Combinatorial Counting

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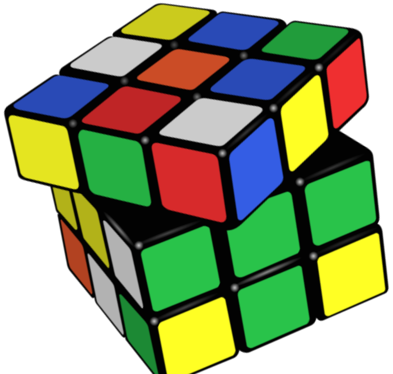
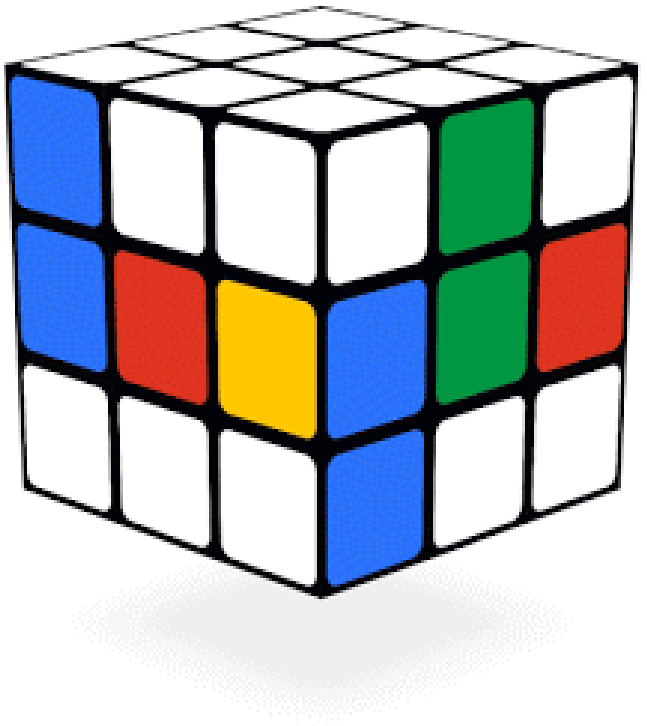




Let's Count!

n balls are put into m bins

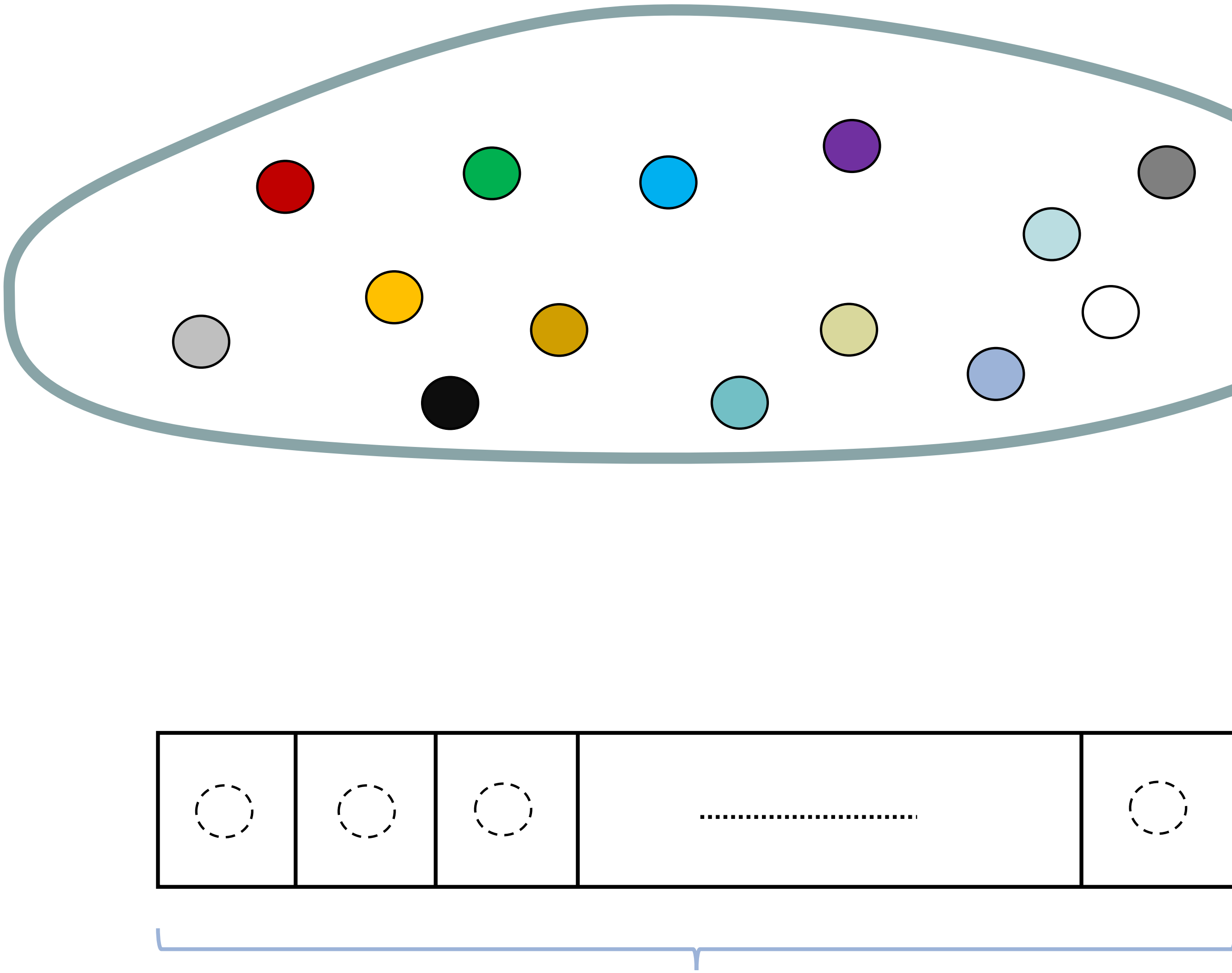
balls per bin	**unrestricted**	$\leq$ **1**	$\geq$ **1**
n distinct balls, m distinct bins.			
n identical balls, m distinct bins.			
n distinct balls, m identical bins.			
n identical balls, m identical bins.			

n balls are put into m bins

balls per bin	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins.			
n identical balls, m distinct bins.			
n distinct balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	
n identical balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	

- Basic counting
- Binomial theorem
- Generalized Binomial theoremSome
- special numbers

We will start with counting the *ordered* objects.



Ordered sequence

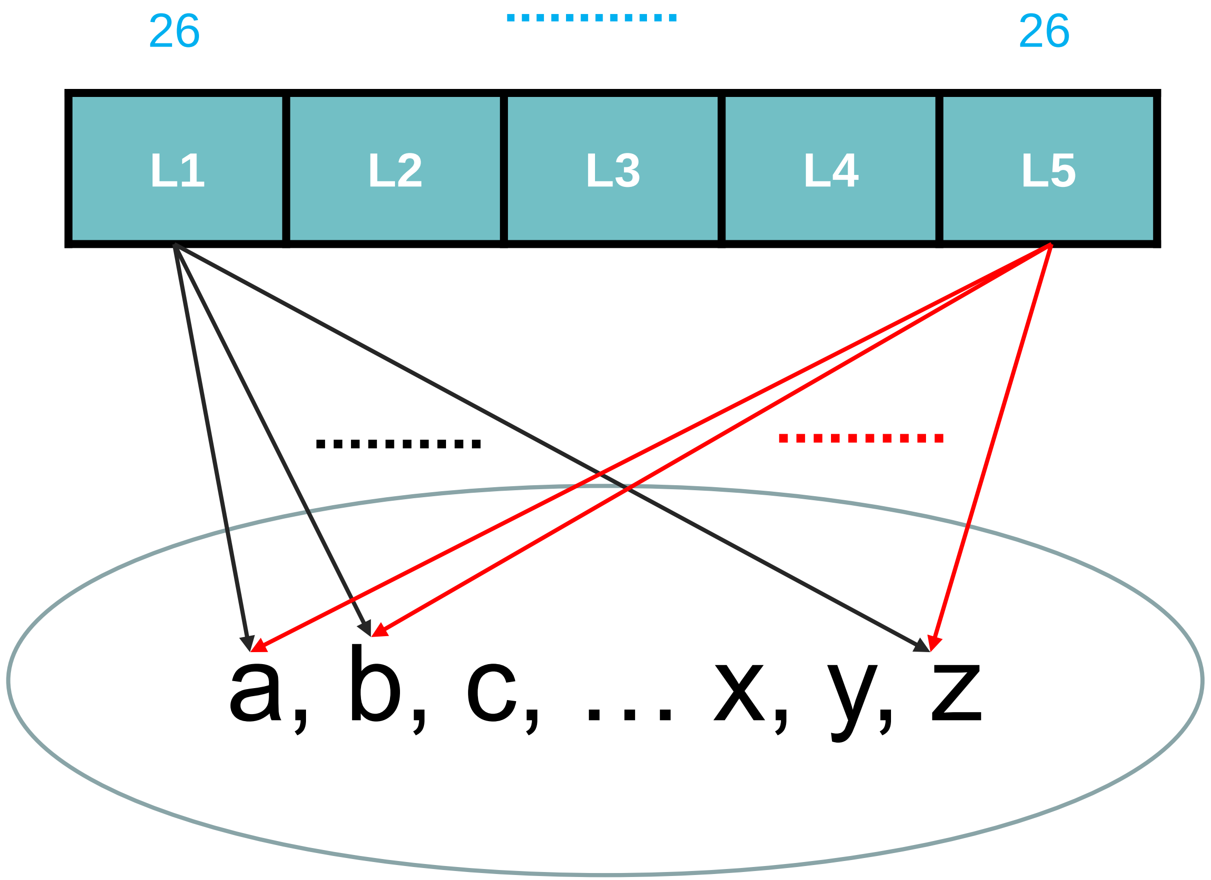
- **Problem1**： How many 5-letter words are there(using the 26-letter English alphabet)？

e. g. abcde, sssdd, …

- **Problem2**： How many distinct 5-letter words are there(using the 26-letter English alphabet) ？

