PARTIALLY ORDERED SETS

A **partially ordered set** or **poset** is a set P and a binary relation $\leq$ such that for all $a, b, c \in P$

1. $a \leq a$ (reflexivity).
2. $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).
3. $a \leq b$ and $b \leq a$ implies $a = b$. (anti-symmetry).

Examples

1. $P = \{1, 2, \ldots, \}$ and $a \leq b$ has the usual meaning.
2. $P = \{1, 2, \ldots, \}$ and $a \leq b$ if a divides b.
3. $P = \{A_1, A_2, \ldots, A_m\}$ where the A_i are sets and $\leq = \subseteq$.

A pair of elements a, b are **comparable** if $a \leq b$ or $b \leq a$. Otherwise they are **incomparable**.

A poset without incomparable elements (Example 1) is a linear or total order.

We write $a < b$ if $a \leq b$ and $a \neq b$.

A **chain** is a sequence $a_1 < a_2 < \cdots < a_s$.

A set A is an **anti-chain** if every pair of elements in A are incomparable.

Thus a Sperner family is an anti-chain in our third example.

Theorem

Let P be a finite poset, then
$\min\{m : \exists$ *anti-chains* $A_1, A_2, \ldots, A_\mu$ *with* $P = \bigcup_{i=1}^{\mu} A_i\}$ *=*
$\max\{|C| : A$ *is a chain*$\}$.

The minimum number of anti-chains needed to cover P is at least the size of any chain, since a chain can contain at most one element from each anti-chain.

We prove the converse by induction on the maximum length μ of a chain. We have to show that P can be partitioned into μ anti-chains.

If $\mu = 1$ then P itself is an anti-chain and this provides the basis of the induction.

So now suppose that $C = x_1 < x_2 < \cdots < x_\mu$ is a maximum length chain and let A be the set of maximal elements of P.

(An element is x *maximal* if $\nexists y$ such that $y > x$.)

A is an anti-chain.

Now consider $P' = P \setminus A$. P' contains no chain of length μ. If it contained $y_1 < y_2 < \cdots < y_\mu$ then since $y_\mu \notin A$, there exists $a \in A$ such that P contains the chain $y_1 < y_2 < \cdots < y_\mu < a$, contradiction.

Thus the maximum length of a chain in P' is $\mu - 1$ and so it can be partitioned into anti-chains $A_1 \cup A_2 \cup \cdots A_{\mu-1}$. Putting $A_\mu = A$ completes the proof. $\square$

Suppose that $C_1, C_2, \ldots, C_m$ are a collection of chains such that $P = \bigcup_{i=1}^{m} C_i$.

Suppose that A is an anti-chain. Then $m \geq |A|$ because if $m < |A|$ then by the pigeon-hole principle there will be two elements of A in some chain.

Theorem

(Dilworth) Let P be a finite poset, then $\min\{m : \exists \text{ chains } C_1, C_2, \ldots, C_m \text{ with } P = \bigcup_{i=1}^{m} C_i\} = \max\{|A| : A \text{ is an anti-chain}\}$.

We have already argued that $\max\{|A|\} \leq \min\{m\}$.

We will prove there is equality here by induction on $|P|$.

The result is trivial if $|P| = 0$.

Now assume that $|P| > 0$ and that μ is the maximum size of an anti-chain in P. We show that P can be partitioned into μ chains.

Let $C = x_1 < x_2 < \cdots < x_p$ be a *maximal* chain in P i.e. we cannot add elements to it and keep it a chain.

Case 1 Every anti-chain in $P \setminus C$ has $\leq \mu - 1$ elements. Then by induction $P \setminus C = \bigcup_{i=1}^{\mu-1} C_i$ and then $P = C \cup \bigcup_{i=1}^{\mu-1} C_i$ and we are done.

Case 2

There exists an anti-chain $A = \{a_1, a_2, \ldots, a_\mu\}$ in $P \setminus C$. Let

- $P^- = \{x \in P : \; x \leq a_i$ for some $i\}$.
- $P^+ = \{x \in P : \; x \geq a_i$ for some $i\}$.

Note that

1. $P = P^- \cup P^+$. Otherwise there is an element x of P which is incomparable with every element of A and so μ is not the maximum size of an anti-chain.
2. $P^- \cap P^+ = A$. Otherwise there exists x, i, j such that $a_i < x < a_j$ and so A is not an anti-chain.
3. $x_p \notin P^-$. Otherwise $x_p < a_i$ for some i and the chain C is not maximal.

