

# Monte Carlo Integration

*Jonathan Navarrete*

*May 20, 2017*

## Introduction

Beyond being able to generate random variables, we should also be able to apply these simulation methods to solve more complex problems. We'll begin with Monte Carlo Integration methods.

Topics to be covered:

1. Classic Monte Carlo Integration
2. Importance Sampling
3. Importance Sampling Resampling

## Monte Carlo Integration

Given a function  $g(x)$  for which we wish to integrate over  $[a, b]$ ,  $\int_a^b g(x)dx$ , we can treat this deterministic problem as a stochastic one to find the solution. Treat  $X$  as if a random variable with density  $f(x)$ , then the mathematical expectation of the random variable  $Y = g(X)$  is

$$E[g(X)] = \int_{\mathcal{X}} g(x)f(x)dx$$

If a random sample  $X_1, \dots, X_n$  is generated from  $f(x)$ , an unbiased estimator of  $E[g(X)]$  is the sample mean.

$$\bar{g}_n = \frac{1}{n} \sum_{i=1}^n g(x_i)$$

As an example, suppose we have a function  $g(x) = 3x^2$  for which we wish to integrate over the interval 0 to 2. We could apply deterministic numerical approximation methods (see R's `integrate`) or we could treat  $x$  as a random variable from a  $Unif(0, 2)$  whose pdf is simply  $f(x) = \frac{1}{2-0} = \frac{1}{2}$ . If we now generate some  $N = 1,000$  random values from  $f(x)$  and evaluate them at  $g(x)$ , then take the mean, we'd be calculating the expected value of  $g(x)$ ,

$$\begin{aligned} v &= \int_0^2 g(x)dx \\ &= \left(\frac{2-0}{2-0}\right) \times \int_0^2 g(x)dx \\ &= 2 \times \int_0^2 g(x)\frac{1}{2}dx \\ &= 2 \times E[g(X)] = 2 \times \int_{-\infty}^{\infty} g(x)f(x)dx \\ &\approx 2 \times \frac{1}{n} \sum_{i=1}^n g(x_i) \approx v \end{aligned}$$

If we now simulate this, we will see we approximate the true solution  $\int_0^2 g(x)dx = 8$

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N = 10000 ## sample size

g <- function(x) { 3*x^2 } ## function of interest, g(x)

X <- runif(n = N, min = 0, max = 2) ## samples from f(x)

v = 2 * mean(g(X)) ## 2 * E[g(x)]

print(v) ## approximately 8

## [1] 7.920124

```

Now, some of you may ask, “why is it that we can use the empirical average to create an estimate?” Well, that’s because the discrete expected value of  $g(x)$  is  $E[g(x)] = g(x_1)P(X = x_1) + \dots + g(x_n)P(X = x_n)$  where  $P(X = x_i) = \frac{1}{n}$  since  $x_i \sim \text{Unif}(0, 2)$ .

Now, to generalize the method above. Given a function  $g(x)$  whose integral is well defined, for which we wish to evaluate at interval  $a$  to  $b$ . Then

$$\begin{aligned}
 & \int_a^b g(x)dx \\
 &= (b-a) \int_a^b g(x) \frac{1}{b-a} dx \\
 &= (b-a) \int_a^b g(x) f(x) dx
 \end{aligned}$$

where  $f(x) = \frac{1}{b-a}$  is  $\text{Unif}(a, b)$ , and  $x \sim \text{Unif}(a, b)$ .

The algorithm is as follows:

1. Find a density  $f(x)$  from which we can sample  $x$  from.
2. Generate  $x_1, \dots, x_n \sim f(x)$
3. Compute  $(b-a) \times \bar{g}_n$ , where  $\bar{g}_n = \frac{1}{n} \sum_{i=1}^n g(x_i)$