Example problems

Problem 1: Consider the infinite series

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\varepsilon)$$

This is a *Fourier series* and converges to ε whenever $\varepsilon \in [0,\pi)$. Let

$$S_k = 2\sum_{n=1}^{k}\frac{(-1)^{n+1}}{n}\sin(n\varepsilon)$$

denote the k'th partial sum. Investigate how large k must be to ensure that $|S_{k-1}-\varepsilon| \leq 10^{-3}$ **and** $|S_k-\varepsilon| \leq 10^{-3}$. That is, the partial sum is within 10^{-3} of ε for **two consecutive terms**.

Denote the resulting value of k by $k_\varepsilon^\star$. Determine $k_\varepsilon^\star$ for $\varepsilon = .05, .1, .15, ..., 1$, and plot $k_\varepsilon^\star$ vs. ε. (Note $k_0^\star = 2$).

Problem 2a: The Laplace distribution with parameter $\theta > 0$ has density

$$p(x) = \frac{\theta e^{-\theta|x|}}{2}$$

We are interested in estimation of θ. Write a program to generate from this distribution, given θ and the sample size. Use this program to estimate

$$E\Big(|X - \text{median}(X)|\Big)$$

as a function of θ, where X has the Laplace distribution with parameter θ. Plot this expectation for $\theta = .5, .6, .7, ..., 10$. On the same axis plot the function $f(\theta) = 1/\theta$, and **deduce an estimator of** θ. Use a sample size of 1000 in each simulation.

Hint: If V is Exponential(θ), and $B = -1, 1$ each with probability $1/2$, then $V \cdot B$ has the a double exponential density with parameter θ.

Problem 2b: Now consider Bayesian estimation of θ. Let θ have a prior distribution that is gamma with $\alpha = 50$, and $\beta = 10$. That is,

$$p(\theta) \propto \theta^{49}e^{-10\theta}$$

Derive an expression proportional to the posterior density, and estimate the posterior mean

$$\int \theta p(\theta|x_1, x_2, ..., x_n)d\theta$$

where $x_1, ..., x_n$ is the observed data. Use importance sampling with $N = 100000$ monte carlo samples to estimate the above expectation using a trial density that is the exponential distribution with rate equal to to $E(|X - \text{median}(X)|)$.

Note: To do this problem you only need the point estimate of $E(\theta|x_1, ..., x_n)$. The ESS for this trial density should be small enough and I am asking you to use a monte carlo sample size large enough so that the error should be negligible, so you need not estimate the standard error (unless you want to!).

Problem 2c: When does the Bayesian estimator outperform the estimator from part a? To investigate this:

1. Specify a value of θ
2. For each value of generate k data sets of sample size n and calculate the estimator from part a, $\hat{\theta}_a$, and the estimator from part b, $\hat{\theta}_b$, for each data set.
3. Calculate the the value $\hat{\theta}_a$ and $\hat{\theta}_b$ for each of the k data sets
4. Save the sample mean from each of the quantities calculated in 3

Intuitively when the true value of θ is closer to the mean of the prior, the bayesian estimator does better. When I ran this simulation I found that for $k = 200$, $n = 50$, when $\theta = 2$

- $E(\hat{\theta}_a) \approx 2.09$
- $E(\hat{\theta}_b) \approx 2.92$

When I ran this simulation I found that for $k = 200$, $n = 50$, when $\theta = 5$

- $E(\hat{\theta}_a) \approx 5.22$
- $E(\hat{\theta}_b) \approx 5.06$

When I ran this simulation I found that for $k = 200$, $n = 50$, when $\theta = 8$

- $E(\hat{\theta}_a) \approx 8.31$
- $E(\hat{\theta}_b) \approx 6.99$

So it appears that the when $\theta = 5$, the bayesian estimator performs better, and is worse when θ is far from 5, which is no surprise. When θ is not 5, the bayes estimator shrinks the estimate toward 5.

How does the amount of shrinkage toward 5 change when you change n?