☆ Robots

For any Matrics A B C

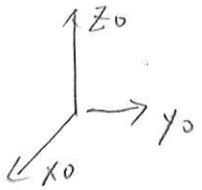
$$\underset{i\times k}{A} = \underset{i\times j}{B}\ \underset{j\times k}{C} \rightarrow \underset{k\times i}{A^T} = \underset{k\times j}{C^T}\ \underset{j\times i}{B^T}$$

proof $A_{ik} = \sum_{j=1}^{n} B_{ij} C_{jk}.$

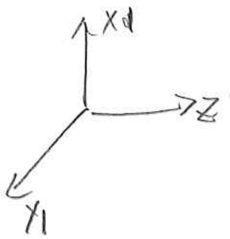
$$(A^T)_{ik} = A_{ki} = \sum_{j=1}^{n} B_{kj} \cdot C_{ji} = \sum_{j=1}^{n} C^T_{ij} B^T_{jk}$$

$\therefore A^T = C^T B^T$

Rotation Matrix



Let $\rightarrow R^0 = (x^0\ y^0\ z^0)$



let $\rightarrow R^1 = (x^1 \cdot y^1\ z^1)$

Define ~~at~~ $R_0^1 = (R^0)^T \cdot R^1 = \underset{3\times 1}{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}} \cdot \underset{1\times 3}{(x^1\ y^1\ z^1)}$

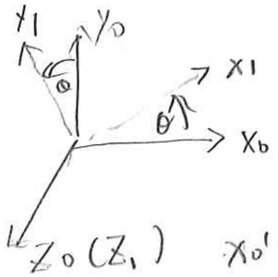
$$= \left[\begin{pmatrix} x^0 \\ y_0 \\ z_0 \end{pmatrix} x^1 \;\middle|\; \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \cdot y^1 \;\middle|\; \begin{pmatrix} x^0 \\ y_0 \\ z_0 \end{pmatrix} \cdot z^1 \right]$$

$$= \left[\begin{array}{c|c|c} x^0 \cdot x_1 & x^0\ y_1 & x_0 \cdot z_1 \\ y_0 \cdot x_1 & y_0\ y^1 & y_0 \cdot z_1 \\ z^0\ x_1 & z_0\ y_1 & z_0 \cdot z_1 \end{array} \right] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

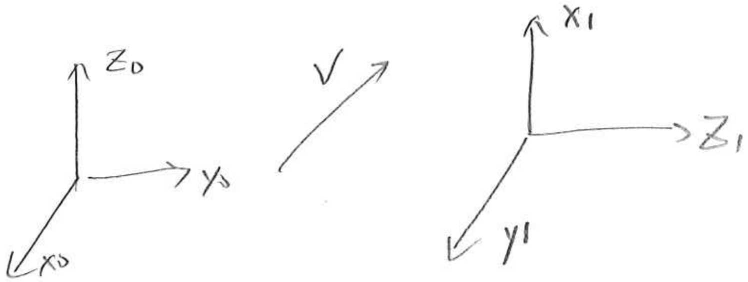
∴ $\rightarrow R_0^1 = (R^1)^T \cdot R^0 \rightarrow R_1^0 = \underline{(R^0)^T \cdot R^1 = R_0^1}$ proved

or $R_1^0 = (R_1)^T \cdot R_0$

* Robots



$$
R_0^1 = \begin{array}{ccc} x_0^1 & y_0^1 & z_0^1 \\ \left[\begin{array}{c|c|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array}\right. & & \left.\right] \end{array}
$$



$$V = V_{0x} X^0 + V_{0y} Y^0 + V_{0z} Z^0 \quad (1)$$

$$V = V_{1x} X^1 + V_{1y} Y^1 + V_{1z} Z^1 \quad (2)$$

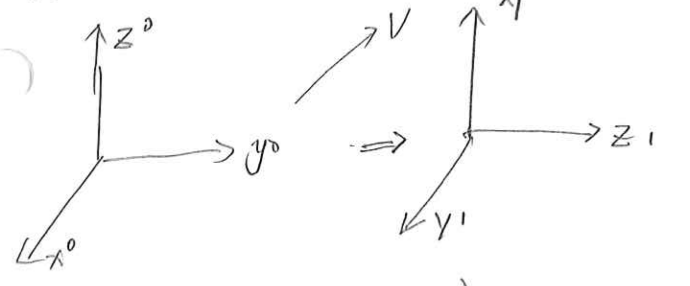
write comp'n of (2) w.r.t. basis 0.

$$\rightarrow V_0 = V_{1x} X_0^1 + V_{1y} Y_0^1 + V_{1z} Z_0^1$$

$$= V_{1x} \begin{pmatrix} X_0^1 x \\ X_0^1 y \\ X_0^1 z \end{pmatrix} + V_{1y} \begin{pmatrix} Y_0^1 x \\ Y_0^1 y \\ Y_0^1 z \end{pmatrix} + V_{1z} \begin{pmatrix} Z_0^1 x \\ Z_0^1 y \\ Z_0^1 z \end{pmatrix}$$

★ Robots.

Vector Coordinate Transformation $\quad V_a = R_a^b V_b$.



Find V_0 if $V_1 = \begin{pmatrix}1\\0\\2\end{pmatrix}$

$$V_0 = R_0^1 V_1 = \underset{3\times 3}{\begin{array}{c} \begin{matrix} x_0^1 & y_0^1 & z_0^1 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}} \cdot \underset{3\times 1}{\begin{pmatrix}1\\0\\2\end{pmatrix}} = \underset{3\times 1}{\begin{pmatrix}0\\2\\1\end{pmatrix}}$$

Orientation of a Basis.

If $V = x^2 \quad \because V_a = R_a^b V_b$

$$\rightarrow \begin{cases} x_0^2 = R_0^1 x_1^2 \\ y_0^2 = R_0^1 y_1^2 \\ z_0^2 = R_0^1 z_1^2 \end{cases}$$

$$R_0^2 = [x_0^2 \mid y_0^2 \mid z_0^2] = [R_0^1 x_1^2 \mid R_0^1 y_1^2 \mid R_0^1 z_1^2]$$

$$= R_0^1 [x_1^2 \mid y_1^2 \mid z_1^2]$$

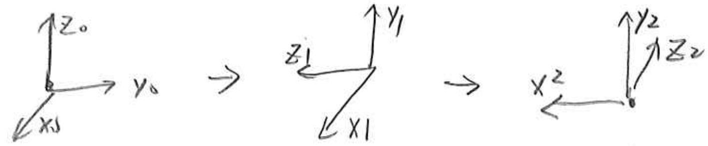
$R_0^2 = R_0^1 R_1^2$, $\quad R_0^3 = R_0^2 \cdot R_2^3 \quad R_0^3 = (R_0^1 \cdot R_1^2) \cdot R_2^3 = R_0^1 \cdot (R_1^2 \cdot R_2^3)$

$R_a^c = R_a^b \cdot R_b^c$ (General Form)

$AB \neq BA$.

