Set Theory
----Paradox & Cardinality

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试讲课调查结果

3.《离散数学》课上是否学习过集合论，如对角线方法等 [多选题]

选项	小计	比例
是	37	38.54%
否	59	61.46%
自学过	2	2.08%
本题有效填写人次	**96**	

Quick review of part I.

Slides are available in canvas.

Brief History of Set Theory

- Georg Cantor(1845-1918)
  - German mathematician
  - Founder of set theory
- Bertrand Russell(1872-1970)
  - British philosopher, logician, mathematician, historian, and social critic.
- Ernst Zermelo(1871-1953)
  - German mathematician, foundations of mathematics and hence on philosophy
- David Hilbert (1862-1943)
  - German mathematician, one of the most influential and universal mathematicians of the 19th and early 20th centuries.
- Kurt Gödel(1906-1978)
  - Austrian American logician, mathematician, and philosopher, 1938: ZF not $\vdash \neg$CH, $\neg$AC .
- Paul Cohen(1934-2007)
  - American mathematician, 1963: ZF not $\vdash$ CH,AC .






Hilbert's twenty-three problems are:

Problem	Brief explanation
1st	The continuum hypothesis (that is, there is no set whose cardinality is strictly between that of the integers and that of the real numbers)
2nd	Prove that the axioms of arithmetic are consistent.
3rd	Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second?
4th	Construct all metrics where lines are geodesics.
5th	Are continuous groups automatically differential groups?
6th	Mathematical treatment of the axioms of physics
7th	Is a^b transcendental, for algebraic $a \neq 0,1$ and irrational algebraic b ?
8th	The Riemann hypothesis ("the real part of any non-trivial zero of the Riemann zeta function is ½") and other prime number problems, among them Goldbach's conjecture and the twin prime conjecture
9th	Find the most general law of the reciprocity theorem in any algebraic number field.
10th	Find an algorithm to determine whether a given polynomial Diophantine equation with integer coefficients has an integer solution.
11th	Solving quadratic forms with algebraic numerical coefficients.
12th	Extend the Kronecker–Weber theorem on abelian extensions of the rational numbers to any base number field.
13th	Solve 7-th degree equation using continuous functions of two parameters.
14th	Is the ring of invariants of an algebraic group acting on a polynomial ring always finitely generated?
15th	Rigorous foundation of Schubert's enumerative calculus.
16th	Describe relative positions of ovals originating from a real algebraic curve and as limit cycles of a polynomial vector field on the plane.
17th	Express a nonnegative rational function as quotient of sums of squares.
18th	(a) Is there a polyhedron which admits only an anisohedral tiling in three dimensions? (b) What is the densest sphere packing?
19th	Are the solutions of regular problems in the calculus of variations always necessarily analytic?
20th	Do all variational problems with certain boundary conditions have solutions?
21st	Proof of the existence of linear differential equations having a prescribed monodromic group
22nd	Uniformization of analytic relations by means of automorphic functions
23rd	Further development of the calculus of variations

What is a set ?

- By *Georg Cantor* in 1870s:

  A set is an unordered collection of objects.

  - The objects are called the *elements*, or *members*, of the set. A set is said to *contain* its elements.

