Intro

Jonathan Navarrete

April 15, 2017

Introduction to Monte Carlo

Monte Carlo methods are methods for generating random variables directly or indirectly from target distributions. We generate random variables to estimate p-values or parameters.

Applications of monte carlo methods are in hypothesis testing and Bayesian computation.

An example of simulation: Gambler's ruin

Consider two gamblers, persons A and B, who start to gamble in a zero-sum game with stakes \$x and \$b-x, respectively. At each round, each gambler puts up a stake of \$h. The probability that A wins a round is p, while the probability that B wins a round is $q = 1 - p$. We wish to compute the probability that A ultimately wins the game. Let us define $v(x, t)$ to the probability that A ultimately wins the game starting with capital \$x on or before the tth round. Similarly, $u(x, t)$ is the probability that B wins the game with their stake of $b - x$ on or before the tth round.

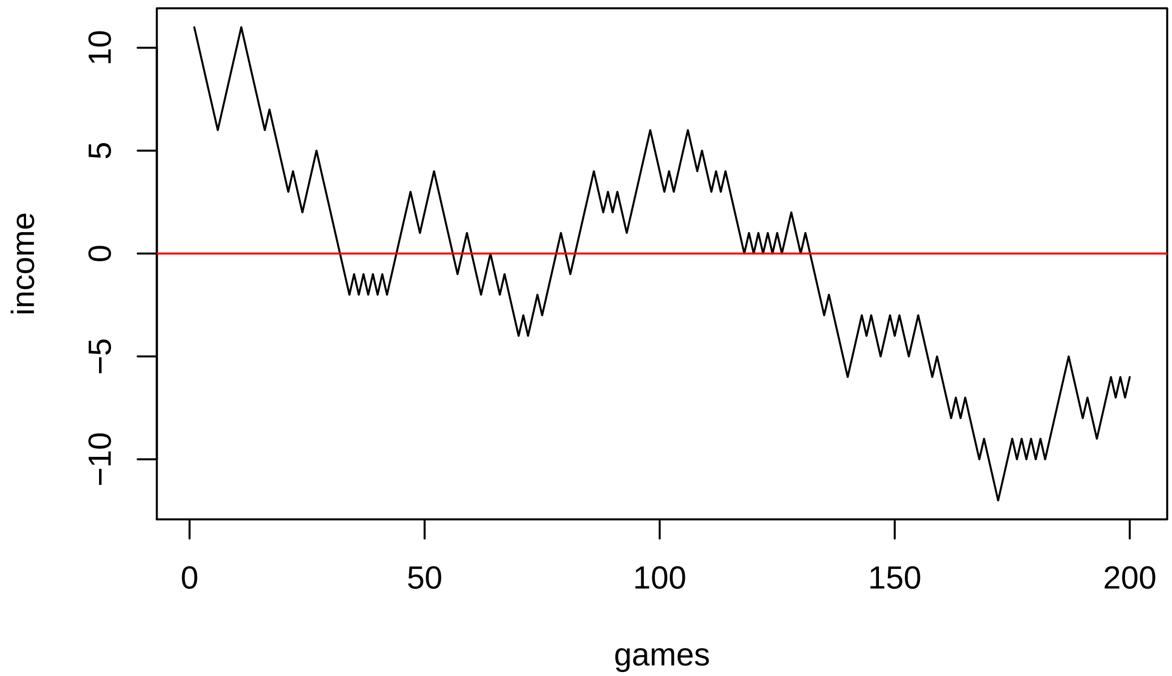
Each of three variables v,u, and w is bounded below by zero and above by 1. Moreover, u and v are nondecreasing in t. w is nonincreasing in t. Thus we can take limits of each of these as t goes to infinity. We shall call these limits v(x), u(x), and w(x), respectively.

Gambler's ruin (fallacy) is the belief that a certain event is *more* likely to occur given the past history. In an experiment where there is a coin toss with probability of seeing heads as 0.5. Each flip of a coin has the same probability of landing on heads regardless of what the previous lands were.