$\therefore R_0^6 = R_0^1 R_1^2 R_2^3 R_3^4 \cdot R_4^5 \cdot R_5^6$.

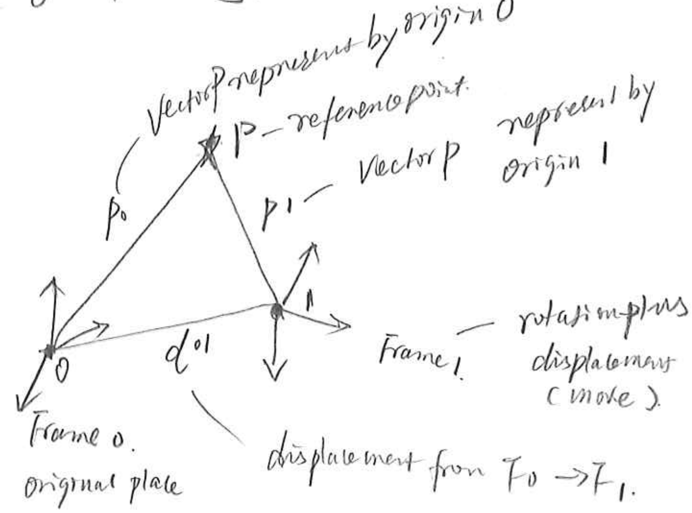
A Robots



$$R_0^2 = R_0^1 R_1^2$$

$$= \begin{array}{c} x_0^1 \quad y_0^1 \quad z_0^1 \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} x_1^2 \quad y_1^2 \quad z_1^2 \\ \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} = \begin{array}{c} x_0^2 \quad y_0^2 \quad z_0^2 \\ \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Homogeneous Transformation



vector. $P^0 = P^1 + d^{01}$

In Frame 0 ⇒ $P_0^0 = P_0^1 + d_0^{01}$

$\longrightarrow \quad P_0^0 = R_0^1 P_1^1 + d_0^{01} \qquad (1)$

Define homogeneous coordinates of point P.

$$P_0 = \begin{pmatrix} P_0^0 \\ 1 \end{pmatrix} \quad (P_0^0 = P_0^1 + d_0^{01}) \qquad P_1 = \begin{pmatrix} P_1^1 \\ 1 \end{pmatrix}$$

Frame 0 $\quad 4\times 1 \qquad$ Frame 1 $\quad 4\times 1$.

$$\rightarrow P_0 = \begin{pmatrix} P_0^0 \\ 1 \end{pmatrix} = \begin{pmatrix} R_0^1 P_1^1 + d_0^{01} \\ \hline 1 \end{pmatrix} = \left[\begin{array}{c|c} \overset{3\times3}{R_0^1} & \overset{3\times1}{d_0^{01}} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \begin{pmatrix} P_1^1 \\ 1 \end{pmatrix}$$

$\rightarrow P_0 = H_0^1 P_1 \qquad H_{0\,4\times4}^1 \quad P_{1\,4\times1}$

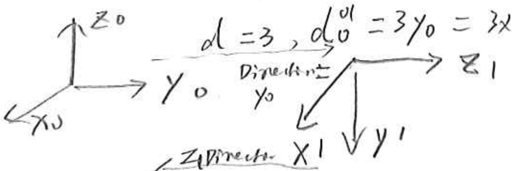
Homogeneous Transformation.

General: $P_a = H_a^b P_b$.

$$H_a^b = \left[\begin{array}{c|c} R_a^b & d_a^{ab} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$$

XYZ Rotate from 0 to 1 coordinate → Value.

eg.



$d = 3,\ d_0^{01} = 3y_0 = 3\times\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\3\\0\end{pmatrix}$

$$\text{Find } H_0^1 = \begin{array}{c} \quad x_0^1 \ \ y_0^1 \ \ z_0^1 \qquad d_0^{01} \\ R_0^1 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$d_1^{10} = 3(-z_1) = 3\begin{pmatrix}0\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\-3\end{pmatrix} \rightarrow H_1^0 = \begin{array}{c} \text{Transpose} \quad x_1^0 \ \ y_1^0 \ \ z_1^0 \qquad d_1^{10} \\ R_1^0 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array} = \left[\begin{array}{c|c} (R_0^1)^T & -(R_0^1)^T d_0^{01} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$$

$$d_1^{10} = R_1^0 d_0^{10} = (R_0^1)^T d_0^{10} = -(R_0^1)^T d_0^{01}$$

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prove: $H_0^1 H_1^0 = H_0^0 = \left[\begin{array}{c|c} R_0^0 & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 000 & 1\end{array}\right] = \left[\begin{array}{ccc|c} 1&0&0&0\\0&1&0&0\\0&0&1&0 \\ \hline 0&0&0 & 1\end{array}\right] = \left[\begin{array}{c|c} 1\;1\;1 & \\ \hline & 1\end{array}\right] = I$

$$\Rightarrow \left.\begin{cases} H_1^0 = \left[\begin{array}{c|c} (R_0^1)^T & -(R_0^1)^T d_0^{01} \\ \hline 000 & 1\end{array}\right] \\ H_0^1 = \left[\begin{array}{c|c} R_0^1 & d_0^{01} \\ \hline 000 & 1\end{array}\right]\end{cases}\right\} \rightarrow H_0^1 \times H_1^0 = \begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix} = I.$$

Homogeneous Coordinates of a Vector.

Given a vector V, let $\bar{V_0} = \begin{pmatrix} V_0 \\ 0 \end{pmatrix}$ and $\bar{V_1} = \begin{pmatrix} V_1 \\ 0\end{pmatrix}$

Then $\bar{V_0} = \begin{pmatrix} V_0 \\ 0 \end{pmatrix} = \begin{pmatrix} R_0^1 V_1 \\ 0 \end{pmatrix} = \left(\begin{array}{c|c} R_0^1 & V_1 \\ \hline 000 & 0\end{array}\right) \rightarrow \bar{V_0} = H_0^1 \cdot V_1$

Trans from 1 to 0 location — Homogeneous Transform

$$\left.\begin{array}{l}\text{Rotation } R^2 = [X^2\ Y^2\ Z^2] \\ \text{Vector } X^2 = \begin{pmatrix} X^2 \\ 0 \end{pmatrix}\end{array}\right\} \rightarrow \begin{cases} X_0^2 = H_0^1 X_1^2 \\ Y_0^2 = H_0^1 Y_1^2 \\ Z_0^2 = H_0^1 Z_1^2 \end{cases}$$

If O^2 is the origin of Frame 2, then $O_0^2 = \begin{pmatrix} d_0^{02} \\ 1 \end{pmatrix}$ and $O_0^2 = H_0^1 O_1^2$

$\rightarrow H_0^2 = [X_0^2 | Y_0^2 | Z_0^2 | O_0^2] = [H_0^1 X_1^2 | H_0^1 Y_1^2 | H_0^1 \cdot Z_1^2 | H_0^1 O_1^2]$

