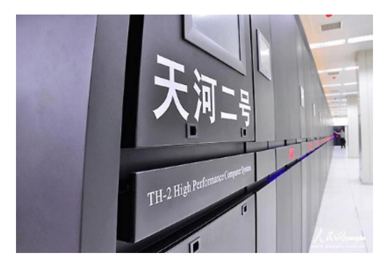
函数的渐进比较:O符号

- 在实际应用中，计算某个问题的精确解可能非常困难。此时，一个可行的方案是：不直接寻找精确解，转而寻找可接受的**估值(*estimate*)**。

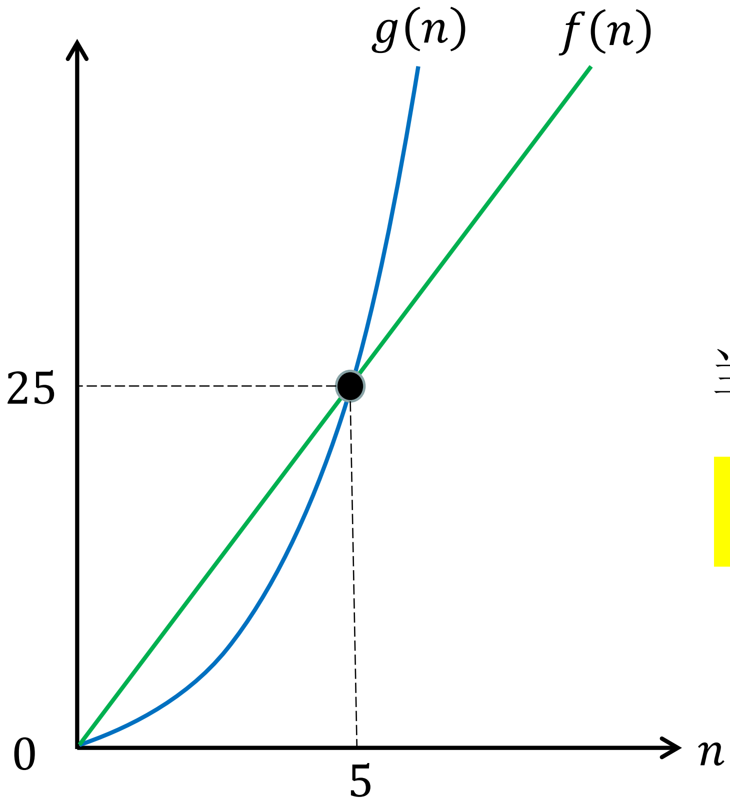
- 精确解困难的原因：
  - 物理设备的差异
  - 数据量激增
  - 精确计算公式复杂
  - 网络环境、自然环境
  - ……

- 1 Bit = Binary Digit
- 8 Bits = 1 Byte
- 1024 Bytes = 1 Kilobyte
- 1024 Kilobytes = 1 Megabyte
- 1024 Megabytes = 1 Gigabyte
- 1024 Gigabytes = 1 Terabyte
- 1024 Terabytes = 1 Petabyte
- 1024 Petabytes = 1 Exabyte
- 1024 Exabytes = 1 Zettabyte
- 1024 Zettabytes = 1 Yottabyte
- 1024 Yottabytes = 1 Brontobyte
- 1024 Brontobytes = 1 Geopbyte



