A 1D WENO SOLVER FOR EULER'S EQUATIONS

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1. Problem formulation.

One-dimensional Euler's equations for gas dynamics are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{1.1a}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial P}{\partial x} = 0 \tag{1.1b}$$

$$\frac{\partial E}{\partial t} + \frac{\partial(u(E+P))}{\partial x} = 0 \tag{1.1c}$$

Sod's shock tube problem [3] is used to test the global first order accuracy of the WENO scheme with Lax-Friedrichs fluxes. For the shock tube problem, Neumann boundary conditions are ideal to simulate the physical flow. However, a periodic boundary condition was adopted for the ease of implementation. Though a periodic boundary condition has no physical meaning for the shock tube problem, the finite speed of propagation of the characteristics allows for the investigation of the flow in a restricted region $x \in (-0.5, 0.5)$ of a larger computational domain $x \in (-1, 1)$, which is devoid of the influence of the periodic boundary condition.

2. Numerical tests.

Prior to testing the WENO scheme on the shock tube problem, the design order of accuracy of the scheme was tested on smooth solutions of one-dimensional advection equation (2.1) with periodic boundary conditions. A Gaussian distribution was used as the initial condition and the numerical solution was computed after one cycle of advection. A third-order WENO scheme with Lax-Friedrichs fluxes was used along with the 3-rd order Total Variation Diminishing Runge Kutta (TVDRK) time integrator. Figure 2.1 shows the log-log ∞-norm absolute error of the numerical solution versus Δx and it establishes expected third-order design accuracy of the chosen scheme. The results can be obtained by running the python file "advection-weno3-tvd3.py".

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{2.1}$$

3. Solution to Sod's shock tube problem.

In Sod's shock tube, a diaphragm separates two ideal gases with different density and pressure, and zero velocity everywhere. The diaphragm can be modelled as a contact discontinuity at the midpoint of the domain ($x = 0$). Once the diaphragm breaks three characteristics emerge from the point of discontinuity: a rarefaction wave into the high density gas, a contact discontinuity and a shock discontinuity into the low density gas. The initial conditions for the left and right states of the ideal gases with specific heat ratio $\gamma = 1.4$, are:

$$\begin{pmatrix} \rho_L \\ P_L \\ u_L \end{pmatrix} = \begin{pmatrix} 1.0 \\ 1.0 \\ 0.0 \end{pmatrix}, \quad \begin{pmatrix} \rho_R \\ P_R \\ u_R \end{pmatrix} = \begin{pmatrix} 0.125 \\ 0.1 \\ 0.0 \end{pmatrix}$$

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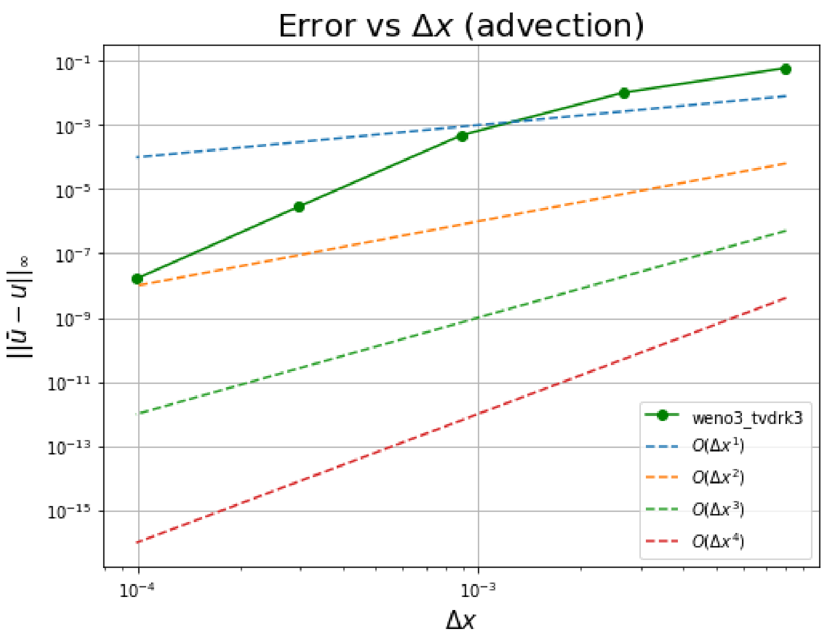


FIG. 2.1. *log-log* ∞ *norm absolute error vs* Δx

where ρ is density, P is pressure and u is velocity. The analytical solution to the problem was obtained from [1]. Third-order WENO scheme and Lax-Friedrichs flux with third-order TVDRK time integrator was used to solve the Sod's shock tube problem and solutions were obtained for $t = 0.2s$. Due to the presence of discontinuities, the numerical solution is expected to have first-order accuracy in L^1 norm. Figure 3.1 shows that the implementation achieves the expected order of accuracy.

A slowly growing zone of non-convergence, starting at contact discontinuity, is expected. Figure 3.2 is a space-time diagram of the two-resolution estimate of the empirical order of convergence (EOC) for density. From Figure 3.2, a good order of convergence can be observed away from the shock-interacting regions while the order of convergence seems to be poor inside such regions, as per the prediction. In addition, it can be observed that the order of convergence, in the regions where the smooth rarefaction wave interacts, is also less than 1. Figure 3.3 shows the analytical and numerical results for density ρ, pressure P and velocity v at the final time $t = 0.2s$ of the computation. The results can be obtained by running the python file "sod-weno3-tvdrk-3.py".

4. WENO Reconstruction.

WENO (Weighted essentially non-oscillatory) reconstruction is an approximation procedure that aims to achieve high order accuracy in smooth regions and resolving shocks or other discontinuities sharply and in an essentially non-oscillatory fashion. This is done by choosing candidate stencils, whose contribution to the design order approximation of the solution, is estimated by adjusting the weights associated with each stencil based on smoothness of solution in each stencil. This helps in avoiding spurious oscillations near discontinuities (Gibbs phenomenon). Though the implementation is straight forward for one-dimensional cases, complexity increases for multi-dimensional and unstructured meshes [2]. Also,

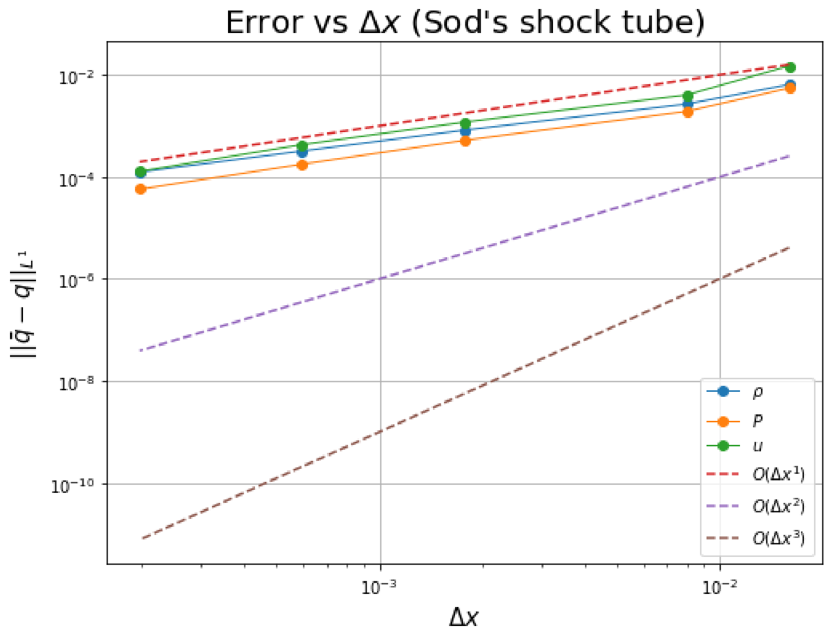


FIG. 3.1. *Log-log plot of L^1 error vs Δx (3rd-order WENO, 3rd-order TVDRK)*

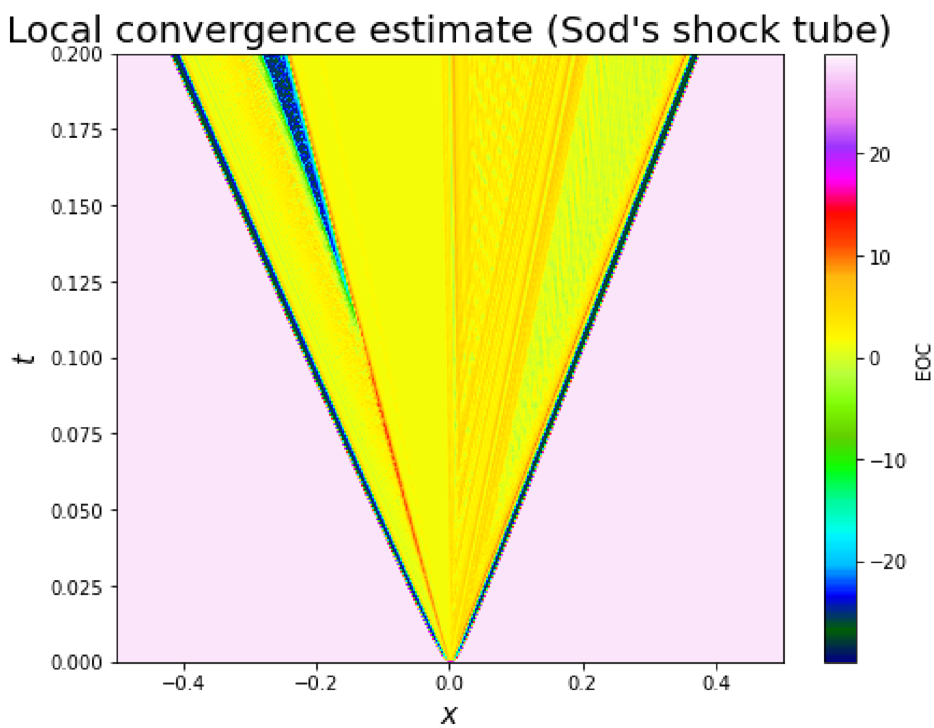
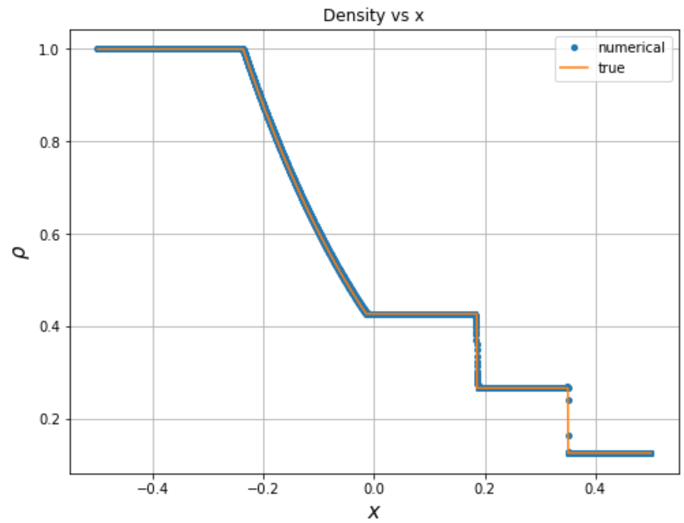
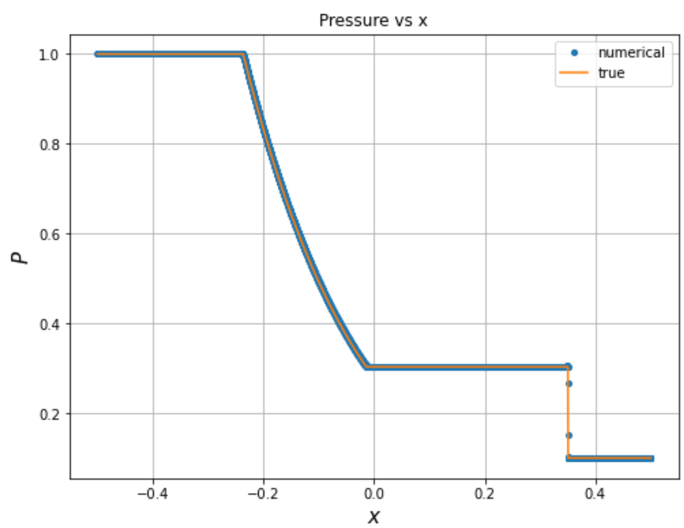


