1 Problem 1

Consider the target density

$$f(x) = \frac{1}{1 + \pi^2(x-1)^2}, \quad x \in R$$

1. Write a Random walk Metropolis-Hastings algorithm to sample from this density.
   - Initialize the chain at $x^{(0)}$.
   - Propose a new value $Y_t = x^{(t)} + \epsilon_t$, where $\epsilon_t \sim N(0, s^2)$
   - The next value in the chain is

   $$x^{(t+1)} = \begin{cases} Y_t & w.p.\ min\left(1, \frac{1+\pi^2(Y_t-1)^2}{1+\pi^2(x^{(t)}-1)^2}\right) \\ x^{(t)} & otherwise \end{cases}$$

2. What about your algorithm in part a can be manipulated for tuning? Describe how you would tune the algorithm, stating clearly what the objective would be in tuning, how you would diagnose having achieved the objective, and what the consequences would be for a poorly tuned or un-tuned algorithm.

   The scale of the proposal, s is how we tune the algorithm. For a unidimensional target, we want the acceptance rate to be near .5. Decrease the scale if the acceptance rate is too low and increase it if it is too hight. This allows us to move around the distribution and have an acceptable level of autocorrelation. A poorly tuned algorithm would show a meandering history plot, autocorrelation a few lags out, and perhaps a lumpy density plot.

2 Problem 2

In Example 5.16, your book introduced the genetic linkage problem in connection with the EM algorithm. You dealt with a modification of this model for EM in HW 5 and revisited it in HW 7 where you applied a Gibbs Sampler to the model. The original article by Gelfand and Smith considers a slightly more complex situation. Assume there are n = 22 animals categorized as Y = (14, 1, 1, 1, 5) with cell probabilities $p = (\theta/4+1/8, \theta/4, \eta/4, \eta/4+3/8, (1-\theta-\eta)/2)$, so characterized by two unknown parameters θ and η whose sum is smaller than 1. In the augmented data model, we separate out the $\theta/4$ and 1/8 in the first category and also separate out the $\eta/4$ and 3/8 in the fourth category, so the model is $X = (X_1, \ldots, X_7) \sim multinomial(22; \theta/4, 1/8, \theta/4, \eta/4, 3/8, (1-\theta-\eta)/2$, so Y augmented through addition of X_1 and X_5.

1. Write out the full joint distribution of X, θ, and η when $Beta(1,1)$ prior distributions are used on θ and η subject to $\theta + \eta \leq 1$.

   $$f(x,\theta,\eta) \propto \binom{22}{x_1\ 14-x_1\ 1\ 1\ x_5\ 1-x_5\ 5}(\theta/4)^{x_1}(1/8)^{14-x_1}(\theta/4)^1(\eta/4)^1(\eta/4)^{x_5}(3/8)^(1-x_5)((1-\theta-\eta)/2)^5$$

2. Deduce the full conditional distributions for θ, η, X_1, and X_5. Note that if Z has a $beta(a, b)$ distribution right-truncated at c, then $f(x) \propto z^{a-1}(1-c-z)^{b-1}$.

   - $\theta|X, \eta \sim beta(x_1 + 2, 6) \;\; \theta < 1 - \eta$
   - $\eta|X, \theta \sim beta(x_5 + 2, 6) \;\; \eta < 1 - \theta$
   - $x_1|\theta, \eta \sim binomial(14, \frac{2\theta}{2\theta+1})$ (the probability of success is $(\theta/4)/(\theta/4+1/8)$)
   - $x_5|\theta, \eta \sim binomial(1, \frac{2\eta}{2\eta+3})$

3. Describe how to implement the Gibbs Sampler for this model Choose initial values for θ and η. Draw iteratively from

   - $x_1^{(t)}|\theta^{(t)}, \eta^{(t)} \sim binomial(14, \frac{2\theta^{(t)}}{2\theta^{(t)}+1})$
   - $x_5^{(t)}|\theta^{(t)}, \eta^{(t)} \sim binomial(1, \frac{2\eta^{(t)}}{2\eta^{(t)}+3})$
   - $\theta^{(t+1)}|X, \eta^{(t)} \sim beta(x_1^{(t)} + 2, 6) \;\; \theta < 1 - \eta^{(t)}$
   - $\eta^{(t+1)}|X, \theta^{(t+1)} \sim beta(x_5^{(t)} + 2, 6) \;\; \eta < 1 - \theta^{(t+1)}$

   Discard iterations before reaching stationarity

3 Problem 3

The objective is to maximize $h(x) = [cos(20x) + sin(50x)]^2$ on the interval (0,1). The following R code gives a simulated annealing algorithm for the optimization.

```
h=function(x){(cos(20*x)+sin(50*x))^2}
x=runif(1)
hval=hcur=h(x)
diff=iter=1
while(diff>epsilon)
  {
  prop=x[iter]+runif(1,-1,1)*scale
  if (              ){

    x=c(x,prop)
  }
  else{x=c(x,x[iter])}
  hcur=h(x[iter+1])
  hval=c(hval,hcur)
  diff=max(hval)-max(hval[1:(iter/2)])
  iter=iter+1
  }
```

1. Fill in the missing part of the code
2. Comment the code to indicate the structure of the algorithm.