Exercises for the 4th Morning

Exercise 0

In this exercise we consider the Metropolis-Hasthings (MH) algorithm. Assume the target density is a standard normal:

$$\pi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2).$$

Furthermore, assume that the proposals are normally distributed centred at the current value and with standard deviation σ:

$$q(x, y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y-x)^2\right).$$

1. Determine the acceptance probability.
2. The resulting Markov chain is irreducible and aperiodic, why?

The following is pseudo code for implementing the MH algorithm:

choose inital value $x^{(0)}$
for i = *1,...,n* **do**
  generate proposal $y \sim q(x^{(i-1)}, y)$
  generate $u \sim Unif[0, 1]$
  calculate $H(x^{(i-1)}, y) = (\pi(y)q(y, x^{(i-1)}))/(\pi(x^{(i-1)})q(x^{(i-1)}, y))$
  if $u < H(x^{(i-1)}, y)$ **then**
    set $x^{(i)} = y$
  else
    set $x^{(i)} = x^{(i-1)}$
  end
end

3. In the MH algorithm the proposal y is accepted with probabilty $\min\{1, H(x, y)\}$. How does the code acheive this?
4. Implement the code in `R` in such a way that it returns a realisation of the Markov chain. Make a trace plots and histrograms of the output. How large does t have to be for the histrogram to look a standard normal. What effect does the value of σ have?
5. Use your code to estimate the probability $P(x \leq -1)$. The correct value is obtain in `R` by `pnorm(-1)`.
6. Assume now that the proposal kernel is

$$q(x, y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}y^2).$$

   What is the interpretation of this proposalkernel? Find the resulting acceptance probability.

Exercise 1

Assume the transition kernel $P(x, A)$ specifies a Markov chain with invariant density $\pi(x)$.

1. Show that if $X^{(t)} \sim \pi(x)$ then $X^{(t+1)} \sim \pi(x)$.
   Hint: You need to show that $P(X^{(t+1)} \in A) = P(A)$.
2. Argue that if $X^{(t)} \sim \pi(x)$ then $X^{(t+n)} \sim \pi(x)$ for all $n \geq 0$.

Exercise 2

Assume that $(X^{(0)}, X^{(1)}, \ldots)$ is an irreducible Markov chain with invariant distribution given by $\pi(x)$. Assume further, that we are given a function h, so that $\mu = \int h(x)\pi(x)dx$ exists. Recall the definition

$$\hat{\mu}_n = \frac{1}{n+1} = \sum_{t=m}^{m+n} h(X^{(t)}).$$

1. Assuming that $x^{(0)} \sim \pi(x)$, show that $\hat{\mu}_n$ is an *unbiased* estimator of μ, i.e. show that $E[\hat{\mu}_n] = \mu$.

Exercise 3 Consider the "two box" target density from the slides

$$\pi(x) = \frac{1}{2}\mathbb{1}\left[|x+1| \leq \frac{1}{2}\right] + \frac{1}{2}\mathbb{1}\left[|x-1| \leq \frac{1}{2}\right]$$

Furthermore, use the following proposal kernel:

$$q(x, y) = \frac{1}{2\delta}\mathbb{1}\left[|y + x| \leq \delta\right],$$

where $\delta > 0$.

1. What is the interpretation of the proposal kernel?
2. Determine the acceptance probability.
3. For what values of δ is the resulting Markov chain irreducible.
4. The Markov chain is aperiodic, why?

Exercise 4

Consider a Markov chain on a discrete state-space $\Omega = \{0, 1\}$ with transition kernel given by

$$P(0, \{1\}) = P(1, \{0\}) = 1$$
$$P(0, \{0\}) = P(1, \{1\}) = 0.$$

1. What does this (rather borring) Markov chain look like?

In the discrete case the definition of invariant distribution is

$$\sum_{x \in \Omega} \pi(x)P(x, \{y\}) = \pi(y).$$

2. Find the invariant distribution of this Markov chain.
3. What is the conditional distribution of $X^{(t)}$ for any $t > 0$ when $X^{(0)} = 0$?