Generating Function

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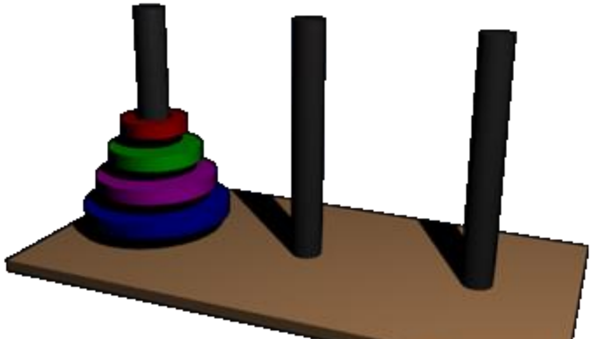


- Problem: How many ways are there to pay the amount of 21 doublezons if we have
  - 6 one-doublezon coins;
  - 5 two-doublezon coins;
  - 4 five-doublezon coins.
- Solution:

$$\boldsymbol{i_1 + i_2 + i_3 = 21} \quad \textbf{(*)}$$

$$\boldsymbol{i_1 \in \{0,1,2,3,4,5,6\};\ i_2 \in \{0,2,4,6,8,10\};\ i_3 \in \{0,5,10,15,20\}.}$$

$$\boldsymbol{(1+x+x^2+x^3+\cdots+x^6)(1+x^2+x^4+x^6+x^8+x^{10})}$$
$$\boldsymbol{\cdot\,(1+x^5+x^{10}+x^{15}+x^{20})}$$

The coefficient of $\boldsymbol{x^{21}}$
= the number of solutions of (*).

Recall

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 \cdots \binom{n}{n}x^n$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \sum_{k=0}^{n}\binom{n}{k} = 2^n$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \cdots = \sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0$$

将第一个式子左右两边求导，再将x = 1 代入即可

$$\boxed{\sum_{k=0}^{n} k\binom{n}{k} = n\,2^{n-1}}$$

Generating function

- $(a_0, a_1, a_2, \ldots)$ be a sequence of real numbers, then the generating function of this sequence is defined as

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Example:

$(1,1,1,\ldots)$ $\qquad a(x) = 1 + x + x^2 + \cdots = \dfrac{1}{1-x}$

Generalized binomial theorem

$$\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$$

If r is a negative integer

$$\boxed{\binom{r}{k}} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$$

$$= (-1)^k \frac{(-r)(-r+1)(-r+2)\cdots(-r+k-1)}{k!}$$

$$\boxed{= (-1)^k \binom{-r+k-1}{k} = (-1)^k \binom{-r+k-1}{-r-1}}$$

$$\boxed{\frac{1}{(1-x)^n} = \binom{n-1}{n-1} + \binom{n}{n-1} x + \binom{n+1}{n-1} x^2 + \cdots \binom{n+k-1}{n-1} x^k + \cdots}$$

More examples

$(a_0, a_1, a_2, \dots)$ $\quad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(0, a_0, a_1, a_2, \dots)$ $\quad a(x) = 0 + a_0 x + a_1 x^2 + \cdots = \boxed{x \cdot a(x)}$

$(0, 1, \frac{1}{2}, \frac{1}{3}, \dots)$ $\quad a(x) = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\ln(1 - x) \quad -1 < x < 1$

$(1, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \dots)$ $\quad a(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$

Operations with Sequences -Addition

$(a_0, a_1, a_2, \ldots)$ $\qquad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(b_0, b_1, b_2, \ldots)$ $\qquad b(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$(a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots)$ $\qquad a(x) + b(x)$

两个序列和的生成函数等于两个序列生成函数的和

Constant linear expansion

$(a_0, a_1, a_2, \ldots)$ $\qquad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(\alpha a_0, \alpha a_1, \alpha a_2, \ldots)$ $\qquad \alpha \cdot a(x)$

在一个序列上乘以一个常数得到新序列的生成函数等于在原序列生成函数上乘以同一个常数

Shifting-right

$(a_0, a_1, a_2, \ldots)$ $\qquad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(\underbrace{0, 0, \cdots, 0}_{n}, a_0, a_1, a_2, \ldots)$ $\qquad \boxed{x^n \cdot a(x)}$