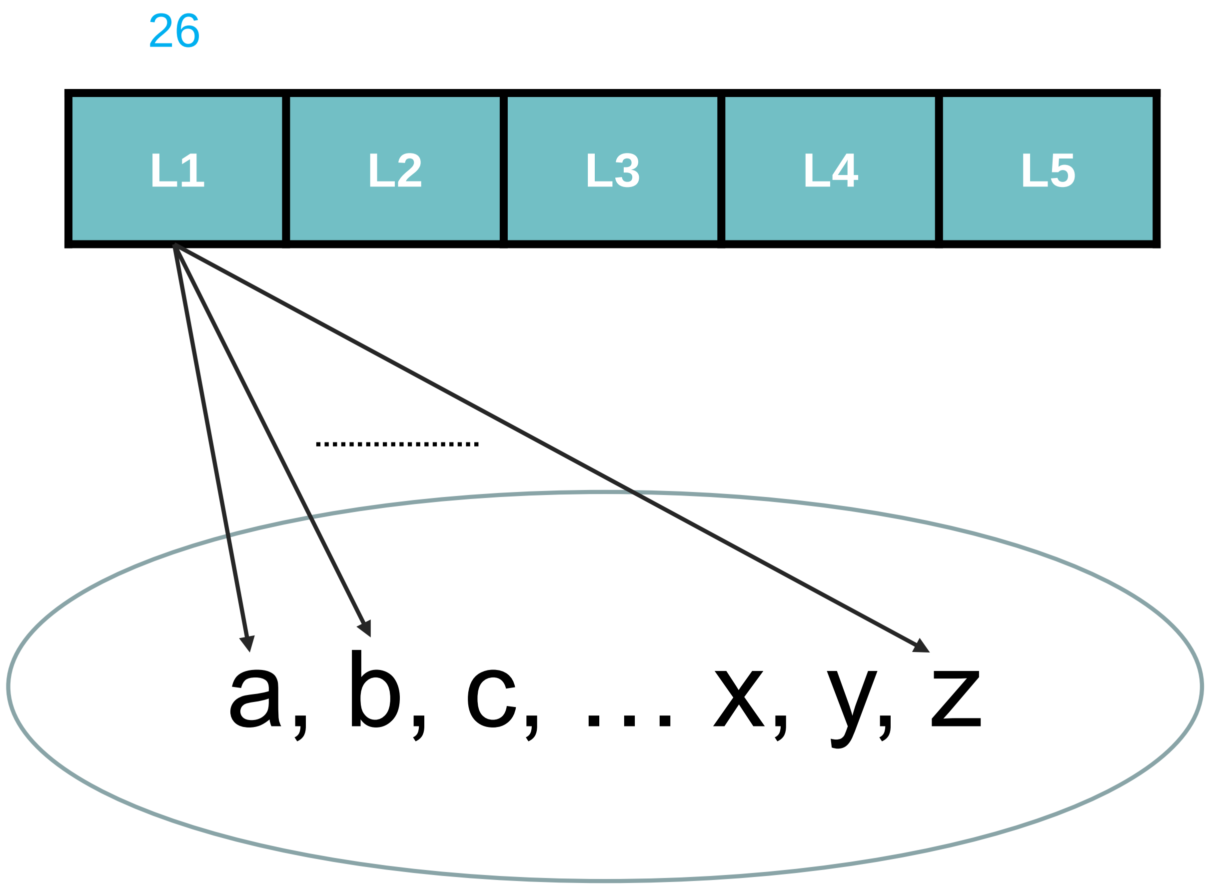
e. g. abcde, ~~sssdd~~, …

5-letter words

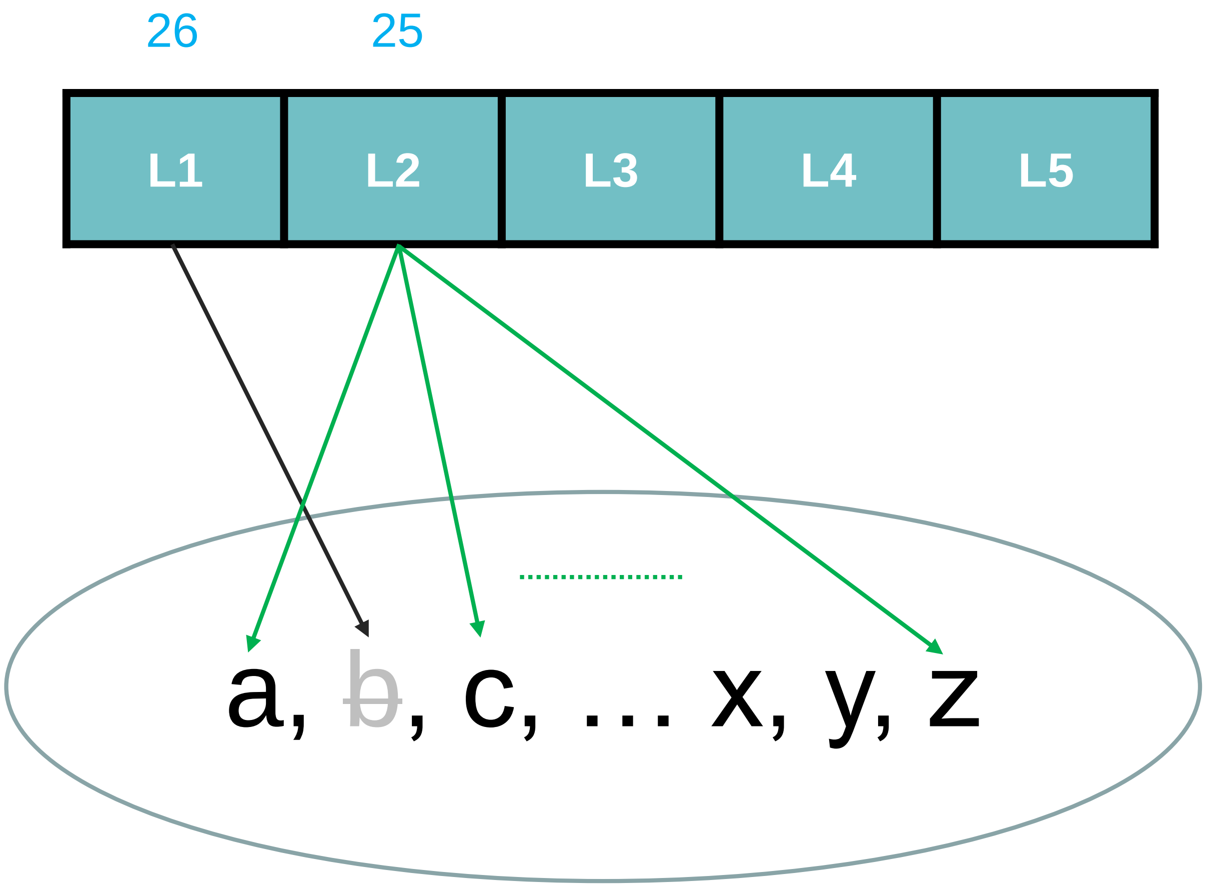


$$26 \times 26 \times 26 \times 26 \times 26 = 26^5$$

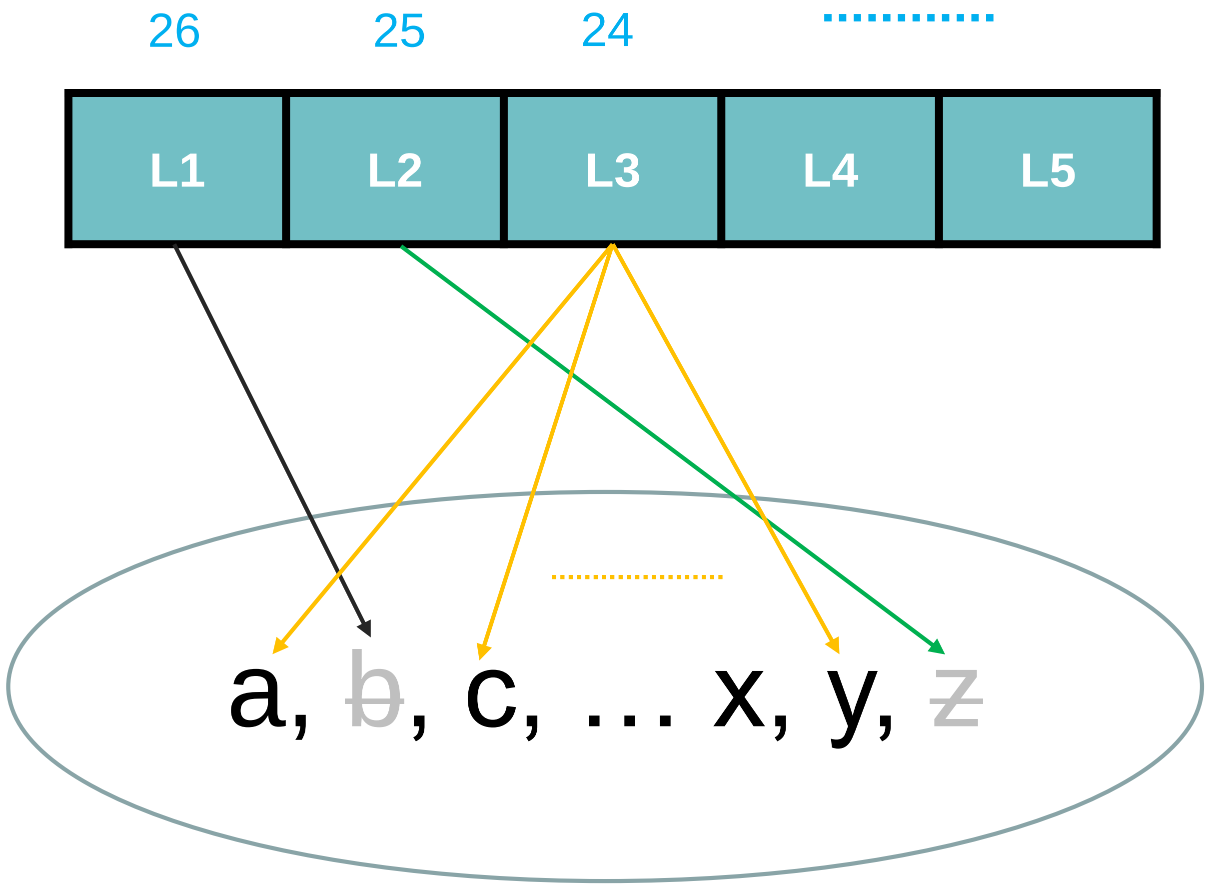
Distinct 5-letter words



Distinct 5-letter words



Distinct 5-letter words



$$26 \times 25 \times 24 \times 23 \times 22$$

Proof by induction

Goal: show that $P(x)$ is true for any $x \in \omega$

① Check that $P(0)$ is true;

② Suppose that $P(k)$ is true; // Induction hypothesis

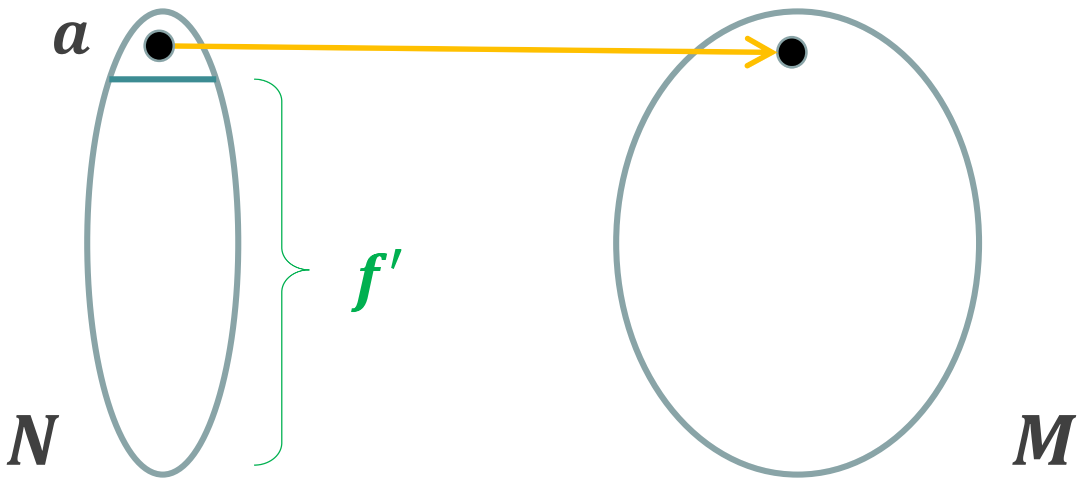
③ Prove that $P(k+1)$ is true.

The generalization of Problem 1

- Proposition1： Let N be an n-element set, and M be an m-element set, with $n \geq 0, m \geq 1$. Then the number of all possible mappings $f: N \to M$ is m^n.

- Proof: (By induction on n)
  - $n = 0$: $f = \emptyset$; $m^0 = 1$ 。
  - Suppose the results works for $n = k$;
  - If $n = k + 1$:

$n = k + 1$ ， take any $a \in N$:



$$m \cdot m^{n-1} = m^n$$

n balls are put into m bins

balls per bin	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins.	m^n		
n identical balls, m distinct bins.			
n distinct balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	
n identical balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	

The generalization of Problem 2

