Runtime and Big-O Notation

Learning Objectives

- Identify the worst case and best case inputs of functions
- Compare the function families that characterize different functions
- Calculate a specific function or algorithm's efficiency using Big-O notation

Efficiency = Time

We'll talk about efficiency a lot in this class. Why do we care?

Computers are fast, but they can still take time to do complex actions. Faster algorithms can save lives, increase company profits, and reduce user frustration.

A major goal of computer scientists is not just to make algorithms that work, but algorithms that work efficiently.

Linear Search vs. Binary Search

Comparing Linear vs. Binary Search

Recall our comparison of linear search vs. binary search in the previous lecture. How can we compare these two algorithms at an abstract level?

We could run both on the same input and time them. However, how quickly a program runs varies based on lots of factors (the implementation, the machine, which other programs are running, etc.)

Instead, we'll choose some meaningful action that occurs in the program and count the number of actions the program takes on a given input.

Counting the number of actions

What actions might we count? Some lines of code may compose multiple operations into one line, and some actions may take longer than others to execute on the computer's hardware.

Instead of trying to count every action the computer takes, we choose some specific action and count how many times the algorithm runs that action based on the size of the input.

For example, in linear or binary search we can count the total number of comparisons that the algorithms make to find an item based on the number of items in the list.

Linear vs. Binary Search: Search for 66

```
def linSearch(lst, target):
    if len(lst) == 0:
        return False
    elif lst[0] == target:
        return True
    else:
        return linSearch(lst[1:], target)
```

How many list elements are compared to 66?

linear search: 9 times
binary search: 4 times

```
def biSearch(lst, target):
    if lst == [ ]:
        return False
    else:
        mid = len(lst) // 2
        if lst[mid] == target:
            return True
        elif target < lst[mid]:
            return biSearch(lst[:mid], target)
        else: # lst[mid] < target
            return biSearch(lst[mid+1:], target)
```

							1st	4th	3rd		2nd			
12	25	32	37	41	48	58	60	66	73	74	79	83	91	95

Best Case, Worst Case

Best Case and Worst Case

To truly compare the algorithms, it isn't enough to test them on a random example. We want to know how they'll do in the best case and in the worst case. Those cases are defined based on the inputs to the function.

Best case: an input of size n that results in the algorithm taking the least steps possible.

Worst case: an input of size n that results in the algorithm taking the most steps possible.

Best Case and Worst Case – Linear Search

What's the best case for linear search?

Answer: a list where the item we search for is in the first position

What's the worst case for linear search?

Answer: a list where the item we search for is not in the list.

Best Case and Worst Case – Binary Search

You do: what's the best case input and worst case input for binary search if we're counting comparisons?

Best Case/Worst Case Actions

How many actions do we perform in the best case?

- For both linear search and binary search, there's just one comparison – a list of any length in which it finds the item with the first comparison.

How many actions in the worst case?

- In linear search, we have to check every single element. If the list has n elements, we do n comparisons.
- What about binary search?

Worst Case Action Count – Binary Search

Each call to binary search compares one item of the list. How many recursive calls (and therefore comparisons) do we make to binary search for different length lists?

List size	Number of recursive calls
1	1
2^2-1 = 3	2
2^3-1 = 7	3
2^4-1 = 15	4
2^5-1 = 31	5
2^k - 1	k
n	$\log_2(n)$

When the input length doubles, linear search does twice as many comparisons.

But, when the input length doubles, binary search does just one more comparison!

Calculating Efficiency

Our implementation of binary search only looks better than our implementation of linear search because we only count comparisons.

Slicing a list also takes additional work, as the computer needs to create a copy of the list. Our recursive implementations of linear and binary search both slice the list on every call.

This is inefficient – we're doing more work than we need to! A better approach would be to pass the reference of the original list and change the indexes checked instead of changing the list itself.