- Notation: $a \in A$

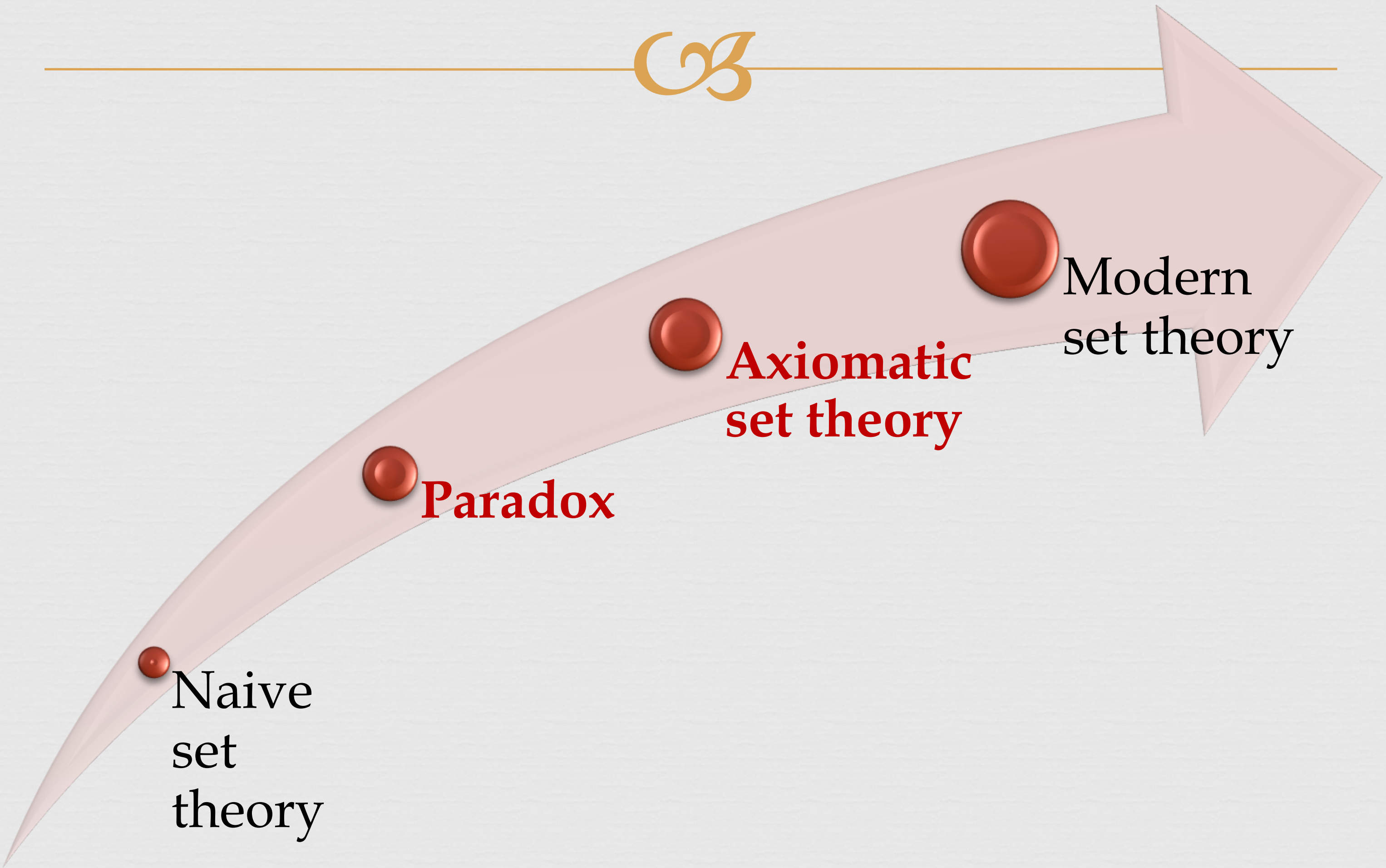
  - Meaning that: a is an element of the set A, or,

  Set A contains a .

Key points one should know of

- Set operations
  - $A \cup B$, $A \cap B$, $A - B$, $\bar{A}$, $A \oplus B$, $P(A)$
- Set identity laws
- Set applications
  - Relation
    - ✓ Ordered pairs, $A \times B$, Relation, Equivalence relation, Partition
  - Function
    - ✓ Onto function/Surjective function
    - ✓ Injective function/One-to-one function/Single-rooted
    - ✓ Bijective function

