INVERSE AKASH DISTRIBUTION AND IT'S APPLICATIONS

ABSTRACT: A new one parameter lifetime distribution named "Inverse Akash distribution" for modeling lifetime data has been introduced. Some important statistical properties of the proposed distribution including its shape characteristics of the density, hazard rate function, survival function, stochastic ordering, entropy measure, stress-strength reliability have been discussed. The maximum likelihood estimation of its parameter has been discussed. Two real data sets were employed in illustrating the usefulness of the new distribution and it was found that the Inverse Akash distribution provides a better fit when compared to the Inverse Exponential distribution and Inverse Lindley distribution.

Keywords: Inverse Akash distribution, Stress-strength reliability model, Statistical properties, Maximum likelihood estimator and Goodness of fit

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1. INTRODUCTION

Modeling and analyzing lifetime data are crucial in many applied sciences including medicine, engineering, insurance and finance, amongst others (Shanker, 2015). There are a number of continuous distributions for modeling lifetime data such as Exponential, Akash, Lindley, Gamma, Lognormal and Weibull and their generalizations (Shanker, 2015). The Exponential, Lindley and the Weibull distributions are more popular than the Gamma and the lognormal distributions because the survival functions of the Gamma and the lognormal distributions cannot be expressed in closed forms and both require numerical integration (Shanker, 2015). Though each of Exponential, Akash and Lindley distributions have one parameter, the Akash and Lindley distributions have one advantage over the exponential distribution, that the exponential distribution have constant hazard rate function whereas the Akash and Lindley distributions have monotonically decreasing hazard rate functions (Shanker, 2015). A detailed study about its various mathematical properties, estimation of parameter and application showing the superiority of Lindley distribution over exponential distribution for the waiting times before service of the bank customers has been done by (Ghitany *et al*. 2008). Hassain (2006), Shanker *et al*. (2013), Sharma *et al.* (2015), Alkarni (2015) have generalized, extended, modified and detailed the applications of the Lindley distribution in reliability and other fields of knowledge.

Hussain (2006) has shown that the Lindley distribution is important for studying stress-strength reliability modeling, there are many situations in the modeling of real lifetime data where the Lindley distribution may not be suitable from a theoretical or applied point of view. It has been shown that the Akash distribution as a probability model can be a better model than the well-known Lindley and exponential distributions in some particular cases (Shanker, 2015).

The probability density function (pdf) and the cumulative distribution function (cdf) of Akash (2015) distribution are given by;

$$f(x;\theta) = \frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}\ x > 0, \theta > 0 \tag{1}$$

$$F(x;\theta) = 1 - \left[1 + \frac{\theta x(\theta_x+2)}{\theta^2+2}\right]e^{-\theta x}\ ,\ x > 0, \theta > 0 \tag{2}$$

The density (1) is a two-component mixture of an exponential distribution having scale parameter θ and a gamma distribution having shape parameter 3 and scale parameter θ with their mixing proportions $\frac{\theta^2}{\theta^2+2}$ and $\frac{2}{\theta^2+2}$ respectively. A detailed study about its various mathematical properties, estimation of parameter and application showing the superiority of Akash distribution over Lindley distribution by Shanker *et al.* (2015).

Since, the Akash distribution is only applicable of modeling to the monotonic increasing hazard rate data, its applicability is restricted to the data that show non-monotone shapes (bathtub and upside-down bathtub) for their hazard rate. Though various article has found in literature that address the analysis of bathtub shape data, limited attention has been paid to the analysis of upside-down bathtub shape data.

