if(u<H) x[i]=y else x[i]=x[i-1]

plot(x,type="s")
hist(x, freq=FALSE, breaks=50)
curve(dnorm(x), -4, 4, add=TRUE) ## add normal pdf to histogram
```

5. The estimate is obtain by `mean(x<=-1)`
6. The proposal is independent of the current state x. Furthermore, the proposal is the same at the target density $\pi(x)$. The acceptance probability is in this case $a(x, y) = 1$.

Exercise 1

1. If $P(X^{(t+1)} \in A) \sim \pi(x)$ it must hold that $P(X^{(t+1)} \in A) = \Pi(A)$:

$$\begin{aligned} P(X^{(t+1)} \in A) &= \int_R P(X^{(t+1)} \in A | X^{(t)} = x)\pi(x)dx \\ &= \int_R P(x, A)p(x)dx = \Pi(A) \end{aligned}$$

2. The answer follows by induction: From question 1 we have that, if $X^{(t)} \sim \pi(x)$ then $X^{(t+1)} \sim \pi(x)$, this in turn implies that $X^{(t+2)} \sim \pi(x)$ etc.

Exercise 2

1. First notice that, since $x^{(0)} \sim \pi(x)$ we have that $x^{(t)} \sim \pi(x)$ for all $t \in \mathbf{N}$. Hence $\mathbf{E}[h(X^{(t)})] = \mu$ for all $t \geq 0$. Using the usual rules for expectation we obtain

$$\mathbf{E}[\hat{\mu}_n] = \mathbf{E}\left[\frac{1}{n+1}\sum_{t=m}^{m+n} h(X^{(t)})\right] = \frac{1}{n+1}\sum_{t=m}^{m+n} \mathbf{E}[h(X^{(t)})] = \frac{1}{n+1}(n+1)\mu = \mu.$$

Exercise 3

1. The proposal is uniformly distributed in the interval $[-x-\delta; -x+\delta]$. That is, if $0.5 \leq x \leq 1.5$ (x is inside the "positive box") then the interval $[-x-\delta; -x+\delta]$ intersects the "negative box" ($-1.5 \leq x \leq -0.5$).
2. Assuming that x is inside a "box", i.e. that $\pi(x) > 0$, then the acceptance probability is $\mathbb{1}[|y+1| \leq \frac{1}{2}] + \mathbb{1}[|y-1| \leq \frac{1}{2}]$. This implies that any proposal inside any of the two boxes will be accepted.
3. The Markov chain is irreducible for all $\delta > 0$.
4. If x is sufficient close to the sides of the two boxes there is a positive probability that the proposal y will be outside the two boxes and hence rejected. This implies that the Markov chain cannot be periodic.

Exercise 4

1. This Markov chain alternates between 1 and 0.
2. Consider the definition for invariant distribution in the case where $y = 0$:

$$\begin{aligned} \pi(0)P(0, \{0\}) + \pi(1)P(1, \{0\}) &= \pi(0) \\ \pi(0) \cdot 0 + \pi(1) \cdot 1 &= \pi(0) \\ \pi(1) &= \pi(0) \end{aligned}$$

   As we require $\pi(0) + \pi(1) = 1$ the only solution is $\pi(0) = \pi(1) = \frac{1}{2}$.
3. We have

$$P(X^{(t)} = 0 | X^{(0)} = 0) = \begin{cases} 1 & \text{if } t = 2i \text{ for some } i \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

   Clearly the Markov chain does not converge. As π is an invariant distribution, and the Markov chain is irreducible we still have that the law of large numbers holds. Hence, the Markov chain can be used for estimating expectations.