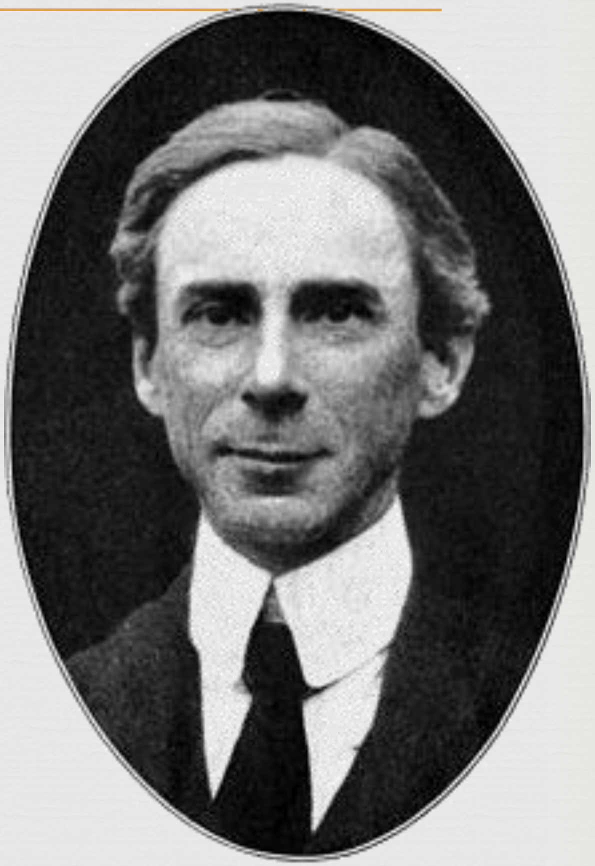
Paradox	• **Paradox and ZFC**
Equinumerosity	• Equinumerosity
Cardinal Numbers	• Ordering
Infinite Cardinals	• Countable sets


Naive
set
theory
Paradox
Axiomatic
set theory
Modern
set theory

Russell`s paradox

- Bertrand Russell(1872-1970)
- British philosopher, logician, mathematician, historian, and social critic.
- In 1950 Russell was awarded the Nobel Prize in Literature, "*in recognition of his varied and significant writings in which he champions humanitarian ideals and freedom of thought*."
- ***What I have lived for?***

  Three passions, simple but overwhelmingly strong, have governed my life: the longing for love, the search for knowledge, and unbearable pity for the suffering of mankind....



Barber Paradox[1918]

- Suppose there is a town with just one male barber. The barber shaves *all* and *only* those men in town who do not shave themselves.
- Question: Does the barber shave himself?
  - If the barber does NOT shave himself, then he MUST abide by the rule and shave himself.
  - If he DOES shave himself, according to the rule he will NOT shave himself.

Formal Proof

- **Theorem** There is no set to which every set belongs. [Russell, 1902]

Proof:

Let A be a set; we will construct a set not belonging to A. Let

$$B=\{x \in A \mid x \notin x\}$$

We claim that $B \notin A$. we have, by the construction of B.

$$B \in B \text{ iff } B \in A \text{ and } B \notin B$$

If $B \in A$, then this reduces to

$B \in B$ iff $B \notin B$, Which is impossible, since one side must be true and the other false. Hence $B \notin A$

Natural Numbers in Set Theory

- Constructing the natural numbers in terms of sets is part of the process of

"Embedding mathematics in set theory"

John von Neumann

- December 28, 1903 – February 8, 1957. Hungarian American mathematician who made major contributions to a vast range of fields:
  - Logic and set theory
  - Quantum mechanics
  - Economics and game theory
  - Mathematical statistics and econometrics
  - Nuclear weapons
  - Computer science

Natural numbers

- By *von Neumann*:

Each natural number is the set of all smaller natural numbers.