Function Families

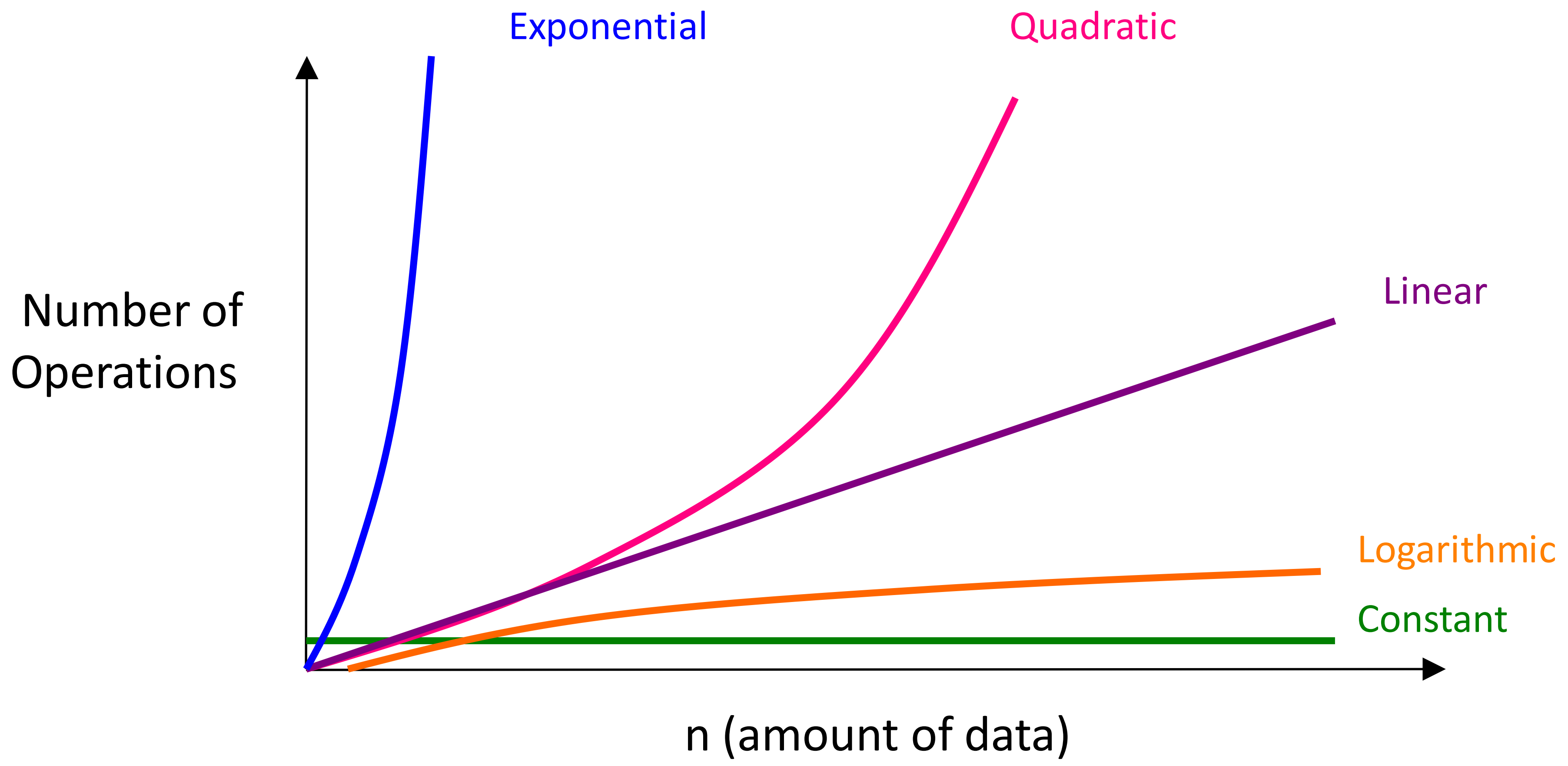
Function Families

When we count the actions taken by algorithms, we don't really care about one-off operations; we care about actions that are related to the size of the input.