$= H_0^1 [X_1^2\ Y_1^2\ Z_1^2\ O_1^2] = H_0^1 H_1^2.$

$\Rightarrow \underline{H_0^2 = H_0^1 H_1^2.}$

$\rightarrow$ General Form $H_a^c = H_a^b H_b^c$

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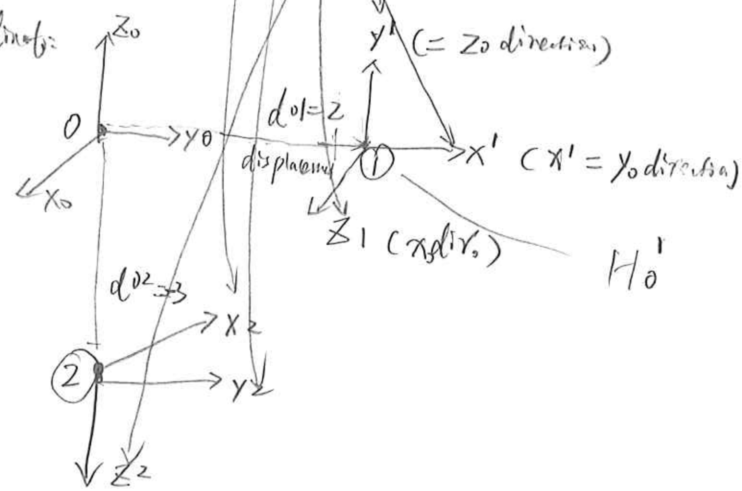
eg. Given $H_0^1 = \begin{array}{cccc} x_0^1 & y_0^1 & z_0^1 & d_0^{01} \text{ — distance} \\ \end{array} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ Displacement, $H_1^2 = \begin{array}{cccc} x_1^2 & y_1^2 & z_1^2 & d_1^{12} \end{array} \left[\begin{array}{ccc|c} 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -3 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ → find H_0^2

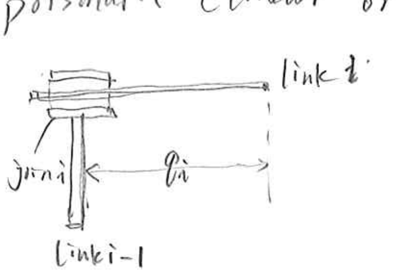
$$H_0^2 = H_0^1 H_1^2 = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right] \rightarrow H_0^2$$

eg. Draw coordinate:

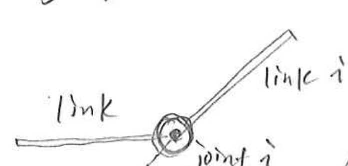


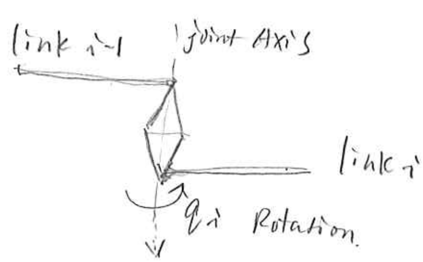
Robot Kinematics:

a) prismatic (linear or sliding joint)

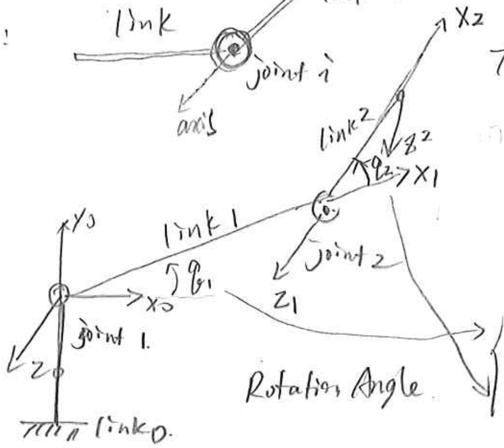


b) Revolute joint





eg



$q_1 = X_1$ vs. X_0

$q_2 = X_2$ vs X_1

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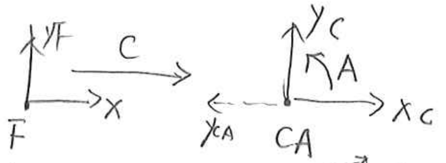
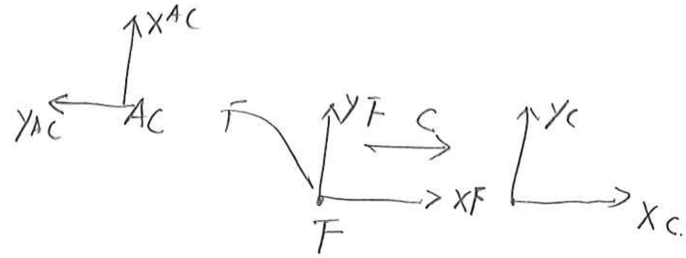
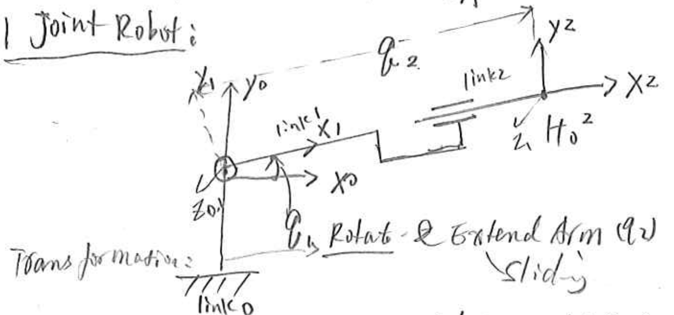
Supose an H-matrix, C locats a moving body M with respect to (wrt) a fixed frame F.

Let A be on H-matrix → Then $\begin{cases} CA \text{ moves } M \text{ by } A \text{ wrt } C. \\ CA \text{ moves } M \text{ by } A \text{ wrt } F. \end{cases}$



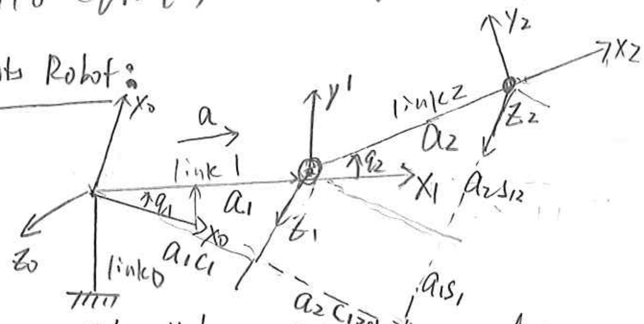


1 Joint Robot:

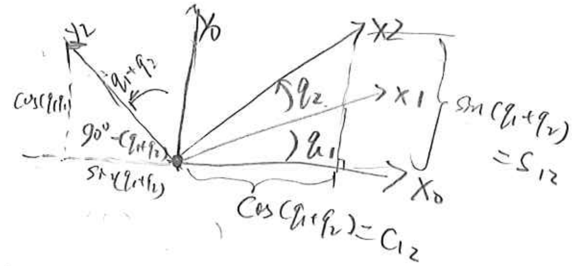


$$H_0^2(q_1, q_2) = H_0^1(q_1) \cdot H_1^2(q_2).$$

2 Joints Robot:



$\begin{cases} \cos q_1 = c_1 \\ \sin q_1 = s_1 \\ \cos(q_1+q_2) = C_{12} \end{cases}$