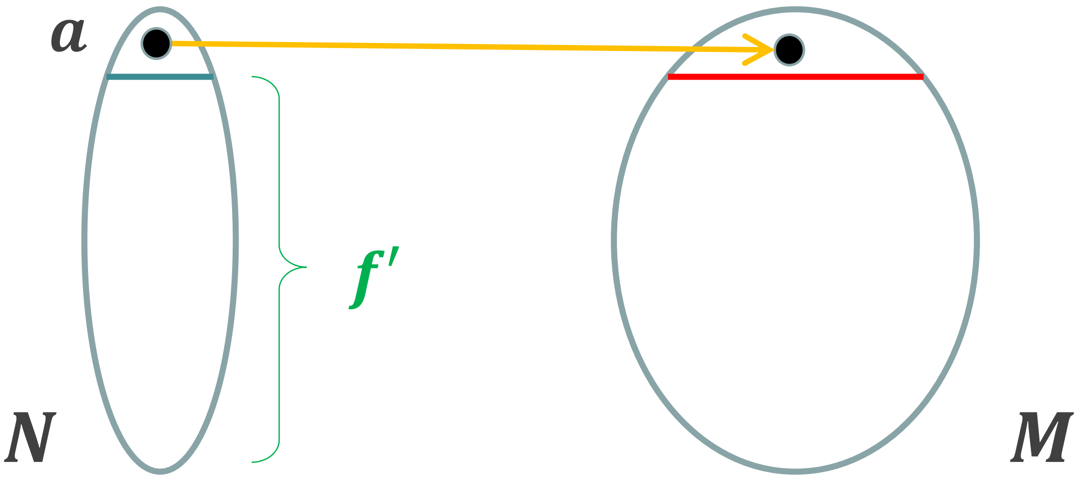
- Proposition2： Let N be an n-element set, and M be an m-elemnt set, with $n, m \geq 0$. Then there exist exactly

$$m(m-1)\ldots(m-n+1)=\prod_{i=0}^{n-1}(m-i)$$

one-to-one mappings from N into M.

- Proof: (By induction on n)
  - $n=0$: $f=\emptyset$. The value of an empty product is defined as 1.
  - Suppose the results works for $n=k$;

– for $n = k + 1$, take any $a \in N$:



$$m(m-1)\ldots(m-n+1)$$

Falling factorial notation

$$\boxed{(x)_n} = x^{\underline{n}} = \boxed{x(x-1)\cdots(x-n+1)}$$

n balls are put into m bins

balls per bin	**unrestricted**	$\le$ **1**	$\ge$ **1**
n distinct balls, m distinct bins.	m^n	$\boxed{(m)_n}$	
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Application 1: Counting the different subsets

Given set X , $|X| = n$, then X has exactly 2^n subsets $(n \geq 0)$.

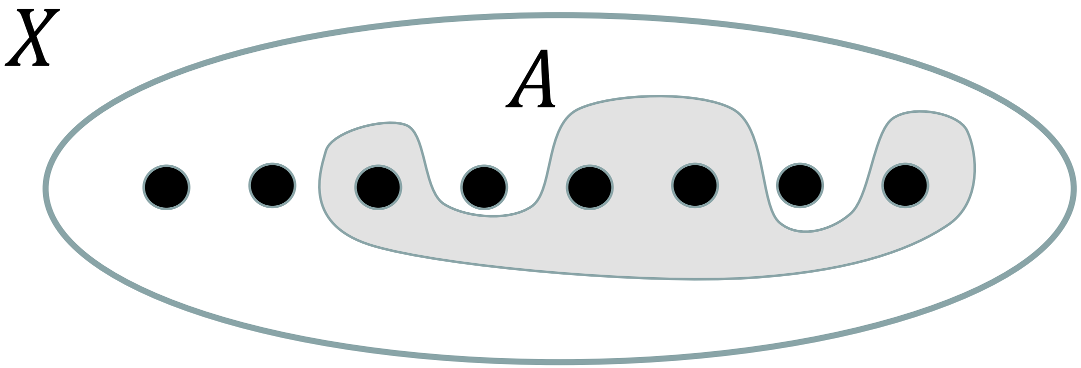
- Proof[1]: By induction on n. (Exercise)
- Proof[2]:

for any $A \subseteq X$, define $f_A: X \to \{0,1\}$ as

$$\boxed{f_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}}$$

Characteristic function

$$f_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$



f_A 0 0 1 0 1 1 0 1

There exists a bijective relation between the subsets of X and $f: X \to \{0,1\}$ (Recall: Equinumerous).