在一个序列前面添加n个0，得到新序列的生成函数等于原序列生成函数乘以x^n

Shifting-left

$(a_0, a_1, a_2, \dots)$ $\qquad$ $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(a_3, a_4, a_5, \dots)$ $\qquad$ $\dfrac{a(x) - a_0 - a_1 x - a_2 x^2}{x^3}$

Substituting $-\alpha x$

$(a_0, a_1, a_2, \ldots)$ $\quad$ $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(a_0, \alpha a_1, \alpha^2 a_2, \ldots)$ $\quad$ $a(\alpha x)$

Example:

$(1,1,1,\ldots)$ $\quad$ $a(x) = \dfrac{1}{1-x}$

$(1,2,4,8,\ldots)$ $\quad$ $a(2x) = \dfrac{1}{1-2x}$

$(a_0, 0, a_2, 0, a_4, 0 \ldots)$ $\quad$ $\dfrac{1}{2}(a(x) + a(-x))$

Substituting $-x^n$

$(a_0, a_1, a_2, \ldots)$ $\quad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(a_0, 0,0, a_1, 0,0, a_2, 0,0, \ldots)$ $\quad a_0 + a_1 x^3 + a_2 x^6 + \cdots = a(x^3)$

Example:

$(1,1,2,2,4,4,8,8, \ldots)$ i.e., $a_n = 2^{\lfloor n/2 \rfloor}$ $\quad \dfrac{1+x}{1-2x^2}$

$(1,2,4,8, \ldots)$ $\quad \dfrac{1}{1-2x}$

$(1,0,2,0,4,0,8, \ldots)$ $\quad \dfrac{1}{1-2x^2}$

$(0,1,0,2,0,4,0,8, \ldots)$ $\quad \dfrac{x}{1-2x^2}$

Differentiation

$(a_0, a_1, a_2, \dots)$ $\qquad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(a_1, 2a_2, 3a_3, \dots)$ $\qquad a'(x)$

Example:

$(1^2, 2^2, 3^2, 4^2, \dots)$ i.e., $a_k = (k+1)^2$

$(1,1,1,1, \dots)$ $\quad \dfrac{1}{1-x}$

$(1,2,3,4, \dots, k+1, \dots)$ $\quad \left(\dfrac{1}{1-x}\right)' = \dfrac{1}{(1-x)^2}$

$(2 \cdot 1, 3 \cdot 2, 4 \cdot 3, \dots, (k+2)(k+1), \dots$ $\quad \left(\dfrac{1}{1-x}\right)'' = \dfrac{2}{(1-x)^3}$

Differentiation

$(a_0, a_1, a_2, \dots)$ $\quad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(a_1, 2a_2, 3a_3, \dots)$ $\quad \boxed{a'(x)}$

Example:

$(1^2, 2^2, 3^2, 4^2, \dots)$ i.e., $a_k = (k+1)^2$ $\quad \boxed{\dfrac{2}{(1-x)^3} - \dfrac{1}{(1-x)^2}}$

$(1,1,1,1,\dots)$ $\quad \dfrac{1}{1-x}$

$(1,2,3,4,\dots,\underline{k+1,}\dots)$ $\quad \left(\dfrac{1}{1-x}\right)' = \dfrac{1}{(1-x)^2}$

$(2\cdot 1, 3\cdot 2, 4\cdot 3, \dots, \underline{(k+1)^2 + k + 1}, \dots$ $\quad \left(\dfrac{1}{1-x}\right)'' = \dfrac{2}{(1-x)^3}$

Integration

$(a_0, a_1, a_2, \dots)$ $\qquad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(0, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2, \frac{1}{4}a_{3,} \dots)$ $\qquad \int_0^x a(t)dt$

Multiplication/Convolution

$(a_0, a_1, a_2, \ldots)$ $\qquad a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$(b_0, b_1, b_2, \ldots)$ $\qquad b(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$(c_0, c_1, c_2, \ldots)$ $\qquad a(x) \cdot b(x)$

$$c_0 = a_0 b_0$$

$$c_1 = a_0 b_1 + a_1 b_0$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$$

$$\vdots$$

$$\boxed{c_k = \sum_{i,j \geq 0; i+j=k} a_i b_j}$$

Solving Recurrence

- Recurrence relation
  Define g_i recursively.
- Manipulation:
  New equivalence concerning $G(x)$.
- Solving
  Closed form for $G(x)$.
- Expanding
  New form for g_i.

