

Homework 9

Problem 1. What is the expected number of trees with k vertices in $G \in \mathcal{G}(n, p)$?

Solution. By Cayley's formula and the linearity of expectation, it is $\binom{n}{k} k^{k-2} p^{k-1}$ \square

Problem 2. Show that, for constant $p \in (0, 1)$, almost no graph in $\mathcal{G}(n, p)$ has a separating complete subgraph.

Solution. This is an application of the 'almost always true of property $P_{i,j}$ '.

Detail:

- **Separating subgraph:** Given $G = (V, E)$, and some $X \subseteq V \cup E$, we call X a separating subgraph if there exists two vertices $u, v \in V(G - X)$ such that u, v are in the same component of G , while u, v lie in two disconnected components of $G - X$ (i.e., X separates u and v).
- **Separating complete subgraph:** If the above subgraph X is also a complete graph.

Now consider a graph $G = (V, E)$ with property $\mathcal{P}_{2,1}$. We claim that a graph with property $\mathcal{P}_{2,1}$ has the following property: For any pair of vertices $u, v \in G$, there exists a pair of vertices w_1, w_2 such that

$$(w_1, u) \in E, \quad (w_1, v) \in E$$

$$(w_2, u) \in E, \quad (w_2, v) \in E$$

$$(w_1, w_2) \notin E.$$

To prove the claim: consider vertices u, v and an arbitrary vertex x . By property $\mathcal{P}_{2,1}$, there exists a vertex w_1 which is neighbor to u and v , but not to x . Now using property $\mathcal{P}_{2,1}$ again (with x replaced by w_1) it follows that there exists a vertex w_2 which is neighbor to u and v , but not to w_1 . Thus the claim holds.

Finally, consider a complete subgraph $H \subset G$ and two arbitrary vertices u and v in $G - V(H)$. By the claim above, there are two non-adjacent vertices w_1 and w_2 in G which are both neighbors of both u and v . Since H is complete, it follows that w_1 and w_2 cannot both belong to H , therefore remove H will not separate u and v . In another word, H does not separate G . The statement now follows since almost all graphs in $\mathcal{G}(n, p)$ have property $\mathcal{P}_{2,1}$ for any constant $p \in (0, 1)$. \square

Problem 3. Show that if almost all $G \in \mathcal{G}(n, p)$ have a graph property \mathcal{P}_1 and almost all $G \in \mathcal{G}(n, p)$ have a graph property \mathcal{P}_2 , then almost all $G \in \mathcal{G}(n, p)$ have both properties.

Solution. The portion of the graphs have both properties equals 1 minus the portion of the graphs which does not have property \mathcal{P}_1 or \mathcal{P}_2 . However the portion of the graph does not have property \mathcal{P}_1 or \mathcal{P}_2 is bounded by the sum of the portion of the graphs does not have property \mathcal{P}_1 and the the portion of the graphs does not have property \mathcal{P}_2 , which both tend to 0 as n approaches ∞ . The claim in the question then follows. \square

Problem 4. Consider $\mathbf{G}(n, p)$ with $p = \frac{1}{3n}$.

1. Use the second moment method to show that with high probability there exists a simple path of length 10.

Solution.(1)

Directly use the theorem “ For every constant $p \in (0, 1)$ and every graph H , almost every $G \in \mathcal{G}(n, p)$ contains an induced copy of H .” \square

Solution.(2) [Use the second moment method]

$$I_\ell = \begin{cases} 1 & \ell \text{ is a length 10 simple path} \\ 0 & \text{otherwise} \end{cases}$$

X is the number of simple path, then $X = \sum_\ell I_\ell$.

The expectation would be $E(X) = \frac{1}{2}(n)_{11} \times p^{10}$. Thus

$$(E(X))^2 = \frac{1}{4} [(n)_{11}]^2 \times p^{20} \quad (\star)$$

(Note (\star) is about $\Theta(n^2)$.)

Then to calculate $E(X^2)$.

$$E(X^2) = E\left[\left(\sum_\ell I_\ell\right)^2\right] = E\left[\sum_\ell I_\ell \sum_{\ell'} I_{\ell'}\right] = \sum_{\ell, \ell'} E(I_\ell I_{\ell'}) \quad (\Delta)$$

ℓ and ℓ' will be independent to each other unless they have common vertices or edges. We use $k = |\ell \cap \ell'|$ to stand for the number of common edges between ℓ

and ℓ' , and $s = |\ell \cap \ell'|$ to stand for the number of common vertices used by ℓ and ℓ' . Obviously $(0 \leq k \leq 10) \wedge (0 \leq s \leq 11) \wedge (k \geq 1 \rightarrow s \geq k + 1)$.

(Δ) can be divided into the following subcases:

1. $k = 0$

$$(a) \ s = 0: \sum_{\ell, \ell'} E(I_\ell I_{\ell'}) = \frac{1}{8} (n)_{22} \times p^{20} \leq (E(X))^2;$$

(b) $1 \leq s \leq 11$:

for each s , $\sum_{\ell, \ell'} E(I_\ell I_{\ell'}) = c \cdot (n)_{22-s} \times (p)^{20} = \mathbf{o}((E(X))^2)$, where c is a constant number.

2. $1 \leq k \leq 10$ ($2 \leq s \leq 11$)

The general formula of each of these cases (constant many) would be

$$\begin{aligned} \sum_{\ell, \ell'} E(I_\ell I_{\ell'}) &= d \cdot (n)_{22-s} \times p^{20-k} \\ &\leq d \cdot (n)_{22-s} \times p^{20-(s-1)} \\ &= d \cdot (n)_{22-s} \times p^{21-s} \\ &= \mathbf{o}((E(X))^2) \end{aligned}$$

Combining the above results we get that $\text{Var}(X) = E(X^2) - (E(X))^2 = \mathbf{o}((E(X))^2)$.

□

Problem 5. (Optional)

1. Prove that the threshold for the existence of cycles in $\mathcal{G}(n, p)$ is $p = \frac{1}{n}$.
2. Search the World Wide Web to find some real world graphs in machine readable form or data bases that could automatically be converted to graphs.
 - (a) Plot the degree distribution of each graph.
 - (b) Compute the average degree of each graph.
 - (c) Count the number of connected components of each size in each graph.
 - (d) Describe what you find.
3. Create a simulation (an animation) to show the evolution of the $\mathcal{G}(n, p)$ (Erdős-Rényi) random graph as its density p is gradually increased. Observe the phase transitions for trees of increasing orders, followed by the emergence of the giant component, etc.

Problem 6. (*Optional*)

Prove that ‘the disappearance of isolated vertices in $\mathbf{G}(n, p)$ ’ has a sharp threshold of $\frac{\ln n}{n}$.

Solution. John’s book, Theorem 8.6. - -+

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