$$H_0^1 = \begin{array}{c} \begin{array}{cccc} x_0^1 & y_0^1 & z_0^1 & d_0 \end{array} \\ \left[\begin{array}{ccc|c} \cos q_1 & -\sin q_1 & 0 & a\cos q_1 \\ \sin q_1 & \cos q_1 & 0 & a\sin q_1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array} \rightarrow d_0^{01} = a_1 x_0^1$$

$$H_0^2(q_1 \cdot q_2) = H_0^1(q_1) H_1^2(q_2)$$

$$H_0^2(q_1, q_2) = \begin{array}{c} \begin{array}{cccc} x_0^2 & y_0^2 & z_0^2 & d_0^2 \end{array} \\ \left[\begin{array}{ccc|c} C_{12} & -S_{12} & 0 & a_1c_1+a_2c_{12} \\ S_{12} & C_{12} & 0 & a_1s_1+a_2s_{12} \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{l} x_0 \\ y_0 \\ z_0 \\ \ \end{array}$$

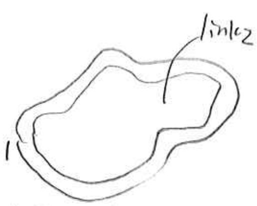
☆ Robots

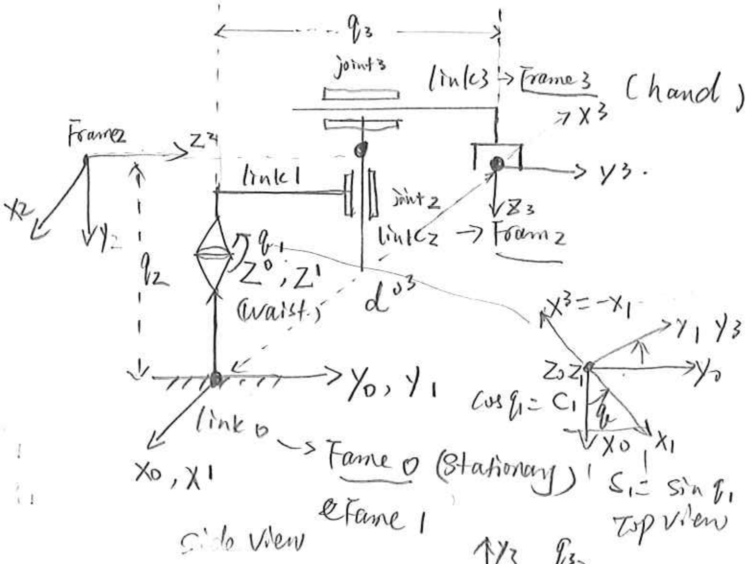
C) Cilindral Robot :

$H_0^3 = H_0^1 \cdot H_1^2 \cdot H_2^3$

$H_0^3 (q_1, q_2, q_3)$

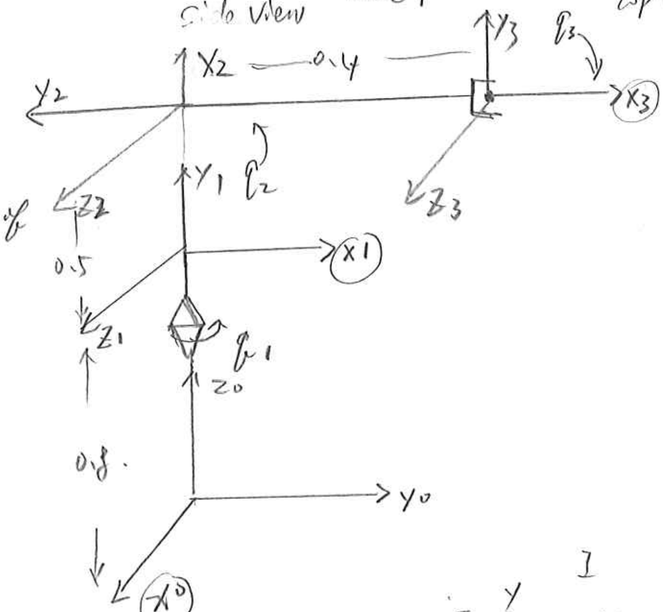
Joint 2 :
(Fr. Above) link 1





$$H_0^3 = \begin{array}{cccc|l} X_0^3 & Y_0^3 & Z_0^3 & d_0^{03} & \\ \left[\begin{matrix} -C_1 & -S_1 & 0 \\ -S_1 & C_1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{matrix}\right. & & & \left.\begin{matrix} -q_3 S_1 \\ q_3 C_1 \\ q_2 \\ 1 \end{matrix}\right] & \begin{matrix} d_{0x}^{03} = q_3 Y_{0x}^3 \\ d_{0y}^{03} = q_3 Y_{0y}^3 \\ d_{0z}^{03} \end{matrix} \end{array}$$

Side View



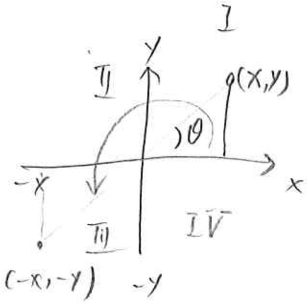
eg: At the pose shown. what are the values of
$q_1 \quad q_2 \quad q_3$?

→ $q_1 = 90°$, $q_2 = 90$; $q_3 = -90°$

what is H_0^3 at the pose?

what is H_0^3 when $q_1 = q_2 = q_3 = 0$?

$$\theta = \text{atan2}(y, x) = \begin{cases} \text{atan}\left(\frac{y}{x}\right) & \text{if } x > 0. \\ \text{atan}\left(\frac{y}{x}\right) \pm 180° & \text{if } x < 0 \end{cases}$$