$0 = \emptyset$

$1 = \{0\} = \{\emptyset\}$

$2 = \{0,1\} = \{\emptyset, \{\emptyset\}\}$

$3 = \{0,1,2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

……

Some properties from the first four natural numbers

$0 = \emptyset$

$1 = \{0\} = \{\emptyset\}$

$2 = \{0,1\} = \{\emptyset, \{\emptyset\}\}$

$3 = \{0,1,2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

$0 \in 1 \in 2 \in 3 \in \cdots$

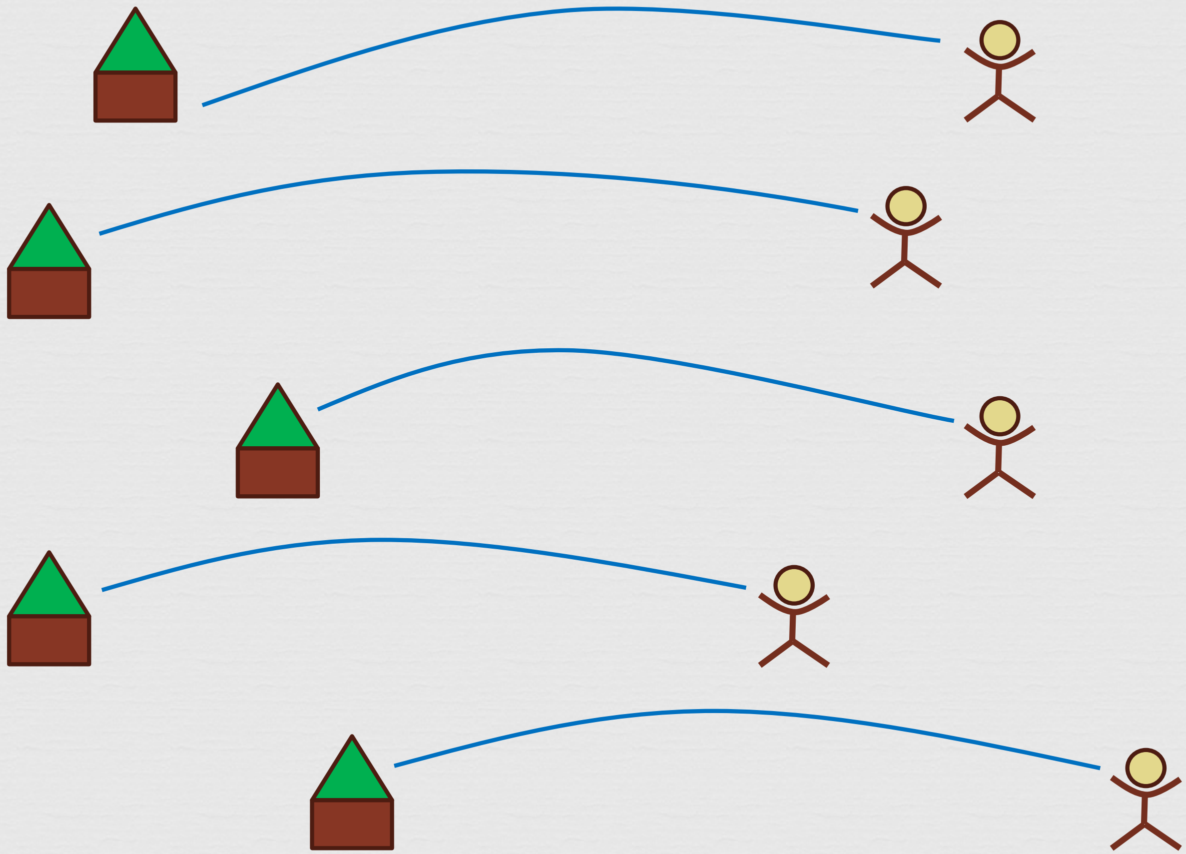
$0 \subseteq 1 \subseteq 2 \subseteq 3 \subseteq \cdots$

Paradox	• Paradox and ZFC
Equinumerosity	• **Equinumerosity**
Cardinal Numbers	• Ordering
Infinite Cardinals	• Countable sets

Motivation

- To discuss the **size** of sets. Given two sets A and B, we want to consider such questions as:
  - Do A and B have the same size?
  - Does A have more elements than B?

Example



Equinumerosity

- **Definition** A set $\mathcal{A}$ is ***equinumerous*** to a set $\mathcal{B}$ (written $\mathcal{A} \approx \mathcal{B}$) iff there is a one-to-one function from $\mathcal{A}$ onto $\mathcal{B}$.

