

Homework 6

Problem 1. Which of the following statements about graph G and H are true?

1. G and H are isomorphic if and only if for every map $f : V(G) \rightarrow V(H)$ and for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.
2. G and H are isomorphic if and only if there exists a bijection $f : E(G) \rightarrow E(H)$.
3. If there exists a bijection $f : V(G) \rightarrow V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$, then G and H are isomorphic.
4. If G and H are isomorphic, then there exists a bijection $f : V(G) \rightarrow V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$.
5. If G and H are isomorphic, then there exists a bijection $f : E(G) \rightarrow E(H)$.
6. G and H are isomorphic if and only if there exists a map $f : V(G) \rightarrow V(H)$ such that for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.
7. Every graph on n vertices is isomorphic to some graph on the vertex set $\{1, 2, \dots, n\}$.
8. Every graph on $n \geq 1$ vertices is isomorphic to infinitely many graphs.

Solution. 4,5,7,8.

□

Problem 2. Two simple graphs $G = (V, E)$ and $G' = (V', E')$. A map $f : V \rightarrow V'$. Now if f satisfies:

- i) It is a bijective function;
- ii) $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$;

Then we say that graph G and G' are isomorphic to each other. We use $G \cong G'$ to stand for the isomorphism relation.

Consider the following questions:

1. $G = K_n$ (Recall: K_n is a clique with n vertices), $g : V \rightarrow V'$ is a function which only satisfies requirement ii). Prove that G' must contain a subgraph which is a clique with n -vertices.
2. $G = K_{n,m}$ (Recall: $K_{n,m}$ is the so-called complete bipartite graphs), g is the same as in question 1. What will be the simplest G' that is related to G under the new relation.

Solution.

1. Two different vertices in G must be mapped to different vertices in G' . For if $u, v \in V$ are mapped to the same vertex $\omega \in V'$, the edge $\{u, v\} \in E$ cannot be reflected in E' , for G' is a simple graph.
2. G' can be just one edge with two incident vertices.

□

Problem 3. How many graphs on the vertex set $\{1, 2, \dots, 2n\}$ are isomorphic to the graph consisting of n vertex-disjoint edges (i.e. with edge set $\{\{1,2\}, \{3,4\}, \dots, \{2n-1, 2n\}\}$)?

Solution. $\frac{(2n \cdot (2n-1))((2n-2) \cdot (2n-3)) \cdots (2 \cdot 1)}{2^n \cdot n!} = (2n-1)(2n-3) \cdots 5 \cdot 3.$

□

Problem 4. Construct an example of a sequence of length n in which each term is some of the numbers $1, 2, \dots, n-1$ and which has an even number of odd terms, and yet the sequence is not a graph score. Show why it is not a graph score.

Solution. E.g. $(1, 1, 3, 3, 4)$. Use the Score theorem to prove that it cannot be a graph score.

□

Problem 5. Let G be a graph with 9 vertices, each of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or at least 6 vertices of degree 5.

Solution. x be the number of vertex in G with $\deg_G(x) = 6$. Obviously $x \geq 5$ or $x \leq 4$.

1. If $x \geq 5$ then the first part of the argument is true.

2. Otherwise ($x \leq 4$). As the other vertices in graph G are of degree 5, there are at least $9 - x \geq 5$ such vertices. According to the hand-shake lemma, there must be even number of odd-degree vertices. Thus there should be at least 6 vertices with degree 5.

□

Problem 6. Given a sequence (d_1, d_2, \dots, d_n) of positive integers (where $n \geq 1$):

- (i) There exists a tree with score (d_1, d_2, \dots, d_n) .
- (ii) $\sum_{i=1}^n d_i = 2n - 2$.

Prove that (i) and (ii) are equivalent.

Solution.

- 1. (i) \Rightarrow (ii) is obvious.
- 2. To prove (ii) \Rightarrow (i):

By induction on the number n .

For $n = 1, 2$ the implication holds trivially, so let $n > 2$. Suppose the implication holds for any $n - 1$ long positive sequence $(d_1, d_2, \dots, d_{n-1})$ with $\sum_{i=1}^{n-1} d_i = 2(n - 1) - 2$.

For the induction step, consider an length n positive sequence $\ell = (d_1, d_2, \dots, d_n)$ with $\sum_{i=1}^n d_i = 2n - 2$:

Since the sum of the d_i is smaller than $2n$, there exists an i with $d_i = 1$. w.l.o.g. we assume $d_1 = 1$. With a similar argument we can also conclude that there must exist some index j such that $d_j \geq 2$. We take $k = \min\{j \mid d_j \geq 2\}$.

Now the sequence $\ell = (d_1, d_2, \dots, d_k, \dots, d_n) = (1, d_2, \dots, d_k - 1 + 1, \dots, d_n)$, we can derive a new sequence $\ell' = (d_2, \dots, d_k - 1, \dots, d_n)$. Obviously ℓ' is a $n - 1$ length sequence (all positive) with the summation to be $2n - 2 - 1 + 1 = 2(n - 1) - 2$. Then according to the induction hypothesis, there exists a tree \mathcal{T}' which corresponds to ℓ' .

Then $\mathcal{T} = (V(\mathcal{T}') \cup \{v_1\}, E(\mathcal{T}') \cup \{v_1, v_k\})$ is the tree which witnesses the validity of the sequence ℓ .

BE CAREFUL: Why is the following ‘proof’ of the implication (ii) \Rightarrow (i) insufficient (or, rather, makes no sense)? We proceed by induction on n . The base case $n = 1$ is easy to check, so let us assume that the implication holds for some $n \geq 1$. We want to prove it for $n + 1$. If $D = (d_1, d_2, \dots, d_n)$ is a sequence of positive integers with $\sum_{i=1}^n d_i = 2n - 2$, then we already know that there exists a tree T on n vertices with D as a score. Add another vertex v to T and connect it to any vertex of T by an edge, obtaining a tree T' on $n + 1$ vertices. Let D' be the score of T' . We know that the number of vertices increased by 1, and the sum of degrees of vertices increased by 2 (the new vertex has degree 1 and the degree of one old vertex increased by 1). Hence the sequence D' satisfies condition (ii) and it is a score of a tree, namely of T' . This finishes the inductive step. \square