Introduction: Monte Carlo Methods

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Introduction to Monte Carlo

Monte Carlo methods are methods for generating random variables directly or indirectly from target distributions. We generate random variables to estimate p-values or parameters.

Applications of monte carlo methods are in hypothesis testing and Bayesian computation.

Simulation: Gambler's ruin

Consider two gamblers, persons A and B, who start to gamble in a zero-sum game with stakes \$x and \$b-x, respectively. At each round, each gambler puts up a stake of \$h. The probability that A wins a round is p, while the probability that B wins a round is $q = 1$ - p. We wish to compute the probability that A ultimately wins the game. Let us define $v(x, t)$ to the probability that A ultimately wins the game starting with capital \$x on or before the tth round. Similarly, $u(x, t)$ is the probability that B wins the game with their stake of $b - x$ on or before the tth round.

Each of three variables v,u, and w is bounded below by zero and above by 1. Moreover, u and v are nondecreasing in t. w is nonincreasing in t. Thus we can take limits of each of these as t goes to infinity. We shall call these limits v(x), u(x), and w(x), respectively.

Gambler's ruin, pt.2

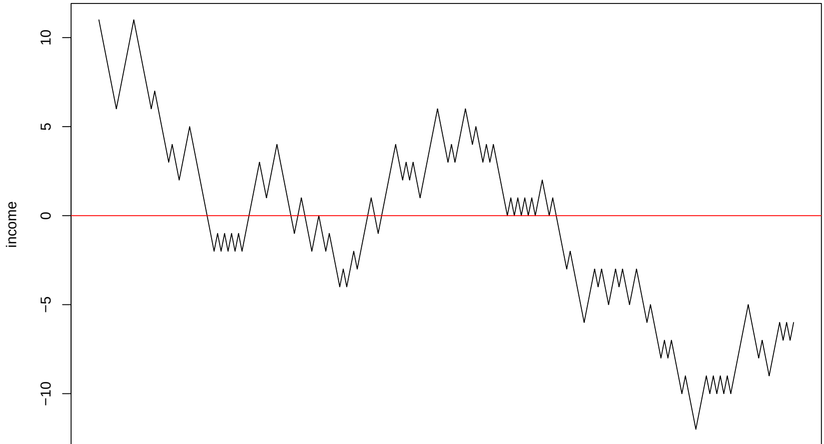
Gambler's ruin (fallacy) is the belief that a certain event is *more* likely to occur given the past history. In an experiment where there is a coin toss with probability of seeing heads as 0.5. Each flip of a coin has the same probability of landing on heads regardless of what the previous lands were.

Imagine a gambler on a roulete table. Say the gambler starts with \$10. In this game, the gambler "wins" when they earn a total of \$20 (that is they must play the game until they've earned \$10 on top of their starting \$10). For each game, there is a probability of winning, $p = 0.473$. Then, can we see how many turns until he/she wins (or loses)?

Gambler's ruin, pt.3

```
set.seed(678)
N = 200
income = 10
games = 2*(runif(N)<0.473) - 1 ## generate 1s and -1s
out = cumsum(games) + income

plot(1:N, out, type = "l", xlab = "games", ylab = "income")
abline(h = 0, col = "red")
```



Monte Carlo Hypothesis Testing

There are two ways that the Chi-squared test is used:

1. comparing the observed distribution to some theoretical distribution pre-specified ahead of time: to test the *Goodness of fit* of the theoretical distribution to the observations;
2. testing for *independence* between different factors (which, technically, is just a specific theoretical distribution, with some extra parameters that must be estimated from the data).

To review the Chi-squared test, follow the *link*

Monte Carlo Hypothesis Testing, pt. 2

However, a disadvantage of the chi-square test is that it requires a sufficient sample size in order for the chi-square approximation to be valid. When cell counts are low, say, below 5 asymptotic properties do not hold well. Therefore, a simple chi-squred test may report an invalid p-value which would increase a Type I error rate.

Monte Carlo Hypothesis Testing, pt. 3

A solution is to use Monte Carlo simulation to generate samples from the null distribution in order to estimate a more accurate p-value to our hypothesis.

```
study = matrix(data = c(21, 2,
                        15, 3), nrow = 2, ncol = 2, byrow = TRUE,
               dimnames = list(c("surgery", "radiation"),
                               c("controlled", "not controlled")))

print(study)
```

```
##           controlled not controlled
## surgery           21              2
## radiation         15              3
```

Set up some functions in order to generate our Chi-squared statistic and Monte Carlo p-value.

