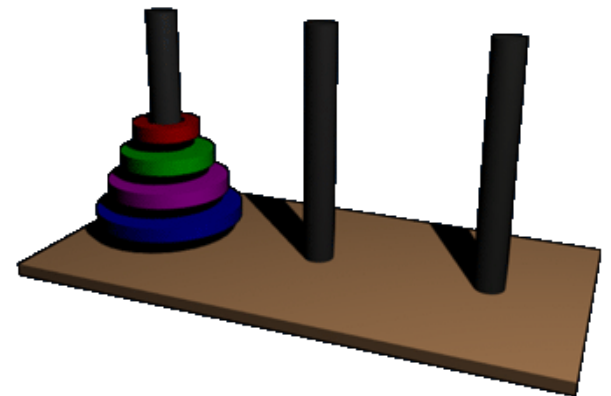
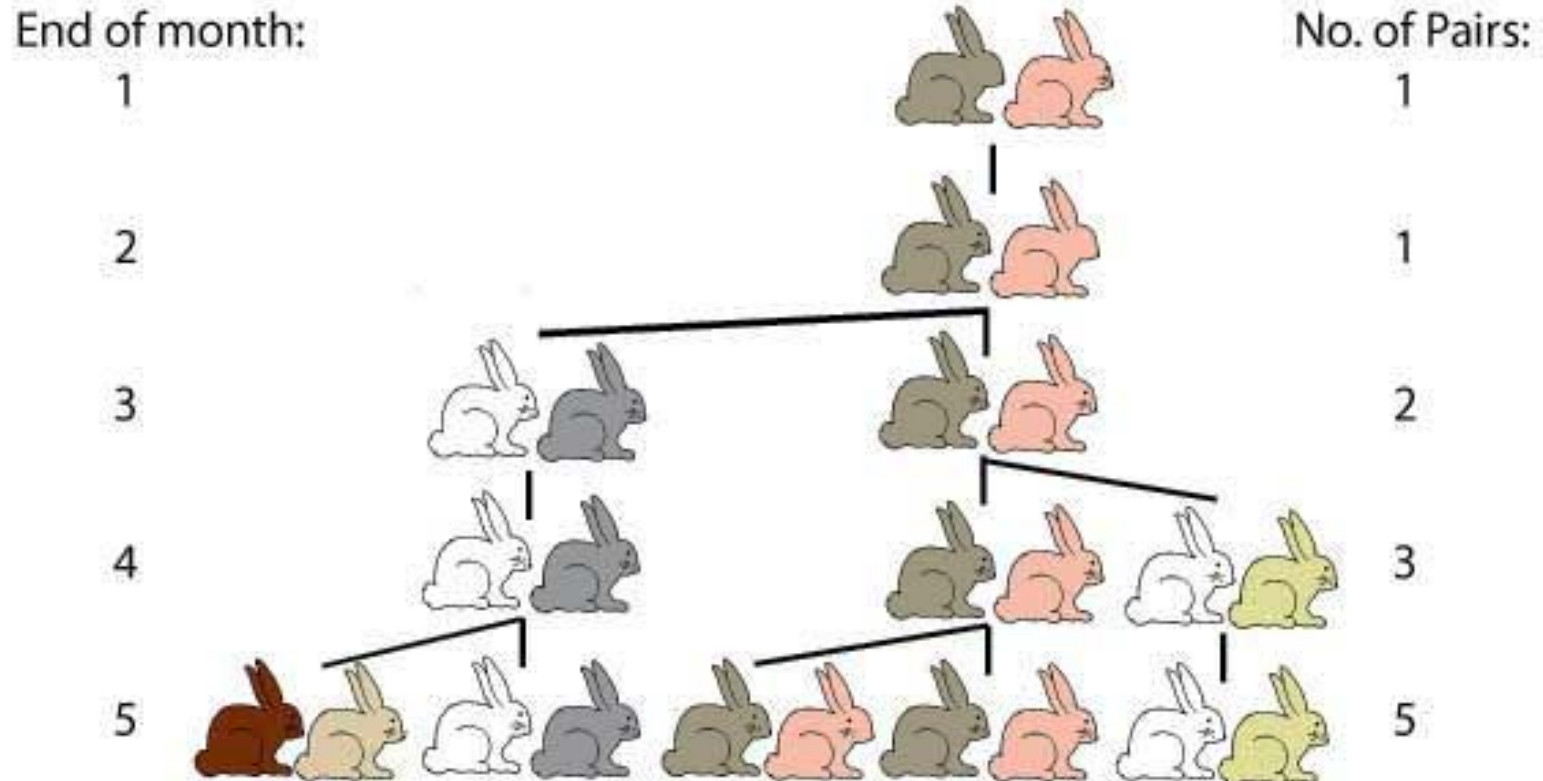


