Problem 1: Suppose you entered the following code into R

```
X <- matrix( rnorm(10000), 2000, 5)
X[,1] <- sqrt(5)*X[,1]
M <- apply(X,1,mean)
print( var(M) )
```

What will be the approximate output of this code?

1. `.2`
2. `1`
3. `9`
4. `.36`

Problem 2: Explain in words the overall goal of the following program (do not detail line-by-line what this program is doing).

```
I <- rep(0, 1000)
mu_D <- 1
for(j in 1:1000)
{

   X1 <- rnorm(20, mean=mu_D)
   X2 <- rnorm(20, mean=0)
   D <- mean(X1)-mean(X2)
   V <- ( var(X1) + var(X2) )/20
   CI <- c( D - 1.96*sqrt(V), D + 1.96*sqrt(V) )
   if( (CI[1] < mu_D) & (CI[2] > mu_D) ) I[j] <- 1 else I[j] <- 0
}
print( mean(I) )
```

Problem 3: Consider Newton-Raphson optimization of the function $f(x) = -(x-k)^2$ where k is a known constant. Regardless of the starting value, where will you be after a single iteration? Justify your answer.

Problem 4: Consider bisection to find the root of the function $f(x) = \tan(x^2) - 2/3$, and you know that $(1/2, 1)$ is a bracket for f. After a single iteration of bisection, what will be your new bracket? ($f(1/2) = -.411, f(3/4) = -.036, f(1) = .891$)

Problem 5: Suppose $X, Y \sim \text{Poisson}(\lambda_1)$ and $Z \sim \text{Poisson}(\lambda_2)$ where all of these variables are independent of each other.

- Explicitly calculate $\text{cov}(X+Y, X+Z)$
- Write a short R function that estimates $\text{cov}(X+Y, X+Z)$ based on a sample correlation. The sample size, n, as well as λ_1, λ_2 should be inputs to the function

Problem 6: Predict the approximate output of the following code:

```
X <- rnorm(1000)
cov(X, abs(X))
```

Problem 7: Based on a sample $X_1, ..., X_n$ from an exponential(λ) distribution there are two intuitive estimators of λ:

$$\hat{\theta}_1 = \frac{1}{\overline{X}}$$

and

$$\hat{\theta}_2 = \sqrt{\frac{1}{\hat{\sigma}^2}}$$

where $\overline{X}$ is the sample mean and $\hat{\sigma}^2$ is the unbiased sample variance. Fill in the blanks, denoted by `****`, in the following code to estimate the MSEs for these two estimators when $n = 20$ and $\lambda = 1$.

```
ThetaHat_1 <- rep(0, 1000)
ThetaHat_2 <- rep(0, 1000)

for(j in 1:1000)
{

   X <- rexp(20, rate=1)
   ThetaHat_1[j] <- ****
   ThetaHat_2[j] <- ****

}

MSE_1 <- mean( (ThetaHat_1 - ****)^2 )
MSE_2 <- mean( (ThetaHat_2 - ****)^2 )
```

Problem 8: Suppose there are two competing estimators, $\hat{\theta}_1$ and $\hat{\theta}_2$, for a parameter θ, and that $\hat{\theta}_1$ is unbiased. Does this mean that $\hat{\theta}_1$ will necessarily outperform $\hat{\theta}_2$ in terms of MSE? If so, then why? If not, what must be true about $\hat{\theta}_2$?

Problem 9: Let X be a single observation that has density

$$p_\theta(x) = \sqrt{\frac{\theta}{2\pi}} \frac{\exp(-\theta/2x)}{x^{3/2}}$$

where θ is an unknown parameter and $x > 0$. We are interested in calculating the MLE for θ using Newton Raphson. If we begin with an initial guess of $\theta_0 = .1$ where will we be after a single iteration? Treat x as a known constant in the likelihood function.

Problem 10: The function

$$p(x) = \sin(x)$$

is a probability density for for $x \in (0, \pi/2)$.

1. Describe an inversion method to sample random variables with this density and write a short R program to implement it.
2. Set up a rejection sampling method to sample from $p(x)$ using a trial density, g, that is the uniform density on the interval $(0, \pi/2)$.
   - What is the optimal criterion for accepting a draw, in terms of the auxiliary uniform(0,1) random variable, U, that must be generated as well as the X that is generated from g?
   - What is the optimal acceptance rate attainable?

Problem 11: Define I to be

$$\int_{-\pi}^{\pi} \sinh(x) dx$$

where $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

1. Explain why this I must evaluate to 0.
2. Describe how this I can be written as an expected value of some quantity with respect to the Uniform($-\pi$, π) density and write a short program to estimate this expectation.

Problem 12: Consider random number generation from the following density:

$$p(x) = \frac{1}{\sqrt{4\pi}} \left(e^{-(x-2)^2} + e^{-(x+2)^2} \right),$$

a 1/2 mixture between the $N(-2, 1/2)$ and $N(2, 1/2)$ densities, write a short program generate from this density.

Hint: Variables having the density p can be thought of as being $N(-2, 1/2)$ with probability 1/2 and $N(2, 1/2)$ with probability 1/2.

Problem 13: Suppose you want to evaluate the integral

$$I = \int_0^1 \sin(e^{1+x^2}) dx$$

Write a **function** to estimate this integral by importance sampling using the beta(α, β) distribution, where α, β, and the monte carlo sample size are inputs to the function.

Describe how you would choose α and β to optimize the accuracy of the integral approximation.

Problem 14: Consider calculating the variance of a random variable with density

$$p(x) = \frac{e^{-x}}{(1+e^{-x})^2}$$

that is (since this variable has mean zero), you want to calculate

$$\int_{-\infty}^{\infty} x^2 \frac{e^{-x}}{(1+e^{-x})^2} dx$$

but are unable to sample from p. Write a short **function** to execute importance sampling using the $N(0, \sigma^2)$ to estimate this integral. This function should take σ^2 and the monte carlo sample size as arguments.

How would you optimally choose σ^2?

Problem 15: Take the function

$$f(x, y) = \exp(x^2 + y^2 - 4xy + 1)$$

Starting from $x_0 = (1, 1)$ where will you be after a single iteration of Newton-Raphson?

Problem 16: Let $X_1, ..., X_n \sim N(0, \sigma^2)$. We want to carry out Bayesian estimation and place an exponential(λ) prior on σ^2.

- Write down the expression for something proportional to the posterior distribution of σ^2
- Describe a rejection sampling algorithm to sample from $p(\sigma^2 | X_1, ..., X_n)$, using the Gamma($\alpha$, β) as your trial distribution.

Note: λ, α, β should all be treated as known constants in your description of the algorithm.