Homework 4

Problem 1. *1. Determine the coefficient of* x^{50} *in* $(x^7 + x^8 + x^9 + x^{10} + \cdots)^6$

2. Determine the coefficient of x^3 *in* $(2 + x)^{\frac{3}{2}}/(1 - x)$

3. Determine the coefficient of x^4 *in* $(2 + 3x)^5 \sqrt{1 - x}$

Solution.

1. We know that $x^7+x^8+x^9+x^{10}+\cdots$ is the generating function of the sequence $(0, 0, 0, 0, 0, 0, 0, 1, 1, 1, \cdots)$. We can get the closed form of this generating function, which is $\frac{x^7}{1-x}$. Thus, the original expression can be rewritten as $\left(\frac{x^7}{1-x}\right)^6 = \frac{x^{42}}{(1-x)^6}$. Since $(1 - x)^{-6} = \binom{5}{5} + \binom{6}{5}x + \binom{7}{5}x^2 + \cdots + \binom{5+k}{5}x^k + \cdots$, the coefficient of x^{50} is $\binom{13}{5}$.

2. We can rewrite the expression as $\sum_{k=0}^{\infty} \binom{3/2}{k} 2^{3/2-k} x^k (1 + x + x^2 + \cdots)$, so the coefficient of x^3 is $\sum_{k=0}^{3} \binom{3/2}{k} 2^{3/2-k} = 2^{3/2} + \frac{3}{2} \times 2^{1/2} + \frac{3}{8} \times 2^{-1/2} - \frac{1}{16} \times 2^{-3/2}$

3. We can rewrite the expression as $\sum_{i=0}^{\infty} \binom{5}{i} 2^{5-i} (3x)^i \cdot \sum_{j=0}^{\infty} \binom{1/2}{j} (-x)^j$, the the coefficient of x^4 is $\sum_{k=0}^{4} \binom{5}{k} 2^{5-k} 3^k \binom{1/2}{4-k} (-1)^{4-k}$

□

Problem 2. *Find generating functions for the following sequences (express them in a closed form, without infinite series!):*

1. $0, 0, 0, 0, -6, 6, -6, 6, -6, \cdots$.

2. $1, 0, 1, 0, 1, 0, \cdots$.

3. $1, 2, 1, 4, 1, 8 \cdots$

Solution.

1. The generating function of $(1, 1, 1, \cdots)$ is $1+x+x^2+\cdots$, and the closed form of the generating function is $\frac{1}{1-x}$, so the closed form of the generating function of $(1, -1, 1, -1, \cdots)$ is $\frac{1}{1+x}$. The generating function of $(-6, 6, -6, 6, \cdots)$ is $\frac{-6}{1+x}$. Thus the generating function of $(0, 0, 0, 0, -6, 6, -6, 6, -6, \cdots)$ is $\frac{-6x^4}{1+x}$.

2. The generating function of $(1, 1, 1, 1, \cdots)$ is $1 + x + x^2 + x^3 + \cdots$ and the generating function of $(1, -1, 1, -1, \cdots)$ is $1 - x + x^2 - x^3 + \cdots$. Thus, the generating function of $(1, 0, 1, 0, 1, 0, \cdots)$ is $\frac{1}{2}(\frac{1}{1-x} + \frac{1}{1+x}) = \frac{1}{1-x^2}$.

3. The generating function of $(1, 2, 4, 8, \cdots)$ is $\frac{1}{1-2x}$, so the generating function of $(1, 0, 2, 0, 4, 0, 8, \cdots)$ is $\frac{1}{1-2x^2}$. Since the generating function of $(1, 0, 1, 0, \cdots)$ is $\frac{1}{1-x^2}$, so the generating function of $(0, 1, 0, 1, \cdots)$ is $\frac{x}{1-x^2}$. Thus, the generating function of $(1, 1, 2, 1, 4, 1, 8, \cdots)$ is $\frac{x}{1-x^2}+\frac{1}{1-2x^2}$, finally we have the generating function of $(1, 2, 1, 4, 1, 8, \cdots)$ is $\left(\frac{x}{1-x^2} + \frac{1}{1-2x^2} - 1\right)/x = -\frac{2x^3+2x^2-2X-1}{(1-x^2)(1-2x^2)}$.

□

Problem 3. *Let a_n be the number of ordered triples $\langle i, j, k\rangle$ of integer numbers such that $i \geq 0, j \geq 1, k \geq 1$, and $i + 3j + 3k = n$. Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$ and calculate a formula for a_n.*

Solution. The value of a_n is equal to the coefficient of x^n in the result of $(1 + x + x^2 + x^3 + \cdots)(x^3 + x^6 + x^9 + \cdots)(x^3 + x^6 + x^9 + \cdots)$. Thus, the generating function of a_n is

$$(1 + x + x^2 + x^3 + \cdots)(x^3 + x^6 + x^9 + \cdots)(x^3 + x^6 + x^9 + \cdots)$$

Since we can rewrite the expression as $\frac{1}{1-x} \cdot \frac{x^3}{1-x^3} \cdot \frac{x^3}{1-x^3}$, a formula for a_n can be

$$a_n = \begin{cases} 0 & n < 6 \\ (-1)^{\frac{n-6}{3}}\binom{n/3}{2} & n \geq 6 \text{ and } n \equiv 0 \text{ (mod3)} \\ (-1)^{\frac{n-7}{3}}\binom{(n-1)/3}{2} & n \geq 6 \text{ and } n \equiv 1 \text{ (mod3)} \\ (-1)^{\frac{n-8}{3}}\binom{(n-2)/3}{2} & n \geq 6 \text{ and } n \equiv 2 \text{ (mod3)} \end{cases}$$

□

Problem 4. *Express the n^{th} term of the sequences given by the following recurrence relations*

1. $a_0 = 2, a_1 = 3, a_{n+2} = 3a_n - 2a_{n+1}$ $(n = 0, 1, 2, \ldots)$.

2. $a_0 = 1, a_{n+1} = 2a_n + 3$ $(n = 0, 1, 2, \ldots)$.

Solution.

1. The characteristic function is $x^2 + 2x - 3 = 0$, there are two different solutions, which are $x_1 = -3, x_2 = 1$. Thus, we have $a_n = c_1(-3)^n + c_2(1)^n$. Since $a_0 = 2, a_1 = 3$, we can figure out taht $c_1 = -1/4, c_2 = 9/4$, so $a_n = -\frac{1}{4}(-3)^n + \frac{9}{4}$.

2. The homogeneous part is $x = 2$, to find one specific solution for the recurrence relation, we try $a_n = p2^n + s$, then we have $p2^{n+1} + s = 2 \cdot p2^n + 2s + 3$, so $s = -3$. Since $a_0 = 1$ we have $(c_1 + p)2^0 - 3 = 1$, so $(c_1 + p) = 4$. Finally, we have $a_n = 2^{n+2} - 3$.

$\square$

Problem 5. *Solve the recurrence relation* $a_{n+2} = \sqrt{a_{n+1}a_n}$ *with initial conditions* $a_0 = 2, a_1 = 8$ *and find* $\lim_{n\to\infty} a_n$.

Solution. Let $b_n = \log_2 a_n$, then we have $2b_{n+2} = b_{n+1} + b_n$, from this recurrence relation we can figure out that $b_n = -\frac{4}{3}(-\frac{1}{2})^n + \frac{7}{3}$. Thus, we have $a_n = 2^{-\frac{4}{3}(-\frac{1}{2})^n+\frac{7}{3}}$ and $\lim_{n\to\infty} a_n = 2^{\frac{7}{3}}$. $\square$