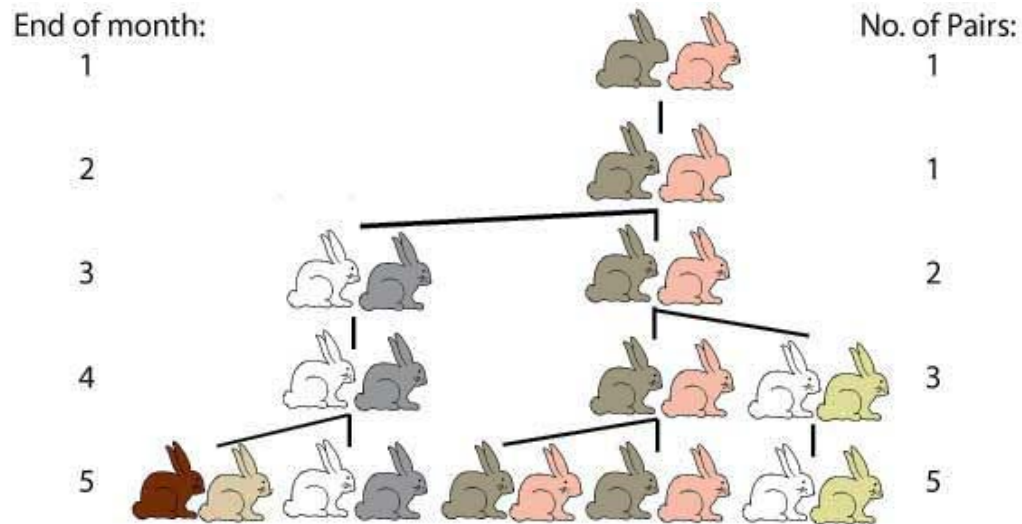
Recurrence relation

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- Fibonacci and his rabbits [1202]





$$f_0 = 0, \quad f_1 = 1, \quad f_2 = 1, \quad f_3 = 2, \quad f_4 = 3$$

$$f_{13} = ?$$

$$f_n = f_{n-1} + f_{n-2}$$

Fibonacci Sequence

- $f_n = f_{n-1} + f_{n-2}$
- $F(x) = f_0 + f_1x + f_2x^2 + \dots + f_{n-2}x^{n-2} + f_{n-1}x^{n-1} + f_nx^n + \dots$
- $xF(x) = f_0x + f_1x^2 + f_2x^3 + \dots + f_{n-2}x^{n-1} + f_{n-1}x^n + \dots$
- $x^2F(x) = f_0x^2 + f_1x^3 + f_2x^4 + \dots + f_{n-2}x^n + f_{n-1}x^{n+1} + \dots$

$$F(x) - xF(x) - x^2F(x) = f_0 + (f_1 - f_0)x$$

$$\begin{aligned} F(x) &= \frac{x}{1 - x - x^2} \\ &= \frac{a}{1 - \lambda_1 x} + \frac{b}{1 - \lambda_2 x} \end{aligned}$$