Application2：Counting the permutations

- ***Permutation***：A bijective mapping of a finite set X to itself is called a permutation of the set X.
- Recall：Bijective functions.

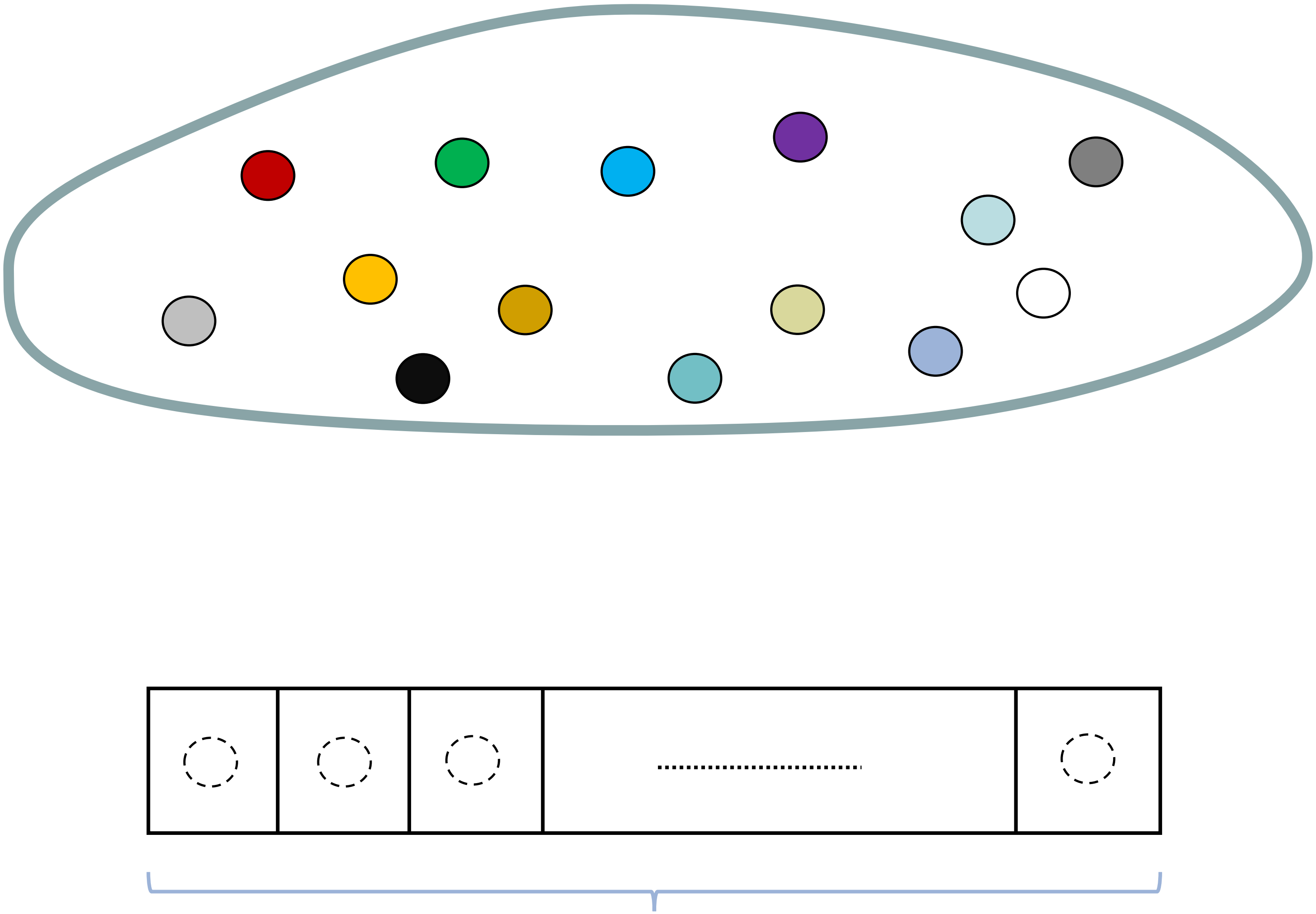
Counting permutations-Factorial

Given set X , $|X| = n$, then there are $n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$ different permutations on set X.

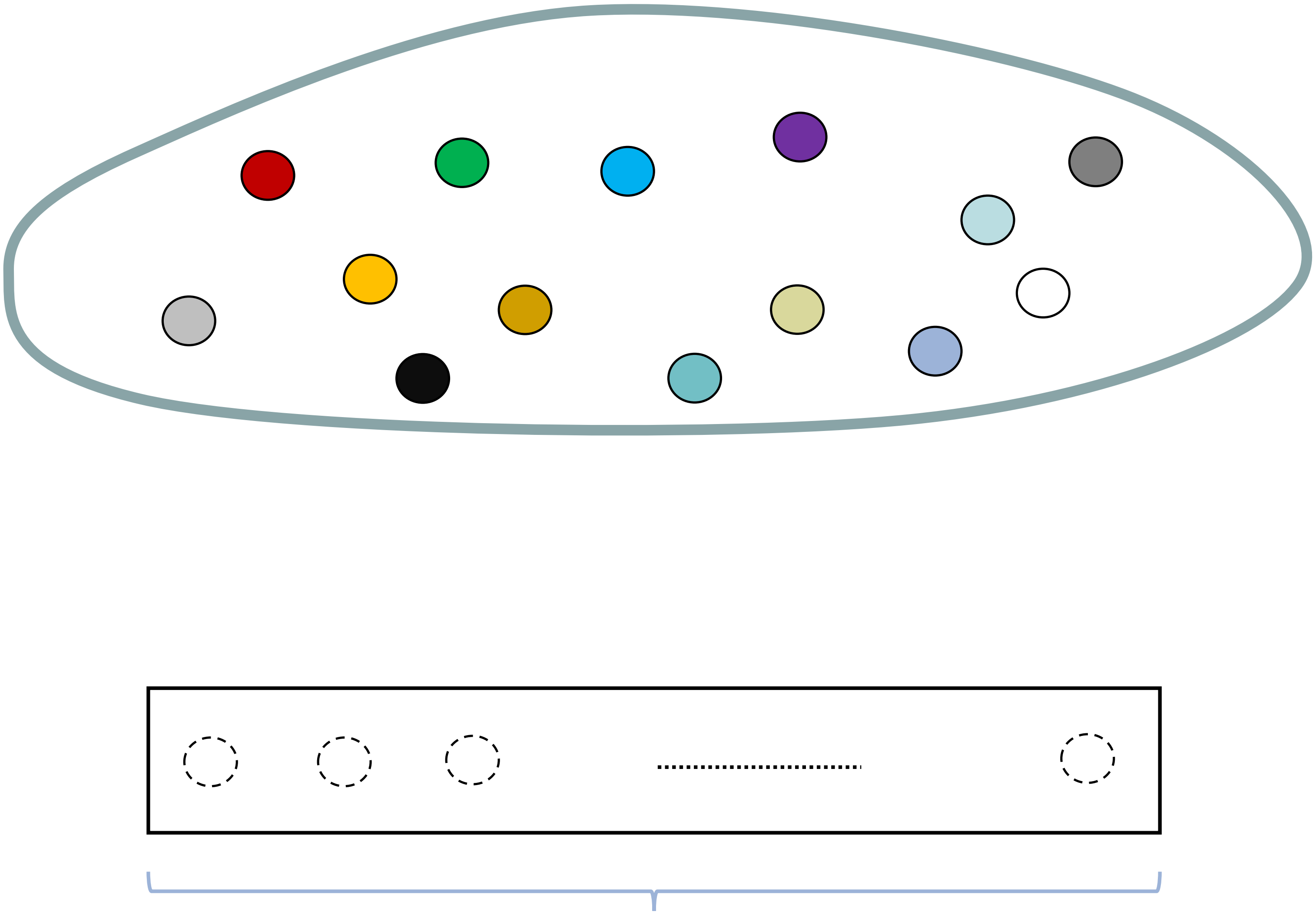
n factorial:

$$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 = \prod_{i=1}^{n} i.$$

- So far, we considered ordered sequences.
- What about the un-ordered occasion?



Ordered sequence



Un-ordered set

Problem 3： counting k-element subsets

Given set X , $|X| = n$, $n \geq k \geq 0$ ， how many different subsets of X contains exactly k elements？

e. g. $X = \{a, b, c\}$ ， $k = 2$ 。

Then: $\{a, b\}$ ， $\{a, c\}$ ， $\{b, c\}$. Three 2-size subsets.

Convention: $\binom{X}{k}$ VS. $\left|\binom{X}{k}\right|$

e. g. $\binom{X}{k} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$, $\left|\binom{X}{k}\right| = 3.$

- Proposition： For any finite set X with $|X| = n$, the number of all k-element subsets is

$$\left|\binom{X}{k}\right| = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k(k-1)\cdot\ldots\cdot2\cdot1}.$$

- Proof： (Double counting!)

