Solutions for the 2nd Morning

1. Let $\underline{x} = (x_1, x_2, \dots, x_n)$ denote the vector of observations. The posterior for the mean is

$$\begin{aligned}
\pi(\mu|\underline{x}) &\propto \pi(\underline{x}|\mu)\pi(\mu) \\
&= \Big(\frac{\tau}{2\pi}\Big)^{\frac{n}{2}} \exp\Big(-\frac{1}{2}\tau\sum_{i=1}^{n}(x_i-\mu)^2\Big)\sqrt{\frac{\tau_0}{2\pi}}\exp\Big(-\frac{1}{2}\tau_0\sum_{i=1}^{n}(\mu-\mu_0)^2\Big) \\
&\propto \exp\Big(-\frac{1}{2}\tau\sum_{i=1}^{n}\mu^2 + \tau\mu\sum_{i=1}^{n}x_i - \frac{1}{2}\tau_0\mu^2 + \tau_u\mu_0\mu\Big) \\
&\propto \exp\Big(-\frac{1}{2}(\tau_0+n\tau)\mu^2 + (\tau\sum_{i=1}^{n}x_i + \tau_0\mu_0)\mu\Big) \\
&\propto \exp(-\frac{1}{2}\tau_1\mu^2 + \tau_1\mu_1\mu)
\end{aligned}$$

Comparing this to equation (2) we see that $\mu|\underline{x} \sim N(\mu_1, \tau_1)$, where

$$\tau_1 = \tau_0 + n\tau \quad \text{and} \quad \mu_1 = \frac{\tau_1\mu_1}{\tau_1} = \frac{\tau\sum_{i=1}^{n}x_i + \tau_0\mu_0}{\tau_0+n\tau} = \frac{\tau n\bar{x} + \tau_0\mu_0}{\tau_0+n\tau}$$

In posterior for the precision is

$$\begin{aligned}
\pi(\tau|\underline{x}) &\propto \pi(\underline{x}|\tau)\pi(\tau) \\
&= \Big(\frac{\tau}{2\pi}\Big)^{\frac{n}{2}} \exp\Big(-\frac{1}{2}\tau\sum_{i=1}^{n}(x_i-\mu)^2\Big)\frac{\tau^{\alpha-1}e^{\tau/\beta}}{\Gamma(\alpha)\beta^\alpha} \\
&\propto \tau^{\frac{n}{2}+\alpha-1}\exp\Big(-\tau\Big(\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2 + \frac{1}{\beta}\Big)\Big)
\end{aligned}$$

Comparing this to the density of a gamma distributed random variable we see that $\tau|\underline{x} \sim \text{Gamma}(\alpha_1, \beta_1)$, where

$$\alpha_1 = \frac{n}{2} + \alpha - 1 \quad \text{and} \quad \beta_1 = \frac{1}{\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2 + \frac{1}{\beta}}.$$

2. Assume a priori that $p \sim Be(\alpha, \beta)$. Then we need to solve

$$E[p] = \frac{\alpha}{\alpha+\beta} = \frac{1}{3} \quad \text{and} \quad V[p] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{32}.$$

From the first equation we obtain $\beta = 2\alpha$. Intersting this in the second equation and isolating α gives $\alpha = \frac{55}{27}$, which in turn implies that $\beta = \frac{110}{27}$. Observing $x = 8$ success in $n = 20$ trail. From Section 2.1 it follows that $p|x \sim Be(x+\alpha, n-x+\beta) = Be(8+\frac{55}{27}, 14+\frac{110}{27})$.

3. Observing x_1 successes in n_1 trail gives posteior $p|x_1 \sim Be(\alpha_1, \beta_1)$,where $\alpha_1 = x_1 + \alpha$ and $\beta_1 = n_1 - x_1 + \beta$. Now use this as our prior, and assume we observe a further x_2 successes in the next n_2 trails. The posterios is then $p|x_1, x_2 \sim Be(x_2+\alpha_1, n_2-x_2+\beta_1)$. Notice that $\alpha_1+\beta_1 = n_1+\alpha+\beta$. With these calcualtions in mind, we can, in some sense, interpret $\alpha+\beta$ as representing the number of experiments that our prior knowledge corresponds to.

4. A priori we assume $\lambda \sim \text{Gamma}(\alpha, \beta)$, i.e.

$$\pi(\lambda) = \frac{\lambda^{\alpha-1}e^{-\lambda/\beta}}{\Gamma(\alpha)\beta^\alpha}.$$

The posterior is then

$$\begin{aligned}
\pi(\lambda|x) &\propto \pi(x|\lambda)\pi(\lambda) \\
&= \frac{e^{-\lambda}\lambda^x}{x!}\frac{\lambda^{\alpha-1}e^{-\lambda/\beta}}{\Gamma(\alpha)\beta^\alpha} \\
&\propto \lambda^{x+\alpha-1}e^{-\lambda(1+1/\beta)}.
\end{aligned}$$

Hence $\lambda|x \sim \text{Gamma}(x+\alpha, \beta/(1+\beta))$.
A more common situation is when $x \sim Pois(\lambda t)$, which coresponds to x being the random number of events in a Poisson process with rate λ on an interval of length t. In this case the posterior is $\lambda|x \sim \text{Gamma}(x+\alpha, \beta/(1+t\beta))$. Here the postrior mean and variance are

$$E[\lambda|x] = \frac{(x+\alpha)\beta}{1+t\beta} = \frac{x\beta}{1+t\beta} + \frac{\alpha\beta}{1+t\beta} \quad \text{and} \quad V[\lambda|x] = \frac{(x+\alpha)\beta^2}{(1+t\beta)^2}.$$

Now, as t increases, $E[\lambda|x]$ will tend towards x/t which is the usual estimator.