Imagine a gambler on a roulete table. Say the gambler starts with \$10. In this game, the gambler "wins" when they earn a total of \$20 (that is they must play the game until they've earned \$10 on top of their starting \$10). For each game, there is a probability of winning, p = 0.473. Then, can we see how many turns until he/she wins (or loses)?

```r
set.seed(678)
N = 200
income = 10
games = 2*(runif(N)<0.473) - 1 ## generate 1s and -1s
out = cumsum(games) + income

plot(1:N, out, type = "l", xlab = "games", ylab = "income")
abline(h = 0, col = "red")
```




```
GamblersRuin = function(i){
  income = 10
  n = 0
  while(!(income %in% c(0,20))){
    n = n + 1 ## number of runs till ruin or success
    x = runif(1)
    if(x <= 0.473){
      income = income + 1
    } else{
      income = income - 1
    }
  }
  return(c(n,income))
}

GamblersRuin()
```

```
## [1] 154   0
```

```
out = lapply(X = 1:100, FUN = GamblersRuin)
out = do.call(rbind, out)

## percentage of success
sum(out[,2] == 20 )
```

```
## [1] 27
```

Hypothesis Testing

There are two ways that the Chi-squared test is used:

1. comparing the observed distribution to some theoretical distribution pre-specified ahead of time: to test the *Goodness of fit* of the theoretical distribution to the observations;
2. testing for *independence* between different factors (which, technically, is just a specific theoretical distribution, with some extra parameters that must be estimated from the data).

To review the Chi-squared test, follow the *link*

Data	Cancer Controlled	Cancer not Controlled	Total
Surgery	21	2	23
Radiation	15	3	18
Total	36	5	41

However, a disadvantage of the chi-square test is that it requires a sufficient sample size in order for the chi-square approximation to be valid. When cell counts are low, say, below 5 asymptotic properties do not hold well. Therefore, a simple chi-squred test may report an invalid p-value which would increase a **Type I error** rate. A solution is to use Monte Carlo simulation to generate samples from the null distribution in order to estimate a more accurate p-value to our hypothesis.

```
study = matrix(data = c(21, 2,
                        15, 3), nrow = 2, ncol = 2, byrow = TRUE,
               dimnames = list(c("surgery", "radiation"),
                               c("controlled", "not controlled")))

print(study)
```

```
##           controlled not controlled
## surgery           21              2
## radiation         15              3
```

Set up some functions in order to generate our Chi-squared statistic and Monte Carlo p-value.

```
## set up

## function will generate chi-squared statistics
## using the expected distribution of the data
simulateChisq <- function(B, E, sr, sc){
    results = numeric(B)
    for(i in 1:B){
        dat = unlist(r2dtable(1, sr, sc))
        M = matrix(dat, ncol = length(sc), nrow = length(sr))
        val = sum( sort( (M - E)^2 / E, decreasing = TRUE))
        results[i] = val
    }
    return(results)
}

## this will produce chi-squared test
ChisqTest <- function(data, Simulations){
    ## data should be a 2X2 matrix
    x = data
    B = Simulations
    n <- sum(x)
    sr <- rowSums(x)
```

```r
    sc <- colSums(x)
    E <- outer(sr, sc, "*")/n ## ORDER MATTERS
    dimnames(E) <- dimnames(study)
    tmp <- simulateChisq(B, E, sr, sc)
    Stat <- sum(sort((x - E)^2/E, decreasing = TRUE))
    pval <- (1 + sum(tmp >=  Stat))/(B + 1)
    df = 2 ## only option for this example
    rawPVal = pchisq(q = Stat, df = df, lower.tail = FALSE)
    out = list(PearsonStat = Stat, MonteCarloPVal = pval, rawPVal = rawPVal)
    return(out)
}
```

We then generate our test statistics.

```r
set.seed(123)

results <- ChisqTest(study, 10000)

print(results)
```

```
## $PearsonStat
## [1] 0.5991546
##
## $MonteCarloPVal
## [1] 0.6417358
##
## $rawPVal
## [1] 0.7411314
```

```r
## compare against chisq.test()
```

Though our ultimate decision to support the null hypothesis of dependence is not a surprise, our results show that the Monte Carlo p-value is greater than the raw p-value obtained from the calculated χ^2 statistic indicating more support for the null hypothesis. Readers should compare these results against R's `chisq.test` function.