Binomial coefficients

- $\boxed{\binom{n}{k}} = \left|\binom{X}{k}\right| = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k(k-1)\cdot\ldots\cdot 2\cdot 1}$

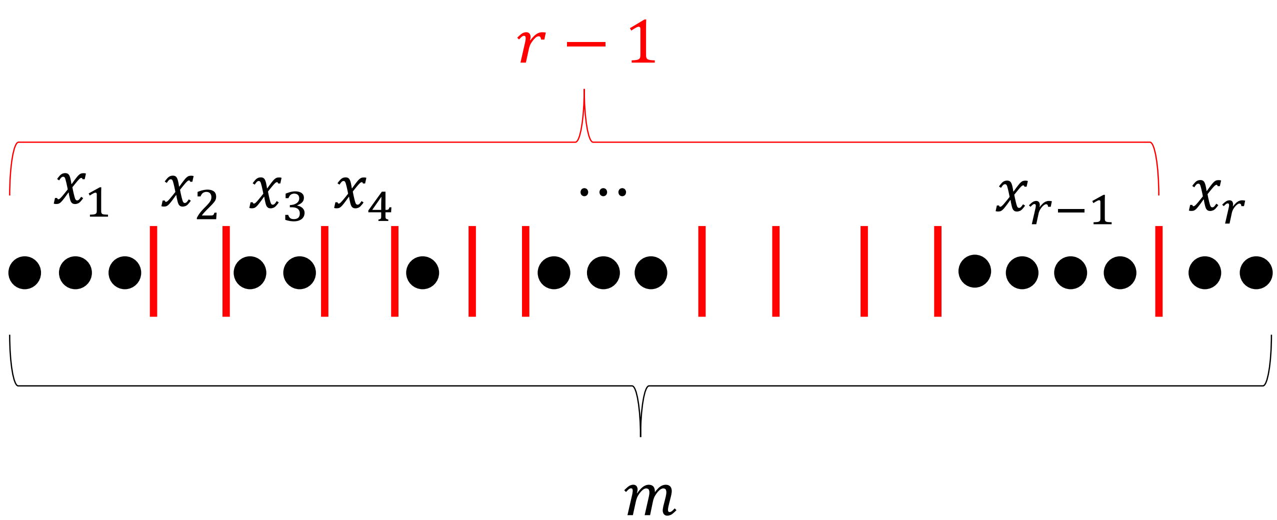
$$= \frac{\prod_{i=0}^{k-1}(n-i)}{k!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-k+1)\cdot(n-k)\cdot\ldots\cdot 1}{k(k-1)\cdot\ldots\cdot 2\cdot 1\cdot(n-k)\cdot\ldots\cdot 1}$$

$$= \boxed{\frac{n!}{k!\cdot(n-k)!}}$$

Application: counting non-negative solutions.

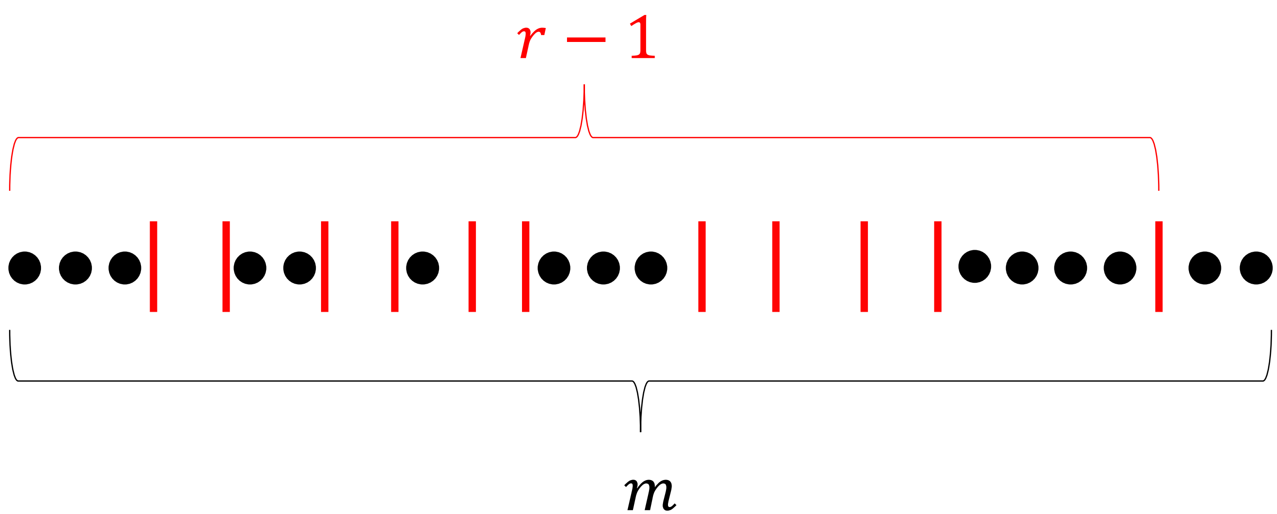
$m \geq r \geq 0$, the equation $x_1 + x_2 + \cdots + x_r = m$ has $\boxed{\binom{m+r-1}{r-1}}$ non-negative integers solutions of the form $(x_1, x_2, \ldots, x_r)$.



$$\boldsymbol{x_1 = 3, x_2 = 0, x_3 = 2, x_4 = 0, \cdots, x_{r-1} = 4, x_r = 2}$$

Application: counting non-negative solutions.

$m \geq r \geq 0$, the equation $x_1 + x_2 + \cdots + x_r = m$ has $\binom{m+r-1}{r-1}$ non-negative integers solutions of the form $(x_1, x_2, \ldots, x_r)$.



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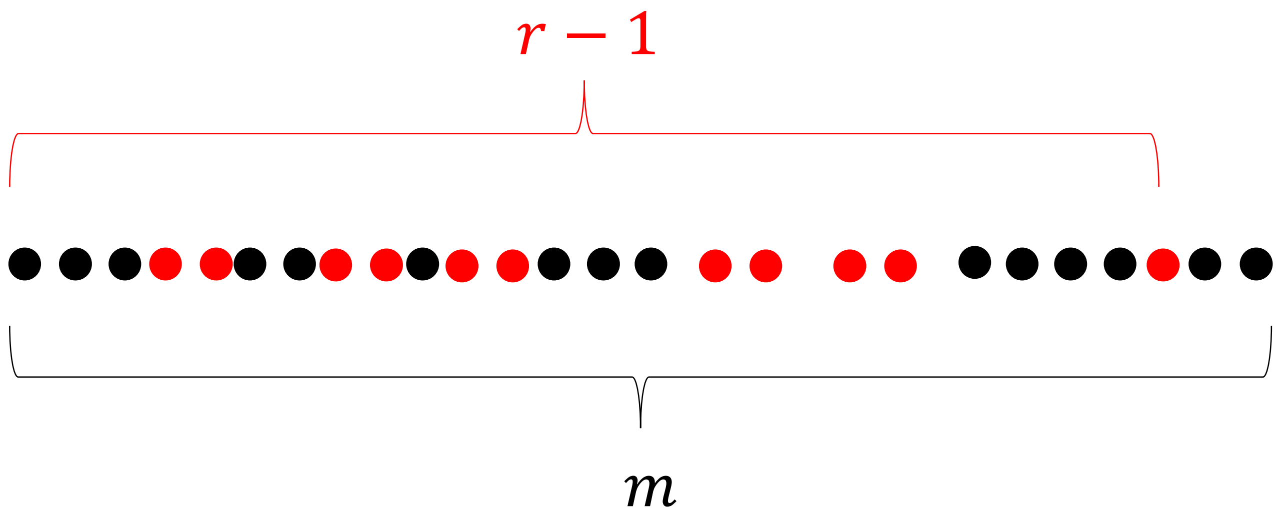
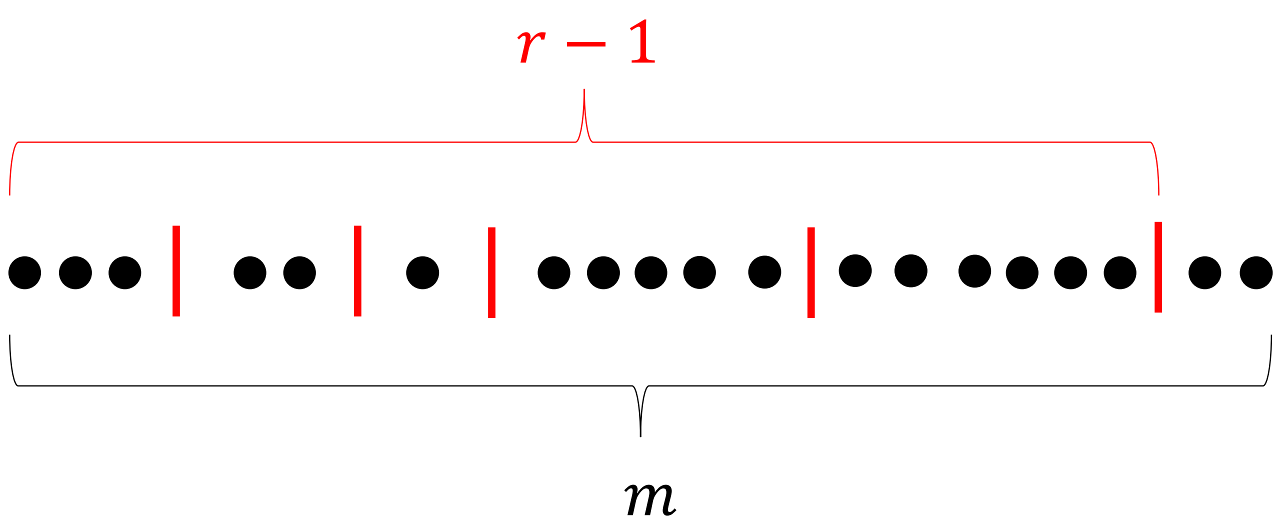