Application

- A box contains 30 red, 40 blue, and 50 green balls; balls of the same color are indistinguishable. How many ways are there of selecting a collection of 70 balls from the box?
- Solution:

$$(1 + x + x^2 + \cdots + x^{30}) \cdot (1 + x + x^2 + \cdots + x^{40}) \cdot (1 + x + x^2 + \cdots + x^{50})$$

$$= \left(\frac{1 - x^{31}}{1 - x}\right) \cdot \left(\frac{1 - x^{41}}{1 - x}\right) \cdot \left(\frac{1 - x^{51}}{1 - x}\right)$$

$$= \frac{1}{(1 - x)^3}(1 - x^{31})(1 - x^{41})(1 - x^{51})$$

$$(1 + x + x^2 + \cdots + x^{30}) \cdot (1 + x + x^2 + \cdots + x^{40}) \cdot (1 + x + x^2 + \cdots + x^{50})$$

$$= \frac{1}{(1-x)^3}(1 - x^{31})(1 - x^{41})(1 - x^{51})$$

$$\boxed{\frac{1}{(1-x)^n} = \binom{n-1}{n-1} + \binom{n}{n-1} x + \binom{n+1}{n-1} x^2 + \cdots \binom{n+k-1}{n-1} x^k + \cdots}$$

$$= \left(\binom{2}{2} + \binom{3}{2} x + \binom{4}{2} x^2 + \cdots\right)(1 - x^{31} - x^{41} - x^{51} + \cdots)$$

Thus the coefficient of x^{70} is：

$$= \binom{70+2}{2} - \binom{70+2-31}{2} - \binom{70+2-41}{2} - \binom{70+2-51}{2}$$

$$= 1061$$

Application

- Fibonacci Number

$$f_0 = 0, \quad f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2}$$

$$F(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_{n-2} x^{n-2} + f_{n-1} x^{n-1} + f_n x^n + \cdots$$

Fibonacci Sequence

- $f_n = f_{n-1} + f_{n-2}$
- $F(x) = f_0 + f_1 x + \boxed{f_2 x^2} + \cdots + f_{n-2}x^{n-2} + f_{n-1}x^{n-1} + \boxed{f_n x^n} + \cdots$
- $xF(x) = \quad f_0 x + \boxed{f_1 x^2} + f_2 x^3 + \cdots\cdots\cdots + f_{n-2}x^{n-1} + \boxed{f_{n-1}x^n} + \cdots$
- $x^2F(x) = \quad \boxed{f_0 x^2} + f_1 x^3 + f_2 x^4 + \cdots\cdots\cdots\cdots\cdots + \boxed{f_{n-2}x^n} + f_{n-1}x^{n+1} + \cdots$

$$F(x) - xF(x) - x^2F(x) = f_0 + (f_1 - f_0)x$$

$$F(x) = \frac{x}{1 - x - x^2}$$

$$= \frac{a}{1-\lambda_1 x} + \frac{b}{1-\lambda_2 x} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1+\sqrt{5}}{2}x} - \frac{1}{1-\frac{1-\sqrt{5}}{2}x}\right)$$

