Stat 591 – Homework 03 **Due: Wednesday 10/16**

Your group should submit a write-up that includes solutions the problems stated below, along with any relevant pictures/graphs or computer code/output.

1. Let $(X_1, \ldots, X_n) \mid \theta \overset{\text{iid}}{\sim} \mathsf{N}(\theta, 1)$ and consider an improper flat prior for θ, i.e., the prior density is $\pi(\theta) = 1$.

   (a) If $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathsf{N}(\theta^\star, 1)$, show that the posterior is consistent.

   (b) Design a simulation study that demonstrates the posterior consistency result you derived. Explain the results of your simulation.

   Hints:

   - Fix $\theta^\star = 0$ and $\varepsilon = 0.01$, say, and define
     $$Q_n = \Pi(\theta : |\theta - \theta^\star| > \varepsilon \mid X_1, \ldots, X_n),$$
     a random variable as a function of $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathsf{N}(\theta^\star, 1)$.
   - For a given n, get a sample of size M from the distribution of Q_n by simulating M sequences $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathsf{N}(\theta^\star, 1)$ and computing the corresponding Q_n for each; take $M = 1000$, say.
   - Repeat this experiment for several values of n, ranging from relatively small to relatively large.
   - Summarize the results by plotting, say, the 50th, 75th, and 95th percentiles of the Q_n distribution as a function of n.

2. Problem 4.2 in [GDS], page 119.

   Hint: Let $\theta^\star = \theta_i$. Because the problem is discrete, it suffices to show that
   $$\Pi(\theta = \theta_i \mid X_1, \ldots, X_n) \to 1 \quad \text{in } \mathsf{P}_{\theta_i}\text{-probability}.$$
   Look at the hint given in [GDS]. If $r = i$, then $Z_r \equiv 0$; if $r \neq i$, then Z_r converges in P_{θ_i}-probability to $-K(\theta_i, \theta_r)$, where $K(\theta_i, \theta_r) = \int \log\{f_{\theta_i}(x)/f_{\theta_r}(x)\} f_{\theta_i}(x)\,dx$ is the *Kullback–Leibler divergence* of $f_{\theta_r}(x)$ from $f_{\theta_i}(x)$. By Jensen's inequality, we know that $K(\theta_i, \theta_r)$ is strictly positive for $r \neq i$.

3. Problem 4.3 in [GDS], page 119.

   Hint: For given $\varepsilon > 0$, define a sequence of random variables
   $$Q_n(X_1, \ldots, X_n) = \Pi(|\theta - \theta^\star| > \varepsilon \mid X_1, \ldots, X_n)$$
   and a sequence of subsets of the sample space
   $$A_n = \{(x_1, x_2, \ldots) : |\hat\theta_n(x_1, \ldots, x_n) - \theta^\star| \leq \varepsilon/2\}.$$
   Dropping the dependence on $(X_1, \ldots, X_n)$ in the notation, write
   $$Q_n = Q_n \cdot I_{A_n} + Q_n \cdot I_{A_n^c},$$
   where, e.g., $I_{A_n} = I_{A_n}(X_1, \ldots, X_n)$ is the indicator of A_n. Now show that both terms in this sum go to zero as $n \to \infty$ in $\mathsf{P}_{\theta^\star}$-probability.

4. Problem 4.4 in [GDS], page 119.

   Hint: Note that [GDS] defines $L_n(\theta)$ to be the log-likelihood function, what I would usually write as $\ell_n(\theta)$. Then

$$\frac{L_n(\theta) - L_n(\theta^\star)}{n} = (\bar{X} - \theta^\star)(\theta - \theta^\star) - \frac{(\theta - \theta^\star)^2}{2}.$$

   For a fixed real number a, define the function $g_a(z) = az - z^2/2$. Argue that this function is negative for $|z| > 2|a|$. Now note that the right-hand side of the above display is $g_a(z)$ with $a = \bar{X} - \theta^\star$ and $z = \theta - \theta^\star$. You'll need the law of large numbers to control $\bar{X} - \theta^\star$.

5. Recall Problem #4 in Homework 2, the one on multinomial with a Dirichlet prior and the agreement application. Argue that the posterior distribution of $n^{1/2}(\kappa - \hat{\kappa}_n)$ is asymptotically normal,[1] and write down an expression for the variance in the normal approximation. Compare this to the results for the asymptotic distribution of the MLE $\hat{\kappa}_n$ you found in Homework 1.

[1] In this problem, the parameter θ is four-dimensional but, as I said in class, all the posterior convergence theorems hold, with obvious adjustments, for any finite-dimensional problem.