Monte Carlo Hypothesis Testing, pt. 4

```
## set up

## function will generate chi-squared statistics
## using the expected distribution of the data
simulateChisq <- function(B, E, sr, sc){
    results = numeric(B)
    for(i in 1:B){
      ## review r2dtable documentation
        dat = unlist(r2dtable(1, sr, sc))
        M = matrix(dat, ncol = length(sc), nrow = length(sr))
        val = sum( sort( (M - E)^2 / E, decreasing = TRUE))
        results[i] = val
    }
    return(results)
}
```

Monte Carlo Hypothesis Testing, pt. 5

```
ChisqTest <- function(data, Simulations){
    x = data
    B = Simulations
    n <- sum(x)
    sr <- rowSums(x)
    sc <- colSums(x)
    E <- outer(sr, sc, "*")/n ## ORDER MATTERS
    dimnames(E) <- dimnames(study)
    tmp <- simulateChisq(B, E, sr, sc)
    Stat <- sum(sort((x - E)^2/E, decreasing = TRUE))
    pval <- (1 + sum(tmp >=  Stat))/(B + 1)
    rawPVal = pchisq(q = Stat, df = 2, lower.tail = FALSE)
    out = list(PearsonStat = Stat,
               MonteCarloPVal = pval,
               rawPVal = rawPVal)
    return(out)
}
```

Monte Carlo Hypothesis Testing, pt. 6

We then generate our test statistics.

```
set.seed(123)

results <- ChisqTest(study, 10000)

print(results)
```

```
## $PearsonStat
## [1] 0.5991546
##
## $MonteCarloPVal
## [1] 0.6417358
##
## $rawPVal
## [1] 0.7411314
```

```
## compare against chisq.test()
```

Though our ultimate decision to support the null hypothesis of dependence is not a surprise, our results show that the Monte Carlo p-value is greater than the raw p-value obtained from the calculated χ^2 statistic indicating more support for the null hypothesis. Readers should compare these results against R's `chisq.test` function.

Bayesian Example

Here is an example taken from *Bayesian Ideas and Data Analysis* by Christensen et al.

$$y|\theta \sim Bin(2430, \theta) \text{ and } \theta \sim Beta(12.05, 116.06)$$

This is a beta-binomial problem. There is a beta prior distribution on θ. Beta is conjugate to the binomial distribution, see: Conjugate priors. Bayesian anaysis uses prior information combined with observed data to update a probability distribution, posterior distribution, from which we can obtain a probability value. The new probability distribution, posterior, describes knowledge about the unkown parameter θ from historical beliefs (e.g. previous experiments, reports, etc.) and current observed data.

Bayesian Example, pt. 2

The Bayesian data model is then

$$y|\theta \sim Bin(n, \theta) \text{ and } \theta \sim Beta(a, b)$$

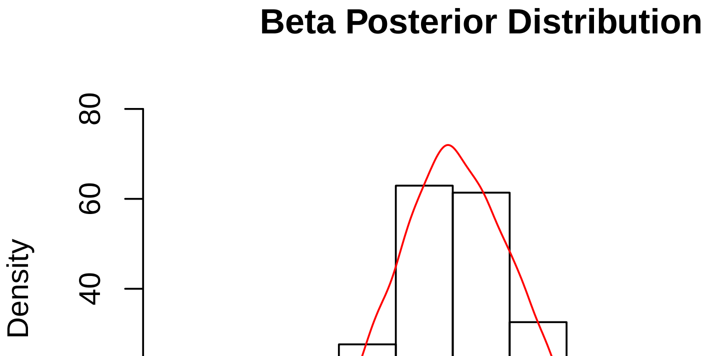
The resulting posterior distribution is then

$$\theta|y \sim Beta(y + a, n - y + b)$$

Bayesian Example, pt. 3

We can now simulate the posterior distribution

```
N = 10^4
set.seed(123)
x = rbeta(n = N, shape1 = 219 + 12.05, shape2 = 2430 - 219 + 116.06)
d = density(x)
hist(x = x, probability = TRUE,
      main = "Beta Posterior Distribution",
     xlab = "x (posterior values)", ylab = "Density",
     ylim = c(0,80))
lines(x = d$x , y = d$y, type = "l", col = 2)
```



Bayesian Example, pt. 4

We can tell the VP that the true probability lies between 7.9% and 10.2%, with median probability of 9%.