Homework 6

Problem 1. *Prove that*

(a) $\left(1+\frac{1}{n}\right)^n \leq e$ *for all* $n \geq 1$.

(b) $\left(1+\frac{1}{n}\right)^{n+1} \geq e$ *for all* $n \geq 1$.

(c) Using (a) *and* (b), *conclude that* $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Solution.

(a) $\left(1+\frac{1}{n}\right)^n \leq \left(e^{\frac{1}{n}}\right)^n = e$.

(b) $\left(1+\frac{1}{n}\right)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1} = \left(\frac{1}{\frac{n}{n+1}}\right)^{n+1} = \left(\frac{1}{1-\frac{1}{n+1}}\right)^{n+1} \geq \left(e^{\frac{1}{n+1}}\right)^{n+1} = e$.

(c) $\because \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}} = 1 \therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}$.

While $e \overset{(b)}{\leq} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) \overset{(a)}{\leq} e\cdot\lim_{n\to\infty}\frac{1}{n} = e$.

□

Problem 2. *Prove* Bernoulli's inequality*: for each natural number n and for every real* $x \geq -1$, *we have* $(1+x)^n \geq 1+nx$.

Solution. Apply binomial theorem to the left. □

Problem 3. *Which of the following statements about graph G and H are true?*

1. *G and H are isomorphic if and only if for every map* $f : V(G) \to V(H)$ *and for any two vertices* $u, v \in V(G)$, *we have* $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.

2. *G and H are isomorphic if and only if there exists a bijection* $f : E(G) \to E(H)$.

3. *If there exists a bijection* $f : V(G) \to V(H)$ *such that every vertex* $u \in V(G)$ *has the same degree as* $f(u)$, *then G and H are isomorphic.*

4. *If G and H are isomorphic, then there exists a bijection* $f : V(G) \to V(H)$ *such that every vertex* $u \in V(G)$ *has the same degree as* $f(u)$.

5. *If G and H are isomorphic, then there exists a bijection* $f : E(G) \to E(H)$.

6. *G and H are isomorphic if and only if there exists a map* $f : V(G) \to V(H)$ *such that for any two vertices* $u, v \in V(G)$*, we have* $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.

7. *Every graph on n vertices is isomorphic to some graph on the vertex set* $\{1, 2, \ldots, n\}$.

8. *Every graph on* $n \geq 1$ *vertices is isomorphic to infinitely many graphs.*

Solution. 4,5,7,8. □

Problem 4. *How many graphs on the vertex set* $\{1, 2, \ldots, 2n\}$ *are isomorphic to the graph consisting of n vertex-disjoint edges (i.e. with edge set {{1,2},{3,4},..., {2n-1,2n}}?*

Solution. $\frac{(2n\cdot(2n-1))((2n-2)\cdot(2n-3))\cdots(2\cdot1)}{2^n\cdot n!} = (2n-1)(2n-3)\cdot\cdots\cdot 5\cdot 3.$

□

Problem 5. *Construct an example of a sequence of length n in which each term is some of the numbers* $1, 2, \ldots, n-1$ *and which has an even number of odd terms, and yet the sequence is not a graph score. Show why it is not a graph score.*

Solution. E.g. $(1, 1, 3, 3, 4)$. Use the *Score theorem* to prove that it cannot be a graph score. □

Problem 6. *Let G be a graph with 9 vertices, each of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or at least 6 vertices of degree 5.*

Solution. x be the number of vertex in G with $deg_G(x) = 6$. Obviously $x \geq 5$ or $x \leq 4$.

1. If $x \geq 5$ then the first part of the argument is true.

2. Otherwise ($x \leq 4$). As the other vertices in graph G are of degree 5, there are at least $9 - x \geq 5$ such vertices. According to the hand-shake lemma, there must be even number of odd-degree vertices. Thus there should be at least 6 vertices with degree 5.

□