Sets, relations and functions

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- Basic set theory
- Relation
- Function

Brief History of Set Theory

- Georg Cantor(1845-1918)
  - German mathematician
  - Founder of set theory
- Bertrand Russell(1872-1970)
  - British philosopher, logician, mathematician, historian, and social critic.
- Ernst Zermelo(1871-1953)
  - German mathematician, foundations of mathematics and hence on philosophy
- David Hilbert (1862-1943)
  - German mathematicia, one of the most influential and universal mathematicians of the 19th and early 20th centuries.
- Kurt Gödel(1906-1978)
  - Austrian American logician, mathematician, and philosopher. ZFC not $\vdash \neg$CH .
- Paul Cohen(1934-2007)
  - American mathematician, 1963: ZFC not $\vdash$ CH,AC .

Hilbert's twenty-three problems are:

Problem	Brief explanation
1st	The continuum hypothesis (that is, there is no set whose cardinality is strictly between that of the integers and that of the real numbers)
2nd	Prove that the axioms of arithmetic are consistent.
3rd	Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second?
4th	Construct all metrics where lines are geodesics.
5th	Are continuous groups automatically differential groups?
6th	Mathematical treatment of the axioms of physics
7th	Is a^b transcendental, for algebraic $a \neq 0,1$ and irrational algebraic b ?
8th	The Riemann hypothesis ("the real part of any non-trivial zero of the Riemann zeta function is ½") and other prime number problems, among them Goldbach's conjecture and the twin prime conjecture
9th	Find the most general law of the reciprocity theorem in any algebraic number field.
10th	Find an algorithm to determine whether a given polynomial Diophantine equation with integer coefficients has an integer solution.
11th	Solving quadratic forms with algebraic numerical coefficients.
12th	Extend the Kronecker–Weber theorem on abelian extensions of the rational numbers to any base number field.
13th	Solve 7-th degree equation using continuous functions of two parameters.
14th	Is the ring of invariants of an algebraic group acting on a polynomial ring always finitely generated?
15th	Rigorous foundation of Schubert's enumerative calculus.
16th	Describe relative positions of ovals originating from a real algebraic curve and as limit cycles of a polynomial vector field on the plane.
17th	Express a nonnegative rational function as quotient of sums of squares.
18th	(a) Is there a polyhedron which admits only an anisohedral tiling in three dimensions? (b) What is the densest sphere packing?
19th	Are the solutions of regular problems in the calculus of variations always necessarily analytic?
20th	Do all variational problems with certain boundary conditions have solutions?
21st	Proof of the existence of linear differential equations having a prescribed monodromic group
22nd	Uniformization of analytic relations by means of automorphic functions
23rd	Further development of the calculus of variations

Sp

What is a set ?

- By *Georg Cantor* in 1870s:

  A set is an unordered collection of objects.

  - The objects are called the *elements*, or *members*, of the set. A set is said to *contain* its elements.

- Notation: $a \in A$

  - Meaning that: a is an element of the set A, or,

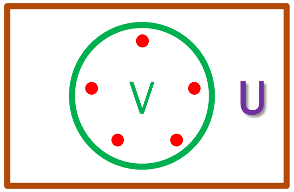
    Set A contains a .

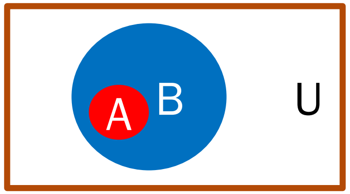
- Important:

  - Duplicates do not matter. multiset里面的重复元素是需要计数的
  - Order does not matter.