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$


Homogeneous linear recurrence of k^{th} degree with constant coefficients

$$h_n = a_{k-1}h_{n-1} + a_{k-2}h_{n-2} + \cdots + a_1h_{n-k-1} + a_0h_{n-k}$$

Characteristic polynomial of the above recurrence

$$p(x) = x^k - a_{k-1}x^{k-1} - a_{k-2}x^{k-2} - \cdots - a_1x - a_0 = 0$$

$$x^k - a_{k-1}x^{k-1} - a_{k-2}x^{k-2} - \cdots - a_1x - a_0 = 0 \quad (\star)$$


$$(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_k) = 0$$

$$x^k - a_{k-1}x^{k-1} - a_{k-2}x^{k-2} - \cdots - a_1x - a_0 = 0 \quad (\star)$$

$$(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_k) = 0$$

- If $\lambda_i \neq \lambda_j$ whenever $i \neq j$ 无重根

Then $h_n = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_k\lambda_k^n$

有重根

- If $p(x) = (x - \lambda_1)^{s_1}(x - \lambda_2)^{s_2} \cdots (x - \lambda_q)^{s_q}$

Then $h_n = (c_{11} + c_{12}n + \cdots + c_{1s_1}n^{s_1-1})\lambda_1^n$
 $+ (c_{21} + c_{22}n + \cdots + c_{2s_2}n^{s_2-1})\lambda_2^n$
 \vdots
 $+ (c_{q1} + c_{q2}n + \cdots + c_{qs_q}n^{s_q-1})\lambda_q^n$

Example

- $f_n = f_{n-1} + f_{n-2} \quad (n \geq 2)$
 $f_0 = 0, f_1 = 1, f_2 = 1$

- $x^2 - x - 1 = 0 \quad (\star)$

- $x = \frac{1 \pm \sqrt{5}}{2}$

- $f_n = a \left(\frac{1 + \sqrt{5}}{2} \right)^n + b \left(\frac{1 - \sqrt{5}}{2} \right)^n$

- $f_0 = 0 \Rightarrow a + b = 0 \Rightarrow a = -b$

$$f_1 = 1 \Rightarrow a \left(\frac{1 + \sqrt{5}}{2} \right) + b \left(\frac{1 - \sqrt{5}}{2} \right) = 1 \Rightarrow a = \frac{1}{\sqrt{5}}$$