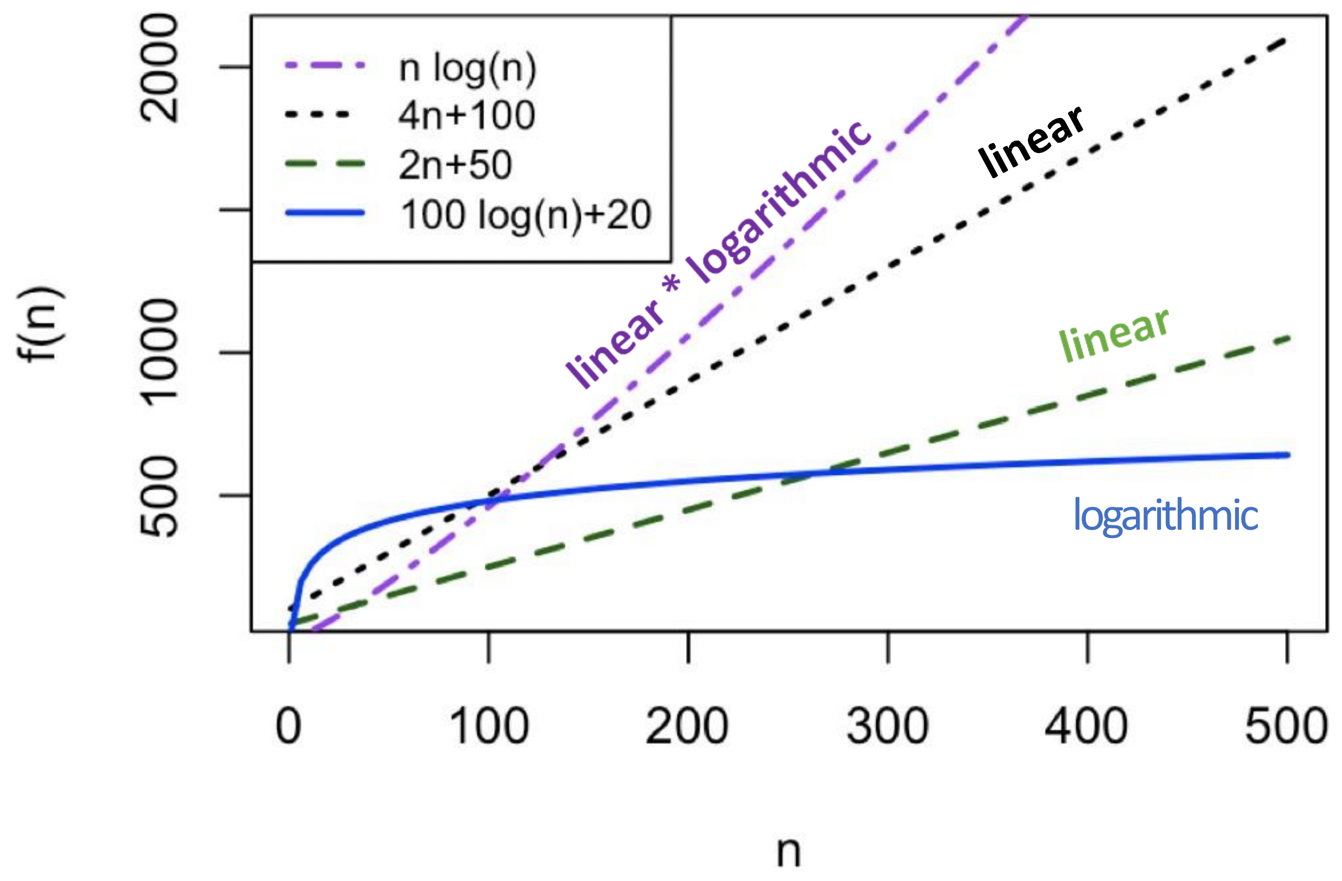
In math, a function family is a set of equations that all grow at the same rate as their inputs grow. For example, an equation might grow linearly or quadratically.

