Homework 9

Problem 1. *What is the expected number of trees with k vertices in $G \in \mathcal{G}(n, p)$?*

Solution. According to *Caley's formula*, the number of different trees with k vertices is k^{k-2}. By the linearity of expectation, we have the expected number of trees with k vertices in $G \in \mathcal{G}(n, p)$ is $\binom{n}{k} k^{k-2} p^{k-1}$. □

Problem 2. *Show that, for constant $p \in (0, 1)$, almost no graph in $\mathcal{G}(n, p)$ has a separating complete subgraph.*

Solution. Let X be a separating subgraph of G and assume X separates two vertices u and v, then for any vertex $w \in V$, if $(u, w) \in E$ and $(w, v) \in E$, we have $w \in X, (u, w) \in X$ and $(w, v) \in X$. As n approches ∞, the number of vertices in X will also approaches ∞. X is also a random graph, thus it has the property $P_{i,j}$, so it can not be a complete graph. □

Problem 3. *Show that if almost all $G \in \mathcal{G}(n, p)$ have a graph property $\mathcal{P}_1$ and almost all $G \in \mathcal{G}(n, p)$ have a graph property $\mathcal{P}_2$, then almost all $G \in \mathcal{G}(n, p)$ have both properties.*

Solution. By definition, we know that the probability that G dose not have the property $\mathcal{P}_1$ tends to be 0 as n approaches ∞, the same for $\mathcal{P}_2$. Thus, the probability that G has neither of the two properties tends to be 0 as n approaches ∞. The probability that G has both properties equals 1 minus the probability that G has neither of the two properties, thus it will tend to be 1 as n approaches ∞. □

Problem 4. *Consider* $\mathbf{G}(n, p)$ *with* $p = \frac{1}{3n}$.

1. *Use the second moment method to show that with high probability there exists a simple path of length 10.*

Solution. For any two different vertices i and j, we define

$$I_{ij} = \begin{cases} 1 & if\ there\ exists\ a\ path\ of\ length\ 10\ between\ i\ and\ j \\ 0 & otherwise \end{cases}$$

Let $x = \sum_{i<j} I_{ij}$, then we have

$$\begin{array}{rcl} E(x^2) & = & E(\sum_{i<j} I_{ij} \cdot \sum_{i<j} I_{ij}) \\ & = & E(\sum_{i<j} I_{ij} \cdot \sum_{k<l} I_{kl}) \\ & = & \sum_{i<j,k<l} E(I_{ij} I_{kl}) \\ & = & \sum_{i<j,k<l} E(I_{ij} I_{kl}) + \sum_{\{i,j,k\},i<j} E(I_{ij} I_{ik}) + \sum_{i<j} E(I_{ij}^2) \\ & = & \binom{n}{4}\left(9!\binom{n-2}{9}p^{10}\right)^2 + \binom{n}{3}\left(9!\binom{n-2}{9}p^{10}\right)^2 + \binom{n}{2}\left(9!\binom{n-2}{9}p^{10}\right) \end{array}$$

and $E(x) = E(\sum_{i<j} I_{ij}) = \sum_{i<j} E(I_{ij}) = 9!\binom{n}{2}\binom{n-2}{9}p^{10}$. Thus, $E^2(x) = \left(9!\binom{n}{2}\binom{n-2}{9}p^{10}\right)^2$ and we have $E(x^2) \leq E^2(x)(1 + o(1))$. By second moment method, we know that with high probability there exists a simple path of length 10. □

Problem 5. *(Optional)*

1. *Prove that the threshold for the existence of cycles in $\mathcal{G}(n, p)$ is $p = \frac{1}{n}$.*

2. *Search the* World Wide Web *to find some real world graphs in machine readable form or data bases that could automatically be converted to graphs.*

   (a) *Plot the degree distribution of each graph.*

   (b) *Compute the average degree of each graph.*

   (c) *Count the number of connected components of each size in each graph.*

   (d) *Describe what you find.*

3. *Create a simulation (an animation) to show the evolution of the $\mathcal{G}(n, p)$ (Erdös-Rényi) random graph as its density p is gradually increased. Observe the phase transitions for trees of increasing orders, followed by the emergence of the giant component, etc.*

Problem 6. *(Optional)*

Prove that 'the disappearance of isolated vertices in $\mathbf{G}(n, p)$*' has a sharp threshold of* $\frac{\ln n}{n}$.

Proof. We define that

$$I_i = \begin{cases} 1 & \text{if } v_i \text{ is an isolated vertex} \\ 0 & \text{else} \end{cases}$$

Let $x = \sum_{i=1}^{n} I_i$, by the linearity of expectation, we have $E(x) = n(1-p)^{n-1}$. Since we believe the threshold to be $\frac{\ln n}{n}$, consider $p = c\frac{\ln n}{n}$. Then, we have

$$\lim_{n\to\infty} E(x) = \lim_{n\to\infty} n(1 - c\frac{\ln n}{n})^n = \lim_{n\to\infty} ne^{-c\ln n} = \lim_{n\to\infty} n^{1-c}$$

If $c > 1$, $E(x)$ tends to go to zero as n approaches infinity. By the first moment method, we have that almost all graphs have no isolated vertices. On the other hand, if $c < 1$, we have $E(x^2) = E(\sum_{i=1}^{n} I_i \cdot \sum_{i=1}^{n} I_i) = \sum_{i=1}^{n} E(I_i^2) + 2\sum_{i<j} E(I_i I_j)$. Since I_i equals 0 or 1, $I_i^2 = I_i$, thus we have

$$E(x^2) = E(x) + n(n-1)E(I_i I_j) = E(x) + n(n-1)(1-p)^{2(n-1)-1}$$

.

$$\frac{E(x^2)}{E^2(x)} = \frac{n(1-p)^{n-1} + n(n-1)(1-p)^{2(n-1)-1}}{n^2(1-p)^{2(n-1)}} = \frac{1}{n(1-p)^{n-1}} + \left(1 - \frac{1}{n}\right)\frac{1}{1-p}$$

For $p = c\frac{\ln n}{n}$ with $c < 1$, we have $\lim_{n\to\infty} \frac{E(x^2)}{E^2(x)} = 1 + o(1)$, thus by the second moment method, we have that almost all graphs have an isolated vertex. Thus, $\frac{\ln n}{n}$ is a sharp threshold for the disappearance of isolated vertices. $\square$