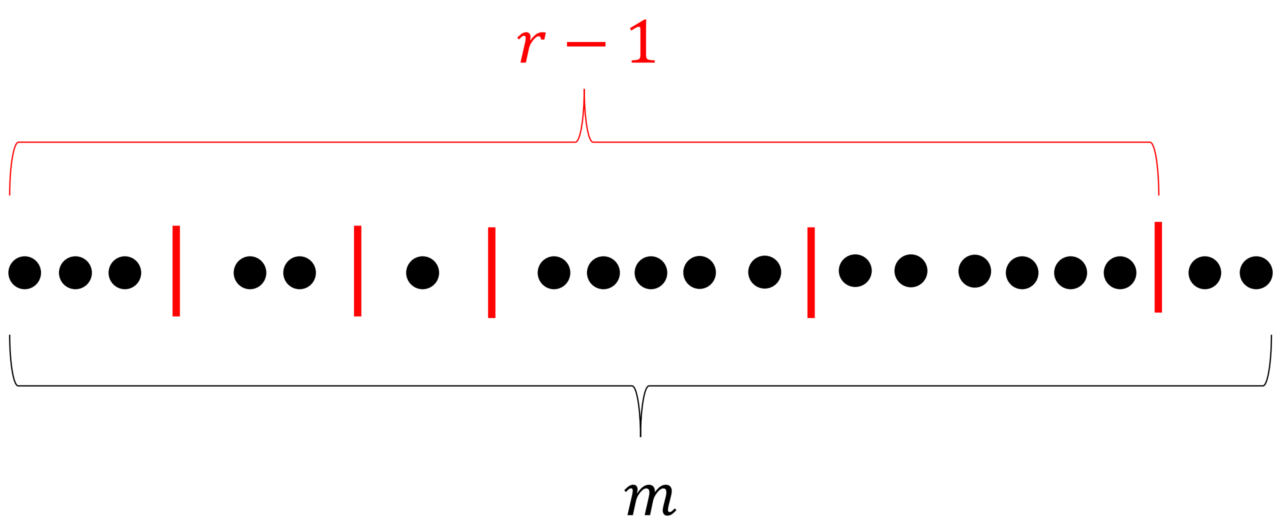
Question: counting positive solutions.

$m \geq r \geq 0$, the equation $x_1 + x_2 + \cdots + x_r = m$ has _____ positive integers solutions of the form $(x_1, x_2, \ldots, x_r)$.



Question: counting positive solutions.

$m \geq r \geq 0$, the equation $x_1 + x_2 + \cdots + x_r = m$ has $\binom{m-1}{r-1}$ positive integers solutions of the form $(x_1, x_2, \ldots, x_r)$.

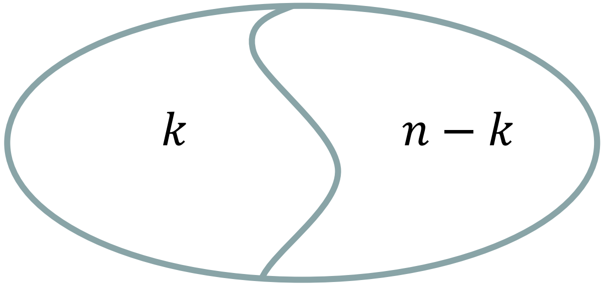


Basic Properties

$$\binom{n}{k} = \binom{n}{n-k}$$

- Proof[1]:

- Proof[2]:

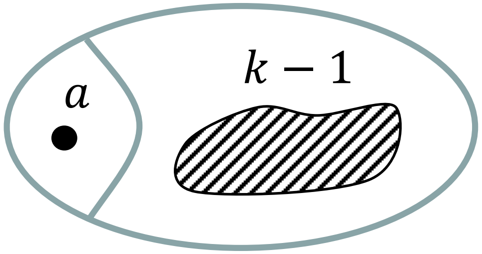


Pascal's Identity:

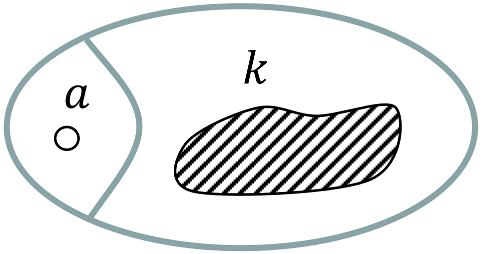
$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

- Proof:

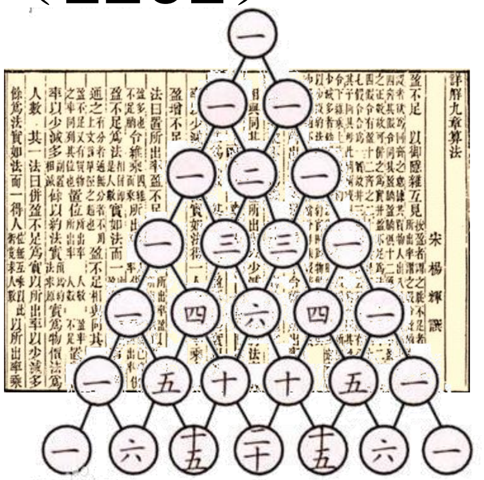
$$\binom{n-1}{k-1}$$



$$\binom{n-1}{k}$$



Pascal's Triangle （1654）/ 杨辉三角（1261）



$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

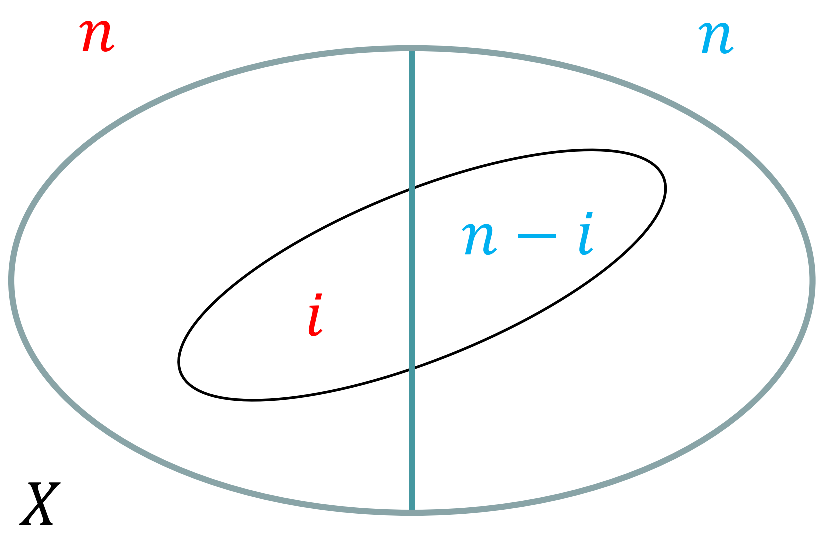
Exercise

$$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

$$\sum_{k=0}^{n} \binom{m+k-1}{k} = \binom{n+m}{n}$$

$$\boxed{\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}}$$

- Proof: $\sum_{i=0}^{n} \binom{n}{i}^2 = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$



Vandermonde's identity/convolution

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

The general form

$$\binom{n_1 + \cdots + n_p}{m} = \sum_{k_1+\cdots+k_p=m} \binom{n_1}{k_1} \binom{n_2}{k_2} \cdots \binom{n_p}{k_p}$$

n balls are put into m bins

balls per bin	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins.	m^n	$(m)_n$	
n identical balls, m distinct bins.	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	
n identical balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	

Multiset Coefficient

- The number of multisets of cardinality k, with elements taken from a finite set of cardinality n, is called the **multiset coefficient** or **multiset number**.
- $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$

转化为x_1+x_2+...+x_n = k的非负整数解

$$= \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} = \frac{n^{\overline{k}}}{k!}$$

n balls are put into m bins

balls per bin	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins.	m^n	$(m)_n$	
n identical balls, m distinct bins.	$\left(\binom{m}{n}\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	
n identical balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	

- Basic counting
- Binomial theorem
- Generalized Binomial theorem
- Some special numbers

Binomial theorem

- ***Binomial Theorem*:** for any non-negative integer n, we have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

- Proof： Exercise
- Applications：
  - $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$ (take $x = 1$)
  - $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \cdots = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$
  - $2\left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots\right] = 2^n$

