Homework 1

Problem 1. *Show the Venn-diagram representation for the following sets:*

(a) $(A \cup B) - C$

(b) $\overline{A \oplus (B \cap C)}$

Problem 2. *For any sets A, B and C, prove that*

$$A \cup B = A \cup C, A \cap B = A \cap C \text{ implies } B = C.$$

Solution. (Proof by contradiction)

Suppose $B \neq C$. As B and C are symmetric, then without loss of generality, take $x \in B - C$:

1. if $x \in A$: then $x \in A \cap B$ and $x \notin A \cap C$;
2. if $x \notin A$: then $x \in A \cup B$ and $x \notin A \cup C$.

neither of the above case could be true. Thus the assumption $B \neq C$ is not correct, which leads to $B = C$. □

Problem 3. *Show that a nonempty set has the same number of odd subsets (i.e., subsets with an odd number of elements) as even subsets.*

Solution. As the set S is non-empty, there is some $a \in S$. Now consider all the subsets of $S' = S - \{a\}$. Let X_0 be the set of even subsets of S', X_1 be the set of odd subsets of S'.

We use the symbol X_i^a to stand for the set getting by adding a into every element of X_i, where $i = 0, 1$. Obviously $|X_i^a| = |X_i|$.

To the original set S, $X_0 \cup X_1^a$ is the set of even subsets of S. Similarly $X_1 \cup X_0^a$ is the set of odd subsets of S. It is easy to prove that x_i and x_{1-i}^a are disjoint (i.e., their intersection is empty). $|X_0 \cup X_1^a| = |X_1 \cup X_0^a|$ by the previous problem. □

Problem 4. A, B, C *are three sets. and two functions* $g : A \to B$, $f : B \to C$

a) *If $f \circ g$ is an injective function and g is surjective, show that f is injective.*

b) *If $f \circ g$ is an surjective function and f is injective, show that g is surjective.*

(Note that $f \circ g(x) = f(g(x))$.)

Proof. a) for any $f(x) = f(y)$ with $x, y \in B$, as g is surjective, there exit $u, v \in A$ such that $g(u) = x, g(v) = y$. Now $f(x) = f(y) \Rightarrow f(g(u)) = f(g(v))$. Since $f \circ g$ is injective, we have $u = v$. It follows that $x = g(u) = g(v) = y$. Thus f is injective.

b) suppose g is not surjective and take $b \in B - Ran(g)$. As f is injective we know that $f(x) \neq f(b) \in C$ for any $x \in B \wedge x \neq b$. Then we will have that $f(b) \notin Ran(f \circ g)$ which betrays that $f \circ g$ is surjective.

□

Problem 5. *$\mathcal{R}$ is a binary relation,*

1. *Show that $\mathcal{R}$ is symmetric iff $\mathcal{R}^{-1} \subseteq \mathcal{R}$.*
2. *Show that $\mathcal{R}$ is transitive iff $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$.*

Proof. We prove the the first statement and omit the second one.

$$\begin{array}{rl} & \mathcal{R} \text{ is symmetric} \\ \Longleftrightarrow & \forall x \forall y (x \mathcal{R} y \longrightarrow y \mathcal{R} x) \\ \Longleftrightarrow & \forall x \forall y (y \mathcal{R}^{-1} x \longrightarrow y \mathcal{R} x) \\ \Longleftrightarrow & \mathcal{R}^{-1} \subseteq \mathcal{R} \end{array}$$

□

Problem 6. *Prove that $\mathcal{P}(A) \approx 2^A$, where A is any set and $2^A = \{f \mid f : A \to \{0, 1\}$ is a function.$\}$*

Solution. Check the slides. □

Problem 7. *A and B are countable sets. Prove that*

1. *$A \cup B$ is countable*
2. *$A \times B$ is countable*

Solution.(Hint) As $A \preceq \omega$ and $B \preceq \omega$, suppose $f : A \to \omega$, and $g : B \to \omega$ are both injective functions.

Then

1. $$h(x) = \begin{cases} 2 \cdot f(x) & x \in A \\ 2 \cdot g(x) + 1 & x \in B - A \end{cases}$$

   and then prove that $h(x)$ is injective.

2. $h(\langle x, y \rangle) = \langle f(x), g(y) \rangle$.

   Then prove that $h(x)$ is injective.

   Function h shows $A \times B \preceq \omega \times \omega$.

   As we know $\omega \times \omega \approx \omega$, we finally get $A \times B \preceq \omega$.

$\square$