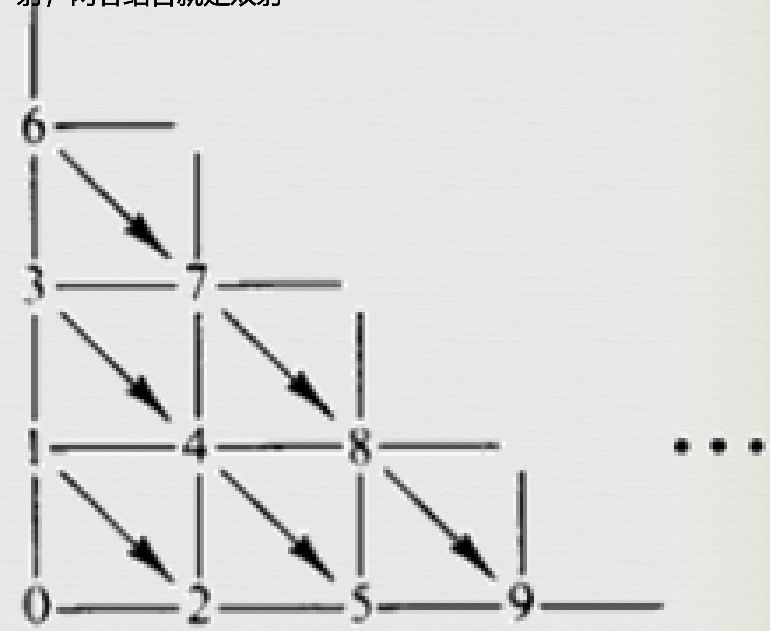
- A one-to-one function from $\mathcal{A}$ onto $\mathcal{B}$ is called a *one-to-one correspondence* between $\mathcal{A}$ and $\mathcal{B}$.

Example: $\boxed{\omega \times \omega \approx \omega}$

- The set $\omega \times \omega$ is equinumerous to ω. There is a function J mapping $\omega \times \omega$ one-to-one onto ω.

