Sirindhorn International Institute of Technology
Thammasat University

MAS117: MATHEMATICS II
2/2023 FINAL EXAMINATION

Name .... **ID** .... **Section** .... **Seat No.** ....

Instructors: Dr. Adisak Seesanea

Date and Time: Tuesday, May 7, 2024, 13:30-16:30 (3 hours)

General Instructions:

- This exam paper has 19 pages, including this page. It consists of 7 main problems with a total score of 70 points. One optional bonus problem of 10 points is provided.
- This is a closed-book exam. No calculators allowed.
- You are not allowed to be out of the examination room during the examination. Going to the restroom may result in a score deduction.
- Write your name, student ID, section, and seat number clearly on each exam sheet.
- You are supposed to complete the exam independently. Academic honesty is fundamental in this course.
- The examination paper cannot be taken out of the examination room. A violation will result in a zero score for the examination.
- Your solutions will be carefully graded based on the written methods rather than the final answers. Please make sure that your solutions are logically presented and readable.
- Your total score will be weighted to 30% of your final grade as indicated in the course syllabus.

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Bonus	
Total	

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Problem 1. [10 points]

Let $f(x, y) = x^2 e^{2y}$.

(1a). Find ∇f and $\nabla f(1, \ln 2)$.

(1b). Find a unit vector $\mathbf{u} \in \mathbb{R}^2$ in the direction in which f increases most rapidly at $(1, \ln 2)$. Determine the rate of change of f at $(1, \ln 2)$ in that direction.

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(1c). Find a unit vector $\mathbf{v} \in \mathbb{R}^2$ such that $D_{\mathbf{v}} f(1, \ln 2) = 0$.

(1d). Find an equation for the tangent plane and parametric equations for the normal line to the surface $z = f(x, y)$ at the point $(1, \ln 2, 4)$.

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Problem 2. [**10 points**]

Given a function $f(x, y) = 4x - 3x^3 - 2xy^2$.

(2a). Compute first and second order partial derivatives of f.

(2b). Find the critical points of f.

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(2c). Employ the Second Derivative Test to locate all local extrema and saddle points (if exist) of f .

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Problem 3. [10 points]

Let's say you have a plan to spend the last 24 hours before the MAS117 exam for a comprehensive study on lecture notes and final review problems. Assume the following:

- Without any preparation, you would earn 10 out of 80 points on the exam–probably from the bonus problem.
- Your exam score will increase by $x(40-x)$ points if you study your lecture notes for x hours; and increase by $y(36-y)$ points if you solve the final review problems for y hours.
- Due to exhaustion and panic, you will lose $(x+y)^2$ points.

How many hours should you spend on studying lecture notes and final review problems in order to maximize your exam score? What is the maximum exam score you can obtain?

<u>**Hint**</u>: Employ the Method of Lagrange's Multiplier to optimize the score function

$$f(x,y) = 10 + x(40-x) + y(36-y) - (x+y)^2$$

subjects to time constraint $g(x,y) = x + y = 24$ with $x, y \geq 0$.

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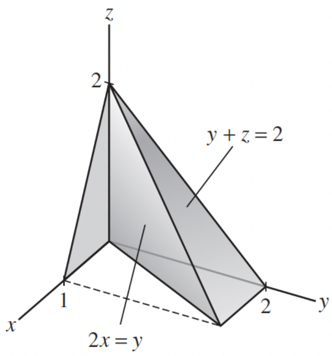
Problem 4. [8 points]

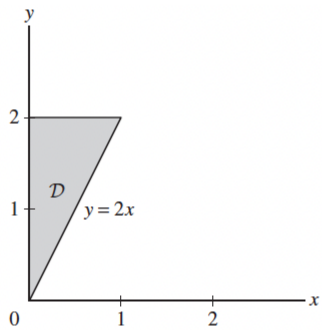
Use a triple integral to find the volume of the solid $\mathcal{G}$ that is bounded by planes

$$z = 2 - y, \qquad y = 2x, \qquad x = 0 \qquad \text{and} \qquad z = 0.$$



Hint: The projection of G on the xy-plane is the triangular region

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 \, : \, 0 \leq x \leq 1 \, \text{ and } \, 2x \leq y \leq 2\}.$$



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Problem 5. [10 points]

For each $\alpha, \beta \in \mathbb{R}$, define a vector field

$$\mathbf{F}(x, y) = (y^3 + \alpha x^2 y + 1)\,\mathbf{i} + (2x^3 + \beta x y^2 + 2)\,\mathbf{j}.$$

(5a). Find all the values of α and β for which $\mathbf{F}$ is conservative.