According to Bourguignon *et al*. (2014), considering the fact that all inverse distribution possesses the upside-down bathtub shape for their hazard rates function, we propose an inverted version of the Akash distribution that can be effectively used to model the upside-down bathtub shape hazard rate data. If a random variable Y has a Akash distribution AD(θ), then the random variable X = ($^1/_Y$) is said to be follow the inverse Akash distribution having a scale parameter θ with its probability density function (Pdf), is defined by

$$f(x;\theta) = \frac{\theta^3}{\theta^2+2}\left(\frac{1+x^2}{x^4}\right)e^{-\frac{\theta}{x}}, x > 0, \theta > 0 \tag{3}$$

It is denoted by IAD(θ). The cumulative distribution function (Cdf) of inverse Akash distribution is given by

$$F(x;\theta) = \left[1 + \left(\frac{\theta+2x}{\theta^2+2}\right)\frac{\theta}{x^2}\right] e^{-\frac{\theta}{x}}, x > 0, \theta > 0 \tag{4}$$

Since this continuous distribution has the nice closed form expressions for the Cdf, hazard function as well as stress-strength reliability, its relevance for survival analysis can never be denied in the literate.

The aim of this paper is in two phases. The first phase is to derive and study the properties of inverse Akash distribution. The second phase is to compare the inverse Akash distribution with both inverse exponential distribution and inverse Lindley distribution.

2. PROPERTIES OF INVERSE AKASH DISTRIBUTION

3.1 Shape characteristics of the density

The first derivation of (3) is given by

$$\frac{d}{dx} f(x) = -\left(\frac{\theta^3}{\theta^2+2}\right)\frac{e^{-\frac{\theta}{x}}}{x^6}\left(2x^3 - (\theta x - 4)x - \theta\right) \qquad (5)$$

Figure (1) shows the various shapes of inverse Lindley density for different choices of θ and also indicates that the density function of inverse Akash density is uni-model in x.

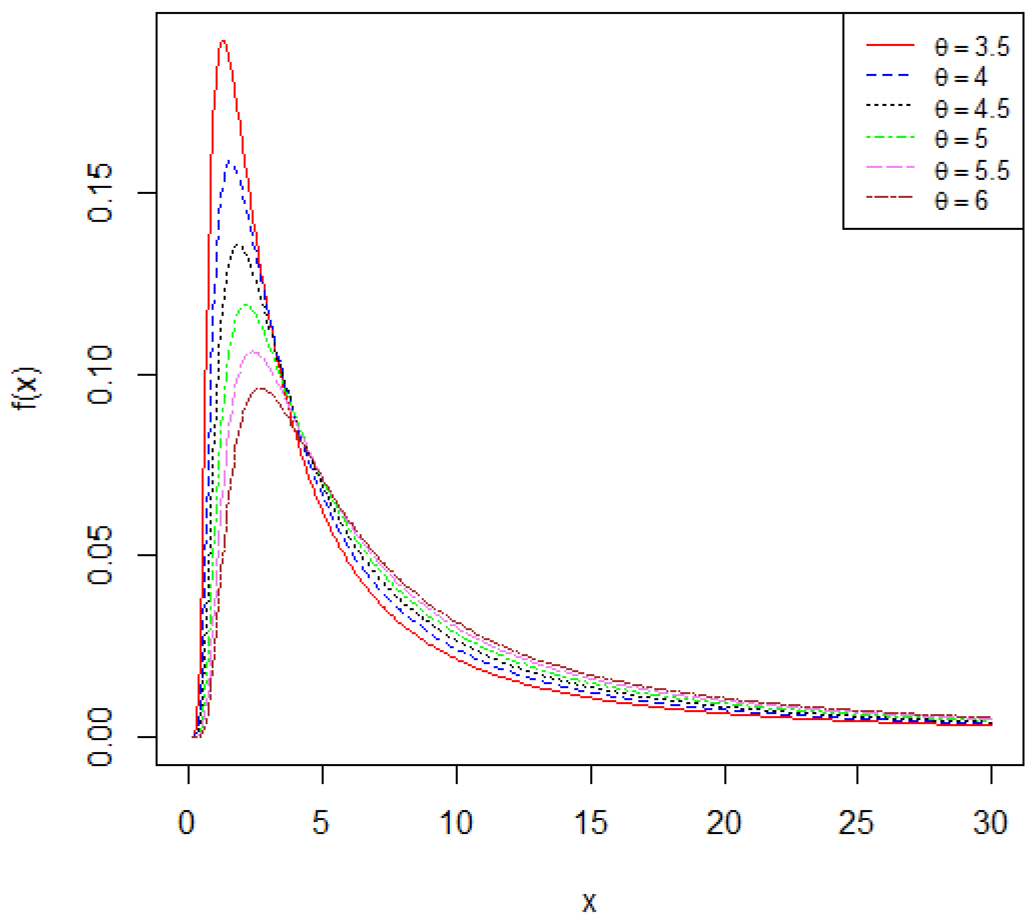

Figure 1. Probability density functions of IAD(θ) for different values of θ.

3.2 Survival function

The survival function is given by

$$s(x) = 1 - F(x) = 1 - \left[1 + \left(\frac{\theta+2x}{\theta^2+2}\right)\frac{\theta}{x^2}\right] e^{-\frac{\theta}{x}} \qquad (6)$$

3.3 Hazard rate function

In general, the hazard rate function is given as:

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\theta^3(1+x^2)}{x^2\left[x^2(\theta^2+2)\left(e^{\frac{\theta}{x}}-1\right)-\theta(\theta+2x)\right]} \qquad (7)$$

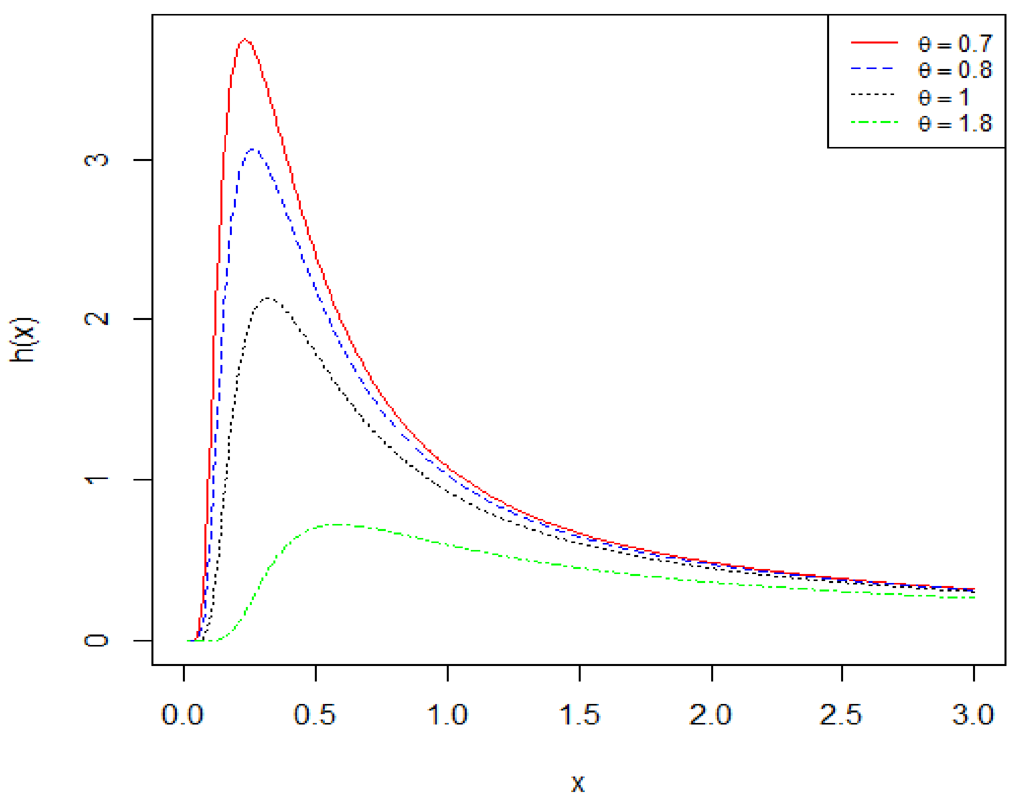

Figure 2. Hazard rate functions of IAD(θ) for different values of θ.

The monotonicity of hazard rate function of IAD(θ), for different values of the parameter θ, has been showed graphically in Figure (2). Clearly, the hazard function of inverse Akash distribution is also a uni-model function in x.

3.4 Stochastic ordering

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behaviour. According to Shanker (2015), a random variable X is said to be smaller than a random variable Y in the;

i. Stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x.
ii. Hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x.
iii. Mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \leq m_Y(x)$ for all x.
iv. Likelihood ratio order $(X \leq_{lr} Y)$ if $\left(\frac{f_X(x)}{f_Y(x)}\right)$ decreases in x.

The following results due to Shaked *et al.* (1994) are well known for establishing stochastic ordering of distributions

$$
\begin{array}{c}
(X \leq_{lr} Y) \Rightarrow (X \leq_{hr} Y) \Rightarrow (X \leq_{mrl} Y) \\
\Downarrow \\
(X \leq_{st} Y)
\end{array}
$$

The inverse Akash distributions are ordered with respect to the strongest likelihood ratio ordering as shown in the following theorem.

Theorem: Let $X \sim IAD(\theta_1)$ and $Y \sim IAD(\theta_2)$.

If $\theta_1 \geq \theta_2$, then $(X \leq_{lr} Y)$ and hence $(X \leq_{hr} Y), (X \leq_{mrl} Y)$ and $(X \leq_{st} Y)$

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^3(\theta_2^2+2)e^{-\frac{\theta_1}{x}}}{\theta_2^3(\theta_1^2+2)e^{-\frac{\theta_2}{x}}}$$

$$= \frac{\theta_1^3(\theta_2^2+2)}{\theta_2^3(\theta_1^2+2)} e^{-\frac{(\theta_1-\theta_2)}{x}}; x > 0 \qquad (8)$$

$$\log\frac{f_X(x)}{f_Y(x)} = \log\left[\frac{\theta_1^3(\theta_2^2 + 2)}{\theta_2^3(\theta_1^2 + 2)}\right] - \left[\frac{(\theta_1 - \theta_2)}{x}\right]$$

This gives $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = \frac{(\theta_1 - \theta_2)}{x^2}$ (9)

Thus, for $\theta_1 \geq \theta_2$, $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} > 0$. This means that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

3.5 Entropy measure

Entropy of a random variable X is a measure of variation of uncertainty. A popular entropy measure is Renyi entropy (1961). If X is a continuous random variable having probability density function$f(.)$, then Renyi entropy is defined as

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)\,dx\right\}$$

Where$\gamma > 0 \; and \; \gamma \neq 1$

For the Inverse Akash distribution, the Renyi entropy measure is defined by;

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\int_0^{\infty}\frac{\theta^{3\gamma}}{(\theta^2 + 2)^{\gamma}}\left[\frac{(1 + x^2)^{\gamma}}{x^{4\gamma}}\right]e^{-\frac{\theta\gamma}{x}}dx$$

We know that $(1 + z)^j = \sum_{\mathrm{j}=0}^{\infty}\binom{\gamma}{j}z^j$ and $\int_0^{\infty} e^{-\frac{b}{x}}x^{-a-1}dx = \frac{\Gamma(\mathrm{a})}{\mathrm{b}^{\mathrm{a}}}$

$$= \frac{1}{1-\gamma}\log\left[\frac{\theta^{3\gamma}}{(\theta^2 + 2)^{\gamma}}\sum_{\mathrm{j}=0}^{\infty}\binom{\gamma}{j}\int_0^{\infty}\frac{e^{-\frac{\theta\gamma}{x}}}{x^{4\gamma-2j}}\,\mathrm{dx}\right]$$

$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\frac{\theta^{3\gamma}}{(\theta^2+2)^{\gamma}}\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\Gamma(4\gamma-2j-1)}{(\theta\gamma)^{4\gamma-2j-1}}\right]$ (10)