Basic notions

- $a \in A$ a is an element of the set A.
- $a \notin A$ a is NOT an element of the set A.
- Set of sets {{a,b},{1, 5.2}, k} collection
- $\emptyset$ the empty set, or the null set, is set that has no elements.
- $A \subseteq B$ subset relation. Each element of A is also an element of B.
- $A = B$ equal relation. $A \subseteq B$ and $B \subseteq A$.
- $A \neq B$
- $A \subset B$ strict subset relation. If $A \subseteq B$ and $A \neq B$
- $|A|$ cardinality of a set, or the number of distinct elements.
- Venn Diagram





Examples

- $a \in \{a, e\ i, o, u\}$
- a ∉{{a}}
- ∅ ∉∅
- ∅ ∈ {∅} ∈ {{∅}}
- {3,4,5}={5,4,3,4}
- ∅⊆S
- ∅ ⊂{∅ }
- S ⊆S
- |{3, 3, 4, {2, 3},{1,2,{f}} }|=4

Set Operations

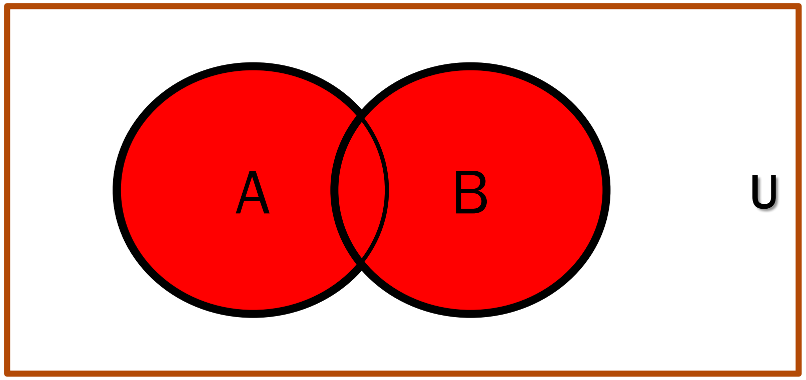
- Union
- Intersection
- Difference
- Complement
- Symmetric difference
- Power set

Union

- **Definition** Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or both.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

- Example: $\{1,3,5\} \cup \{1,2,3\} = \{1,2,3,5\}$
- Venn Diagram representation

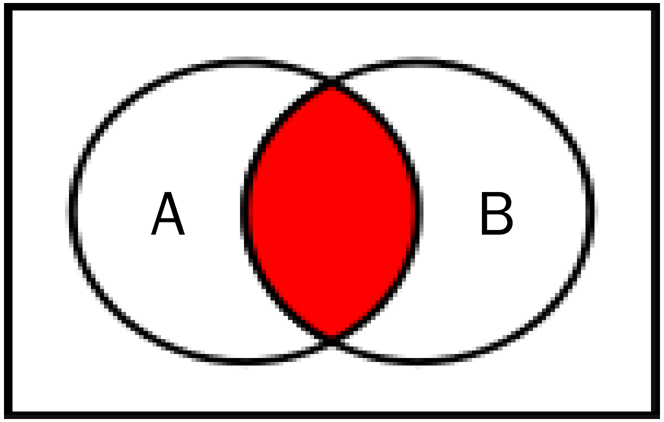


Intersection

- **Definition** Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set that containing those elements in both A and B.

$$A \cap B=\{x \mid x \in A \text{ and } x \in B\}$$

- Example: $\{1,3,5\} \cap \{1,2,3\}=\{1,3\}$
- Venn Diagram Representation

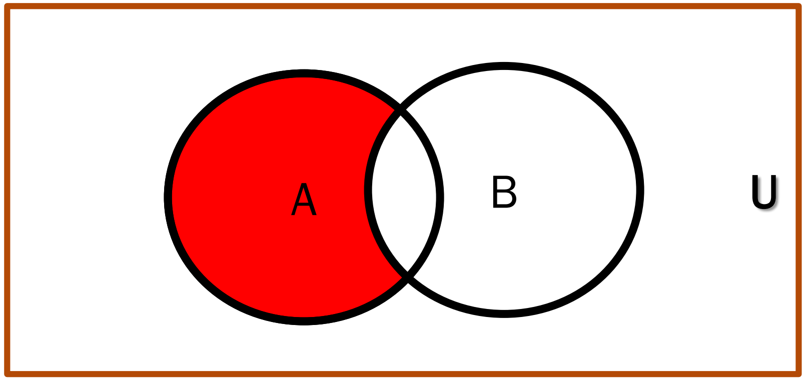


Difference

- **Definition** Let A and B be sets. The difference of the sets A and B, denoted by A - B, is the set that containing those elements in A but not in B.

