Homework 5

Problem 1. *Fill in the blanks with either true (✓) or false (×)*

$f(n)$	$g(n)$	$f = O(g)$	$f = \Omega(g)$	$f = \Theta(g)$
$2n^3 + 3n$	$100n^2 + 2n + 100$	×	✓	×
$50n + \log n$	$10n + \log\log n$	✓	✓	✓
$50n \log n$	$10n \log\log n$	×	✓	×
$\log n$	$\log^2 n$	✓	×	×
$n!$	5^n	×	✓	×

Problem 2. 1. *Find two functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.*

2. *Furthermore, we say a function $h : \mathbb{R} \to \mathbb{R}$ is* monotonically increasing *if it satisfies the property '$x \leq y \Rightarrow h(x) \leq h(y)$'.*
   Find two monotonically increasing functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.

(Please give the detailed proof that your functions satisfy the requirements.)

Solution.

1. $f(x) = \cos x$ and $g(x) = \sin x$. If $f(x) = O(g(x))$, that is, $\cos x = O(\sin x)$. By the definition of O notation, we have there exits a constant x_0 and a constant C, such that, for any $x \geq x_0$, there will be $|\cos x| \leq C \cdot \sin x$. Suppose we choose $x_1 \geq x_0$, and $\cos x_1 \neq 0$, since $|\cos x_1| \leq C \cdot \sin x_1$, when $x_2 = x_1 + \pi \geq x_0$, we have $|\cos x_1| \leq -C \cdot \sin x_1$. However this can not be true, since $\cos x_1 \neq 0$. Thus $f(x) \neq O(g(x))$. We can prove $g(x) \neq O(f(x))$ in a similiar way.

2. $f(x) = e^{x+\sin x}$ and $g(x) = e^{x+\cos x}$. First, since $(x+\sin x)' = 1+\cos x \geq 0$, thus $x+\sin x$ is monotonically increasing, then it would be easy to say $f(x)$ is also monotonically increasing. We can prove $g(x)$ is monotonically increasing in a similiar way. Second, if $f(x) = O(g(x))$, then by definition, we have there exits a constant x_0 and a constant C, such that, for any $x \geq x_0$, there will be $|e^{x+\sin x}| \leq C \cdot e^{x+\cos x}$, similiar as (1), we can prove it is not true, thus $f(x) \neq O(g(x))$, again we can prove $g(x) \neq O(f(x))$.

□

Problem 3. *Prove that*

(a) $\left(1+\frac{1}{n}\right)^n \leq e$ *for all* $n \geq 1$.

(b) $\left(1+\frac{1}{n}\right)^{n+1} \geq e$ *for all* $n \geq 1$.

(c) Using (a) *and* (b), *conclude that* $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Proof.

(a) A well known inequality is that $1+x \leq e^x$, if we let $x = \frac{1}{n}$, we have $\left(1+\frac{1}{n}\right)^n \leq \left(e^{1/n}\right)^n = e$.

(b) $\left(1+\frac{1}{n}\right)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1} = \left(\frac{1}{1-\frac{1}{n+1}}\right)^{n+1} \geq \left(e^{\frac{1}{n+1}}\right)^{n+1} = e$.

(c) From (a), (b) we can see that $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n \leq e$ and $e \leq \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n+1}$. Since

$$\lim_{n\to+\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}} = \lim_{n\to+\infty}\frac{1}{1+\frac{1}{n}} = 1$$

Thus $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n \leq e \leq \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n$, so $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

□

Problem 4. *Prove* Bernoulli's inequality*: for each natural number n and for every real* $x \geq -1$, *we have* $(1+x)^n \geq 1+nx$.

Proof. We prove this statement by induction on n.
Basis step. $n = 0$, for every real $x \geq -1$, we have $(1+x)^0 = 1 \geq (1+0\times x) = 1$.
Induction hypothesis. Assume when $n = k$, we have $(1+x)^k \geq 1+kx$ for every $x \geq -1$.
Proof of induction step. When $n = k+1$, since $x \geq -1$,we have $(1+x) \geq 0$ and $(1+x)^{k+1} = (1+x)(1+x)^k \geq (1+x)(1+kx) = 1+(k+1)x+kx^2 \geq 1+(k+1)x$. Thus, we can say $(1+x)^n \geq 1+nx$ for each natural number n and for every real $x \geq -1$. □

Problem 5. *Prove that for $n = 1, 2, \ldots,$ we have*

$$2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1.$$

Proof. We can prove this statement by induction on n.