3. STRESS-STRENGTH RELIABILITY AND MAXIMUM LIKELIHOOD ESTIMATION

Let Y and X be independent stress and strength random variables that follow Inverse Akash distribution with parameter $\theta_1 \, and \, \theta_2$ respectively. Then, the stress-strength reliability R is defined as

$$R = P[Y<X] = \int_0^\infty P[Y < X \mid X = x] \, f_x(x) dx = \int_0^\infty f(x,\theta_1) \, F(x,\theta_2) dx$$

$$= \int_0^\infty \left(\left[1 + \left(\frac{\theta_2 + 2x}{\theta_2^2 + 2} \right) \frac{\theta_2}{x^2} \right] e^{-\frac{\theta_2}{x}} \right) \frac{\theta_1^3}{\theta_1^2 + 2} \left(\frac{1 + x^2}{x^4} \right) e^{-\frac{\theta_1}{x}} dx$$

$$= \frac{\theta_1^3}{\theta_1^2 + 2} \int_0^\infty \left(\frac{1 + x^2}{x^4} \right) e^{-\frac{(\theta_1+\theta_2)}{x}} dx + \frac{\theta_1^3 \theta_2}{(\theta_1^2 + 2)(\theta_2^2 + 2)} \int_0^\theta \left(\frac{1 + x^2}{x^4} \right) \left(\frac{\theta_2 + 2x}{x^2} \right) e^{-\frac{(\theta_1+\theta_2)}{x}} dx$$

$$= \frac{\theta_1^3}{\theta_1^2 + 2} \left[\int_0^\infty x^{-4} e^{-\frac{(\theta_1+\theta_2)}{x}} dx + \int_0^\infty x^{-2} e^{-\frac{(\theta_1+\theta_2)}{x}} dx \right] + \frac{\theta_1^3 \theta_2}{(\theta_1^2 + 2)(\theta_2^2 + 2)} \int_0^\infty \left(\frac{\theta_2 + \theta_2 x^2 + 2x + 2x^3}{x^6} \right) e^{-\frac{(\theta_1+\theta_2)}{x}} dx$$

Using the definitions of IAD and Inverse Gamma distribution, we get the expression for the stress-strength reliability as;

$$R = \left(\frac{\theta_1^3}{\theta_1^2+2} \right) \left[\frac{\Gamma(3)}{(\theta_1+\theta_2)^3} + \frac{\Gamma(1)}{(\theta_1+\theta_2)} \right] + \frac{\theta_1^3 \theta_2}{(\theta_1^2+2)(\theta_2^2+2)} \left[\theta_2 \left(\frac{\Gamma(5)}{(\theta_1+\theta_2)^5} + \frac{\Gamma(3)}{(\theta_1+\theta_2)^3} \right) + 2 \left(\frac{\Gamma(4)}{(\theta_1+\theta_2)^4} + \frac{\Gamma(2)}{(\theta_1+\theta_2)^2} \right) \right]$$

$$R = \frac{\theta_1^3 \{ [2+(\theta_1+\theta_2)^2][(\theta_2^2+2)(\theta_1+\theta_2)^2] + \theta_2 [24\theta_2 + 2\theta_2(\theta_1+\theta_2)^2 + 12(\theta_1+\theta_2) + 2(\theta_1+\theta_2)^3] \}}{(\theta_1^2+2)(\theta_2^2+2)(\theta_1+\theta_2)^5} \qquad (11)$$

Since R is the Stress-Strength Reliability function with parameters θ_1 and θ_2, we need to obtain the maximum likelihood estimators (MLEs) of θ_1 and θ_2 to compute the maximum likelihood estimation R under Invariance property of the maximum likelihood estimation.

Suppose $X_1, X_2, \dots, X_n$ is a Strength randon variable sample from Inverse Akash distribution (θ_1) and $Y_1, Y_2, \dots, Y_m$is a Stress random sample from Inverse Akash distribution (θ_2). Thus, the likelihood function based on the observed sample is given by;

$$L\left(\theta_1, \theta_2 / \underline{x}, \underline{y}\right) = \frac{\theta_1^{3n}\theta_2^{3m}}{\left(\theta_1^2+2\right)^n\left(\theta_2^2+2\right)^m} \prod_{i=1}^{n}\left(\frac{1+x_i^2}{x_i^4}\right) \prod_{j=1}^{m}\left(\frac{1+y_j^2}{y_j^4}\right) e^{-(\theta_1 S_1+\theta_2 S_2)} \tag{12}$$

Where, $S_1 = \sum_{i=1}^{n}\frac{1}{x_i}$, $\quad S_2 = \sum_{j=1}^{m}\frac{1}{y_j}$

