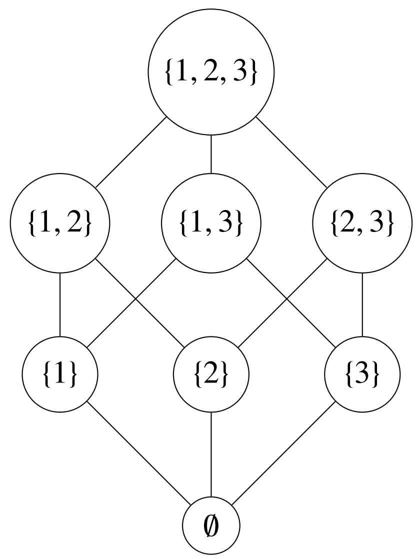
Homework 2

Problem 1. *Draw the Hasse diagram of the set of all subsets of* 1, 2, 3 *ordered by inclusion.*

Solution.



□

Problem 2. *Let*$(X, \preceq_1)$, $(Y, \preceq_2)$ *be (partially) ordered sets. We say that they are* isomorphic *if there exists a bijection* $f : X \to Y$ *such that for every* $x, y \in X$, *we have* $x \preceq_1 y$ *if and only if* $f(x) \preceq_2 f(y)$.

1. *Draw Hasse diagrams for all nonisomorphic 3-element posets.*
2. *Prove that any two n-element linearly ordered sets are isomorphic.*
3. *Prove that* $(\mathbb{N}, \leq)$ *and* $(\mathbb{Q}, \leq)$ *are not isomorphic. (where* $\mathbb{N}$ *is the set of natural numbers,* $\mathbb{Q}$ *is the set of rational numbers,* $\leq$ *is the usual 'less or equal to' between numbers).*

Solution.

1. 

2. Let X, Y be two linearly ordered sets each with n elements. Since X is a linearly ordered set, every pair of elements in X are comparable, so dose Y. For the sake of convenience, we assume that $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. ($x_i, y_i$ is the ith element in X and Y.) Now we can construct the bijection function f, which creats a one-to-one correspondence with x_i and y_i, that is, $f(x_i) = y_i$ and $f^{-1}(y_i) = x_i$. It is easy to see that for every $x_i, x_j \in X$, we have $x_i \leq_1 x_j$ if and only if $f(x_i) \leq_2 f(x_j)$. Thus, any two n-element linearly ordered sets are isomorphic.

3. If $(\mathbb{N}, \leq)$ and $(\mathbb{Q}, \leq)$ are isomorphic, then there exists a bijection f and for every $x, y \in X$, we have $x \leq_1 y$ if and only if $f(x) \leq_2 f(y)$. Suppose that $f(0) = p, f(1) = q$ (p, q are rational numbers), since $0 < 1$, we have $p < q$, further more $p < \frac{p+q}{2}$ and $\frac{p+q}{2} < q$. Since f is a bijection, there must be a natural number, say i, that has a one-to-one correspondence with $\frac{p+q}{2}$, but since $p < \frac{p+q}{2}$ and $\frac{p+q}{2} < q$, we have $f^{-1}(p) < f^{-1}(\frac{p+q}{2})$ and $f^{-1}(\frac{p+q}{2}) < f^{-1}(q)$, so $0 < i$ and $i < 1$, there is no such a natural number satisfying this. Thus we can conclude that $(\mathbb{N}, \leq)$ and $(\mathbb{Q}, \leq)$ are not isomorphic.

$\square$

Problem 3. *Prove or disprove: If a partially ordered set $(X, \leq)$ has a single minimal element, then it is a smallest element as well.*

Solution. This statement is wrong. Consider the set $\{a\} \cup \mathbb{Q}$ (a is just the character 'a' in the alphabet) and the partial ordering $\leq$ on this set. ($\leq$ is the usual 'less or equal to' between numbers.) Since the element a in this set is incomparable to any other element, thus a is a minimal element. It is easy to see that a is the only minimal element, but a is not a smallest element. Thus the statement is wrong. $\square$

Problem 4. *Let $(X, \leq)$ and $(X', \leq')$ be partially ordered sets. A mapping $f : X \to X'$ is called an embedding of $(X, \leq)$ into $(X', \leq')$ if the following conditions hold:*

- *f is an injective mapping;*
- *$f(x) \leq' f(y)$ if and only if $x \leq y$.*

Now consider the following problem

a) *Describe an embedding of the set $\{1, 2\} \times \mathbb{N}$ with the lexicographic ordering into the ordered set $(\mathbb{Q}, \leq)$.*

b) *Solve the analog of a) with the set* $\mathbb{N} \times \mathbb{N}$ *(ordered lexicographically) instead of* $\{1, 2\} \times \mathbb{N}$.

Solution.

a) We can find a function that $f(\langle i, n\rangle) = i - \frac{1}{2^n}$ where $i \in \{1, 2\}$ and $n \in \mathbb{N}$, this function maps any element in the set $\{1, 2\} \times \mathbb{N}$ with a rational number in $\mathbb{Q}$. For any $\langle i_1, n_1\rangle, \langle i_2, n_2\rangle$ in the set $\{1, 2\} \times \mathbb{N}$, if $\langle i_1, n_1\rangle \neq \langle i_2, n_2\rangle$, we have $f(\langle i_1, n_1\rangle) \neq f(\langle i_2, n_2\rangle)$, thus f is an injective mapping. We also have $f(\langle i_1, n_1\rangle) \leq f(\langle i_2, n_2\rangle)$ if and only if $\langle i_1, n_1\rangle \leq \langle i_2, n_2\rangle$ (in the lexicographic order). Thus, we can say f is an embedding of the set $\{1, 2\} \times \mathbb{N}$ with the lexicographic ordering into the ordered set $(\mathbb{Q}, \leq)$.

b) We can use the same injective function f as in (a).

□

Problem 5. *Prove the following strengthening of the* ***Erdös-Szekeres Lemma****: Let* κ, ℓ *be natural numbers. Then every sequence of real numbers of length* $\kappa\ell + 1$ *contains an nondecreasing subsequence of length* $\kappa + 1$ *or a decreasing subsequence of length* $\ell + 1$.

Solution. Proof by contradiction. Let the $\kappa\ell + 1$ elements in the sequence be denoted as $a_1, a_2, \ldots, a_{\kappa\ell+1}$. Suppose the argument is wrong, then the length of every nondecreasing subsequence is less than or equal to κ, and the length of every decreasing subsequence is less than or equal to ℓ.

We denote the length of the longest nondecreasing subsequence starts with a_i as x_i and the length of the longest decreasing subsequence starts with a_i as y_i. For each element in the sequence, we have a ordered pair $\langle x_i, y_i\rangle$. Since there are $\kappa\ell + 1$ elements, we have $\kappa\ell + 1$ ordered pairs, we also have $1 \leq x_i \leq \kappa$ and $1 \leq y_i \leq \ell$. Thus there must be two ordered pairs, say $\langle x_i, y_i\rangle, \langle x_j, y_j\rangle$, which are exactly the same.

We say this should not happen, if a_j is after a_i in the sequence, and if $a_i \leq a_j$, then we should have $x_i > x_j$, if $a_i \geq a_j$, then we should have $y_i > y_j$. If a_j is before a_i in the sequence, and if $a_i \leq a_j$, then we should have $y_j > y_i$, if $a_i \geq a_j$, then we should have $x_j > x_i$. Thus $\langle x_i, y_i\rangle$ can not be the same as $\langle x_j, y_j\rangle$. □