- $f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$

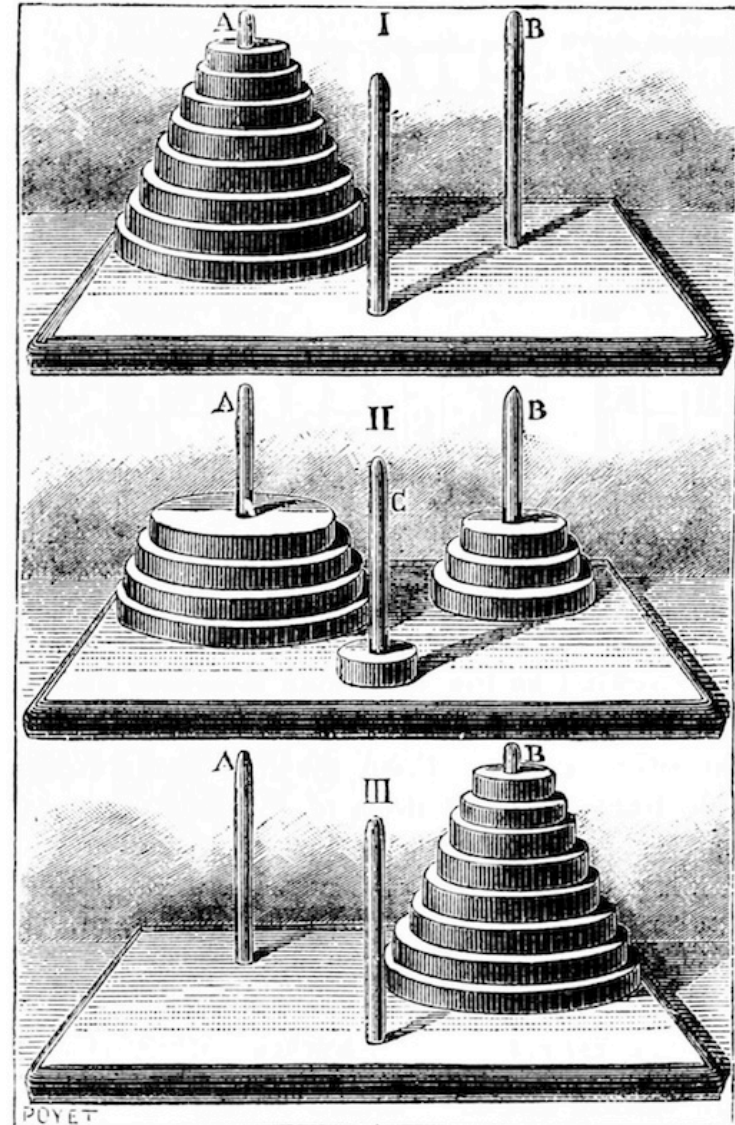
Example

- $h_n = -h_{n-1} + 3h_{n-2} + 5h_{n-3} + 2h_{n-4} \quad (n \geq 4)$
 $h_0 = 1, h_1 = 0, h_2 = 1, h_3 = 2$
- $x^4 + x^3 - 3x^2 - 5x - 2 = 0 \quad (\star)$
- $x = -1, -1, -1, 2$
- $h_n = c_1(-1)^n + c_2n(-1)^n + c_3n^2(-1)^n + c_42^n$
- | | | | |
|-----------|-----------------------------|-------|-----------|
| $(n = 0)$ | c_1 | + | $c_4 = 1$ |
| $(n = 1)$ | $-c_1 - c_2 - c_3 + 2c_4$ | $= 0$ | |
| $(n = 2)$ | $c_1 + 2c_2 + 4c_3 + 4c_4$ | $= 1$ | |
| $(n = 3)$ | $-c_1 - 3c_2 - 9c_3 + 8c_4$ | $= 2$ | |
- $c_1 = \frac{7}{9}, c_2 = -\frac{3}{9}, c_3 = 0, c_4 = \frac{2}{9}$
- $h_n = \frac{7}{9}(-1)^n - \frac{3}{9}n(-1)^n + \frac{2}{9}2^n$

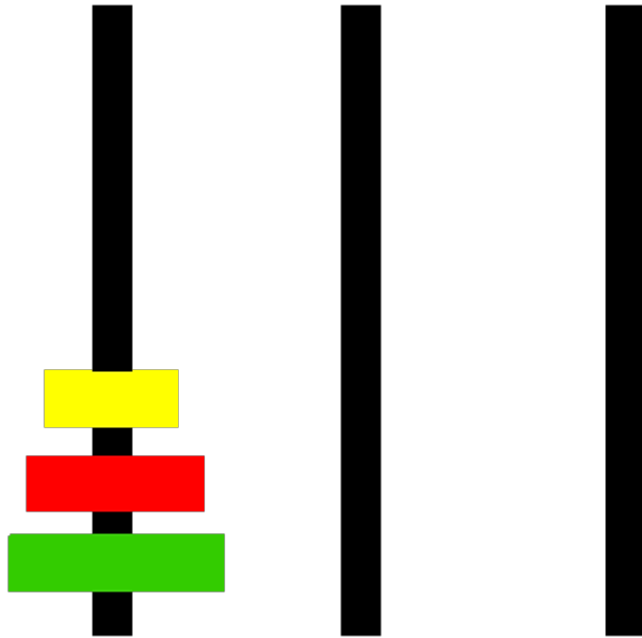
- Tower of Hanoi [Édouard Lucas, 1883]

Game rules:

1. Only one disk can be moved at a time.
2. Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack i.e. a disk can only be moved if it is the uppermost disk on a stack.
3. No disk may be placed on top of a smaller disk.