Applying the inductive hypothesis to P^- ($|P^-| < |P|$ follows from 3) we see that P^- can be partitioned into μ chains $C_1^-, C_2^-, \ldots, C_\mu^-$.

Now the elements of A must be distributed one to a chain and so we can assume that $a_i \in C_i^-$ for $i = 1, 2, \ldots, \mu$.

a_i must be the maximum element of chain C_i^-, else the maximum of C_i^- is in $(P^- \cap P^+) \setminus A$, which contradicts 2.

Applying the same argument to P^+ we get chains $C_1^+, C_2^+, \ldots, C_\mu^+$ with a_i as the minimum element of C_i^+ for $i = 1, 2, \ldots, \mu$.

Then from 2 we see that $P = C_1 \cup C_2 \cup \cdots \cup C_\mu$ where $C_i = C_i^- \cup C_i^+$ is a chain for $i = 1, 2, \ldots, \mu$. $\square$

(i) Another proof of

Theorem

Erdős and Szekerés

$a_1, a_2, \ldots, a_{n^2+1}$ *contains a monotone subsequence of length* $n+1$.

Let $P = \{(i, a_i) : 1 \leq i \leq n^2+1\}$ and let say $(i, a_i) \leq (j, a_j)$ if $i < j$ and $a_i \leq a_j$.

A chain in P corresponds to a monotone increasing subsequence. So, suppose that there are no monotone increasing sequences of length $n+1$. Then any cover of P by chains requires at least $n+1$ chains and so, by Dilworths theorem, there exists an anti-chain A of size $n+1$.

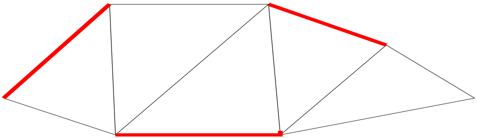
Let $A = \{(i_t, a_{i_t}) : \ 1 \leq t \leq n+1\}$ where $i_1 < i_2 \leq \cdots < i_{n+1}$.

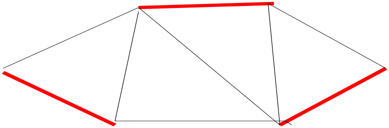
Observe that $a_{i_t} > a_{i_{t+1}}$ for $1 \leq t \leq n$, for otherwise $(i_t, a_{i_t}) \leq (i_{t+1}, a_{i_{t+1}})$ and A is not an anti-chain.

Thus A defines a monotone decreasing sequence of length $n+1$. $\square$

Matchings in bipartite graphs

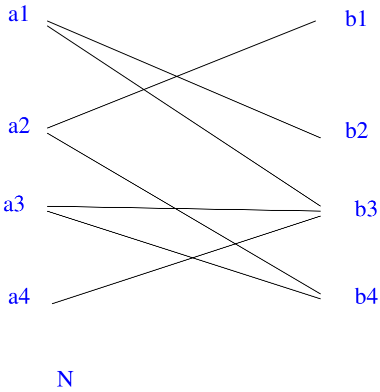
Re-call that a matching is a set of vertex disjoint edges.





P

Let $G = (A \cup B, E)$ be a bipartite graph with bipartition A, B.
For $S \subseteq A$ let $N(S) = \{b \in B : \exists a \in S, (a, b) \in E\}$.

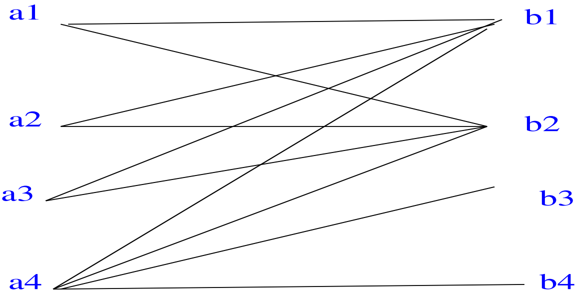


Clearly, $|M| \leq |A|, |B|$ for any matching M of G.

Theorem

(Hall) G contains a matching of size $|A|$ iff

$$|N(S)| \geq |S| \qquad \forall S \subseteq A.$$



$N(\{a_1, a_2, a_3\}) = \{b_1, b_2\}$ and so at most 2 of a_1, a_2, a_3 can be saturated by a matching.

If G contains a matching M of size $|A|$ then $M = \{(a, f(a)) : \; a \in A\}$, where $f : A \to B$ is a $\boxed{\text{1-1 function}}$.

But then,

$$|N(S)| \geq |f(S)| = S$$

for all $S \subseteq A$.

Let $G = (A \cup B, E)$ be a bipartite graph which satisfies Hall's condition. Define a poset $P = A \cup B$ and define $<$ by $a < b$ only if $a \in A, b \in B$ and $(a, b) \in E$.