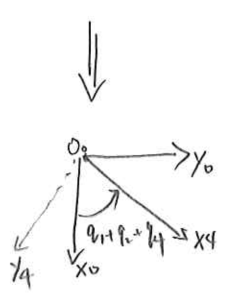


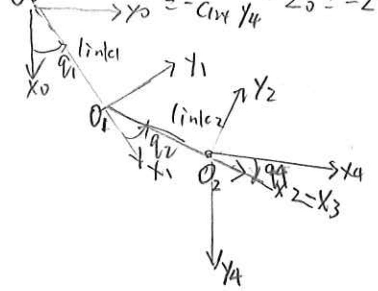
E) 4 Joints Robot

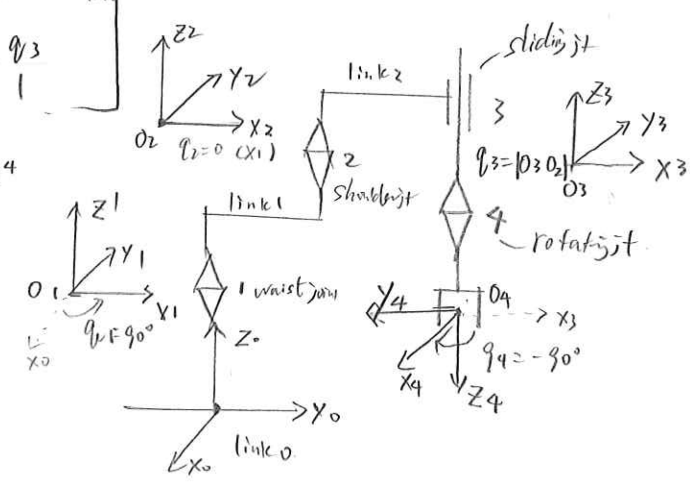
$$H_0^4 (q_1, q_2, q_3, q_4) = \begin{array}{cccc} X_0^4 & Y_0^4 & Z_0^4 & d_0^{04} \\ \end{array} \left[\begin{array}{ccc|c} C_{124} & S_{124} & 0 & 0.5C_1 + 0.4C_{12} \\ S_{124} & -C_{124} & 0 & .5S_1 + 0.4S_{12} \\ 0 & 0 & -1 & q_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$Y_0 = Y^4 \times \sin(90 + \theta) = -C_{124} Y_4$ $\quad Z_0 = -Z^4$

Top View:

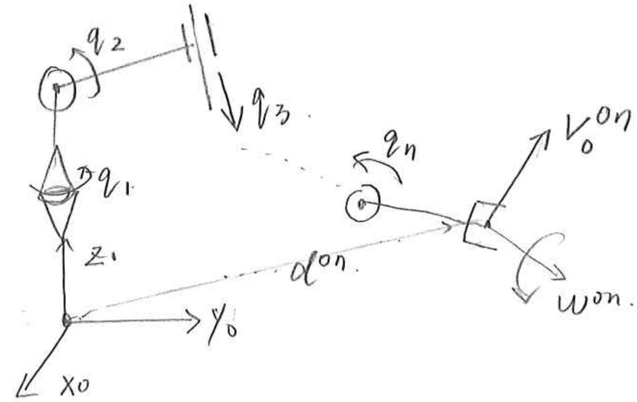






A Robots

3 Robot Velocity Kinetic $\rightarrow$ Find V_0^{on} & ω_0^{on}
(q) ($\dot{q}$)

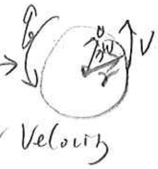


$$q = \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{pmatrix}$$

3.1 linear Velocity $V_0^{on} = \dot{d}_0^{on} = \dfrac{d(d_0^{on})}{dt}$

From forward kin:

$$d_0^{on}(q) = \begin{pmatrix} X(q) \\ Y(q) \\ Z(q) \end{pmatrix} = \begin{pmatrix} X(q_1 q_2 \cdots q_n \\ Y(q_1 q_2 \cdots q_n \\ Z(q_1 q_2 \cdots q_n \end{pmatrix}$$

$V = \omega \circ R \rightarrow$ 

linear Velocity, Angular Velocity

chain Rule

1D: $V_0^{on} = \dot{d}_0^{on} = \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix}$, $\dot{x} = \dfrac{d}{dt}(q_0(t), q_1(t) \cdots q_n(t)) \rightarrow \dfrac{dx}{dt} = \dfrac{dx}{dq_1} \cdot \dfrac{dq_1}{dt}$ (1)

for q_1 changes (1 joint move only at a time) other qs stay constant

$\xrightarrow[\text{muti variables}]{\Sigma q}$ $\dot{x} = \dfrac{dx}{dq_1} \cdot \dfrac{dq_1}{dt} + \dfrac{dx}{dq_2} + \dfrac{dq_2}{dt} + \cdots + \dfrac{dx}{dq_n} \cdot \dfrac{dq_n}{dt}$ $\xrightarrow{\text{Matrix Form}}$ $\dfrac{dq}{dt} = \dot{q} = w$.

$$= \left[\frac{\partial x}{\partial q_1} + \frac{\partial x}{\partial q_2} + \cdots \frac{\partial x}{\partial q_n} \right] \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

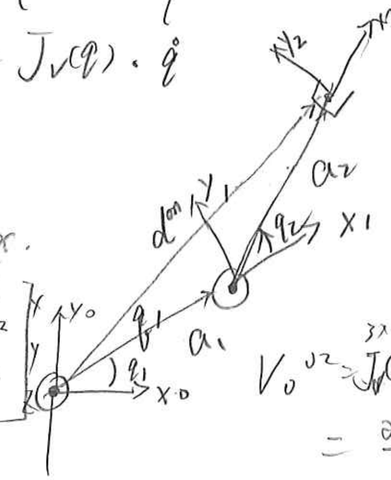
label: $\dfrac{\partial d_0^{o1}}{q_1}$ $\dfrac{\partial d_0^{o1}}{\partial q_2}$ $\cdots$ $\dfrac{\partial d_0^{o1}}{\partial q_n}$

$\xrightarrow{3D:}$
$$= \begin{bmatrix} \frac{\partial x}{\partial q_1} + \frac{\partial x}{\partial q_2} + \cdots \frac{\partial x}{\partial q_n} \\ \frac{\partial y}{\partial q_1} + \frac{\partial y}{\partial q_2} + \cdots \frac{\partial y}{\partial q_n} \\ \frac{\partial z}{\partial q_1} + \frac{\partial z}{\partial q_2} + \cdots \frac{\partial z}{\partial q_n} \end{bmatrix} \dot{q} \rightarrow \frac{d(d_0^{on})}{dt} = \frac{\partial d_0^{o1}}{\partial q} \cdot \frac{dq}{dt}.$$

$$\Longrightarrow V_0^{on} = \underbrace{\left[\frac{\partial d_0^{on}}{\partial q} \right]}_{3 \times n \text{ Vector}} \underbrace{\dot{q}}_{n \times 1} \xrightarrow[\text{Generalized}]{\text{Jacobian}} V_0^{on} = J_v(q) \cdot \dot{q} \quad (3)$$

J matrix, velocity of the joins

$\rightarrow$ Robot is a Visulization of a Jacobian Matrix.



above case

eg: $d_0^{o2} = \begin{pmatrix} X = d_{0x}^{on} \\ Y = d_{0y}^{on} = \\ 0 \end{pmatrix} \begin{pmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{pmatrix} \rightarrow J_v(q) = \begin{array}{cc} d_0^{o2}/\partial q_1 & \partial(d_0^{o2})/\partial q_2 \end{array} \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \end{bmatrix} \begin{array}{c} x \\ y \\ z \end{array}$