When determining which equation family represents the actions taken by an algorithm, we say that n is the size of the input. For a list, that's the number of elements; for a string, the number of characters.

Common Function Families

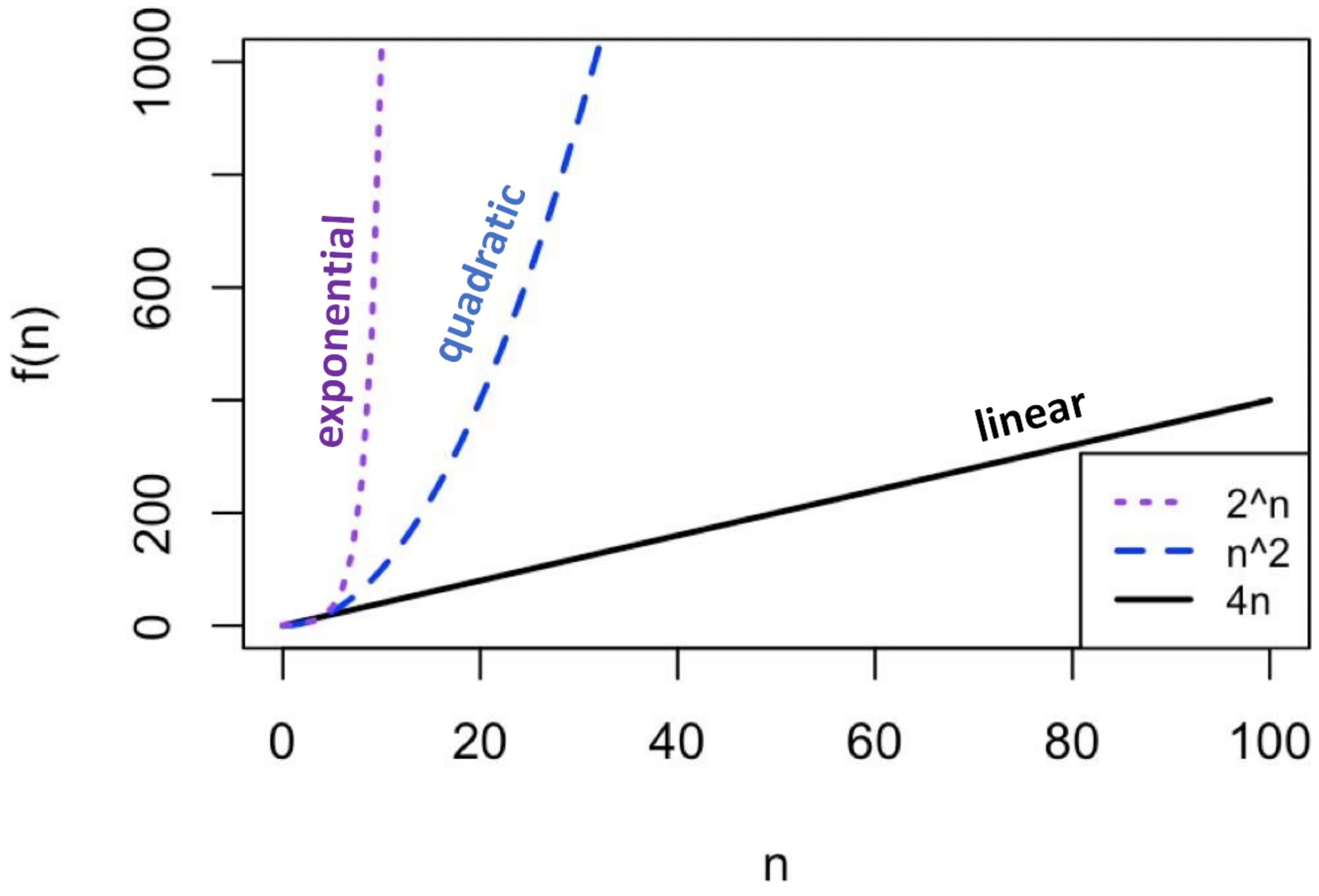


Function Families and Constants



Notice that as n grows, the two linear functions become larger than the logarithmic function and the linear * logarithmic function becomes larger than both linear functions, regardless of the constants.