函数的比较

- 比较：$f(n) = 5n$ 以及 $g(n) = n^2$，其中 $n \in N$ 为自然数。



当$n \to \infty$时，$f(n)$的增长不快于$g(n)$

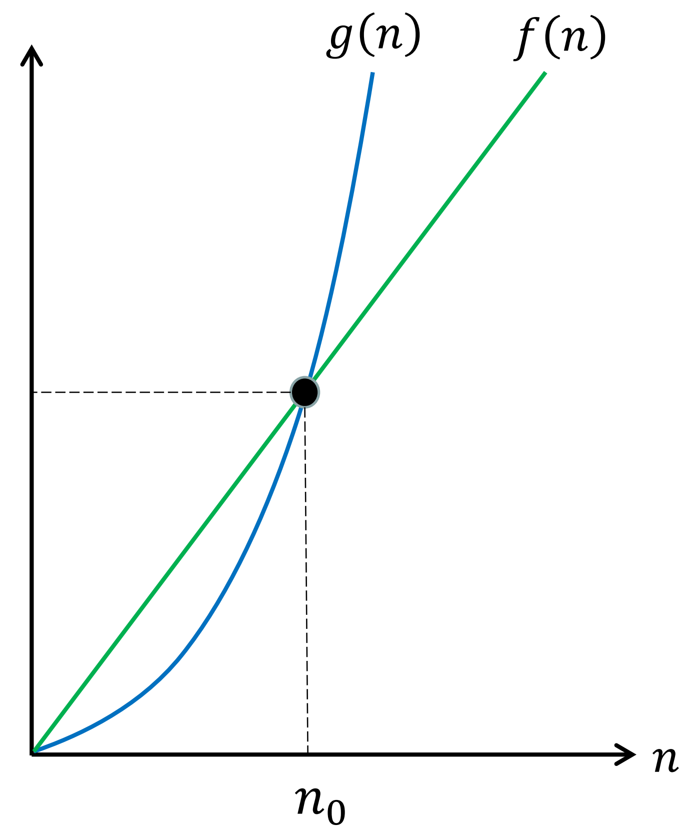
$$f(n) = O(g(n))$$

函数的渐进比较(Asymptotic comparison)

定义： f, g：$N \to R$ 是两个从自然数到实数的单变量方程

$$f(n) = O(g(n))$$

表示存在常数n_0和C，使得对所有$n \geq n_0$，不等式$|f(n)| \leq C \cdot g(n)$ 成立。



直观：f的增长不比g快很多。即：$\lim_{n\to\infty} \frac{f(n)}{g(n)} \nrightarrow \infty$

一些例子

- $1000000 = O(1)$
- $(7n^2 + 6n + 1)(n^3 + 4) = O(n^5)$
- $\binom{n}{2} = n(n-1)/2 = \frac{1}{2}n^2 + O(n) = O(n^2)$
- $0 < \alpha \le \beta \Rightarrow n^\alpha = O(n^\beta)$
- $\forall\ C > 0, a > 1\ n^C = O(a^n)$
- $\forall\ C > 0, \alpha > 0\ (\ln n)^C = O(n^\alpha)$
- 在函数的渐进比较中，部分常用的其他符号如下：

函数的渐进比较

符号	定义	含义
$f(n)=O(g(n))$	$\lim\limits_{n\to\infty}\dfrac{f(n)}{g(n)}\nrightarrow\infty$	f的增长不比g快很多
$f(n)=o(g(n))$	$\lim\limits_{n\to\infty}\dfrac{f(n)}{g(n)}=0$	f的增长远远慢于g
$f(n)=\Omega(g(n))$	$g(n)=O(f(n))$	f的增长至少和g一样快
$f(n)=\Theta(g(n))$	$f(n)=O(g(n))$且 $f(n)=\Omega(g(n))$	f和g几乎是同一数量级
$f(n)\sim g(n)$	$\lim\limits_{n\to\infty}\dfrac{f(n)}{g(n)}=1$	$f(n)$和$g(n)$几乎是一样的

调和级数

- 调和级数(Harmonic number)：

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{n}$$

调和级数估值

- 估计调和级数的值：用数列对调和级数的加项做分类。
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \cdots, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \cdots$

$\left(\frac{1}{2^1}, \frac{1}{2^0}\right]$	$\left(\frac{1}{2^2}, \frac{1}{2^1}\right]$	$\left(\frac{1}{2^3}, \frac{1}{2^2}\right]$	$\left(\frac{1}{2^4}, \frac{1}{2^3}\right]$	$\left(\frac{1}{2^5}, \frac{1}{2^4}\right]$
G_1	G_2	G_3	G_4	G_5

$$
\begin{aligned}
G_k &= \left\{\frac{1}{i} \,\middle|\, \frac{1}{2^k} < \frac{1}{i} \leq \frac{1}{2^{k-1}}\right\} \\
&= \left\{\frac{1}{2^k-1}, \frac{1}{2^k-2}, \frac{1}{2^k-3}, \cdots, \frac{1}{2^{k-1}}\right\}
\end{aligned}
$$

$$\underbrace{1,}_{G_1}\ \underbrace{\frac{1}{2},\frac{1}{3},}_{G_2}\ \underbrace{\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},}_{G_3}\ \underbrace{\frac{1}{8},\frac{1}{9},\cdots,\frac{1}{15},}_{G_4}\cdots$$