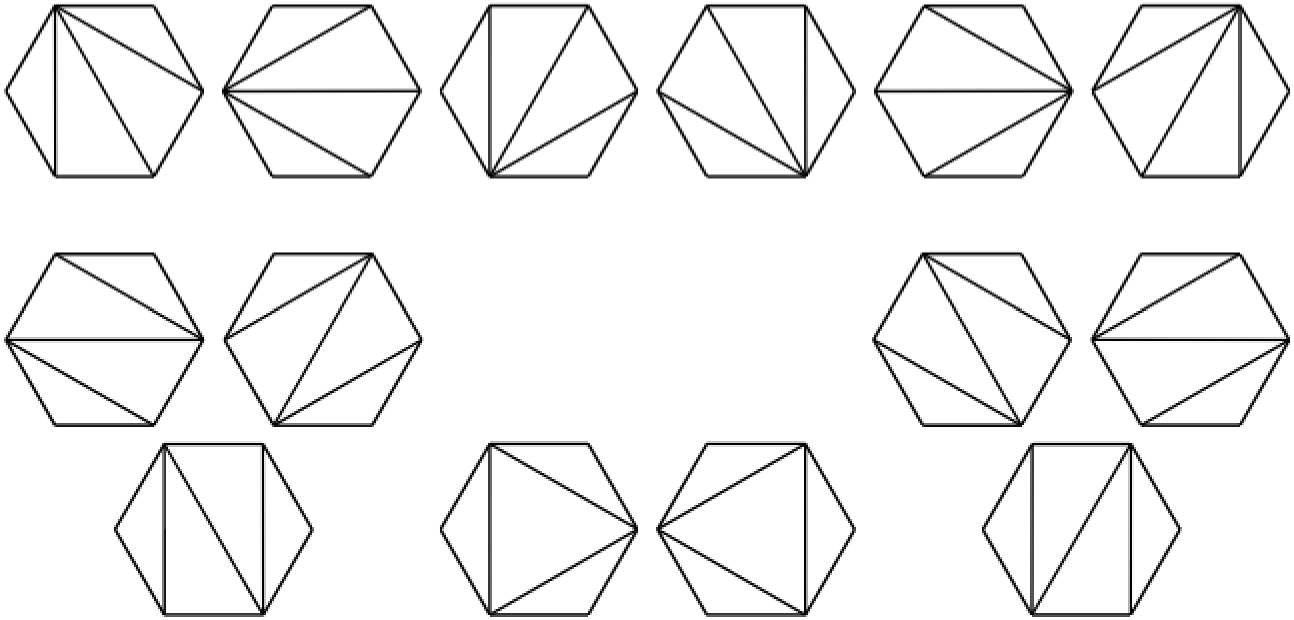
$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

这里就是已经知道closed form
反向求生成函数。也就是我们
要把F(x)展开成a0 + a1x + ...,
这样就可以知道fn了

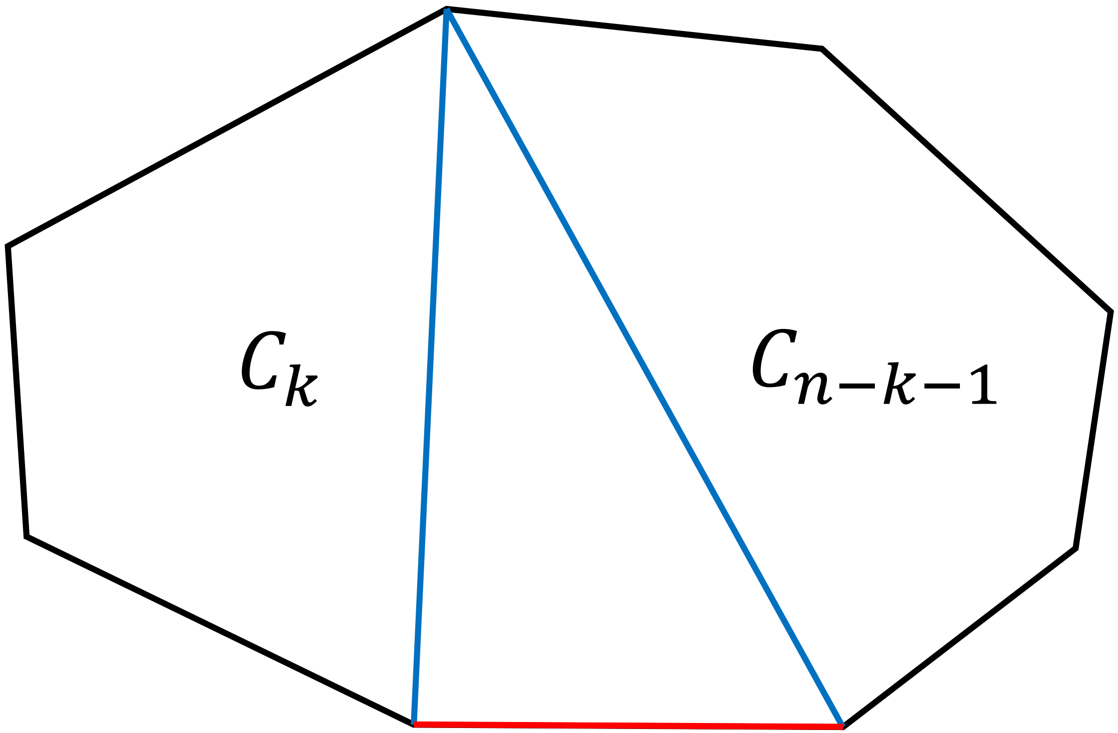
Catalan Number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

- Number of different ways a convex polygon with $\boxed{n+2}$ sides can be cut into triangles by connecting vertices with straight lines (a form of Polygon triangulation).