Function Family Comparisons

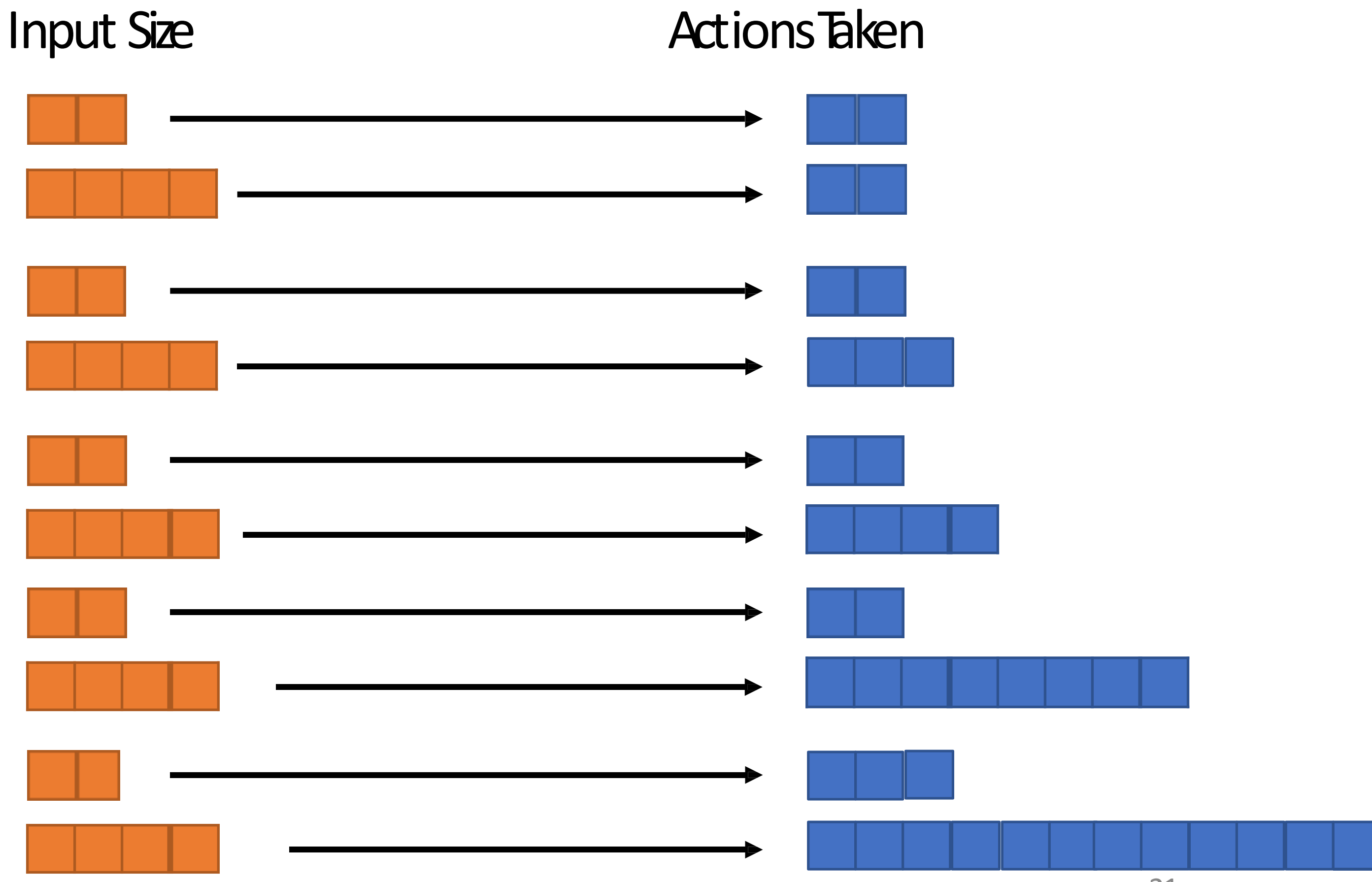


Even for small n, exponential functions quickly skyrocket and quadratic functions grow rapidly compared to linear functions.

Alternate Visualization

Here's another way to think about the function families. Consider what happens when you double the size of the input.

Constant	double input, no change in actions
Logarithmic	double input, +1 action
Linear	double input, double actions
Quadratic	double input, quadruple actions
Exponential	double input, many many more actions!



Big-O

Big-O Notation

When we determine a program or algorithm's runtime, we ignore constant factors and smaller terms. All that matters is the dominant term (the highest power of n), the function family. That is the idea of Big-O notation.

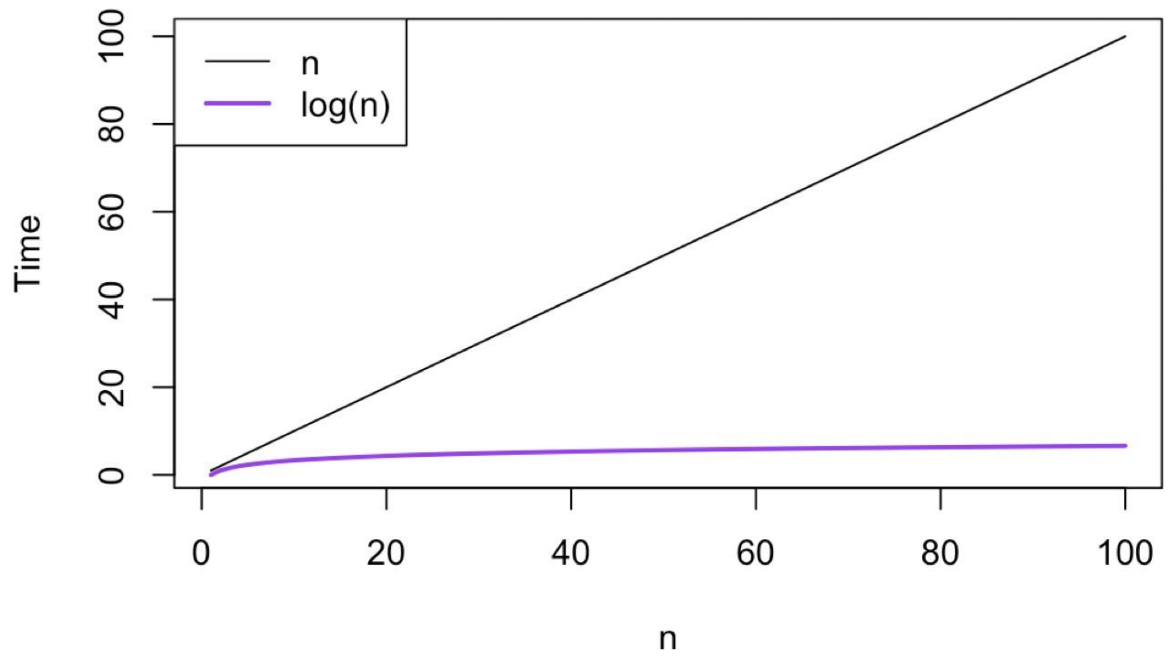
f(n)	**Big-O**
n	O(n)
32n + 23	O(n)
$5n^2 + 6n - 8$	$O(n^2)$
18 log(n)	O(log n)

Unless specified otherwise, the Big-O of an algorithm refers to its worst case run time (computer scientists are pessimists).

Caveat: this is a simplified definition. If you take other CS Classes (e.g. algorithm), you'll learn more about how Big-O actually works.