$V_0^{o2} = \underset{3 \times 2}{J(q)} \, \overset{2}{\dot{q}} = \dfrac{\partial d_0^{o2}}{dq} \cdot \dot{q}$

$x(q_1\ q_2) = a_1 \cos q_1 + a_2 \cos(q_1 + q_2)$

$\dfrac{\partial x}{\partial q_1} = -a_1 \sin q_1 - q_2 \sin(q_1 + q_2) = -a_1 S_1 - a_2 S_{12}$

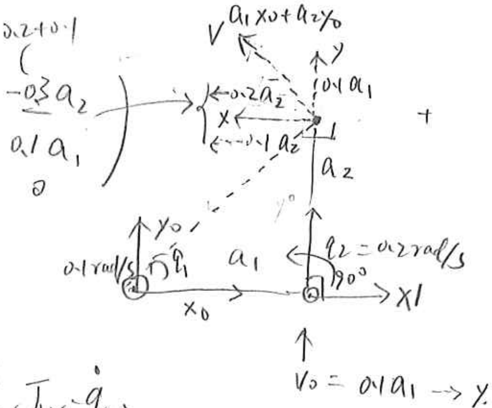
$\dfrac{\partial x}{\partial q_2} = 0 - a_2 \sin(q_1 + q_2) = -q_2 q_{12}$

☆ Robots

b) Find V_0^{22} where $q = \begin{pmatrix} 0 \\ 90^\circ \end{pmatrix}$ ($q_1 = 0^\circ$) and $\dot{q} = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}$ rad/s. (Given)

$q_2 = 90^\circ \rightarrow \begin{cases} \sin 90^\circ = 1 \\ \cos 90^\circ = 0 \end{cases}$

$$J_V(0, 90^\circ) = \begin{bmatrix} -a_2 & -a_2 \\ d_1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow V_0^{0n} = J_V \cdot \dot{q} = \begin{pmatrix} -0.3a_2 \\ 0.1a_1 \\ 0 \end{pmatrix}$$



Effect of $\dot{q}_2 \rightarrow$ let $\dot{q}_1 = 0 \rightarrow V_0^{02} = \begin{pmatrix} -0.2a_2 \\ 0 \\ 0 \end{pmatrix}$

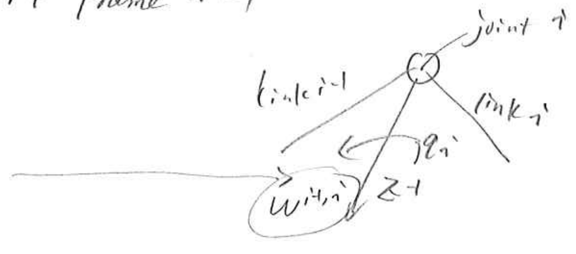
Effect of $\dot{q}_1 \rightarrow$ let $\dot{q}_2 = 0 \rightarrow V_0^{02} =$

$$w^{0n} = w^{01} + w^{12} + \dots + w^{n-1,n} = \sum_{i=1}^{n} w_0^{i-1,i} = \sum_{i=1}^{n} J_{wi}\dot{q}_i$$

3.2 Angular Velocity

Let $w^{i-1,i}$ be the angular velocity of frame i wrt frame $i-1$

If joint i is $\begin{cases} \text{prismatic}, & w^{i-1,i} = 0 \\ \text{revolute}, & w^{i-1,i} = \dot{q}_i Z_0^{i-1} \end{cases}$



$$w_0^{i-1,i} = J_{wi}\dot{q}_i$$

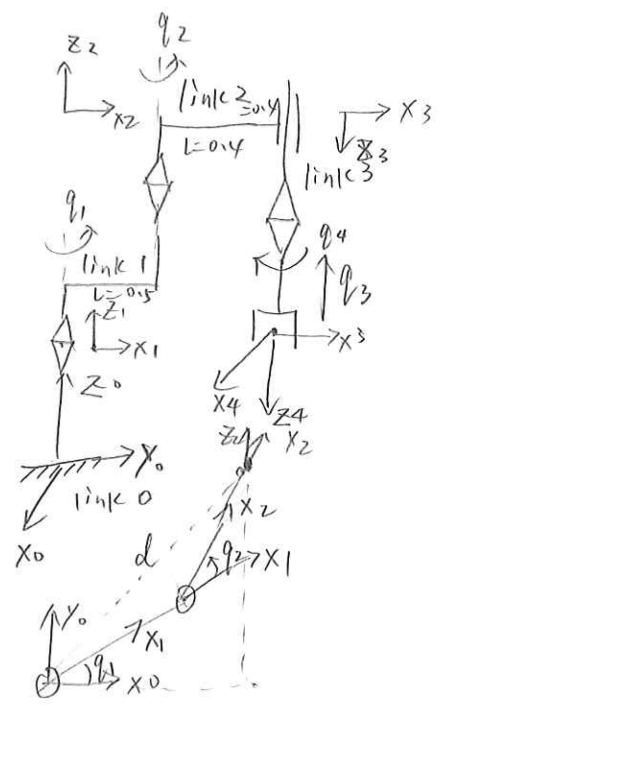
where $J_{wi} = \begin{cases} 0 & \text{if } i \text{ is prismatic} \\ Z_0^{i-1} & \text{if } i \text{ is revolute.} \end{cases} \rightarrow W^{0n} = \sum_{i=1}^{n} J_{wi}\dot{q}_i$

$= [J_{w1} | J_{w2} \dots J_{wn}]$

$$\underset{3\times1}{W_0^{0n}} = \underset{3\times n}{J_w(q)} \cdot \underset{n\times1}{\dot{q}}$$

combine $\Rightarrow$ $\begin{pmatrix} V_0^{0n} \\ W_0^{0n} \end{pmatrix} = \begin{bmatrix} J_V \\ J_w \end{bmatrix} \dot{q} = J\dot{q}$

eg. Find J of the Adapt Manapulator



$$J_w = [Z_0^0 | Z_0^1 | \begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix} | Z_0^3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

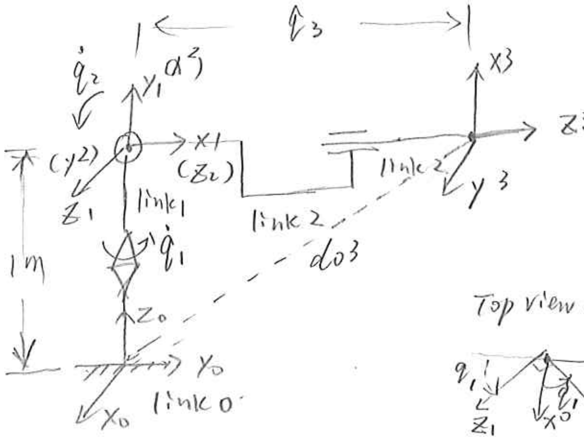
$$W_0^{04} = J_w\dot{q} = \begin{pmatrix} 0+0+0+0 \\ 0+0+0+0 \\ \dot{q}_1+\dot{q}_2+0+\dot{q}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1+\dot{q}_2+\dot{q}_4 \end{pmatrix}$$

$$d_0^{04} = \begin{pmatrix} 0.5C_1 + 0.4C_{12} \\ 0.5S_1 + 0.4S_{12} \\ q_3 \end{pmatrix}$$