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(5b). For those α and β obtained in (**5a**), find a potential function of $\mathbf{F}$, that is, a function $\phi = \phi(x, y)$ such that

$$\mathbf{F} = \nabla\phi.$$

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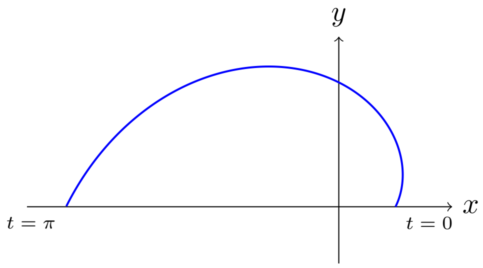
(5c). For those α and β obtained in (**5a**), use any method to compute the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathcal{C}$ is a part of the spiral described by the vector equation

$$\mathbf{r}(t) = (e^t \cos t)\,\mathbf{i} + (e^t \sin t)\,\mathbf{j} \qquad \text{where} \quad 0 \le t \le \pi.$$



Hint: Appealing to the Fundamental Theorem of Line Integrals might be helpful.

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Problem 6. [12 points]

(6a). State Green's theorem, including all hypotheses.

(6b). Consider the vector field

$$\mathbf{F}(x,y) = (\sin^3 x \cos x + e^x x^{55} + \pi y)\,\mathbf{i} + ((\pi+1)x - 2^y y^{99})\,\mathbf{j}.$$

Find $\operatorname{curl}_z \mathbf{F}$, the 2-dimensional curl of F. Determine whether $\mathbf{F}$ is conservative on $\mathbb{R}^2$.

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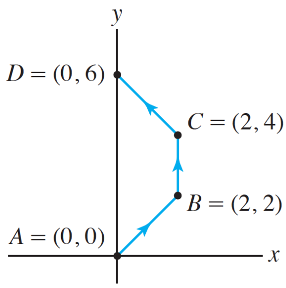
(6c). Let $\mathbf{F}$ be the vector filed defined in (**6b**). Use any method to evaluate the line integral

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathcal{C}_1$ is a non-closed curve consists of line segments from $A(0,0)$ to $B(2,2)$, $B(2,2)$ to $C(2,4)$, and $C(2,4)$ to $D(0,6)$, as shown below.



Hint: Draw the line segment $\mathcal{C}_2$ from $D(0,6)$ to $A(0,0)$. Applying Green's Theorem for the closed curve $\tilde{\mathcal{C}} = \mathcal{C}_1 \cup \mathcal{C}_2$ yields

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = \oint_{\tilde{\mathcal{C}}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \mathrm{curl}_z \mathbf{F} \, dA,$$

where $\mathcal{D}$ is the trapezoidal region enclosed by $\tilde{\mathcal{C}}$.

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Problem 7. [10 points]

A *curved lamina* is an idealized object that is thin enough to be viewed as a surface in 3-space. Let us consider a curved lamina σ which is a part of an elliptic paraboloid

$$z = 16 - x^2 - y^2$$

between the planes $z = 7$ and $z = 12$. Denoted by $\mathcal{R}$ the projection of σ on the xy-plane.

(7a). Sketch the curved lamina σ and its projection $\mathcal{R}$.

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(7b). Suppose σ has density (mass per unit area)

$$f(x,y,z) = \frac{x^2}{16-z}.$$

Compute the mass of this lamina by using a surface integral.

Hint: Given $z = g(x,y)$, one has

$$\iint_{\sigma} f(x,y,z)\, dS = \iint_{\mathcal{R}} f(x,y,g(x,y))\sqrt{z_x^2 + z_y^2 + 1}\, dA.$$

Converting the double integral into polar coordinates might be helpful. Also, recall that $\cos^2\theta = \dfrac{1}{2}(1+\cos 2\theta)$.

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Bonus Problem. [10 points]

Determine whether the statement is true or false. Circle your answer for each statement. No justification is required.

(8a). The directional derivative of $f(x, y)$ at (x_0, y_0) in the direction of $\mathbf{u} = \langle 0, e^2 - 1\rangle$ coincides with the partial derivative $f_y(x_0, y_0)$.

True False

(8b). If $f(x, y)$ is continuous on the ring

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : \pi \le x^2 + y^2 \le 2\pi\},$$

then f must have both global maximum and minimum on $\mathcal{R}$.

True False

(8c). For any continuous functions $g = g(x, y)$, one has $\displaystyle\int_0^1 \int_0^x g(x, y)\, dydx = \int_0^x \int_0^1 g(x, y)\, dxdy.$

True False

(8d). If $\mathcal{C}$ is a smooth curve in $\mathbb{R}^2$ and $f(x, y)$ is a continuous function defined on $\mathcal{C}$, then

$$\int_{\mathcal{C}} f(x, y)\, ds = -\int_{-\mathcal{C}} f(x, y)\, ds.$$

True False

(8e). Given a vector filed $\mathbf{F}(x, y) = f(x, y)\,\mathbf{i} + g(x, y)\,\mathbf{j}$ whose components f and g have continuous first derivatives on region $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : (x, y) \ne (0, 0)\}$. If $f_y = g_x$ on $\mathcal{D}$, then $\mathbf{F}$ must be conservative on $\mathcal{D}$.

True False

Hint: