Homework 9

Problem 1. *What is the expected number of trees with k vertices in $G \in \mathcal{G}(n, p)$?*

Solution. By *Cayley's formula* and the linearity of expectation, it is $\binom{n}{k} k^{k-2} p^{k-1}$ □

Problem 2. *Show that, for constant $p \in (0, 1)$, almost no graph in $\mathcal{G}(n, p)$ has a separating complete subgraph.*

Solution. This is an application of the 'almost always true of property $P_{i,j}$'.

Detail:

- **Separating subgraph**: Given $G = (V, E)$, and some $X \subseteq V \cup E$, we call X a separating subgraph if there exists two vertices $u, v \in V(G - X)$ such that u, v are in the some component of G, while u, v lie in two disconnected components of $G - X$ (i.e., X separates u and v).
- **Separating complete subgraph**: If the above subgraph X is also a complete graph.

Now consider a graph $G = (V, E)$ with property $\mathcal{P}_{2,1}$. We claim that a graph with property $\mathcal{P}_{2,1}$ has the following property: For any pair of vertices $u, v \in G$, there exists a pair of vertices w_1, w_2 such that

$$(w_1, u) \in E, \quad (w_1, v) \in E$$

$$(w_2, u) \in E, \quad (w_2, v) \in E$$

$$(w_1, w_2) \notin E.$$

To prove the claim: consider vertices u, v and an arbitrary vertex x. By property $\mathcal{P}_{2,1}$, there exists a vertex w_1 which is neighbor to u and v, but not to x. Now using property $\mathcal{P}_{2,1}$ again (with x replaced by w_1) it follows that there exists a vertex w_2 which is neighbor to u and v, but not to w_1. Thus the claim holds.

Finally, consider a complete subgraph $H \subset G$ and two arbitrary vertices u and v in $G - V(H)$. By the claim above, there are two non-adjacent vertices w_1 and w_2 in G which are both neighbors of both u and v. Since H is complete, it follows that w_1 and w_2 cannot both belong to H, therefore remove H will not separate u and v. In another word, H does not separate G. The statement now follows since almost all graphs in $\mathcal{G}(n, p)$ have property $\mathcal{P}_{2,1}$ for any constant $p \in (0, 1)$.

□

Problem 3. *Show that if almost all $G \in \mathcal{G}(n,p)$ have a graph property $\mathcal{P}_1$ and almost all $G \in \mathcal{G}(n,p)$ have a graph property $\mathcal{P}_2$, then almost all $G \in \mathcal{G}(n,p)$ have both properties.*

Solution. The portion of the graphs have both properties equals 1 minus the portion of the graphs which does not have property $\mathcal{P}_1$ or $\mathcal{P}_2$. However the portion of the graph does not have property $\mathcal{P}_1$ or $\mathcal{P}_2$ is bounded by the sum of the portion of the graphs does not have property $\mathcal{P}_1$ and the the portion of the graphs does not have property $\mathcal{P}_2$, which both tend to 0 as n approaches ∞. The claim in the question then follows. □

Problem 4. *Consider* $\mathbf{G}(n,p)$ *with* $p = \frac{1}{3n}$.

1. *Use the second moment method to show that with high probability there exists a simple path of length 10.*

Solution. [Use the second moment method]

$I_\ell = \begin{cases} 1 & \ell \text{ is a length 10 simple path} \\ 0 & \text{otherwise} \end{cases}$

X is the number of simple path, then $X = \sum_\ell I_\ell$.

The expectation would be $E(X) = \frac{1}{2}(n)_{11} \times p^{10}$. Thus

$$(E(X))^2 = \frac{1}{4}\,[(n)_{11}]^2 \times p^{20} \qquad (\star)$$

(Note $(\star)$ is about $\Theta(n^2)$.)

Then to calculate $E(X^2)$.

$$E(X^2) = E\left[(\sum_\ell I_\ell)^2\right] = E\left[\sum_\ell I_\ell \sum_{\ell'} I_{\ell'}\right] = \sum_{\ell,\ell'} E(I_\ell I_{\ell'}) \qquad (\Delta)$$

ℓ and ℓ' will be independent to each other unless they have common vertices or edges. We use $k = |\ell \cap \ell'|$ to stand for the number of common edges between ℓ and ℓ', and $s = |\ell \cap \ell'|$ to stand for the number of common vertices used by ℓ and ℓ'. Obviously $(0 \le k \le 10) \wedge (0 \le s \le 11) \wedge (k \ge 1 \to s \ge k+1)$.

(Δ) can be divided into the following subcases:

1. $k = 0$

(a) $s = 0$: $\sum_{\ell,\ell'} E(I_\ell I_{\ell'}) = \frac{1}{8}(n)_{22} \times p^{20} \leq (E(X))^2$;

(b) $1 \leq s \leq 11$:
for each s, $\sum_{\ell,\ell'} E(I_\ell I_{\ell'}) = c \cdot (n)_{22-s} \times (p)^{20} = \mathbf{o}((E(X))^2)$, where c is a constant number.

2. $1 \leq k \leq 10$ ($2 \leq s \leq 11$)
The general formula of each of these cases (constant many) would be

$$\begin{aligned}
\sum_{\ell,\ell'} E(I_\ell I_{\ell'}) &= d \cdot (n)_{22-s} \times p^{20-k} \\
&\leq d \cdot (n)_{22-s} \times p^{20-(s-1)} \\
&= d \cdot (n)_{22-s} \times p^{21-s} \\
&= \mathbf{o}((E(X))^2)
\end{aligned}$$

Combining the above results we get that $Var(X) = E(X^2) - (E(X))^2 = \mathbf{o}((E(X))^2)$.
□

Problem 5. *(Optional)*

1. *Prove that the threshold for the existence of cycles in $\mathcal{G}(n,p)$ is $p = \frac{1}{n}$.*

2. *Search the* World Wide Web *to find some real world graphs in machine readable form or data bases that could automatically be converted to graphs.*

   (a) *Plot the degree distribution of each graph.*
   (b) *Compute the average degree of each graph.*
   (c) *Count the number of connected components of each size in each graph.*
   (d) *Describe what you find.*

3. *Create a simulation (an animation) to show the evolution of the $\mathcal{G}(n,p)$ (Erdös-Rényi) random graph as its density p is gradually increased. Observe the phase transitions for trees of increasing orders, followed by the emergence of the giant component, etc.*

Problem 6. *(Optional)*
Prove that 'the disappearance of isolated vertices in $\mathbf{G}(n,p)$' has a sharp threshold of $\frac{\ln n}{n}$.

Solution. John's book, Theorem 8.6. - -+ □