$$\boxed{A - B} = \{x |\ x \in A \ but \ x \notin B\} = \boxed{A \cap \bar{B}}$$

- Example: {1,3,5}-{1,2,3}={5}
- Venn Diagram Representation

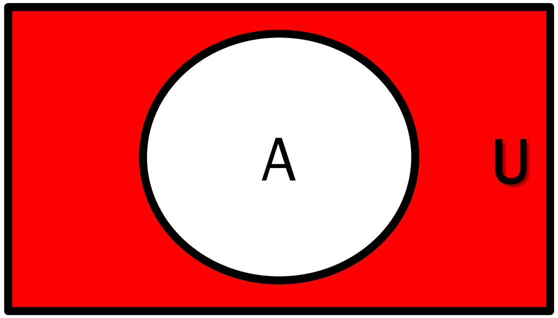


Complement

- **Definition** Let U be the universal set. The complement of the sets A, denoted by $\bar{A}$ or $-A$, is the complement of with respect to U.

$$\bar{A} = \{x \mid x \notin A\} = U - A$$

- Example: -E = O
- Venn Diagram Representation



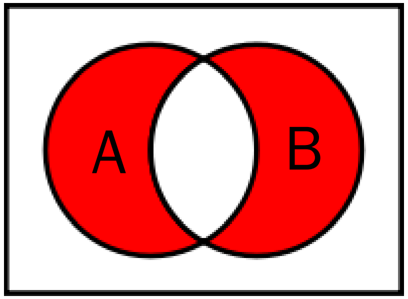
Symmetric difference

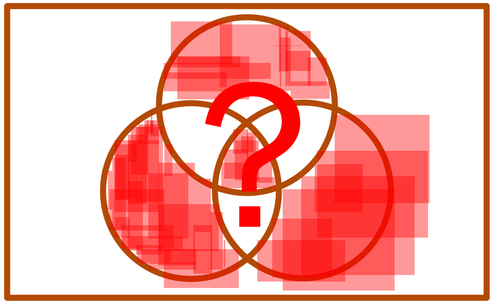
- **Definition** Let A and B be sets. The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in their intersection.

$$\boxed{A \oplus B} = \{x \mid (x \in A \lor x \in B) \land x \notin A \cap B\}$$

$$= \boxed{(A-B) \cup (B-A)}$$

- Venn Diagram: $A \oplus B$ $A \oplus B \oplus c$





The Power Set

- Many problems involves testing all combinations of elements of a set to see if they satisfy some property. To consider all such combinations of elements of a set S, we build a new set that has its members all the subsets of S.
- Definition: Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by P(S) or $\wp$S .
- Example:
  - P({0,1,2})={ϕ, {0},{1},{2}, {0,1},{0,2},{1,2},{0,1,2} }
  - P(Ø)={Ø}
  - P({Ø})={Ø,{Ø}}

Set Identities

1. Identity laws

$$A \cup \emptyset = A \qquad\qquad A \cap U = A$$

2. Domination laws

$$A \cup U = U \qquad\qquad A \cap \emptyset = \emptyset$$

3. Idempotent laws

$$A \cup A = A \qquad\qquad A \cap A = A$$

Set Identities (Cont.)

4. Complementation law

$$\overline{(\overline{A})} = A$$

5. Commutative laws

$$\boxed{\begin{array}{l} A \cup B = B \cup A \\ A \cap B = B \cap A \end{array}}$$

6. Associative laws

$$\boxed{\begin{array}{l} A \cup (B \cup C) = (A \cup B) \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C \end{array}}$$

Set Identities (Cont.)

7. Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

8. De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Set Identities (Cont.)

9. Absorption laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

10. Complement laws

$$A \cup \bar{A} = U$$

$$A \cap \overline{A} = \emptyset$$

Example

Theorem 1 (De Morgan's Law). $\overline{S \cap T} = \bar{S} \cup \bar{T}$ *or* $S \cap T = \overline{\bar{S} \cup \bar{T}}$

Proof. (Proved by Venn Diagram)

$$\boxed{x \in \overline{S \cap T} \Rightarrow x \notin S \cap T}$$

证明S是T的子集:证明x属于S可以推出x属于T

$$\Rightarrow \text{ either } x \notin S \text{ or } x \notin T$$

$$\Rightarrow \text{ either } x \in \bar{S} \text{ or } x \in \bar{T}$$

证明两个集合相等的常见思路：证明x属于S可以推出x属于T；再证明x属于T可以推出x属于S

$$\Rightarrow x \in \bar{S} \cup \bar{T}$$

$$\boxed{x \in \bar{S} \cup \bar{T} \Rightarrow \text{ reverse steps}}$$

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Ordered Pairs

- In set theory $\{1,2\}=\{2,1\}$
- What if we need the object $<1,2>$ that will encode more information:
  - 1 is the first component
  - 2 is the second component
- Generally, we say

$$<x, y> = <u, v> \quad \text{iff} \quad x=u \wedge y=v$$

Cartesian Product

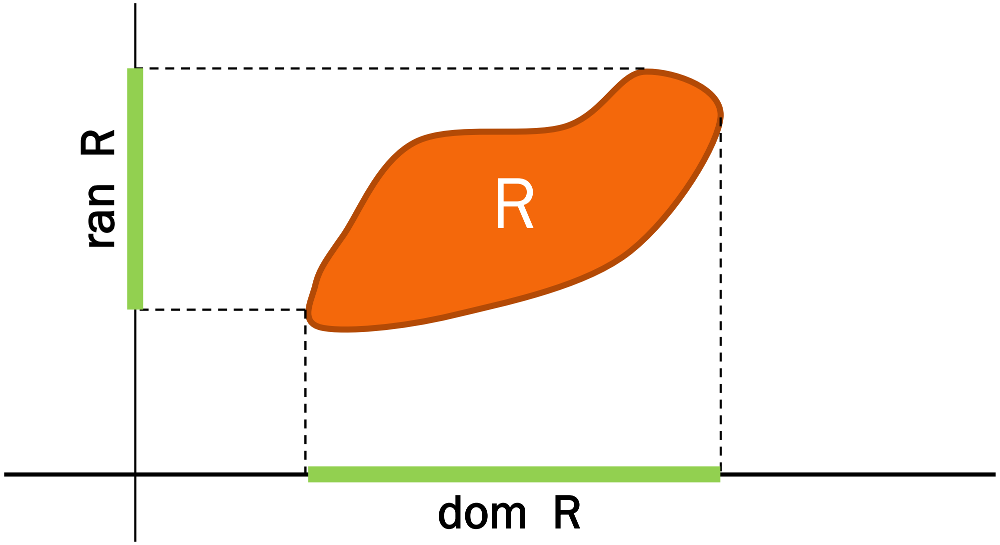
- $A \times B = \{<x,y> \mid x \in A \land y \in B\}$ is the Cartesian product of set A and set B.
- Example

  $A=\{1,2\}$ $B=\{a,b,c\}$

  $A \times B = \{<1,a>,<1,b>,<1,c>,$
  $<2,a>,<2,b>,<2,c>\}$

Relation

- **Definition** A **relation** is a set of ordered pairs.
- **Examples**
  - <={<x,y>∈R×R| x is less than y}
  - M={<x,y> ∈People× People| x is married to y}



A relation as a subset of the plane

More about the binary relation

Let R denote any binary relation on a set x, we say:

- R is reflexive, if $(\forall a \in x)(aRa)$;
- R is symmetric, if $(\forall a, b \in x)(aRb \rightarrow bRa)$;
- R is transitive, if $(\forall a, b, c \in x)[(aRb \wedge bRc) \rightarrow (aRc)]$;

反对称：if (aRb and bRa) --> a = b

Equivalence relation

等价关系

- Definition R is an **equivalence relation on A** iff R is a binary relation on A that is
  - Reflexive
  - Symmetric
  - Transitive

Partition

- **Definition** A ***partition*** π of a set A is a set of nonempty subsets of A that is disjoint and exhaustive. i.e.

