Homework 6

Problem 1. *Which of the following statements about graph G and H are true?*

1. *G and H are isomorphic if and only if for every map $f : V(G) \to V(H)$ and for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.*
2. *G and H are isomorphic if and only if there exists a bijection $f : E(G) \to E(H)$.*
3. *If there exists a bijection $f : V(G) \to V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$, then G and H are isomorphic.*
4. *If G and H are isomorphic, then there exists a bijection $f : V(G) \to V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$.*
5. *If G and H are isomorphic, then there exists a bijection $f : E(G) \to E(H)$.*
6. *G and H are isomorphic if and only if there exists a map $f : V(G) \to V(H)$ such that for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.*
7. *Every graph on n vertices is isomorphic to some graph on the vertex set $\{1, 2, \ldots, n\}$.*
8. *Every graph on $n \geq 1$ vertices is isomorphic to infinitely many graphs.*

Solution. The right statements are $4, 5, 7, 8$. The rest statements are wrong for following reasons.

- For statement 1 it is not "for every map $f : V(G) \to V(H)$".
- For statement 2 "there exists a bijection $f : E(G) \to E(H)$" can not guarantee G and H are isomorphic, otherwise, any two graphs with the same number of edges are isomorphic.
- For statement 3, it is easy to find a counterexample if we treat two triangles as G and treat a hexagon as H.
- For statement 6, f should be a bijection.

□

Problem 2. *Two simple graphs $G = (V, E)$ and $G' = (V', E')$. A map $f : V \to V'$. Now if f satisfies:*

i) It is a bijective function;

ii) $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$;

Then we say that graph G and G' are isomorphic *to each other. We use $G \cong G'$ to stand for the isomorphism relation.*

Consider the following questions:

1. *$G = K_n$ (Recall: K_n is a clique with n vertices), $g : V \to V'$ is a function which only satisfies requirement ii). Prove that G' must contain a subgraph which is a clique with n-vertices.*

2. *$G = K_{n,m}$ (Recall: $K_{n,m}$ is the so-called* complete bipartite graphs*), g is the same as in question 1. What will be the simplest G' that is related to G under the new relation.*

Solution.

1. We first prove that g is an injective function, that is for any $v_1 \in V, v_2 \in V$, if $v_1 \neq v_2$, then $g(v_1) \neq g(v_2)$. Suppose it is not that case, then there exist two vertices, say v_1, v_2 such that $v_1 \neq v_2$ but $g(v_1) = g(v_2)$. Since $v_1 \neq v_2$, we can have the edge $\{v_1, v_2\}$ in G, however $g(v_1) = g(v_2)$, then we can not have the edge $\{g(v_1), g(v_2)\}$. This contradicts *ii*). Thus, we have that g is an injective function. Since g is an injective function, then the n vertices in G are mapped to n different vertices in G', and by *ii*) we have $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$. Since $G = K_n$, every two vertices have an edge between them, thus there must be n different vertices in G' that every two vertices in them are connected by an edge. So G' must contain a subgraph which is a clique with n-vertices.

2. Since g only satisfies requirement *ii*), we can let all the vertices on one side of G be mapped to a vertex and all the vertices on the other side be mapped to another vertex. Thus the simplest G' that is related to G under the new relation is $K_{1,1}$.

□

Problem 3. *How many graphs on the vertex set* $\{1, 2, \ldots, 2n\}$ *are isomorphic to the graph consisting of n vertex-disjoint edges (i.e. with edge set* $\{\{1,2\},\{3,4\},\ldots,\{2n\text{-}1,2n\}\}$*?*

Solution. To do this, we need to divide the $2n$ vertices in to n groups and each group has 2 vertices. We can do this by inserting a plank every two elements in a permutation of the $2n$ vertices, thus creat n boxes, however the order of the n boxes dose not matter, and the order of the two elements in the box also dose not matter. Thus, we can calculate the answer is $\frac{2n!}{n! \cdot 2^n} = (2n-1)!!$. □

Problem 4. *Construct an example of a sequence of length n in which each term is some of the numbers* $1, 2, \ldots, n-1$ *and which has an even number of odd terms, and yet the sequence is not a graph score. Show why it is not a graph score.*

Solution. I find it hard to give a general format, I just come up with a specific counterexample that is $1, 2, 3, 4, 5, 5$ for the case that $n = 6$. By **Score Theorem** we have $1, 2, 3, 4, 5, 5$ is a graph score if and only if $0, 1, 2, 3, 4$ is a graph score. And $0, 1, 2, 3, 4$ is a graph score if and only if $-1, 0, 1, 2$ is a graph score, since $-1, 0, 1, 2$ is not a graph score, so $1, 2, 3, 4, 5, 5$ is not a graph score. □

Problem 5. *Let G be a graph with 9 vertices, each of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or at least 6 vertices of degree 5.*

Solution. Let n denote the number of vertices of degree 6. If $n \geq 5$, then the statement is already true. Now, consider that $n \leq 4$, we say that actually $n \leq 3$. If $n = 4$, then there will be $9 - 4 = 5$ vertices of degree 5. Since the graph has 9 vertices, each of degree 5 or 6, thus this result in that the graph has odd number of vertices, which have odd degree, this contradicts the **hand-shake lemma**. Thus, we have the number of vertices of degree 5 is greater than or equal to $9 - 3 = 6$. So the statement is true. □

Problem 6. *Given a sequence* $(d_1, d_2, \ldots, d_n)$ *of positive integers (where* $n \geq 1$*):*

(i) There exists a tree with score $(d_1, d_2, \ldots, d_n)$*.*

(ii) $\sum_{i=1}^{n} d_i = 2n - 2$*.*

Prove that (i) and (ii) are equivalent.

Solution.

1. $(i) \to (ii)$. If there exists a tree with score $(d_1, d_2, \ldots, d_n)$, then we know that the tree has n vertices and by Euler's formula we know the tree has $n-1$ edges. Thus $\sum_{i=1}^{n} d_i = 2n - 2$.

2. $(ii) \to (i)$. We prove this statement by induction on n.
   Basis step. When $n = 2$, the statement is true.
   Induction hypothesis. Assume that when $n = k$ the statement is true, that is given a sequence $(d_1, d_2, \ldots, d_k)$, if $\sum_{i=1}^{k} d_i = 2k - 2$, then there exists a tree with score $(d_1, d_2, \ldots, d_k)$.
   Proof of induction step. When $n = k+1$, assume that we write the sequence in nondecreasing order, that is $d_1 \leq d_2 \leq \cdots \leq d_{k+1}$. Since $\sum_{i=1}^{k+1} d_i = 2(k+1) - 2$, we can say that there must exists a vertex whose degree is 1, assume that $d_1 = 1$. We can also say that there must exists some vertices whose degree are greater than or equal to 2. Assume that d_j is the first number in the sequence such that $d_j \geq 2$. Now consider the following sequence $(d_2, d_3, \ldots, d_j - 1, \ldots, d_{k+1})$, its length is k and the sum of its items is $2(k+1) - 2 - 1 - 1 = 2k - 2$. Thus, by induction hypothesis, we know that there exists a tree with the score sequence $(d_2, d_3, \ldots, d_j - 1, \ldots, d_{k+1})$, then we add a vertex v_1 and an edge (v_1, v_j) to it, this gives us a tree with exactly the score sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 = 1$. Thus, the statement that $(ii) \to (i)$ is true.

□