Robots.

spherical Manipulator.

a) Find V_0^{03} and w_0^{03} as functions of q & $\dot q$



$$V_0^{03} = J_v(q)\,\dot q$$
$$w_0^{03} = J_w(q)\,\dot q$$

Top view of Frame 0,1

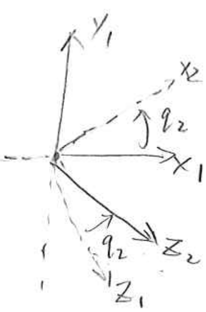
$$d^{03} = Z^0 + q_3 Z^2$$

$d_0^{03} = Z_0^0 + q_3 Z_0^2$, $R_0^2 = R_0^1 R_1^2$, $Z_0^2(q_1, q_2) = R_0^1(q_1)\, Z_1^2\, q_2$,

x_0' y_0' z_0' → Z' vs. { $x_0 = \_\_ \times Z_1$? $y_0 = \_\_ \times Z_1$? $z_0 = \_\_ \times Z_1$? }

$$\rightarrow R_0^1(q_1) = \begin{array}{c} \begin{matrix} x_0' & y_0' & z_0' \end{matrix} \\ \begin{bmatrix} C_1 & 0 & S \\ S_1 & 0 & -C_1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$Z_1^2 = \begin{pmatrix} S_2 \\ -C_2 \\ 0 \end{pmatrix}$$



$$Z_0^2 = R_0^1 Z_1^2 = \begin{pmatrix} C_1 S_2 \\ S_1 S_2 \\ -C_2 \end{pmatrix}$$

$$d_0^{03} = Z_0^0 + q_3 Z_0^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} C_1 S_2 q_3 \\ S_1 S_2 q_3 \\ -C_2 q_3 \end{pmatrix} = \begin{pmatrix} C_1 S_2 q_3 \\ S_1 S_2 q_3 \\ 1 - C_2 q_3 \end{pmatrix}$$

$$\Rightarrow J_v(q) = \begin{pmatrix} \frac{\partial d_0^{03}}{\partial q_1} & \frac{\partial d_0^{03}}{\partial q_2} & \frac{\partial d_0^{03}}{\partial q_3} \\ -S_1 S_2 q_3 & C_1 C_2 q_3 & C_1 S_2 \\ C_1 S_2 q_3 & S_1 C_2 q_3 & S_1 S_2 \\ 0 & S_2 q_3 & -C_2 \end{pmatrix}$$

Direction of Vector:
Right Hand sys:

eg: X cross Y = Z

$a \times (b + c) = a \times b + a \times c.$

$$\Rightarrow J_w(q) = \begin{pmatrix} Z_0^0 & Z_0^1 & \\ 0 & S_1 & 0 \\ 0 & -C_1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

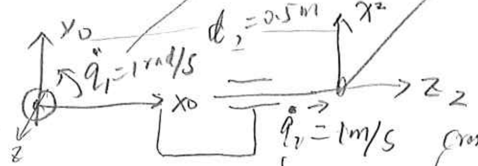
Finding J_v at a given pose:

$$V_0^{0n} = J_v \dot q = \underbrace{J_{v1} \dot q_1}_{=1} + \underbrace{J_{v2} \dot q_2}_{=0} + \cdots \underbrace{J_{vn} \dot q_n}_{=0}$$

$J_{vi} = V_0^{0n}$ when $\dot q_i = 1$ and $\dot q_{k \neq i} = 0$

eg. Z_0 rotate: $w = 1$ rad/s
X_0 extend: $V = \ldots$ m/s

$V_0^2 = 1 Z^0 \times 0.5 X^0$ (X cross)
$= 0.5 Y^0$ m/s $= 0.5 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$



cross

$V_0^2 = 0 Z^0 \times 1 = 1 X_0$ m/s $= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

— For a revolute joint: $J_{vi} = (w \times r)_0 = (1 Z^{i-1} \times d^{i-1,n})_0$

$J_{v1} = (1 Z^0 \times d^{02})_0 = (Z^0 \times 0.5 X^0)_0 = (0.5 Y^0)_0 = 0.5 Y_0^0 = \begin{pmatrix} 0 \\ 0.5 \\ 0 \end{pmatrix}$

— For a primater joint: $J_{vi} = Z_0^{i-1}$.

$$J_v^1 = V_0^{02}\big|_{\dot q_1 = 1, \dot q_2 = 0} = \begin{pmatrix} 0 \\ 0.5 \\ 0 \end{pmatrix}$$

$$J_{v2} = V_0^{02}\big|_{\dot q_1 = 0, \dot q_2 = 1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow J_v = [J_{v1} | J_{v2}] = \begin{array}{c} \begin{matrix} \dot q_1 & \dot q_2 \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

~~A~~ Robots :

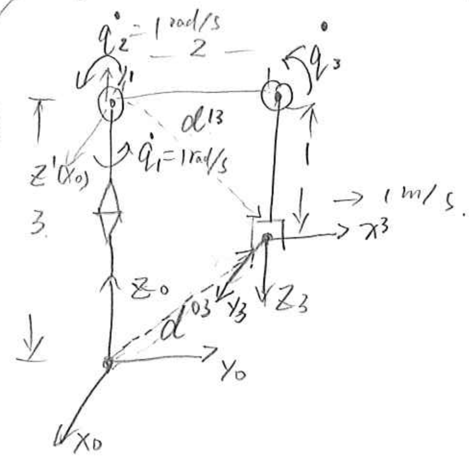
eg. $J_{v1} = (1 z^0 \times d^{0,3})_0 = [z^0 \times (2y^0 + 2z^0)]$

$$= (2 \underbrace{z^0 \times y^0}_{2\cdot(-x_0)} + \underbrace{2 z^0 \times z^0}_{0})_0 = (-2x^2)_0 = \begin{pmatrix}-2\\0\\0\end{pmatrix}$$

$J_{v2} = (1 z^1 \times d^{1,3})_0 = (z^1 \times (2y^0 - 1z^0))_0$

$$= (2z^0 + 1y^0)_0 = \begin{pmatrix}0\\1\\2\end{pmatrix}$$

q_1 rotate to $-x$ direction at $L = 2$.



A Robots

Arm Sigurarities

- Assume 3 arm joints (ignore wrist) $\rightarrow$ $V_0^{03} = \overset{3\times3}{J_v}(q)\,\dot{q}$, $q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

If $J_v(q)$ is not signlar at q, then for any V_0^{03} there is a unique $\dot{q}$ that achieves it:

$\dot{q} = J_v(q)^{-1} V_0^{03}$

otherwise, $J_v(q)$ is signlar :
- let $J_v(q) = 0$.
- some V_0^{03} are not possible.
- some V_0^{03} have many $\dot{q}$ that produce them. — many solutions.
- There are values of $\dot{q} \neq 0$ such tat $V_0^{03} = 0$.

