Stat 591 – Homework 05 Due: Friday 11/15

Your group should submit a write-up that includes solutions the problems stated below, along with any <u>*relevant*</u> *pictures/graphs or computer code/output.*

1. (a) Problem 7.3 in [GDS].
   (b) Problem 7.4 in [GDS].
   (c) Explain the advantage of the Box–Muller method compared to the inverse CDF method for simulating from $\mathsf{N}(0, 1)$.

2. (Based on Problems 7.5 and 7.9 in [GDS].) The goal to simulate $X \sim f(x)$. Suppose that $f(x)$ is a non-standard density, but that there exists a density $g(x)$ that is easy to sample from and satisfies $f(x) \leq Mg(x)$ for some $M \geq 1$. Then the accept–reject method (closely related to Metropolis–Hastings) is as follows:

   STEP 1. Sample $Y \sim g$ and $U \sim \mathsf{Unif}(0, 1)$, independent.
   STEP 2. If $U \leq f(Y)/Mg(Y)$, then accept Y as a sample $X \sim f(x)$; otherwise, reject Y and return to Step 1.

   (a) Show that the output X of the accept–reject method has distribution $f(x)$. The distribution in question is that of Y given that it's accepted, i.e., given that $U \leq f(Y)/Mg(Y)$.
   (b) Find the acceptance probability $\mathsf{P}(U \leq f(Y)/Mg(Y))$. The algorithm is most efficient if the acceptance probability equals 1. Under what conditions can the acceptance probability equal 1? Can this be achieved?
   (c) Consider sampling from $\mathsf{Gamma}(\theta, 1)$. If θ is an integer, then this can be done by sampling θ $\mathsf{Exp}(1)$ random variables and summing them. If θ is not an integer, then it's more difficult. Develop and implement an accept–reject method for simulating from a $\mathsf{Gamma}(\theta, 1)$ when θ is ≥ 2 and a non-integer. Your method cannot use `rgamma` or `qgamma`. Simulate 1000 values from $\mathsf{Gamma}(5.5, 1)$ using your method and draw a histogram with the gamma density overlaid. How is the fit? What is your acceptance rate?

      Hint. For non-integer θ, a gamma proposal with shape $[\theta]$, the integer part, would be OK, especially if θ is large. You can improve on this with proposal $\mathsf{Gamma}([\theta], b)$ where b is chosen so that the mean is θ.

3. Suppose data $X_1, \ldots, X_n$ are iid from a Student-t distribution, with known degrees of freedom ν, and unknown location $\theta \in (-\infty, \infty)$. The pdf for X_1 is

$$f_\theta(x) \propto \Big(1 + \frac{(x-\theta)^2}{\nu}\Big)^{-(\nu+1)/2}, \quad x \in (-\infty, \infty).$$

   This is a location parameter problem, so the invariant prior distribution is a flat prior for θ, with density $\pi(\theta) \propto 1$. If $L_n(\theta)$ is the likelihood function, then the posterior mean

$$\tilde{\theta}_n(X) := \int \theta \pi(\theta \mid X)\, d\theta = \frac{\int \theta L_n(\theta)\, d\theta}{\int L_n(\theta)\, d\theta}$$

is the *Pitman estimator*, and is the "best equivariant estimator" of θ in a decision theoretic sense. Develop and implement an importance sampling strategy to evaluate the posterior mean/Pitman estimator, and test it on a simulated sample of size $n = 50$ with $\nu = 5$ and $\theta = 7$.

Hint. The n is relatively large, so the posterior for θ ought to be close to normal. Use a fatter-tailed version of this normal distribution as your proposal/importance density. Also, you'll need a numerical optimization procedure (e.g., `nlm` in R) to find the MLE and observed information for the normal approximation.

4. Problem 7.11 in [GDS]; you can use my Metropolis–Hastings code. Recall that the exponential distribution is a special case of Weibull, i.e., when $\alpha = 1$. Draw a histogram to visualize the (marginal) posterior distribution for α. Based on this plot, do you think an exponential model would give a reasonable fit for the given data? Explain.

5. The Gibbs sampling strategy given in Example 7.13 of [GDS] is for an extension of the usual one-way ANOVA model. The standard one-way ANOVA model looks the same as in Example 7.13 except that the errors, ε_{ij}, are iid normal with *common* variance σ^2. Derive a Gibbs sampler for this simpler one-way ANOVA model using the same priors as in Example 7.13. The details are similar to those in the text. but the problem is a bit simpler now since there is only one variance.

   Next, consider the following simulated data $Y = (y_{ij})$:

Treatment, i	Replication, j					
1	6.58	6.54	0.61	7.69	2.18	3.84
2	2.48	3.89	2.11	2.46	5.93	5.65
3	1.32	3.27	6.90	5.65	1.81	2.79
4	3.53	3.11	5.58	7.80	6.33	4.72
5	7.01	3.96	4.60	5.47	6.29	1.97

   Implement your Gibbs sampler and simulate from the (marginal) posterior distribution of σ_π^2 for the given data. Plot a histogram of this posterior sample. Does this picture give you any indication of whether there is a significant treatment effect? Explain.