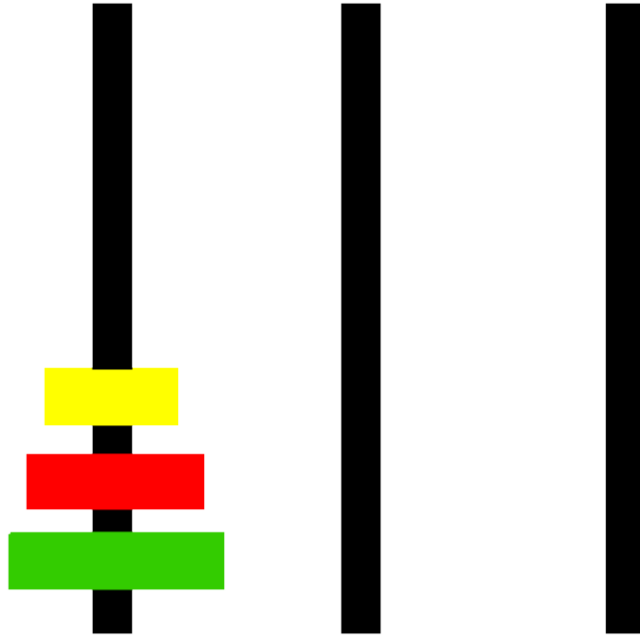


- Tower of Hanoi



$n = 3$

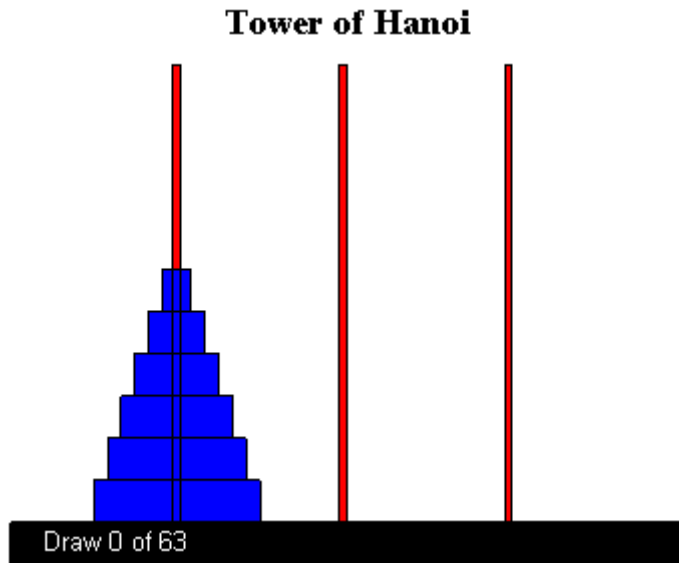
- Tower of Hanoi



$$n = 3$$

$$h_n = 7$$

- Tower of Hanoi



$$n = 5$$

$$h_n = 63$$

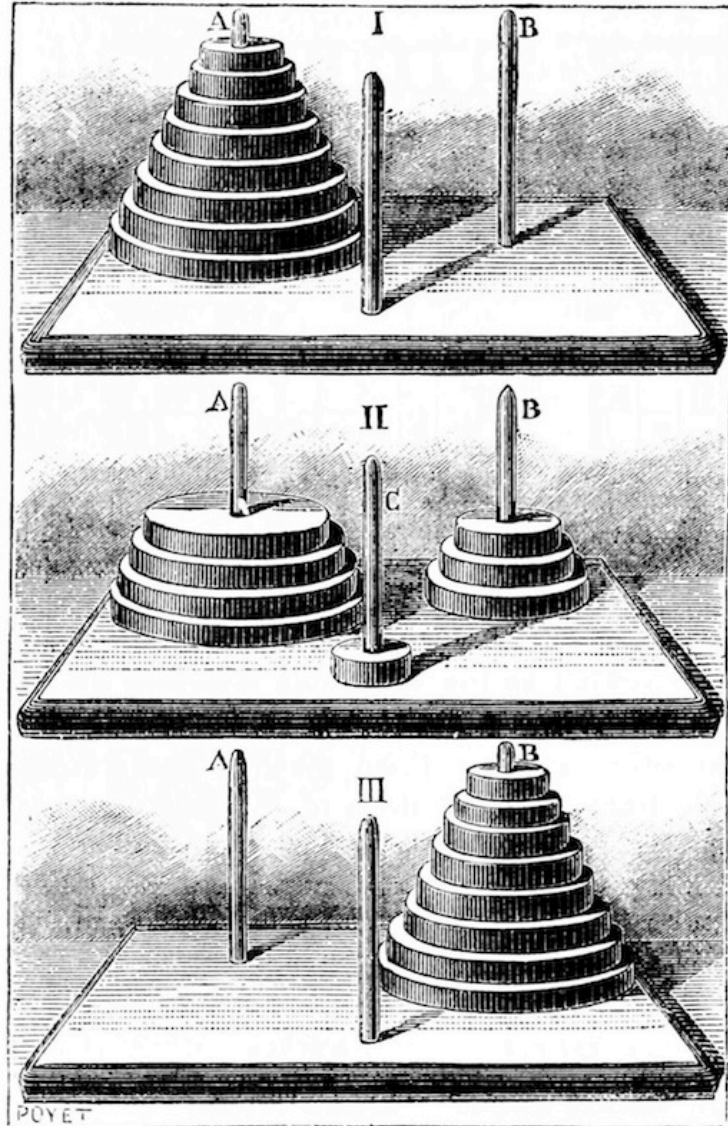
$$h_0 = 0$$

$$h_1 = 1$$

$$h_2 = 3$$

$$\vdots$$

$$h_n = 2h_{n-1} + 1$$



$$h_0 = 0$$

$$h_1 = 1$$

$$h_2 = 3$$

$$\vdots$$

$$h_n = 2h_{n-1} + 1$$

$$= 2(2h_{n-2} + 1) + 1 = 2^2 h_{n-2} + 2 + 1$$

$$= 2^2(2h_{n-3} + 1) + 2 + 1 = 2^3 h_{n-3} + 2^2 + 2 + 1$$

$$\vdots$$

$$= 2^{n-1}(h_0 + 1) + 2^{n-2} + \dots + 2^2 + 2 + 1$$

$$= 2^{n-1} + 2^{n-2} + \dots + 2^2 + 2 + 1$$

$$= 2^n - 1$$

Non-homogeneous linear recurrence of k^{th} degree
with constant coefficients

$$h_n = a_{k-1}h_{n-1} + a_{k-2}h_{n-2} + \cdots + a_0h_{n-k} + b_n$$

Every solution to nonhomogeneous equation is of the form:

Some specific solution + **Solution to homogeneous.**

Some suggestions

- If b_n is of n 's k –degree polynomial, then the specific solution is more likely to be n 's k –degree polynomial as well.
 - If $b_n = c$ try $h_n = r$
 - If $b_n = dn + c$ try $h_n = rn + s$
 - If $b_n = rn^2 + sn + t$ try $h_n = fn^2 + dn + c$