$$G_k=\left\{\frac{1}{i}\,\middle|\,\frac{1}{2^k}<\frac{1}{i}\le\frac{1}{2^{k-1}}\right\}$$
$$=\left\{\frac{1}{2^k-1},\frac{1}{2^k-2},\frac{1}{2^k-3},\cdots,\frac{1}{2^{k-1}}\right\}$$
$$=\left\{\frac{1}{2^{k-1}},\frac{1}{2^{k-1}+1},\frac{1}{2^{k-1}+2},\cdots,\frac{1}{2^k-1}\right\}$$

$$|G_k|=2^{k-1}$$

每一个G_k中的调和级数加项和：

$$\sum_{x\in G_k}x\le|G_k|\max G_k$$
$$=2^{k-1}\cdot\frac{1}{2^{k-1}}$$
$$=1$$
$$\sum_{x\in G_k}x\ge|G_k|\min G_k$$
$$>2^{k-1}\cdot\frac{1}{2^k}$$
$$=\frac{1}{2}$$

$$\underbrace{1,}_{G_1}\ \underbrace{\frac{1}{2},\frac{1}{3},}_{G_2}\ \underbrace{\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},}_{G_3}\ \underbrace{\frac{1}{8},\frac{1}{9},\cdots,\frac{1}{15},}_{G_4}\cdots$$

$$\begin{aligned} G_k &= \left\{\frac{1}{i}\,\middle|\,\frac{1}{2^k} < \frac{1}{i} \le \frac{1}{2^{k-1}}\right\} \\ &= \left\{\frac{1}{2^k-1},\frac{1}{2^k-2},\frac{1}{2^k-3},\cdots,\frac{1}{2^{k-1}}\right\} \\ &= \left\{\frac{1}{2^{k-1}},\frac{1}{2^{k-1}+1},\frac{1}{2^{k-1}+2},\cdots,\frac{1}{2^k-1}\right\} \end{aligned}$$

$$|G_k| = 2^{k-1}$$

每一个G_k中的调和级数加项和：

$$\sum_{x\in G_k} x \le |G_k| \max G_k$$

$$= 2^{k-1}\cdot\frac{1}{2^{k-1}}$$

$$= 1$$

$$\boldsymbol{\frac{1}{2} < \sum_{x\in G_k} x \le 1}$$

$$= \frac{1}{2}$$

$$H_n = \underbrace{1}_{G_1} + \underbrace{\frac{1}{2} + \frac{1}{3}}_{G_2} + \frac{1}{4} + \cdots + \underbrace{\frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}}_{G_t}$$

$$G_k = \left\{ \frac{1}{i} \,\middle|\, \frac{1}{2^k} < \frac{1}{i} \le \frac{1}{2^{k-1}} \right\}$$

$$\frac{1}{2} < \sum_{x \in G_k} x \le 1$$

$$2^{k-1} \le i < 2^k$$

$k = \lfloor \log_2 i \rfloor + 1$ 故 $t = \lfloor \log_2 n \rfloor + 1$

$$H_n \le t \cdot 1 \le \log_2 n + 1$$

$$H_n > (t-1) \cdot \frac{1}{2} \ge \frac{1}{2} \lfloor \log_2 n \rfloor$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}$$

$$\underbrace{1}_{G_0} \quad \underbrace{\frac{1}{2} + \frac{1}{3}}_{G_1} \quad \cdots \quad \underbrace{\frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}}_{G_t}$$

$$G_k = \left\{ \frac{1}{i} \;\middle|\; \frac{1}{2^k} < \frac{1}{i} \le \frac{1}{2^{k-1}} \right\}$$

$$\frac{1}{2} < \sum_{x \in G_k} x \le 1$$

$$2^{k-1} \le i < 2^k$$

$k = \lfloor \log_2 i \rfloor + 1$ 故 $t = \lfloor \log_2 n \rfloor + 1$

$$\left.\begin{aligned} H_n &\le t \cdot 1 \le \log_2 n + 1 \\ H_n &> (t-1) \cdot \frac{1}{2} \ge \frac{1}{2} \lfloor \log_2 n \rfloor \end{aligned}\right\}$$

$$H_n = \Theta(\log_2 n) = \Theta(\ln n)$$