Big-O of Linear Search / Binary Search

Because runtime for linear search is proportional to the length of the list in the worst case, it is O(n). Every time we double the length of the list, binary search does just one more comparison in the worst case; it is O(log n).



Except for very small n, binary search is blazingly faster. Linear search is exponentially slower in the worst case!

Big-O Calculation Strategy

We'll often need to calculate the Big-O of an algorithm or a piece of code to determine how efficient it is and whether we can make it better.

We can determine an algorithm's Big-O by determining how many actions are added if we increase the size of the input. We can often do a rough estimate of actions by just counting the number of statements that will run.

Let's go through a bunch of examples to demonstrate.

O(1) is Constant Time

```
def swap(lst, i, j):
        tmp    =  lst[i]
        lst[i]         =  lst[j]
        lst[j]         =  tmp
```

Does the runtime of this algorithm depend on the number of items in the list?
  Answer: No.

We say that an algorithm is constant time or O(1) when its time does not change with the size of the input.

O(log n) is Logarithmic Time

```
def countDigits(n):
        count  =  0
        while  n  >  0:
            N  =  n  //    10
            count = count + 1
    return count
```

Every time you increase n by a factor of 10, you do the loop one more time. All the operations in the loop are constant time. Analogous to binary search, the algorithm is logarithmic time, or O(log n).

Why? O(log 2n) = O(log n) + 1 - you add one action per doubling of the input.

Even though this is log10(n), we don't include the base in the Big-O notation because a change of base is just a multiplicative factor.

O(n) is Linear Time

```
def countdown(n):
    for i in range(n, -1, -5):
        print(i)
```

If we double the size of n, how many more times do we go through the loop?

Answer: We double the number of times through the loop. That is linear time, or O(n), as it is proportional to the size of n. Stepping by 5 doesn't change the function family.

Note that O(2n) = O(n) + O(n)

O(n^2) is Quadratic Time

```
def multiplicationTable(n):
        for i in range(1, n+1):
                for j in range(1, n+1):
                        print(i, "*", j, "=", i*j)
```

If we double the size of n, we execute the outer loop twice as many times. And for each time we execute the outer loop, we execute the inner loop twice as many times. Generating the table takes 4 times as long. This is quadratictime, or O(n^2).

Every time you add a new element, 1 action is added to each iteration of the inner loop and 1 iteration is added to the outer loop (n+1 actions). That's 2n+1 new actions added. O((n+1)2) = O(n2) + 2n + 1.

$O(2^n)$ is Exponential Time

```
def move(start, tmp, end, num):
    if num == 1:
        return 1
    else:
        moves = 0
        moves   =   moves   +   move(start, end,   tmp,   num  -  1)
        moves   =   moves   +   move(start, tmp,   end,   1)
        moves   =   moves   +   move(tmp, start,   end,   num  -  1)
        return moves
```

This is Towers of Hanoi. Every time we add 1 disc we double the number of moves. That's exponential time, or $O(2^n)$.

$O(2^{n+1}) = O(2^n) + O(2^n)$

For Recursion, Look at the Number of Calls

Is all recursion exponential? Not necessarily! It depends on the number of recursive calls the function will need to make.

```
def countdown(n):
    if n <= 0:
        print("Finished!")
    else:
        print(n)
        countdown(n - 5)
```

Consider the example above. If you call the function on 100, it will make the next call on 95, then 90, etc; 20 total calls will be made. If you double the input, 40 calls will be made. The function is O(n).

Be Careful of Built-in Runtimes!

```
def countAll(lst):
    for i in range(len(lst)):
        count = lst.count(i)
        print(i, "occurs", count, "times")
```

This is actually O(n^2), because each call to lst.count(i)takes O(n) time.

Activity: Calculate the Big-O of Code

Activity: predict the Big-O runtime of the following piece of code.

```
def sumEvens(lst): # n = len(lst)
    result = 0
    for i in range(len(lst)):
        if lst[i] % 2 == 0:
            result = result + lst[i]
    return result
```