The log-Likelihood function is given by;

$$\log L(\theta_1, \theta_2) = 3n\log\theta_1 + 3m\log\theta_2 - n\log(\theta_1^2 + 2) - m\log(\theta_2^2 + 2) - \theta_1 S_1 - \theta_2 S_2 + \sum_{i=1}^{n}\log\left(\frac{1+x_i^2}{x_i^4}\right) + \sum_{j=1}^{m}\log\left(\frac{1+y_j^2}{y_j^4}\right) \tag{13}$$

The Maximum Likelihood Estimators of $\theta_1$and θ_2, say $\hat{q}_1$ and $\hat{q}_2$ respectively can be obtained as the solution of the following eguations;

$$\frac{\partial \log L(\theta_1,\theta_2)}{\partial\theta_1} = \frac{3n}{\theta_1} - \frac{2\theta_1 n}{(\theta_1^2+2)} - S_1 \tag{14}$$

$$\frac{\partial \log L(\theta_1,\theta_2)}{\partial\theta_2} = \frac{3m}{\theta_2} - \frac{2\theta_2 m}{(\theta_2^2+2)} - S_2 \tag{15}$$

From (14) and (15), obtain MLEs as

$$\Rightarrow \theta_1^3 S_1 - \theta_1^2 n + 2\theta_1 S_1 - 6n = 0 \tag{16}$$

$$\Rightarrow \theta_2^3 S_2 - \theta_2^2 m + 2\theta_2 S_2 - 6m = 0 \tag{17}$$

Hence, using the invariance property of the MLE, the maximum likelihood estimator $\hat{R}_{mle}$ of R can be obtained by substituting $\hat{q}_k$ in place of θ_k for k = 1, 2.

$$\hat{R}_{mle} = \left.\frac{\theta_1^3\{[2+(\theta_1+\theta_2)^2][(\theta_2^2+2)(\theta_1+\theta_2)^2]+\theta_2[24\theta_2+2\theta_2(\theta_1+\theta_2)^2+12(\theta_1+\theta_2)+2(\theta_1+\theta_2)^3]\}}{(\theta_1^2+2)(\theta_2^2+2)(\theta_1+\theta_2)^5}\right|_{q_k=\hat{q}_k, k=1,2} \tag{18}$$

4. THE MAXIMUM LIKELIHOOD ESTIMATION OF THE INVERSE AKASH DISTRIBUTION

The use of the MLE involves getting the joint PDF for all the observations. In particular, for independent and identically distributed random sample, a maximum likelihood estimator of a parameter θ for a PDF f(x) is an estimator that maximizes the likelihood function

$L(x_1, x_2, \dots, x_n;\ \theta)$ as a function of θ. (Hoel, 1954).

$$L(x_1, x_2, \dots, x_n; \theta) = \prod_{i=1}^{n} [f(x_i; \theta)]$$

$$= \frac{\theta^{3n}}{(\theta^2 + 2)^n} \prod_{i=1}^{n} \left(\frac{1 + x_i^2}{x_i^4} \right) e^{-\theta S}$$

Where, $S = \sum_{i=1}^{n} \frac{1}{x_i}$

The Log-Likelihood is given by;

$$\log L(\theta) = 3n \log \theta - n \log(\theta^2 + 2) - \theta S + \sum_{i=1}^{n} \log \left(\frac{1 + x_i^2}{x_i^4} \right)$$

The Maximum Likelihood Estimator of θ, say $\hat{q}$ can be obtained as the solution of the equation;

$$\frac{\partial \log L(\theta)}{\partial \theta} = \frac{3n}{\theta} - \frac{2n\theta}{(\theta^2 + 2)} - S \qquad (19)$$

$$\Rightarrow \theta^3 S - \theta^2 n + 2\theta S - 6n = 0 \qquad (20)$$

Notably, the analytic solution of (20) can be obtained. As a consequence, we can apply a numerical method to solve (20). The numerical solution of (20) can be found using R software.

