Approximations for Mean and Variance of a Ratio

Consider random variables R and S where S either has no mass at 0 (discrete) or has support $[0, \infty)$. Let $G = g(R, S) = R/S$. Find approximations for EG and Var(G) using Taylor expansions of $g()$.

For any $f(x, y)$, the bivariate first order Taylor expansion about any $\boldsymbol{\theta} = (\theta_x, \theta_y)$ is

$$f(x, y) = f(\boldsymbol{\theta}) + f'_x(\boldsymbol{\theta})(x - \theta_x) + f'_y(\boldsymbol{\theta})(y - \theta_y) + \mathrm{R} \tag{1}$$

where R is a remainder of smaller order than the terms in the equation.

Switching to random variables with finite means $EX \equiv \mu_x$ and $EY \equiv \mu_y$, we can choose the expansion point to be $\theta = (\mu_\mathbf{x}, \mu_\mathbf{y})$. In that case the first order Taylor series approximation for $f(X, Y)$ is

$$f(X, Y) = f(\boldsymbol{\theta}) + f'_x(\boldsymbol{\theta})(X - \mu_x) + f'_y(\boldsymbol{\theta})(Y - \mu + y) + R \tag{2}$$

The approximation for $E(f(X, Y))$ is therefore

$$\begin{aligned}
E(f(X, Y)) &= E\left[f(\boldsymbol{\theta}) + f'_x(\boldsymbol{\theta})(X - \mu_x) + f'_y(\boldsymbol{\theta})(Y - \mu_y) + R\right] && (3)\\
&\approx E\left[f(\boldsymbol{\theta})\right] + E\left[f'_x(\boldsymbol{\theta})(X - \mu_x)\right] + E\left[f'_y(\boldsymbol{\theta})(Y - \mu_y)\right] && (4)\\
&= E\left[f(\boldsymbol{\theta})\right] + f'_x(\boldsymbol{\theta})E\left[(X - \mu_x)\right] + f'_y(\boldsymbol{\theta})E\left[(Y - \mu_y)\right] && (5)\\
&= E\left[f(\boldsymbol{\theta})\right] + 0 + 0 && (6)\\
&= f(\mu_x, \mu_y) && (7)
\end{aligned}$$

Note that if $f(X, Y)$ is a linear combination of X and Y, this result matches the well-known result from mathematical statistics that $E(aX + bY) = aEX + bEY = a\mu_x + b\mu_y$, and in that case the error of approximation is zero. But with the Taylor series expansion, we have extended that result to non-linear functions of X and Y.

For our example where $f(x, y) = x/y$ the approximation is $E(X/Y) = E(f(X, Y)) = f(\mu_x, \mu_y) = \mu_x/\mu_y$.

The second order Taylor expansion is

$$\begin{aligned}
f(x, y) &= f(\boldsymbol{\theta}) + f'_x(\boldsymbol{\theta})(x - \theta_x) + f'_y(\boldsymbol{\theta})(y - \theta_y) && (8)\\
&+ \frac{1}{2}\left\{f''_{xx}(\boldsymbol{\theta})(x - \theta_x)^2 + 2f''_{xy}(\boldsymbol{\theta})(x - \theta_x)(y - \theta_y) + f''_{yy}(y - \theta_y)^2\right\} + R && (9)
\end{aligned}$$

So a better approximation is for $E[f(X, Y)]$ expanded around $\boldsymbol{\theta} = (\mu_x, \mu_y)$ is

$$E(f(X, Y)) \approx f(\boldsymbol{\theta}) + \frac{1}{2}\left\{f''_{xx}(\boldsymbol{\theta})\mathrm{Var}(X) + 2f''_{xy}(\boldsymbol{\theta})\mathrm{Cov}(X, Y) + f''_{yy}(\boldsymbol{\theta})\mathrm{Var}(Y)\right\}. \tag{10}$$

Note that we again use the fact that $E(X - \mu_x) = 0$, and we now add in the definitions for variance and covariance: Var$(X) = E[(X - \mu_x)^2]$ and Cov$(X) = E[(X - \mu_x)(Y - \mu_y)]$.

For $f(R,S) = R/S$, the derivatives are $f''_{RR}(R,S) = 0, \quad f''_{RS}(R,S) = -S^{-2}, \quad$ and $f''_{SS}(R,S) = \frac{2R}{S^3}$.