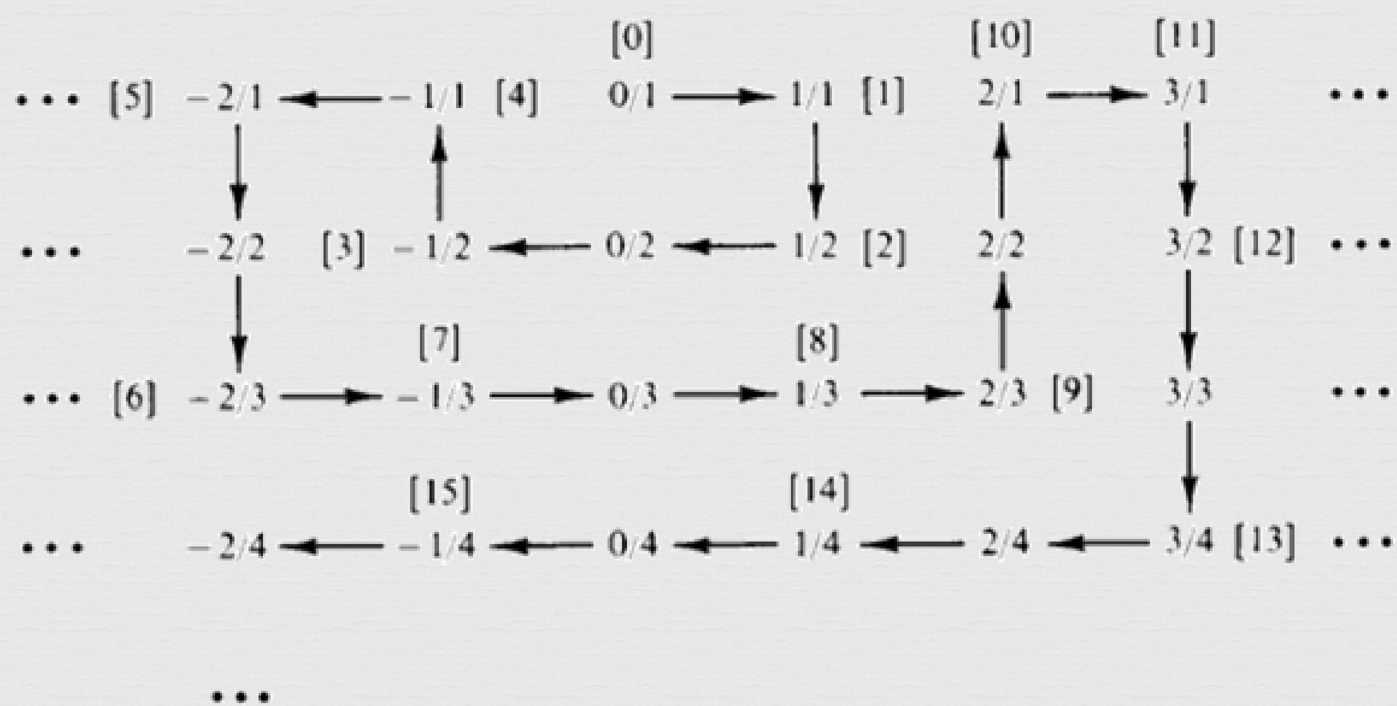
one-to-one是单射，onto是满射，两者结合就是双射

$J(m,n)=((m+n)^2+3m+n)/2$

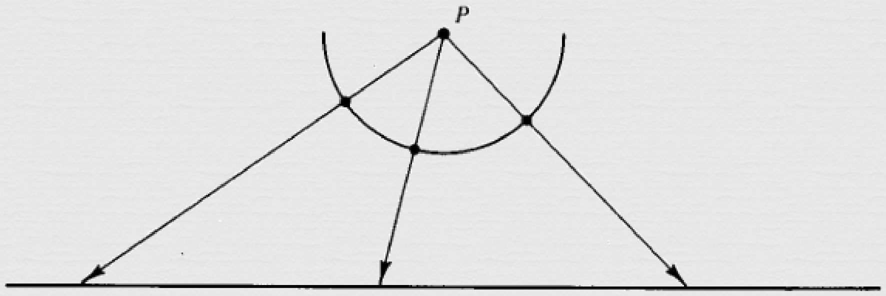


Example: $\omega \approx \mathbb{Q}$

- f: $\omega \longrightarrow \mathbb{Q}$



Example: $\boxed{(0,1) \approx \mathbb{R}}$

- $(0,1)=\{x \in \mathbf{R} \mid 0<x<1\}$, then $(0,1) \approx \mathbf{R}$ 区间也是集合
  - 

  - $f(x)= \tan(\pi(2x-1)/2)$

Examples

- $(0,1) \approx (n,m)$
  - Proof: $f(x) = (n-m)x+m$
- $(0,1) \approx \{x \mid x \in \omega \wedge x>0\} = (0,+\infty)$
  - Proof: $f(x)=1/x -1$
- $[0,1] \approx [0,1)$
  - Proof: $f(x)=x$ if $0 \le x<1$ and $x \ne 1/(2^n)$, $n \in \omega$
    $f(x)=1/(2^{n+1})$ if $x=1/(2^n)$, $n \in \omega$
- $[0,1) \approx (0,1)$
  - Proof: $f(x)=x$ if $0<x<1$ and $x \ne 1/(2^n)$, $n \in \omega$
    $f(0)=1/2$ $x=0$
    $f(x)=1/(2^{n+1})$ if $x=1/(2^n)$, $n \in \omega$
- $[0,1] \approx (0,1)$

Example: $\wp(\mathrm{A}) \approx {}^{\mathrm{A}}2$

- For any set A, we have $P(A) \approx {}^{A}2$.

Proof: Define a function H from $P(A)$ onto ${}^{A}2$ as:

For any subset B of A, $H(B)$ is the characteristic function of B:

$$f_B(x) = \begin{cases} 1 & \text{if} \quad x \in B \\ 0 & \text{if} \quad x \in A - B \end{cases}$$

H is one-to-one and onto.

Theorem

- For any sets A, B and C:
  - $A \approx A$
  - If $A \approx B$ then $B \approx A$
  - If $A \approx B$ and $B \approx C$ then $A \approx C$.

Proof:

Theorem(Cantor 1873)

- The set ω is not equinumerous to the set **R** of real numbers.
- No set is equinumerous to its power set.

2是1的推论

The set ω is not equinumerous to the set **R** of real numbers.

Proof: show that for any functon $f: \omega \rightarrow \mathbf{R}$, there is a real number z not belonging to *ran f*

$$f(0) = 32.4345\ldots,$$
$$f(1) = -43.334\ldots,$$
$$f(2) = 0.12418\ldots,$$
$$\ldots\ldots$$

z: the integer part is 0, and the $(n+1)^{st}$ decimal place of z is 7 unless the $(n+1)^{st}$ decimal place of *f(n)* is 7, in which case the $(n+1)^{st}$ decimal place of z is 6.