- Number of different ways a convex polygon with $n + 2$ sides can be cut into triangles by connecting vertices with straight lines (a form of Polygon triangulation).



$$C_0 = 1, \ C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$C_0 = 1,$$
$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = \sum_{n \geq 0} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$\big(G(x)\big)^2 = \sum_{n \geq 0} \sum_{k=0}^{n} C_k C_{n-k} x^n$$

$$x\big(G(x)\big)^2 = \sum_{n \geq 0} \sum_{k=0}^{n} C_k C_{n-k} x^{n+1}$$

$$= \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-k-1} x^n$$

$$C_0 = 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = \sum_{n\geq 0} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$x\big(G(x)\big)^2 = \sum_{n\geq 1} \sum_{k=0}^{n-1} C_k C_{n-k-1} x^n$$

$$G(x) = \sum_{n\geq 0} C_n x^n = C_0 + \sum_{n\geq 1} \sum_{k=0}^{n-1} C_k C_{n-k-1} x^n$$

$$= 1 + x\big(G(x)\big)^2$$

$$C_0 = 1,$$
$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = 1 + x\big(G(x)\big)^2 \quad \Longrightarrow \quad G(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

$$\lim_{x\to 0} G(0) = C_0 = 1 \quad \Longrightarrow \quad G(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

$$\sqrt{1-4x} = \sum_{n\geq 0}^{\infty} \binom{1/2}{n} (-4x)^n = 1 + \sum_{n\geq 1}^{\infty} \binom{1/2}{n} (-4x)^n$$

$$= 1 + \sum_{n\geq 0}^{\infty} \binom{1/2}{n+1} (-4x)^{n+1} = 1 - 4x \sum_{n\geq 0}^{\infty} \binom{1/2}{n+1} (-4x)^n$$

$$C_0 = 1,$$
$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = \frac{1-\sqrt{1-4x}}{2x}$$

$$\sqrt{1-4x} = 1 - 4x \sum_{n\geq 0}^{\infty} \binom{1/2}{n+1} (-4x)^n$$

$$G(x) = \frac{4x \sum_{n\geq 0}^{\infty} \binom{1/2}{n+1} (-4x)^n}{2x} = 2 \sum_{n\geq 0}^{\infty} \binom{1/2}{n+1} (-4x)^n$$

$$C_0 = 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

$$G(x) = 2 \sum_{n \geq 0}^{\infty} \binom{1/2}{n+1} (-4x)^n$$

$$C_n = 2\binom{1/2}{n+1}(-4)^n = 2\,\frac{(\frac{1}{2})(\frac{1}{2}-1)\cdots(\frac{1}{2}-n)}{(n+1)!}(-4)^n$$

$$= \frac{2^n}{(n+1)!}\prod_{k=1}^{n}(2k-1) \quad = \frac{2^n}{(n+1)!}\prod_{k=1}^{n}\frac{(2k-1)2k}{2k}$$

$$= \frac{1}{n!\,(n+1)!}\prod_{k=1}^{n}(2k-1)2k$$

$$= \frac{(2n)!}{n!\,(n+1)!} \quad = \boldsymbol{\frac{1}{n+1}\binom{2n}{n}}$$

- Number of Dyck words of length $2n$:

A string consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's.