- If b_n is of n 's exponential form, then the specific solution is more likely to be n 's exponential form as well.
 - If $b_n = d^n$ try $h_n = pd^n$

Example

- $h_n = 3h_{n-1} - 4n$ ($n \geq 1$) with $h_0 = 2$
- Homogeneous part: $h_n = 3h_{n-1}$, $x - 3 = 0$ (☆)
- $h_n = c3^n$ ($n \geq 1$)
- Find one specific solution for $h_n = 3h_{n-1} - 4n$ ($n \geq 1$)
 - Try $h_n = rn + s$
 - $rn + s = 3(r(n-1) + s) - 4n$
 - $rn + s = (3r - 4)n + (-3r + 3s)$
 - $\Rightarrow r = 2, s = 3 \Rightarrow h_n = 2n + 3$ ✓
- $h_n = c3^n + 2n + 3$
- ($n = 0$) $2 = c \times 3^0 + 2 \times 0 + 3 \Rightarrow c = -1$
- $h_n = -3^n + 2n + 3$ ($n \geq 0$)

Example

- $h_n = 3h_{n-1} + 3^n$ ($n \geq 1$) with $h_0 = 2$
- Homogeneous part: $h_n = c3^n$
- Find one specific solution for $h_n = 3h_{n-1} + 3^n$ ($n \geq 1$)
 - Try $h_n = p3^n$
 - $p3^n = 3p3^{n-1} + 3^n \Rightarrow p = p + 1 \Rightarrow \text{Impossible!}$
 - Try $h_n = pn3^n$
 - $pn3^n = 3p(n-1)3^{n-1} + 3^n \Rightarrow p = 1 \Rightarrow h_n = n3^n \quad \checkmark$
- $h_n = c3^n + n3^n$
- ($n = 0$) $c(3^0) + 0(3^0) = 2 \Rightarrow c = 2$
- $h_n = 2 \times 3^n + n3^n = (2 + n)3^n$ ($n \geq 0$)