Suppose that the largest anti-chain in P is $A = \{a_1, a_2, \ldots, a_h, b_1, b_2, \ldots, b_k\}$ and let $s = h + k$.

Now

$$N(\{a_1, a_2, \ldots, a_h\}) \subseteq B \setminus \{b_1, b_2, \ldots, b_k\}$$

for otherwise A will not be an anti-chain.

From Hall's condition we see that

$$|B| - k \geq h \text{ or equivalently} |B| \geq s.$$

Now by Dilworth's theorem, P is the union of s chains:

A matching M of size m, $|A| - m$ members of A and $|B| - m$ members of B.

But then

$$m + (|A| - m) + (|B| - m) = s \leq |B|$$

and so $m \geq |A|$. $\square$

Marriage Theorem

Theorem

Suppose $G = (A \cup B, E)$ is k-regular. ($k \geq 1$) i.e. $d_G(v) = k$ for all $v \in A \cup B$. Then G has a perfect matching.

Proof

$$k|A| = |E| = k|B|$$

and so $|A| = |B|$.
Suppose $S \subseteq A$. Let m be the number of edges incident with S. Then

$$k|S| = m \leq k|N(S)|.$$

So Hall's condition holds and there is a matching of size $|A|$ i.e. a perfect matching.

König's Theorem

We will use Hall's Theorem to prove König's Theorem. Given a bipartite graph $G = (A \cup B), E)$ we say that $S \subseteq V$ is a cover if $e \cap S \neq \emptyset$ for all $e \in E$.

Theorem

$$\min\{|S| : S \textit{ is a cover}\} = \max\{|M| : M \textit{ is a matching}\}.$$

Proof One part is easy. If S is a cover and M is a matching then $|S| \geq |M|$. This is because no vertex in S can belong to more than one edge in M.

To begin the main proof, we first prove a lemma that is a small generalisation of Hall's Theorem.

Lemma

Assume that $|A| \leq |B|$. *Let* $d = \max\{(|X| - |N(X)|)^+ : X \subseteq A\}$ *where* $\xi^+ = \max\{0, \xi\}$. *Then*

$$\mu = \max\{|M| : M \text{ is a matching } \} = |A| - d.$$

Proof That $\mu \leq |A| - d$ is easy. For the lower bound, add d dummy vertices D to B and add an edge between each vertex in D and each vertex in A to create the graph Γ. We now find that Γ satisfies the conditions of Hall's Theorem.

If M_1 is a matching of size $|A|$ in Γ then removing the edges of M_1 incident with D gives us a matching of size $|A| - d$ in G. □

Continuing the proof of König's Theorem let $S \subseteq A$ be such that $|N(S)| = |S| - d$.

Let $T = A \setminus S$. Then $T \cup N(S)$ is a cover, since there are no edges joining S to $B \setminus N(S)$.

Finally observe that

$$|T \cup N(S)| = |A| - |S| + |S| - d = |A| - d = \mu.$$

$\square$

Intervals Problem

$I_1, I_2, \ldots, I_{mn+1}$ are closed intervals on the real line i.e. $I_j = [a_j, b_j]$ where $a_j \leq b_j$ for $1 \leq j \leq mn+1$.

Theorem

Either (i) there are $m+1$ intervals that are pair-wise disjoint or (ii) there are $n+1$ intervals with a non-empty intersection

Define a partial ordering $<$ on the intervals by $I_r < I_s$ iff $b_r < a_s$. Suppose that $I_{i_1}, I_{i_2}, \ldots, I_{i_t}$ is a collection of pair-wise disjoint intervals. Assume that $a_{i_1} < a_{i_2} \cdots < a_{i_t}$. Then $I_{i_1} < I_{i_2} \cdots < I_{i_t}$ form a chain and conversely a chain of length t comes from a set of t pair-wise disjoint intervals.
So if (i) does not hold, then the maximum length of a chain is m.

This means that the minimum number of chains needed to cover the poset is at least $\lceil \frac{mn+1}{m} \rceil = n+1$.

Dilworth's theorem implies that there must exist an anti-chain $\{I_{j_1}, I_{j_2}, \ldots, I_{j_{n+1}}\}$.

Let $a = \max\{a_{j_1}, a_{j_2}, \ldots, a_{j_{n+1}}\}$ and $b = \min\{b_{j_1}, b_{j_2}, \ldots, b_{j_{n+1}}\}$.

We must have $a \leq b$ else the two intervals giving a, b are disjoint.

But then every interval of the anti-chain contains $[a, b]$.