1. We first prove $2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$.
   Basis step. When $n = 1$, we have $2\sqrt{2} - 2 < 1$, which is true.
   Induction hypothesis. Assume when $n = k$ the statement is true, that is, $2\sqrt{k+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}}$.
   Proof of induction step. When $n = k+1$, by induction hypothesis, we have $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > 2\sqrt{k+1} - 2 + \frac{1}{\sqrt{k+1}}$. Thus, we just need to prove that $2\sqrt{k+2} - 2 < 2\sqrt{k+1} - 2 + \frac{1}{\sqrt{k+1}}$, that is, $2\sqrt{k+2} < 2\sqrt{k+1} + \frac{1}{\sqrt{k+1}}$. Since

   $$\begin{aligned} 2\sqrt{k+2} < 2\sqrt{k+1} + \tfrac{1}{\sqrt{k+1}} &\iff 2\sqrt{(k+2)(k+1)} < 2(k+1) + 1 \\ &\iff 4(k^2 + 3k + 2) < 4k^2 + 12k + 9 \\ &\iff 8 < 9 \end{aligned}$$

   Thus, we can say $2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$ is right.

2. Then, we prove $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1$.
   Basis step. When $n = 1$, we have $1 \leq 2\sqrt{1} - 1 = 1$, which is true.
   Induction hypothesis. Assume when $n = k$ the statement is true, that is, $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} \leq 2\sqrt{k} - 1$.
   Proof of induction step. When $n = k+1$, by induction hypothesis, we have $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}$. Thus we just need to prove $2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$, that is, $2\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}$. Since

   $$\begin{aligned} 2\sqrt{k} + \tfrac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} &\iff 2\sqrt{k(k+1)} + 1 \leq 2(k+1) \\ &\iff 4(k^2 + k) \leq 4k^2 + 4k + 1 \\ &\iff 0 < 1 \end{aligned}$$

   Thus, we can say $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1$ is right.

From 1 and 2, we can say the statement is true. □

Problem 6.

a) *Show that the product of all primes p with $m < p \le 2m$ is at most $\binom{2m}{m}$.*

b) *Using a), prove the estimate $\pi(x) = O(\frac{x}{\ln x})$, where $\pi(x)$ denote the number of primes not exceeding the number x.*

Solution.

a) For all primes p with $m < p \le 2m$, we have $p \mid 2m!$, and we also have $p \nmid m!$, thus $p \nmid (m!)^2$. Since $2m! = \binom{2m}{m} \times (m!)^2$, so we have For all primes p with $m < p \le 2m$, we have $p \mid \binom{2m}{m}$, that is, the product of all primes p with $m < p \le 2m$ is at most $\binom{2m}{m}$.

b) For any x, we say there will be a natural number k, such that $2^{k-1} < x \le 2^k$. From a) we know $\prod_{m<p\le 2m} p \le \binom{2m}{m}$, where p denotes the number is a prime. Thus, we have $\sum_{m<p\le 2m} \ln p \le \ln\binom{2m}{m}$. Since $\binom{2m}{m} \le 2^{2m}$ and $\sum_{m<p\le 2m} \ln p \ge (\pi(2m) - \pi(m)) \ln m$. So we now have $(\pi(2m) - \pi(m)) \ln m \le 2m \ln 2$, that is, $\pi(2m) - \pi(m) \le 2 \ln 2 \frac{m}{\ln m}$. If $m = 2^h$, we have $\pi(2^{h+1}) - \pi(2^h) \le \frac{2^{h+1}}{h}$. It is easy to know $\pi(2^{h+1}) \le 2^h$, thus we have $(h + 1)\pi(2^{h+1}) - h\pi(2^h) \le 3 \cdot 2^h$. So $\sum_{h=0}^{k-1}((h+1)\pi(2^{h+1}) - h\pi(2^h)) = k\pi(2^k) \le \sum_{h=0}^{k-1} 3 \cdot 2^h = 3 \cdot 2^k$. Thus, $\pi(2^k) \le 3 \cdot \frac{2^k}{k}$, since $2^{k-1} < x \le 2^k$, we have $\pi(x) \le \pi(2^k) \le 3 \cdot \frac{2^k}{k}$, so $\pi(x) = O(\frac{x}{\ln x})$.

□