Homework 4

Problem 1. *1. Determine the coefficient of* x^{50} *in* $(x^7 + x^8 + x^9 + x^{10} + \cdots)^6$

2. Determine the coefficient of x^3 *in* $(2 + x)^{\frac{3}{2}}/(1 - x)$

3. Determine the coefficient of x^4 *in* $(2 + 3x)^5\sqrt{1 - x}$

Solution.

1.
$$\begin{aligned} &= x^{42}(1 + x + x^2 + x^3 + \cdots)^6 \\ &= x^{42}\textstyle\prod_{i=1}^{6}(1 + x + x^2 + x^3 + \cdots)_i \end{aligned}$$

   The coefficient of x^{50} is $\binom{8+6-1}{6-1} = \binom{13}{5}$.

2.
$$\begin{aligned} &= (x + 2)^{3/2}(1 + x + x^2 + \cdots) \\ &= \textstyle\sum_{k=0}^{\infty}\binom{3/2}{k}x^k(2)^{3/2-k}(1 + x + x^2 + \cdots) \end{aligned}$$

   The coefficient of x^3 is $\sum_{k=0}^{3}\binom{3/2}{k}(2)^{3/2-k}$. Then use the Newton formula ....

3.
$$= \textstyle\sum_{k=0}^{5}\binom{5}{k}2^k(3x)^{5-k}\sum_{j=0}^{\infty}\binom{1/2}{j}(-x)^j$$

   The coefficient of x^4 is $\sum_{k=1}^{5}\binom{5}{k}2^k(3)^{5-k}\binom{1/2}{k-1}(-1)^{k-1}$

□

Problem 2. *Find generating functions for the following sequences (express them in a closed form, without infinite series!):*

1. $0, 0, 0, 0, -6, 6, -6, 6, -6, \cdots$.

2. $1, 0, 1, 0, 1, 0, \cdots$.

3. $1, 2, 1, 4, 1, 8\cdots$

Solution.

□

Sequence	Generating Function
$(1, 1, 1, 1, \ldots)$	$\frac{1}{1-x}$
$(1, -1, 1, -1, \ldots)$	$\frac{1}{1+x}$
$(-6, 6, -6, 6, \ldots)$	$\frac{-6}{1+x}$
$(0, 0, 0, 0, -6, 6, -6, 6, \ldots)$	$\frac{-6x^4}{1+x}$
$(1, 0, 1, 0, \ldots)$	$\frac{\frac{1}{1-x}+\frac{1}{1+x}}{2} = \frac{1}{1-x^2}$
$(0, 1, 0, 1, \ldots)$	$\frac{\frac{1}{1-x}-\frac{1}{1+x}}{2} = \frac{x}{1-x^2}$
$(1, 2, 4, 8, \ldots)$	$\frac{1}{1-2x}$
$(2, 4, 8, \ldots)$	$\frac{\frac{1}{1-2x}-1}{x} = \frac{2}{1-2x}$
$(1, 0, 2, 0, 4, 0, 8, \ldots)$	$\frac{1}{1-2x^2}$
$(1, 1, 2, 1, 4, 1, 8, \ldots)$	$\frac{1}{1-2x^2} + \frac{x}{1-x^2}$
$(1, 2, 1, 4, 1, 8, \ldots)$	$\frac{\frac{1}{1-2x^2}+\frac{x}{1-x^2}-1}{x} = -\frac{2x^3+2x^2-2x-1}{(1-2x^2)(1-x^2)}$

Problem 3. *Let a_n be the number of ordered triples $\langle i, j, k\rangle$ of integer numbers such that $i \geq 0, j \geq 1, k \geq 1$, and $i + 3j + 3k = n$. Find the generating function of the sequence $(a_0, a_1, a_2, \ldots$ and calculate a formula for a_n.*

Solution.

$(1 + x + x^2 + x^3 + \cdots)(x^3 + x^6 + x^9 + \cdots)(x3 + x^6 + x^9 + \cdots)$
$= \frac{1}{1-x}\frac{x^3}{1-x^3}\frac{x^3}{1-x^3}$
$= \frac{x^6(1+x+x^2)}{(1-x^3)^3} = x^6(1 + x + x^2)(1 - x^3)^{-3}$.
Then use the generalized binomial theorem. □

Problem 4. *Express the n^{th} term of the sequences given by the following recurrence relations*

1. $a_0 = 2, a_1 = 3, a_{n+2} = 3a_n - 2a_{n+1}$ $(n = 0, 1, 2, \ldots)$.

2. $a_0 = 1, a_{n+1} = 2a_n + 3$ $(n = 0, 1, 2, \ldots)$.

Solution.

1. Characteristic function is $x^2 + 2x - 3 = (x + 3)(x - 1) = 0$.

   Let $f_n = a(-3)^n + b \cdot 1^n$. Then $\begin{cases} 2 &= a + b \\ 3 &= -3a + b \end{cases} \Rightarrow a = -1/4, b = 9/4$.

   $\therefore$ the $n-$th term is f_n.

2. Characteristic function for the homogeneous part is $x = 2$. Take $a_n = p2^n + \lambda$

   $a_0 = 1, a_1 = 5$. Now $\begin{cases} 1 &= p + \lambda \\ 5 &= 2p + \lambda \end{cases} \Rightarrow p = 4, \lambda = -3$.

□

Problem 5. *Solve the recurrence relation* $a_{n+2} = \sqrt{a_{n+1}a_n}$ *with initial conditions* $a_0 = 2, a_1 = 8$ *and find* $\lim_{n\to\infty} a_n$.

Solution. Consider the sequence $b_n = \log_2 a_n$. Then

$$2 \log_2 a_{n+2} = \log_2 a_{n+1} + \log_2 a_n$$

i.e. $2b_{n+2} = b_{n+1} + b_n$. $b_0 = 1, b_1 = 3$. One can find $b_n = (-\frac{4}{3})(-\frac{1}{2})^n + \frac{7}{3}$. $\therefore a_n = 2^{(-\frac{4}{3})(-\frac{1}{2})^n + \frac{7}{3}}$. $\lim_{n\to\infty} a_n = 2^{\frac{7}{3}}$. □