Homework 7

Problem 1. *Find an example to verify the claim that '(pairwise) independence does not imply mutual independence'. Pls give a detailed proof.*

Solution. (by S. Bernstein)

Suppose X and Y are two independent tosses of a fair coin, where we designate 1 for heads and 0 for tails. Let the third random variable $Z = (X + Y) \bmod 2$.

Then jointly the triple $\langle X, Y, Z\rangle$ has the following probability distribution:

$$\langle X, Y, Z\rangle = \begin{cases} \langle 0,0,0\rangle & \text{with probability } 1/4 \\ \langle 0,1,1\rangle & \text{with probability } 1/4 \\ \langle 1,0,1\rangle & \text{with probability } 1/4 \\ \langle 1,1,0\rangle & \text{with probability } 1/4 \end{cases}$$

$i, j, k \in \{0, 1\}$.

It is easy to verify that $Pr(X = i) = Pr(Y = j) = Pr(Z = k) = 1/2$ and $Pr(X = i, Y = j) = Pr(X = i, Z = k) = Pr(Y = j, Z = k) = 1/4$. i.e., X, Y, Z are pairwise independent.

However, $Pr(X = i, Y = j, Z = k) \neq Pr(X = i) \cdot Pr(Y = j) \cdot Pr(Z = k)$. For example, the left side equals $1/4$ for $\langle x, y, z\rangle = \langle 0, 0, 0\rangle$ while the right side equals $1/8$.

In fact, any of $\langle X, Y, Z\rangle$ is completely determined by the first two components. That is as far from independence as random variables can get.

□

Problem 2. *Show that, if $E_1, E_2, \dots, E_n$ are mutually independent, then so are $\overline{E_1}, \overline{E_2}, \dots, \overline{E_n}$.*

Solution. (sketch) It will be enough to prove that for any $2 \leq k \leq n$, and $\{F_1, F_2, \dots, F_k\} \subseteq \{E_1, E_2, \dots, E_n\}$

$$Pr\Big(\bigcap_{i=1}^{k} \overline{F_i}\Big) = \prod_{i=1}^{k} Pr(\overline{F_i})$$

Let $Pr(F_i) = f_i$, then

$$Pr\Big(\bigcap_{i=1}^{k} \overline{F_i}\Big) = 1 - Pr\Big(\bigcup_{i=1}^{k} F_i\Big) = 1 - \sum_{i=1}^{k} f_i + \sum_{1\leq i<j\leq k} f_i f_j - \sum_{1\leq i<j<l\leq k} f_i f_j f_l + \cdots$$

The right hand side of the above equation is $(1 - f_1)(1 - f_2)\cdots(1 - f_k) = \prod_{i=1}^{k} Pr(\overline{F_i})$. □

Problem 3. *The problem on the 37*[st] *page of slide on '*Probability: a quick review*' (i.e.,* the more complicated example*). (What is* $Pr(U|W)$ *?)*

Solution.

$$Pr(U|W) = \frac{Pr(U \cap W)}{Pr(W)} = \frac{Pr(U \cap W)}{Pr(R) \cdot Pr(W|R) + Pr(\neg R) \cdot Pr(W|\neg R)} \quad \text{//Law of total probability}$$

$$= \frac{\Pr(R) \cdot Pr(U \cap W|R) + \Pr(\neg R) \cdot Pr(U \cap W|\neg R)}{Pr(R) \cdot Pr(W|R) + Pr(\neg R) \cdot Pr(W|\neg R)} \quad \text{//Law of total probability}$$

□

Problem 4. *Suppose X and Y are two independent random variables, show that*

$$E(X \cdot Y) = E[X] \cdot E[Y]$$

.

Solution.

$$\begin{aligned} E[X \cdot Y] &= \Sigma_i \Sigma_j (i \cdot j) \cdot Pr((X = i) \cap (Y = j)) \\ &= \Sigma_i \Sigma_j (i \cdot j) \cdot Pr(X = i) \cdot Pr(Y = j)) \\ &= (\Sigma_i i \cdot Pr(X = i)) \left(\Sigma_j j \cdot Pr(Y = j) \right) \\ &= E(X) \cdot E(Y) \end{aligned}$$

□

Problem 5. *A monkey types on a 26 -letter keyboard that has lowercase letters only. Each letter is chosen independently and uniformly at random from the alphabet. If the monkey types 1,000,000 letters. what is the expected number of times the sequence "proof" appears?*

Solution. By the linearity of expectation:

$$E[X] = (1/26)^5 \times (1000000 - 4)$$

□