可用来证明任意有限集合的odd subset和even subset一样多

Pascal's Triangle （1654）/ 杨辉三角（1261）

$$\binom{0}{0}$$

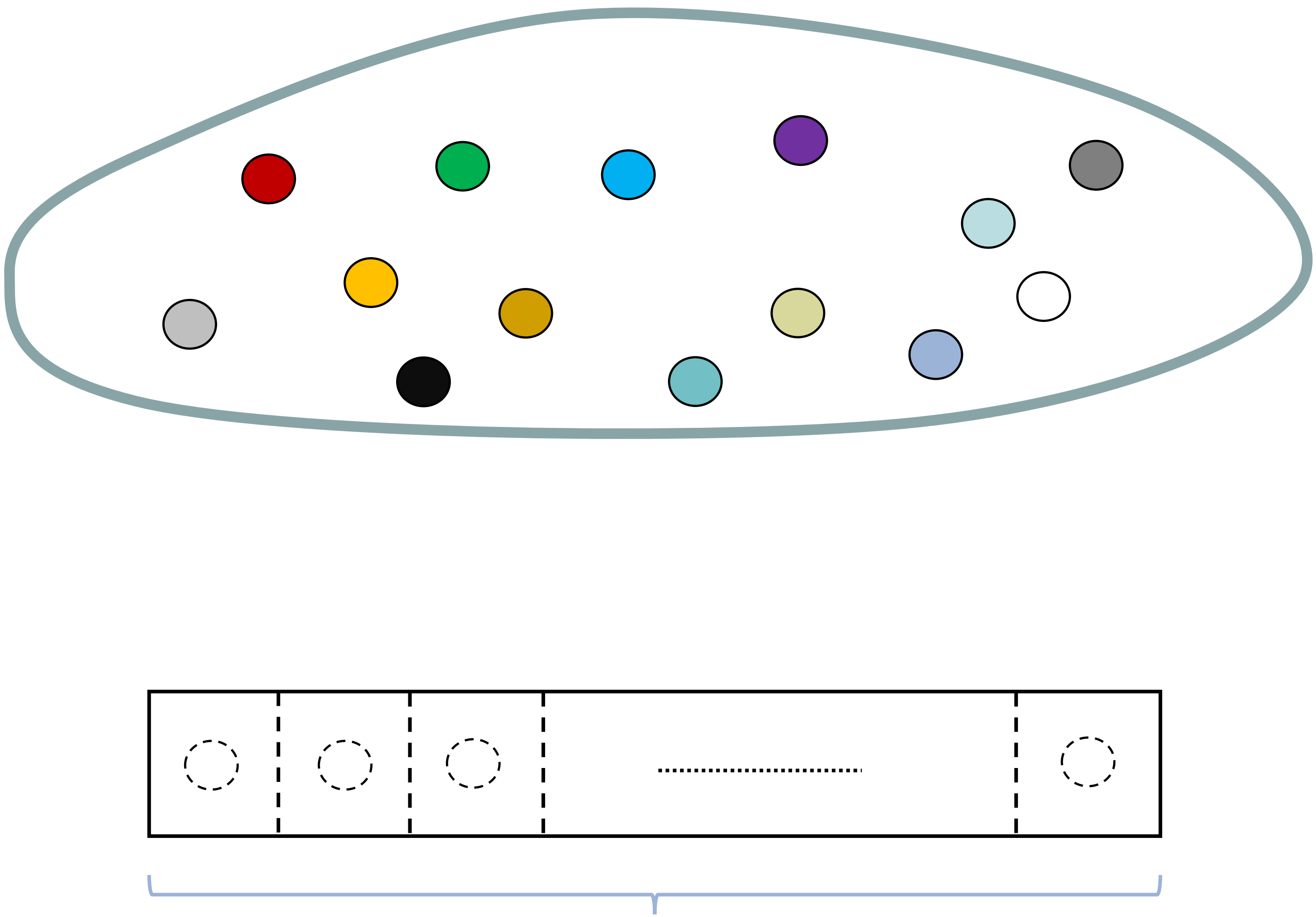
$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

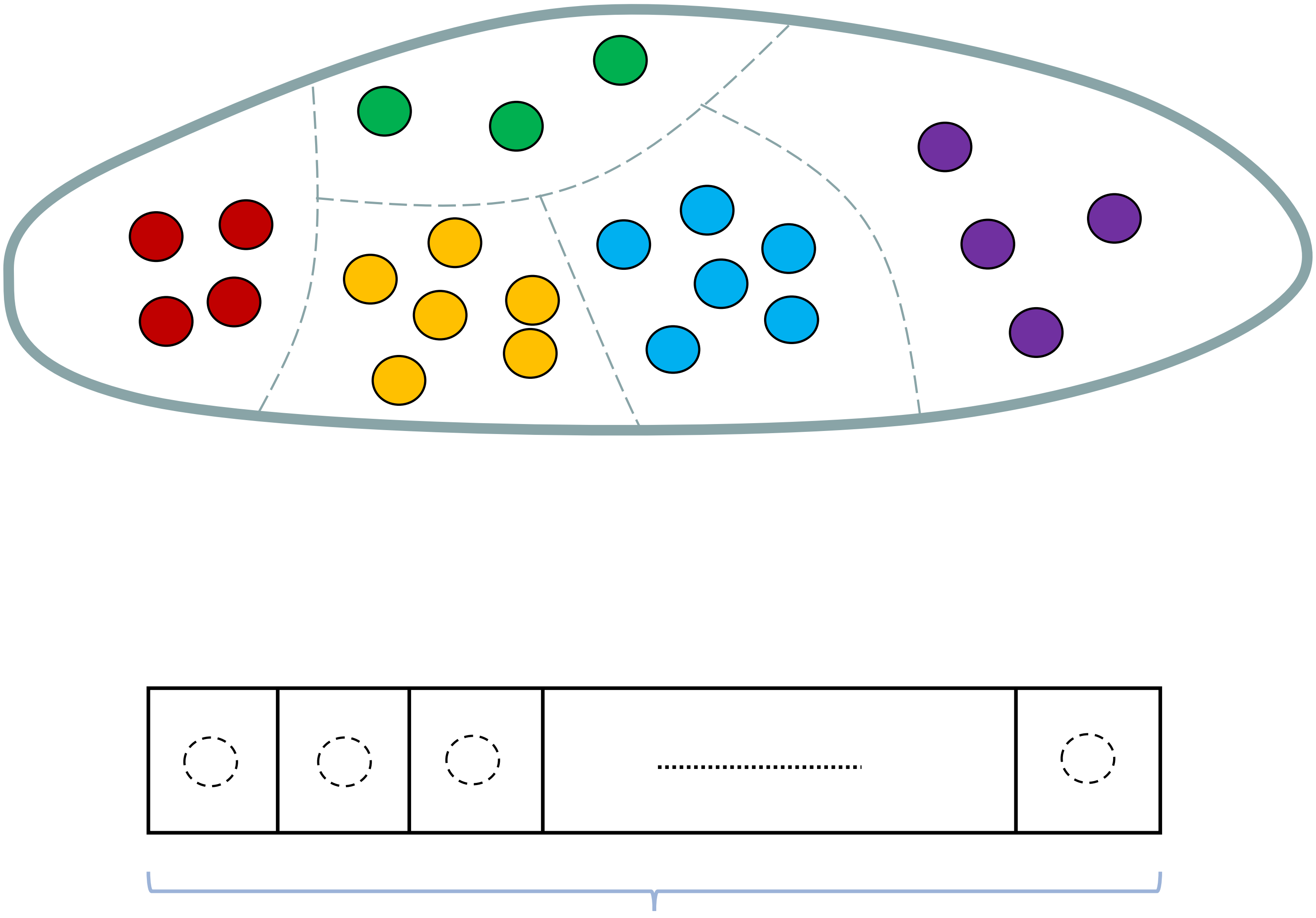
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

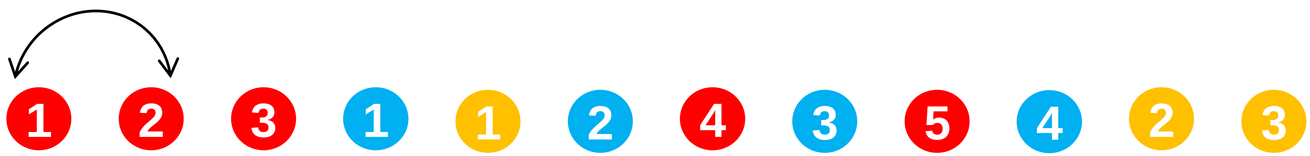


(Un-)Ordered sequence

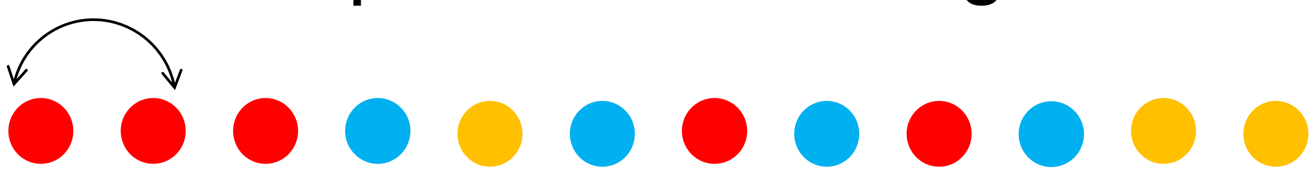


Ordered sequence

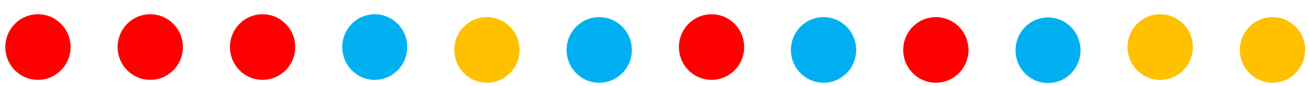
- With 5 different red balls, 3 different yellow balls, 4 different blue balls, we can get $(5+3+4)! = 12!$ different sequences.



- **Question:** With 5 equal red balls, 3 equal yellow balls, 4 equal blue balls, how many different sequences can we get?