Then z is a real number not in *ran f*.

No set is equinumerous to its power set.

Proof: Let $g: A \to \wp(A)$; we will construct a subset B of A that is not in *ran g*. Specifically, let

$$B=\{x \in A \mid x \notin g(x)\}$$

Then $B \subseteq A$, but for each $x \in A$

$$x \in B \text{ iff } x \notin g(x)$$

这里用到的是两个集合相等当且仅当两个集合里面的元素相同

Hence $B \neq g(x)$.

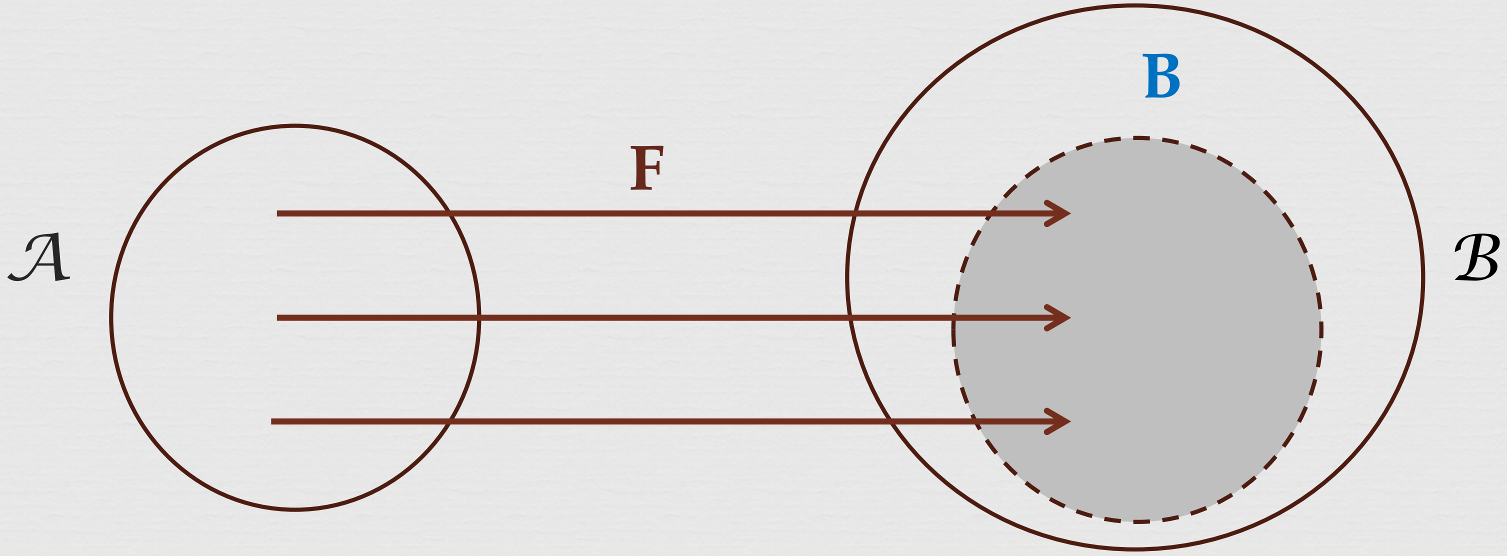
Paradox	• Paradox and ZFC
Equinumerosity	• Equinumerosity
Cardinal Numbers	• **Ordering**
Infinite Cardinals	• Countable sets

Ordering Cardinal Numbers

- **Definition** A set $\mathcal{A}$ is **dominated** by a set $\mathcal{B}$ (written $\mathcal{A} \preccurlyeq \mathcal{B}$) iff there is a ***one-to-one*** function from $\mathcal{A}$ into $\mathcal{B}$. 存在一个从A到B的单射，也就是A里面的元素应该小于B中的元素

Examples

- Any set dominates itself.
- If $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}$ is dominated by $\mathcal{B}$.
- $\mathcal{A} \preccurlyeq \mathcal{B}$ iff $\mathcal{A}$ is equinumerous to some subset of $\mathcal{B}$.