Specifically, when $\boldsymbol{\theta} = (\mu_R, \mu_S)$, we have $f(\boldsymbol{\theta}) = \mu_R/\mu_S, \quad f''_{RR}(\boldsymbol{\theta}) = 0, \quad f''_{RS}(\boldsymbol{\theta}) = -\frac{1}{(\mu_S)^2}, \quad$ and $f''_{SS}(\boldsymbol{\theta}) = \frac{2\mu_R}{(\mu_S)^3}$.

Then an improved approximation of $E(R/S)$ is

$$E(R/S) \equiv E(f(R,S)) \approx \frac{\mu_R}{\mu_S} - \frac{\text{Cov}(R,S)}{(\mu_S)^2} + \frac{\text{Var}(S)\mu_R}{(\mu_S)^3} \tag{11}$$

By the definition of variance, the variance of $f(X,Y)$ is

$$\text{Var}(f(X,Y)) = E\left\{[f(X,Y) - E(f(X,Y))]^2\right\} \tag{12}$$

Using $E(f(X,Y)) \approx f(\boldsymbol{\theta})$ (from above)

$$\text{Var}(f(X,Y)) \approx E\left\{[f(X,Y) - f(\boldsymbol{\theta})]^2\right\} \tag{13}$$

Then using the first order Taylor expansion for $f(X,Y)$ expanded around $\boldsymbol{\theta}$

$$\begin{aligned}
\text{Var}(f(X,Y)) &\approx E\left\{\left[f(\boldsymbol{\theta}) + f'_x(\boldsymbol{\theta})(X-\theta_x) + f'_y(\boldsymbol{\theta})(Y-\theta_y) - f(\boldsymbol{\theta})\right]^2\right\} && (14)\\
&= E\left\{\left[f'_x(\boldsymbol{\theta})(X-\theta_x) + f'_y(\boldsymbol{\theta})(Y-\theta_y))\right]^2\right\} && (15)\\
&= E\left\{f'^2_x(\boldsymbol{\theta})(X-\theta_x)^2 + 2f'_x(\boldsymbol{\theta})(X-\theta_x)f'_y(\boldsymbol{\theta})(Y-\theta_y) + f'^2_y(\boldsymbol{\theta})(Y-\theta_y)^2\right\} && (16)\\
&= f'^2_x(\boldsymbol{\theta})\text{Var}(X) + 2f'_x(\boldsymbol{\theta})f'_y(\boldsymbol{\theta})\text{Cov}(X,Y) + f'^2_y(\boldsymbol{\theta})\text{Var}(Y) && (17)
\end{aligned}$$

Now we return to our example: $f(R,S) = R/S$ expanded around $\boldsymbol{\theta} = (\mu_R, \mu_S)$.

Since $f'_R = S^{-1}, f'_S = \frac{-R}{S^2}$ and $\boldsymbol{\theta} = (\mu_R, \mu_S)$, we now have $f'^2_R(\boldsymbol{\theta}) = \frac{1}{(\mu_S)^2}, \quad f'_R(\boldsymbol{\theta})f'_S(\boldsymbol{\theta}) = \frac{-\mu_R}{(\mu_S)^3}, \quad f'^2_S(\boldsymbol{\theta}) = \frac{(\mu_R)^2}{(\mu_S)^4}$.

and so

$$\begin{aligned}
\text{Var}(R/S) &\approx \frac{1}{(\mu_S)^2}\text{Var}(R) + 2\frac{-\mu_R}{(\mu_S)^3}\text{Cov}(R,S) + \frac{(\mu_R)^2}{(\mu_S)^4}\text{Var}(S) && (18)\\
&= \frac{(\mu_R)^2}{(\mu_S)^2}\left[\frac{\text{Var}(R)}{(\mu_R)^2} - 2\frac{\text{Cov}(R,S)}{\mu_R\,\mu_S} + \frac{\text{Var}(S)}{(\mu_S)^2}\right] && (19)\\
&= \frac{(\mu_R)^2}{(\mu_S)^2}\left[\frac{\sigma_R^2}{(\mu_R)^2} - 2\frac{\text{Cov}(R,S)}{\mu_R\,\mu_S} + \frac{\sigma_S^2}{(\mu_S)^2}\right] && (20)
\end{aligned}$$

Reference: *Kendall's Advanced Theory of Statistics*, Arnold, London, 1998, 6th Edition, Volume 1, by Stuart & Ord, p. 351.

Reference: *Survival Models and Data Analysis*, John Wiley & Sons NY, 1980, by Elandt-Johnson and Johnson, p. 69.