Bayesian Example

Here is an example taken from *Bayesian Ideas and Data Analysis* by Christensen et al.

$$y|\theta \sim Bin(2430, \theta) \; and \; \theta \sim Beta(12.05, 116.06)$$

This is a beta-binomial problem. There is a beta prior distribution on θ. Beta is conjugate to the binomial distribution (see: https://en.wikipedia.org/wiki/Conjugate_prior#Discrete_distributions). Bayesian anaysis uses prior information combined with observed data to update a probability distribution, posterior distribution, from which we can obtain a probability value. The new probability distribution, posterior, describes knowledge about the unkown parameter θ from historical beliefs (e.g. previous experiments, reports, etc.) and current observed data.

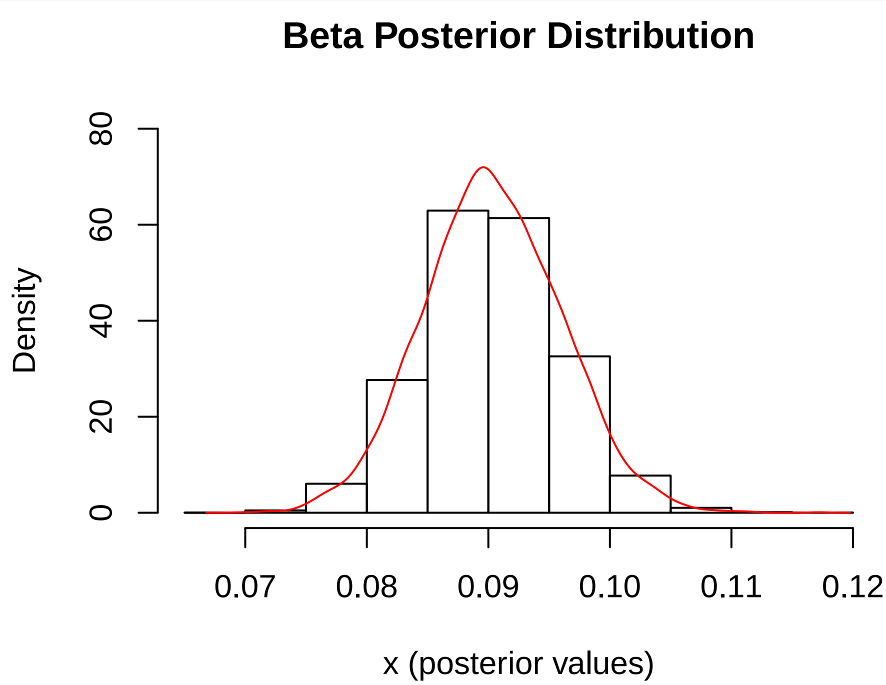
$$y|\theta \sim Bin(n, \theta) \; and \; \theta \sim Beta(a, b)$$

The resulting posterior distribution is then

$$\theta|y \sim Beta(y + a, n - y + b)$$

We can now simulate the posterior distribution

```
N = 10^4
set.seed(123)
x = rbeta(n = N, shape1 = 219 + 12.05, shape2 = 2430 - 219 + 116.06)
d = density(x)
hist(x = x, probability = TRUE,
      main = "Beta Posterior Distribution",
     xlab = "x (posterior values)", ylab = "Density",
     ylim = c(0,80))
lines(x = d$x , y = d$y, type = "l", col = 2)
```




```
print("Median: ")
```

```
## [1] "Median: "
```

```r
print(quantile(x = x, probs = c(0.025, 0.5, 0.975)))
```

```
##       2.5%        50%      97.5%
## 0.07942339 0.09018049 0.10164574
```

Conclusion

We can tell the VP that the true probability lies between 7.9% and 10.2%, with median probability of 9%.