Schröder-Bernstein Theorem

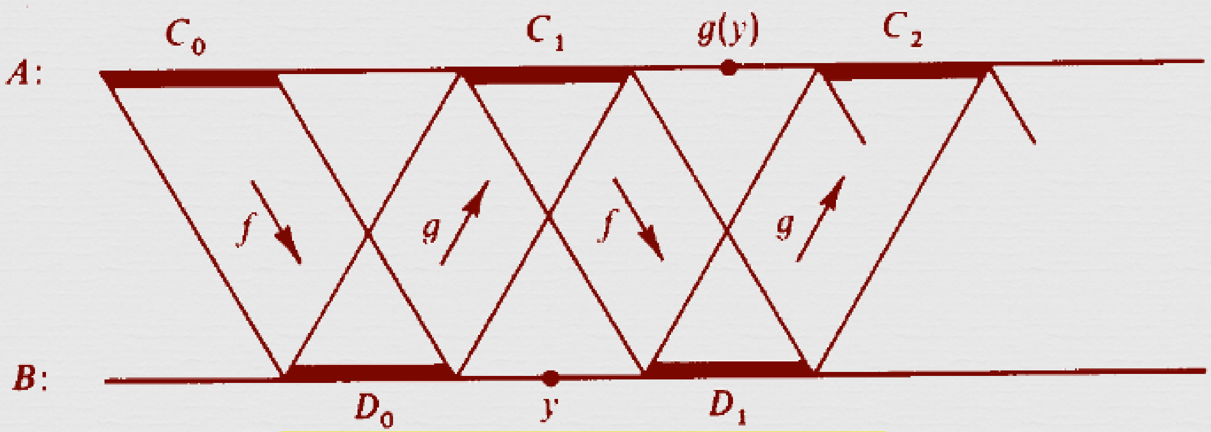
- If $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \approx B$.

Proof:

$f: A \to B$, $g: B \to A$. Define C_n by recursion:

$C_0 = A - ran\ g$ and $C_n^+ = g[f[C_n]]$

$$h(x) = \begin{cases} f(x) & \text{if } x \in C_n \text{ for some } n, \\ g^{-1}(x) & \text{otherwise} \end{cases}$$



$h(x)$ is one-to-one and onto.

Application of the Schröder-Bernstein Theorem

- Example
  - If $A \subseteq B \subseteq C$ and $A \approx C$, then all three sets are equinumerous.
  - The set **R** of real numbers is equinumerous to the closed unit interval $[0,1]$.

- $\aleph_0$ is the *least infinite* cardinal. i.e. $\omega \preccurlyeq A$ for any infinite A.
- $\aleph_0 \cdot 2^{\aleph_0} = ?$

$$2^{\aleph_0} \leq \aleph_0 \cdot 2^{\aleph_0} \leq 2^{\aleph_0} \cdot 2^{\aleph_0} = \boxed{2^{\aleph_0}}$$

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Countable Sets

- **Definition** A set A is countable iff $A \preccurlyeq \omega$,
- Intuitively speaking, the elements in a countable set can *be counted by* means of the natural numbers.
- An equivalent definition: A set A is countable iff either A is finite or $A \approx \omega$.

Example

- ω is countable, as is **Z** and **Q**
- **R** is uncountable
- A, B are countable sets
  - $\forall\ C \subseteq A$, C is countable
  - $A \cup B$ is countable
  - $A \times B$ is countable
- For any infinite set A, $\wp(A)$ is uncountable.

Continuum Hypothesis

- Are there any sets with cardinality between $\aleph_0$ and $2^{\aleph_0}$?
- Continuum hypothesis (Cantor): No.

  i.e., there is no λ with $\aleph_0 < \lambda < 2^{\aleph_0}$.

  Or, equivalently, it says: Every uncountable set of real numbers is equinumerous to the set of all real numbers.

GENERAL VERSION: for any infinite cardinal κ, there is no cardinal number between κ and 2^{κ} .

HISTORY

- Georg Cantor: 1878, proposed the conjecture
- David Hilbert: 1900, the first of Hilbert's 23 problems.
- Kurt Gödel: 1939, $\mathrm{ZFC} \nvdash \neg\mathrm{CH}$.
- Paul Cohen: 1963, $\mathrm{ZFC} \nvdash \mathrm{CH}$.

Thanks!