# Homework 3

**Problem 1.** *Prove the formula*

1. $\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$

2. $\sum_{k=0}^{n} \binom{m+k-1}{k} = \binom{n+m}{n}$

*Solution.*

1. Use the equivalence $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$ iteratively.

2. Note that $\binom{m-1}{0} = \binom{m}{0} = 1$. The rest is just like above.

$\square$

**Problem 2.** *For natural numbers $m \leq n$ calculate (i.e. express by a simple formula not containing a sum) $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k}$.*

*Solution.* $\binom{k}{m}\binom{n}{k} = \frac{k!}{m!(k-m)!} \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{m!(n-m)!} \cdot \frac{(n-m)!}{(k-m)!(n-k)!} = \binom{n}{m}\binom{n-m}{n-k}$.

Thus $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k} = \sum_{k=m}^{n} \binom{n}{m}\binom{n-m}{n-k} = \binom{n}{m} \sum_{k=m}^{n} \binom{n-m}{n-k} = \binom{n}{m} 2^{n-m}$. $\square$

**Problem 3.** *Calculate (i.e. express by a simple formula not containing a sum)*

1. $\sum_{k=1}^{n} \binom{k}{m}\frac{1}{k}$

2. $\sum_{k=0}^{n} \binom{k}{m} k$

*Solution.*

1. It can be verified that $\frac{1}{k}\binom{k}{m} = \frac{1}{m}\binom{k-1}{m-1}$.

   Thus $\sum_{k=1}^{n} \binom{k}{m}\frac{1}{k} = \frac{1}{m} \sum_{k=1}^{n} \binom{k-1}{m-1} = \frac{1}{m}\binom{n}{m}$.

2. It can be verified that $k\binom{k}{m} = (k+1)\binom{k}{m} - \binom{k}{m} = (m+1)\binom{k+1}{m+1} - \binom{k}{m}$.

   Thus $\sum_{k=0}^{n} \binom{k}{m} k = \sum_{k=0}^{n} \left((m+1)\binom{k+1}{m+1} - \binom{k}{m}\right) = (m+1) \sum_{k=0}^{n} \binom{k+1}{m+1} - \sum_{k=0}^{n} \binom{k}{m}$

   $= (m+1)\binom{n+2}{m+2} - \binom{n+1}{m+1} = \cdots$

□

**Problem 4.** *(a) Using* **Problem 1.** *for* $r = 2$*, calculate the sum* $\sum_{i=2}^{n} i(i-1)$ *and* $\sum_{i=1}^{n} i^2$.

*(b) Use* $(a)$ *and* **Problem 1.** *for* $r = 3$*, calculate* $\sum_{i=1}^{n} i^3$.

*Solution.*

1.
$$r = 2: \qquad \binom{2}{2} + \binom{3}{2} + \cdots + \binom{i}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}$$

Thus $\frac{\sum_{i=2}^{n} i(i-1)}{2!} = \binom{n+1}{3} \therefore \sum_{i=2}^{n} i(i-1) = 2\binom{n+1}{3}$

$$r = 1: \qquad \binom{1}{1} + \binom{2}{1} + \cdots + \binom{i}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2}$$

Thus $\therefore \sum_{i=1}^{n} i = \binom{n+1}{2}$.

Finally, $\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} (i(i-1) + i) = \sum_{i=1}^{n} i(i-1) + \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6}$.

2.
$$r = 3: \qquad \binom{3}{3} + \binom{4}{3} + \cdots + \binom{i}{3} + \cdots + \binom{n}{3} = \binom{n+1}{4}$$

Thus $\frac{\sum_{i=3}^{n} i(i-1)(i-2)}{3!} = \binom{n+1}{4}$. $\therefore \sum_{i=3}^{n} i^3 - 3i^2 + 2i = 6\binom{n+1}{4}$,

...

The final result is $\binom{n+1}{2}^2$.

□

**Problem 5.** *Count the number of linear extensions for the following partial ordering:*

*$X$ is a disjoint union of sets $X_1, X_2, \ldots, X_k$ of sizes $r_1, r_2, \ldots, r_k$, respectively. Each $X_i$ is linearly ordered by $\preceq$, and no two elements from the different $X$ are comparable.*

*Solution.* $\binom{r_1+r_2+\cdots+r_k}{r_1, r_2, \cdots, r_k}$. □

**Problem 6.** *There are n married couples attending a dance. How many ways are there to form n pairs for dancing if no wife should dance with their husband.*

*Solution*. It is $D(n)$. □

**Problem 7.** *Count the permutations with exactly k fixed points. (Remark: $\pi$ is a permutation of the set {1,2,..., n}. Call an index i with $\pi(i) = i$ a* fixed point *of the permutation $\pi$.)*

*Solution*. First choose the points that are fixed. It will have $\binom{n}{k}$ possible choices.

The rest is counting the number of permutation without a fixed point, which is $D(n-k)$.

In all, the answer is $\binom{n}{k} \cdot D(n-k)$. □

**Problem 8.** *What is wrong with the following inductive proof that $D(n) = (n-1)!$ for all $n \geq 2$? Can you find a false step in it? For $n = 2$, the formula holds, so assume $n \geq 3$. Let $\pi$ be a permutation of $\{1, 2, ..., n-1\}$ with no fixed point. We want to extend it to a permutation $\pi'$ of $\{1, 2, ..., n\}$ with no fixed point. We choose a number $i \in \{1, 2, ..., n-1\}$, and we define $\pi'(n) = \pi(i), \pi'(i) = n$, and $\pi'(j) = \pi(j)$ for $j \neq i$, n. This defines a permutation of $\{1, 2, ..., n\}$, and it is easy to check that it has no fixed point. For each of the $D(n-1) = (n-2)!$ possible choices of $\pi$, the index i can be chosen in $n-1$ ways. Therefore, $D(n) = (n-2)! \cdot (n-1) = (n-1)!$.*

*Solution*. Basically it says that the extended function $\pi'$ is totally decided by the choice of $i$. However this is not the case: after letting $\pi'(n) = \pi(i), \pi'(i) = n$ could not be the only choice for keeping $\pi'$ bijective. In another word, the construction is an undercount. □