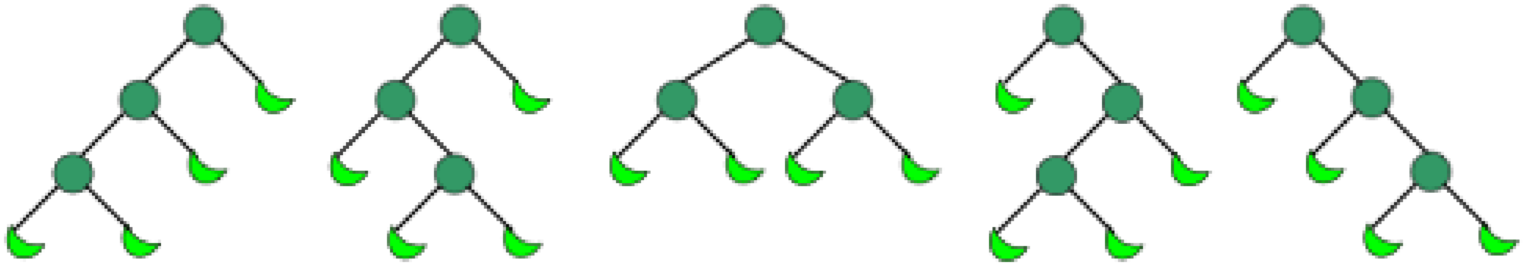
- For example, the following are the Dyck words of length 6:

$$XXXYYY \quad XYXXYY \quad XYXYXY$$
$$XXYYXY \quad XXYXYY$$

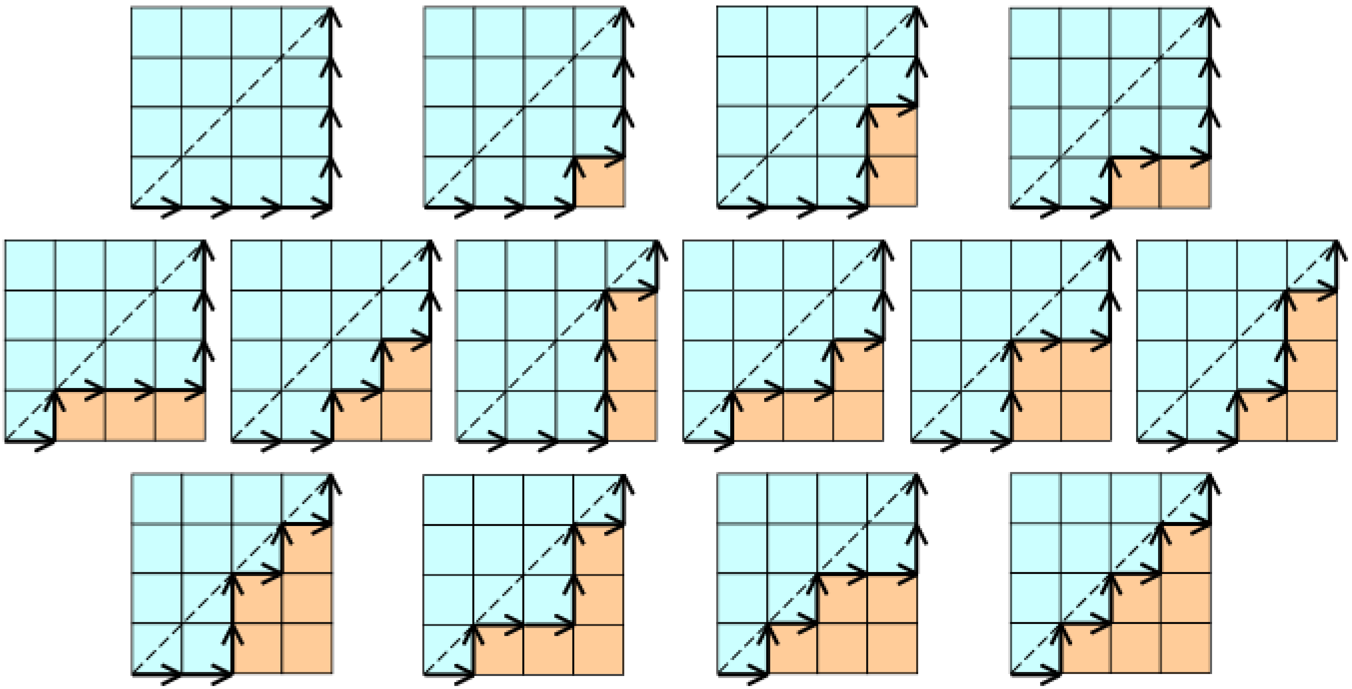
- Number of expressions containing n pairs of parentheses which are correctly matched.

((()))   ()(())   ()()()   (())()   (()())

- Number of full binary trees with $n + 1$ leaves



- Number of monotonic lattice paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal.



Catalan Number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$