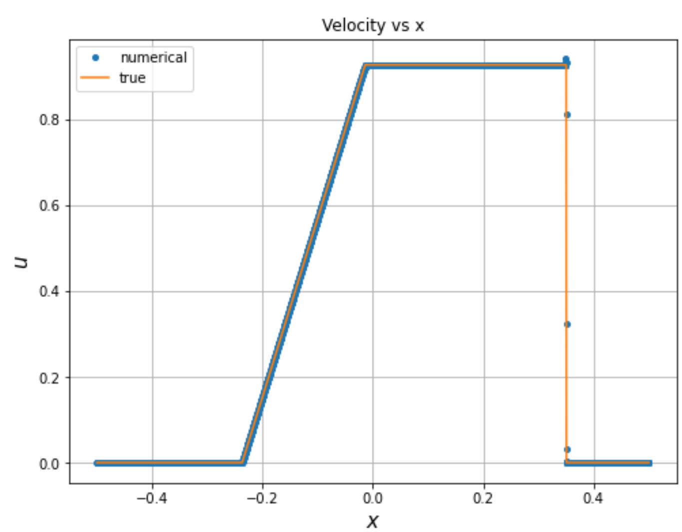
FIG. 3.2. *Space-time graph of empirical order of convergencec(EOC) of density for Sod's shock tube problem.*



(a) Analytical and numerical solution for density at $t = 0.2s$ and $\Delta x = 2e^{-4}$



(b) Analytical and numerical solution for pressure at $t = 0.2s$ and $\Delta x = 2e^{-4}$



(c) Analytical and numerical solution for velocity at $t = 0.2s$ and $\Delta x = 2e^{-4}$

slight oscillations were observed near strong discontinuities for density distribution as shown in the magnified plot Figure 4.1. This could make the convergence to steady state solutions harder [4].

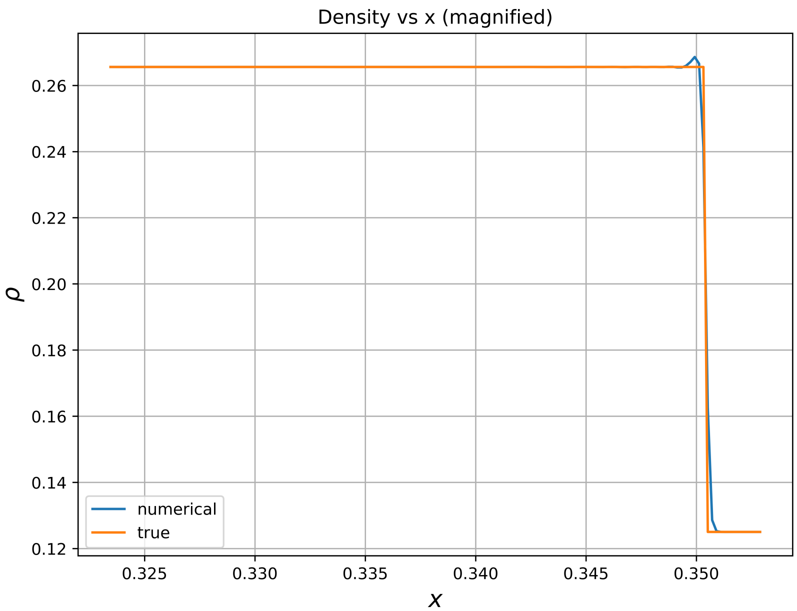


Fig. 4.1. *Zoomed density distribution,* $\Delta x = 2e^{-4}$

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