- **Theorem：** if we have objects of m kinds, k_i indistinguishable objects of ith kind, where $k_1 + k_2 + \cdots + k_m = n$, then the number of distinct arrangements of the objects in a row is $\frac{n!}{k_1!k_2!\ldots k_m!}$. Usually written $\binom{n}{k_1,k_2,\ldots,k_m}$。



$\frac{12!}{5!3!4!}$ 种

- ***Multinomial Theorem*:** For arbitrary real number $x_1, x_2, \ldots, x_m$ and any natural number $n \geq 1$, the following equality holds:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{\substack{k_1 + \cdots k_m = n \\ k_1, \cdots, k_m \geq 0}} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}.$$

- e. g. In $(x + y + z)^{10}$ the coefficient of $x^2 y^3 z^5$ is $\binom{10}{2,3,5} = 2520$.

Basic counting

Binomial theorem

Generalized Binomial theorem

Some special numbers

Newton(1665)'s generalized binomial theorem

Let $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!} = \frac{(r)_k}{k!}$ where r is arbitrary, $k > 0$ is an integer

If x and y are real numbers with $|x| > |y|$

$$(x+y)^r = \sum_{k=0}^{\boxed{\infty}} \binom{r}{k} x^{r-k} y^k$$

$$= x^r + r x^{r-1} y + \frac{r(r-1)}{2!} x^{r-2} y^2 + \frac{r(r-1)(r-2)}{3!} x^{r-3} y^3 + \cdots$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \cdots$$

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

Generally: $r = -s$

$$\boxed{\frac{1}{(1-x)^s} = \sum_{k=0}^{\infty} \binom{s+k-1}{k} x^k}$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 - \frac{63}{256}x^5 + \cdots$$

- Basic counting
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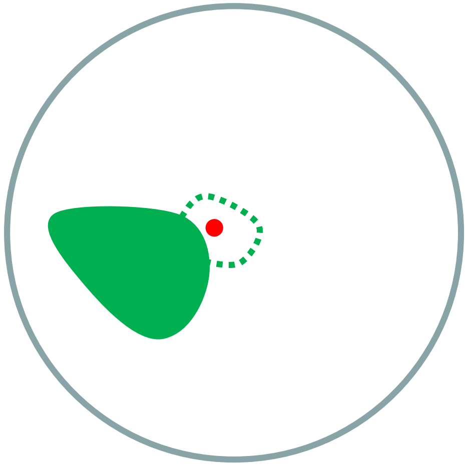
- Some special numbers

Stirling subset numbers

- The second Stirling Numbers $\left\{ n \atop k \right\}$: The number of ways to partition a set of n things into k nonempty subsets.
- e.g. $\left\{ 4 \atop 2 \right\} = 7$
- $\boxed{\left\{ n \atop 2 \right\} = 2^{n-1} - 1}$ why?

注意2^{n-1}的前面没有系数n，这是因为最开始选择的红色圆点虽然有n种选法，但是后面再和其他子集合并的时候，由于集合元素的无序性，会导致重复，所以前面的系数应该去掉



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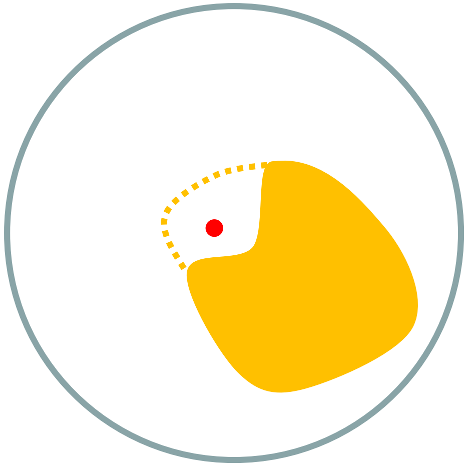

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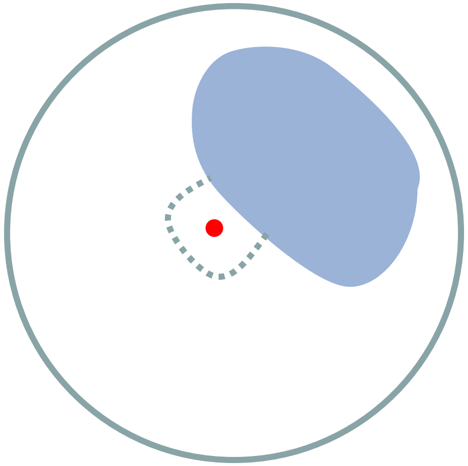
- e.g. $\left\{ 4 \atop 2 \right\} = 7$
- $\left\{ n \atop 2 \right\} = 2^{n-1} - 1$ why?

选取一个元素a，在剩下的n-1个元素里面任意取一个子集，将其与a合并，作为一个子集，剩下的作为另一个集合，这样就得到两个子集，但是在剩下的n-1里面不能全选，否则实际上只有一个集合，没有划分为两个
所以是2^(n-1) - 1



Stirling subset numbers

- The second Stirling Numbers $\left\{ n \atop k \right\}$: The number of ways to partition a set of n things into k nonempty subsets.
- e.g. $\left\{ 4 \atop 2 \right\} = 7$
- $\boxed{\left\{ n \atop k \right\} = k \left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\}}$

先选取一个元素a，现在分成两种情况：
1.a单独作为一个集合，此时需要将剩下的n-1元素划分为k-1个
2.a被合并到某一个集合里面，那么剩下的n-1个元素就需要被分成k个，最后a在这k个里面选一个加进去

Stirling cycle numbers

- The first Stirling Numbers $\left[{n \atop k}\right]$: The number of ways to partition a set of n things into k nonempty cycles.
- $\left[{n \atop k}\right] \geq \left\{{n \atop k}\right\}$, e.g. $\left[{4 \atop 2}\right] = 11$
- $\boxed{\left[{n \atop 1}\right] = (n-1)!}$

相当于n个人围成一圈，可以认为先n!排列，在围成圈，这样就不用管谁先开始

证明思路：n!其实是{1, 2, ..., n}到自身的不同的双射函数的个数，所以我们只要证明任意一个斯特灵数和一个这样的双射函数一一对应就可以了

- $\boxed{\sum_{k=0}^{n} \left[{n \atop k}\right] = n!}$ where $n \in Z^+$.

1 2 3 4
1 4 2 3
唯一决定了{1} {2->4->3->2}这样两个圈

系数n-1是因为插入的选择有n-1个

- $\boxed{\left[{n \atop k}\right] = (n-1) \cdot \left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]}$ Why?

n balls are put into m bins

balls per bin	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins.	m^n	$(m)_n$	先把n个球分成m份，再对这m份全排列 $m!\left\{ n \atop m \right\}$
n identical balls, m distinct bins.	非负整数解 $\binom{n+m-1}{m-1}$	$\binom{m}{n}$	正整数解 $\binom{n-1}{m-1}$
n distinct balls, m identical bins.	$\sum_{k=1}^{m} \left\{ n \atop k \right\}$	$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	$\left\{ n \atop m \right\}$
n identical balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	

Partition of a number

- $P_k(n)$: number of partition the positive integer n into k parts.
- e.g. $P_2(7) = 3 \quad \{\{1,6\},\{2,5\},\{3,4\}\}$
  $P_6(7) = 1 \quad \{\{1,1,1,1,1,2\}\}$
- Number of integral solutions to

$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \ge x_2 \ge \cdots \ge x_k \ge 1 \end{cases}$$

这个序关系排除了多余的解

- $\boxed{P_k(n) = P_{k-1}(n-1) + P_k(n-k)}$ why?

如果x_k = 1，那么就转化为了右边的第一部分，否则可以在第一个式子左右都减去k

n balls are put into m bins

balls per bin	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins.	m^n	$(m)_n$	$m!\left\{ {n \atop m} \right\}$
n identical balls, m distinct bins.	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins.	$\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$	$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	$\left\{ {n \atop m} \right\}$
n identical balls, m identical bins.	$\sum_{k=1}^{m} p_k(n)$	$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	$p_m(n)$

Partition of a number

- $P_k(n)$： number of partition the positive integer n into k parts.

- $\boxed{\sum_{k=1}^{m} p_k(n) = p_m(n+m)}$ why?

理解思路：也是构造一个左右各项一一对应的双射

等式的右边相当于先m个盒子里面每个里面放一个球，再将剩下的n个球分成m个正整数解，这样其实和左边是一个意思

Twelvefold way

The twelve combinatorial objects and their enumeration formulas.

f-class	Any *f*	Injective *f*	Surjective *f*
f	*n*-sequence in *X* x^n	*n*-permutation in *X* $x^{\underline{n}}$	composition of *N* with *x* subsets $x!\left\{ {n \atop x} \right\}$
$f \circ S_n$	*n*-multisubset of *X* $\binom{x+n-1}{n}$	*n*-subset of *X* $\binom{x}{n}$	composition of *n* with *x* terms $\binom{n-1}{n-x}$
$S_x \circ f$	partition of *N* into ≤ *x* subsets $\sum_{k=0}^{x} \left\{ {n \atop k} \right\}$	partition of *N* into ≤ *x* elements $[n \le x]$	partition of *N* into *x* subsets $\left\{ {n \atop x} \right\}$
$S_x \circ f \circ S_n$	partition of *n* into *x* non-negative parts $p_x(n+x)$	partition of *n* into ≤ *x* parts 1 $[n \le x]$	partition of *n* into *x* parts $p_x(n)$

https://en.wikipedia.org/wiki/Twelvefold_way