  (a) no two different sets in π have any common elements, and

  (b) each element of A is in some set in π.

Equivalence class

商集

- If R is an equivalence relation on A, then the quotient set (equivalence class) A/R is defined as

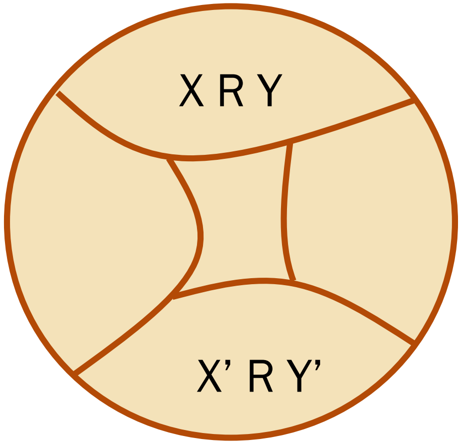
$$A/R=\{ [x]_R \mid \in A \}$$

  Where A/R is read as "A modulo R"

- The ***natural map*** (or canonical map) $\alpha: A \to A/R$ defined by

$$\alpha(x)= [x]_R$$

- **Theorem** Assume that R is an equivalence relation on A. Then the set $\{[x]_R \mid x \in A\}$ of all equivalence classes is a partition of A.



Examples

- Let $\omega = \{0,1,2,\dots\}$; and $\boxed{m \sim n} \Leftrightarrow m - n$ is divisible by 6.

  Then $\sim$ is an equivalence relation on ω. The quotient set $\omega/\sim$ has six members:

  $[0] = \{0,6,12,\dots\}$,

  $[1] = \{1,7,13,\dots\}$,

  ……

  $[5] = \{5,11,17,\dots\}$

- Clique (with self-circles on each node) : a graph in which every edge is presented. Take the existence of edge as a relation. Then the equivalence class decided by such relation over the graph would be clique.

Ordering relations

- Linear order/total order
  - transitive
  - trichotomy
- Partial order
  - reflexive
  - anti-symmetric
  - transitive
- Well order 良序关系一定是全序关系
  - total order
  - every non-empty subset of S has a least element in this ordering.

Basic set theory

Relation

Function

Function

- **Definition** A **function** is a relation *F* such that for each *x* in *dom F* there is only one *y* such that *x F y*. And *y* is called the value of *F* at *x*.
- **Notation** $F(x)=y$
- **Example** $f(x) = x^2$ $f : R \rightarrow R$, $f(2) = 4$, $f(3) = 9$, etc.
- **Composition** $(f \circ g)(x)=f(g(x))$
- **Inverse** The inverse of *F* is the set

  $$F^{-1}=\{\langle u,v\rangle \mid v\ F\ u\}$$

  F^{-1} is not necessarily a function (why?)

Special functions

- We say that *F is a function* *from A into B* or that *F* *maps A into B* (written $F: A \rightarrow B$) iff F is a function, *dom F=A* and *ran F⊆B*.
  - If, in addition, *ran F=B*, then *F* is a function from *A* ***onto*** *B*. *F* is also named a ***surjective function***. 满射
  - If, in addition, for any $x \in dom\ F$, $y \in dom\ F$, with $x \neq y$, $F(x) \neq F(y)$, then *F* is an ***injective function***. or ***one-to-one*** (or ***single-rooted***). 单射
  - *F* is ***bijective function***: *f* is surjective and injective.

References

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Thank you