5. REAL DATA APPLICATION

The IAD was applied to two real life data set (Data(X), and Data(Y)) in order to assess its statistical superiority over other models; the Inverse Exponential distribution and Inverse Lindley distribution, to demonstrate that the theoretical results in the previous sections can be used in practice. The data sets represent the Survival times of two groups of patients suffering from Head and Neck cancer disease. The patients in one group were treated using radiotherapy (RT) whereas the patients belonging to the other group were treated using a combined radiotherapy and chemotherapy (RT+CT).

The data sets are as follows;

Data (X):
6.53,7,10.42,14.48,16.10,22.70,34,41.55,42,45.28,49.40,53.62,63,64,83,84,91,108,
112,129,133,133,139,140,140,146,149,154,157,160,160,165,146,149,154,157,160,
160,165,173,176,218,225,241,248,273,277,297,405,417,420,440,523,583,594,110
1,1146,1417

Data (Y):
12.20,23.56,23.74,25.87,31.98,37,41.35,47.38,55.46,58.36,63.47,68.46,78.26,74.4
7,81.43,84,92,94,110,112,119,127,130,133,140,146,155,159,173,179,194,195,209,
249,281,319,339,432,469,519,633,725,817,1776

First we checked the validity of the inverse Akash distribution for given data sets by using Kologorov-Smirnov goodness-of-fit test(K-S), Akaike information criterion (AIC), Bayesian information criterion (BIC), Negative Log-Likelihood Function (-L), Anderson Darling Test (A*), Cramer-Von Mises Criterion (W*), Standard Error Estimate of the Parameter (SE), and The Estimate of the Parameter.

We compared the applicability of inverse Akash distribution with competing one parameter distributions, inverse Exponential distribution (IED) and inverse Lindley distribution (ILD). Based on real data sets.

Data (X):

The performance of the IAD with respect to the IED and the ILD using the observations in Data(X) is as shown in Table 1:

Table 1: Performance Ratings of Inverse Akash distribution Using Data(X)

Distribution	Estimate	SE	-L	AIC	BIC	KS	W*	A*
IAD	59.2532	7.7671	385.6517	773.3034	775.3638	0.3043	1.1845	5.6598
IED	59.1225	7.7632	385.6871	773.3742	775.4346	0.3048	1.1896	5.6822
ILD	60.0883	7.7649	385.7031	773.4062	775.4666	0.3048	1.1904	5.6863

From Table 1, the IAD has the lowest -L value of 385.6517, the lowest AIC value of 773.3034, the lowest BIC value of 775.3638, the lowest KS value of 0.3034, the lowest W* value of 1.1845 and the lowest A* value of 5.6598 therefore, the IAD provides a better fit than the IED and ILD.

For Data (Y):

The performance of the IAD with respect to the IED and the ILD using the observations in Data (Y) is as shown in Table 2:

Table 2: Performance Ratings of Inverse Akash distribution Using Data (Y)

Distribution	Estimate	SE	-L	AIC	BIC	KS	W*	A*
IAD	76.7452	11.5580	279.5750	561.1500	562.9342	0.08881	0.07589	0.49271
IED	76.7006	11.5631	279.5773	561.1546	562.9388	0.08884	0.07592	0.49292
ILD	77.6754	11.5649	279.5784	561.1568	562.9409	0.08888	0.07596	0.49308

From Table 2, the IAD has the lowest -L value of 279.5750, the lowest AIC value of 561.1500, the lowest BIC value of 562.9342, the lowest KS value of 0.08881, the lowest W* value of 0.07589 and the lowest A* value of 0.49271 therefore, the IAD provides a better fit than the IED and ILD.

6. CONCLUSION

A one parameter lifetime distribution named, “Inverse Akash distribution” has been proposed. Its statistical properties including shape characteristics of density, survival function, hazard rate function, stochastic ordering has been discussed. Further, expressions for entropy measure and, Stress-Strength Reliability of the proposed distribution have been derived. The method of maximum likelihood estimation has also been discussed for estimating its parameter. Finally, the goodness of fit test using –L, KS Statistics, AIC, BIC, W* and A* based on two real lifetime data sets and the applicability and superiority over Inverse Exponential and Inverse Lindley distributions while modeling certain lifetime data have been established.

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