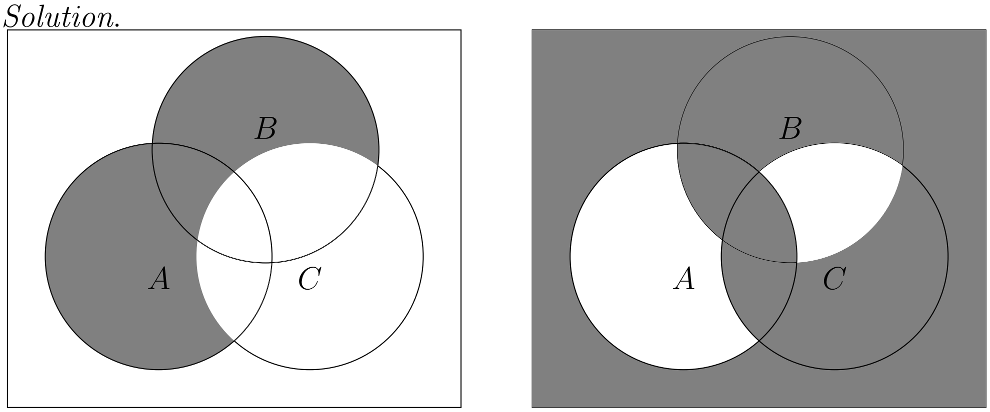
# Homework 1

**Problem 1.** *Show the Venn-diagram representation for the following sets:*

*(a)* $(A \cup B) - C$

*(b)* $\overline{A \oplus (B \cap C)}$

*Solution.*




$\square$

**Problem 2.** *For any sets $A$, $B$ and $C$, prove that*

$$A \cup B = A \cup C, A \cap B = A \cap C \textit{ implies } B = C.$$

*Solution.* If $B \neq C$, then there must be an element, say $x$, such that $x \in B$ but $x \notin C$. (if $x \in C$ but $x \notin B$, it can be proofed in the same way.) Since $A \cup B = A \cup C$, we have $x \in A$. Thus $x \in (A \cap B)$, but $x \notin (A \cap C)$, this contradicts that $A \cap B = A \cap C$. Thus we can conclude that $B = C$. $\square$

**Problem 3.** *Show that a nonempty set has the same number of odd subsets (i.e., subsets with an odd number of elements) as even subsets.*

*Solution.* Assume the set has $n$ elements, it has $2^n$ subsets. These subsets can form $2^{n-1}$ pairs, each pair contains an odd subset and an even subset. The element appears in the very begining of the set is denoted as $a_0$, then each pair is constructed in the flowing way, randomly selecting a subset $B$, if it is $\emptyset$ then pair it with $\{a_0\}$, if it is $\{a_0\}$ pair it with $\emptyset$. For other cases if

$a_0 \notin B$, pair $B$ with the subset that formed by adding $a_0$ at the very begining of $B$. In this way we can construt $2^{n-1}$ pairs, each containing an odd subset and an even subset. Thus, the set has the same number of odd subsets as even subsets. $\square$

**Problem 4.** *$A, B, C$ are three sets. and two functions $g : A \to B$, $f : B \to C$*

a) *If $f \circ g$ is an injective function and $g$ is surjective, show that $f$ is injective.*

b) *If $f \circ g$ is an surjective function and $f$ is injective, show that $g$ is surjective.*

*(Note that $f \circ g(x) = f(g(x))$.)*

*Solution.*

a) If $f$ is not injective, then $\exists y_1, y_2 \in B, y_1 \neq y_2 \wedge f(y_1) = f(y_2)$. Since $g$ is surjective, we have $\forall y \in B \longrightarrow \exists x \in A \wedge g(x) = y$. Thus

$$\exists x_1, x_2 \in A, x_1 \neq x_2 \wedge g(x_1) = y_1, g(x_2) = y_2$$

$$\exists x_1, x_2 \in A, x_1 \neq x_2 \wedge f(g(x_1)) = f(g(x_2))$$

This contradicts that $f \circ g$ is an injective function. Thus we can conclude that $f$ is injective.

b) If $g$ is not surjective, then $\exists y_0 \in B \wedge \forall x \in A, g(x) \neq y_0$. Since $f$ is injective, then $\forall y_1, y_2 \in B, y_1 \neq y_2 \longrightarrow f(y_1) \neq f(y_2)$. Suppose that $f(y_0) = z_0 \in C$, since $\forall x \in A, g(x) \neq y_0$, we have $\forall x \in A, f(g(x)) \neq z_0$. This contradicts that $f \circ g$ is an surjective function. Thus we can conclude that $g$ is surjective.

$\square$

**Problem 5.** *$\mathcal{R}$ is a binary relation,*

1. *Show that $\mathcal{R}$ is symmetric iff $\mathcal{R}^{-1} \subseteq \mathcal{R}$.*
2. *Show that $\mathcal{R}$ is transitive iff $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$.*

*Solution.*

1. If $\mathcal{R}$ is symmetric, then $\forall (x, y) \in \mathcal{R}^{-1} \longrightarrow (y, x) \in \mathcal{R} \longrightarrow (x, y) \in \mathcal{R}$, thus $\mathcal{R}^{-1} \subseteq \mathcal{R}$.

   If $\mathcal{R}^{-1} \subseteq \mathcal{R}$, then $\forall (x, y) \in \mathcal{R} \longrightarrow (y, x) \in \mathcal{R}^{-1} \longrightarrow (y, x) \in \mathcal{R}$, thus $\mathcal{R}$ is symmetric.

2. If $\mathcal{R}$ is transitive, then $\forall (x, y) \in \mathcal{R} \circ \mathcal{R} \longrightarrow \exists z, (x, z) \in \mathcal{R} \wedge (z, y) \in \mathcal{R} \longrightarrow (x, y) \in \mathcal{R}$.

   If $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$, then $\forall (x, z), (z, y) \in \mathcal{R} \longrightarrow (x, y) \in \mathcal{R} \circ \mathcal{R} \longrightarrow (x, y) \in \mathcal{R}$.

□

**Problem 6.** *Prove that $\mathcal{P}(A) \approx 2^A$, where $A$ is any set and $2^A = \{f \mid f : A \to \{0, 1\}$ is a function.$\}$*

*Solution.* To prove $\mathcal{P}(A) \approx 2^A$, we try to find out a one-to-one correspondence $f$ between $\mathcal{P}(A)$ and $2^A$. For any subset $B$ of $A$, we have

$$f(B) = g(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

Thus given any subset $B$ of $A$, we can construct a funcation from $A$ to $\{0, 1\}$, and for any funcation from $A$ to $\{0, 1\}$, we can construct a subset $B$ of $A$.Then we may conclude that $\mathcal{P}(A) \approx 2^A$. □

**Problem 7.** *$A$ and $B$ are countable sets. Prove that*

1. *$A \cup B$ is countable*

2. *$A \times B$ is countable*

*Solution.*

1. Since the cardinality of $A \cup B$ is at most $\aleph_0 + \aleph_0$, which is equal to $\aleph_0$, we can conclude that $A \cup B$ is countable.

2. Since the cardinality of $A \times B$ is at most $\aleph_0 \cdot \aleph_0$, which is equal to $\aleph_0$, we can conclude that $A \times B$ is countable.

□