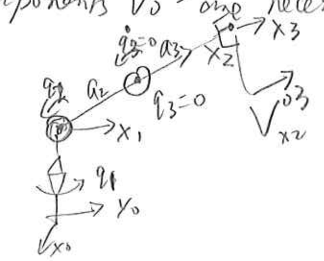
eg.

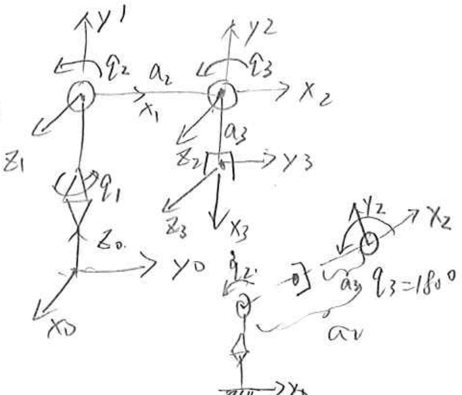
i) Identify the condition on $q \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ that make signlar?

ii) At each signlarity what components V_0^{03} are necessarily zero?

eg. $V_{2x}^{03} = 0$, $q_3 = 0$.

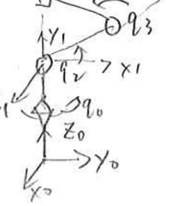
or $V_{3x}^{03} = 0$.

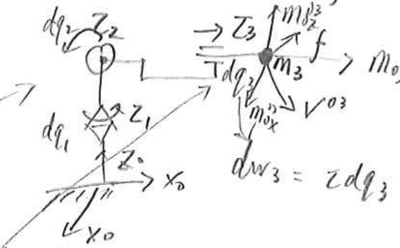




Get $V^{03}_{\cdot} = 0$ with $\dot{q}_1 = 0$ $\rightarrow$ $V_{2y}^{03} = \dot{q}_2(a_2 + a_3) + \dot{q}_3 \cdot a_3 = 0$ $\rightarrow$ $\dot{q}_2 = \dfrac{-a_3}{a_2 + a_3} q_3$.

$V_{1z}^{03} = 0$.





Statics:

Robot Motor Required: Forces & Torques level $\Rightarrow$ for Rotational Joints / extension

f is the force applied by the robot tips. (to some object)

τ is the generalized fore applied at joint i
- a torque if i is revolute
- a force if i is primatic.

problem: Given f and q, find $\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{pmatrix}$

Virtual work done by motors: $dw = dw_1 + dw_2 + \dots dw_n$.

$\dfrac{dw}{dt} = \tau_1 \dfrac{dq_1}{dt} + \tau_2 \dfrac{dq_2}{dt} + \dots \tau_n \dfrac{dq_n}{dt}$ $= \tau_1 dq_1 + \tau_2 dq_2 + \dots \tau_n dq_n$

$\rightarrow$ power input of motors $= \tau_1 \dot{q}_1 + \tau_2 \dot{q}_2 + \dots \tau_n \dot{q}_n = \tau^T \dot{q}$

A Robot

power output at tip $= f_x V_{0x}^{on} + f_{0y} V_{0y}^{on} + f_{z} V_{0z}^{on} = (f_0)^T V_0^{on}$

where $f_x = \begin{pmatrix} f_{0x} \\ f_{0y} \\ f_{0z} \end{pmatrix}$

power in = power out $\rightarrow \underline{\tau^T \dot{q}} = (f_0)^T \underbrace{V_0^{on}} = (f_0)^T \underline{J_v \dot{q}}$

$\rightarrow (\tau^T - f_0^T J_v)\, \dot{q} = 0 \quad \rightarrow r\dot{q} = 0 \quad \because \dot{q}$ is arbitrary $\rightarrow$ let $\dot{q} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rightarrow r_1 = 0$

Now let $\dot{q} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rightarrow r_2 = 0 \quad \cdots \quad r = 0 = \tau^T - f_0^T J_v \rightarrow \tau^T = f_0^T J_v$ (Transpose)

Velocity force $\rightarrow$ Force

$\rightarrow \boxed{\tau_f = J_v^T f_0}$

Motor Force — Tip force

power Transmitted by $m_0^3 = m_{0x}^3 w_{0x}^{03} + m_{0y}^3 w_{0y}^{03} + m_{0z}^3 w_{0z}^{03}$

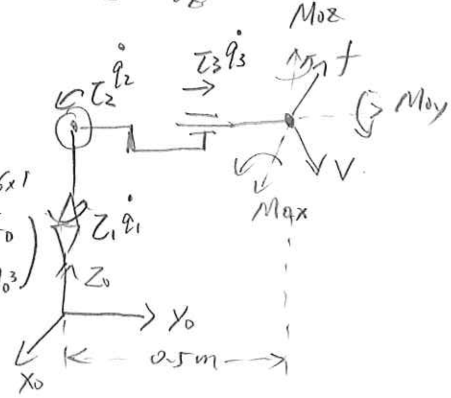
Angular Velocity

Moment $\rightarrow$ Force $\rightarrow$ effect of m_0^n on τ.

$\rightarrow \tau_m = J_w^T \cdot m_0^n$

$\Rightarrow$ Total $\tau = \tau_f + \tau_m = J_v^T f_0 + J_w^T m_0^n \Big\} \rightarrow \tau = \underset{n\times3 \;\; n\times3}{\left[J_v^T \,\vdots\, J_w^T \right]} \underset{6\times1}{\begin{pmatrix} f_0 \\ m_0^3 \end{pmatrix}}$

Let $F_0^n = \begin{pmatrix} f_0 \\ m_0^3 \end{pmatrix}$ wrench



$\rightarrow \tau = \left[\frac{J_v}{J_w} \right]^T \cdot \begin{pmatrix} f_0 \\ m_0 \end{pmatrix} = J^T \cdot F \quad \Rightarrow \underset{n}{\tau} = \underset{n\times6}{J^T} \underset{6\times1}{F}$ (3 joints, 6 degrees of freedom)

Example: At pose shown, with $q_3 = 0.5$m. Find τ if $F_0^3 = (1\,Nm\; 2\,Nm\; 3\,Nm\; 4\,Nm\; 5\,N.m.\; 6\,Nm)^T$

$q_3 = 0.5$ m.

$$J_v = \begin{array}{c} \begin{matrix} \dot{q}_1 & \dot{q}_2 & \dot{q}_3 \end{matrix} \\ \begin{bmatrix} -0.5m & 0 & 0 \\ 0 & 0 & 1m \\ 0 & 0.5m & 0 \end{bmatrix} \end{array}, \quad J_w = \begin{array}{c} \begin{matrix} \dot{q}_1 & \dot{q}_2 & \dot{q}_3 \end{matrix} \\ \begin{bmatrix} 0 & X_1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ Z_0 \end{array} \rightarrow J^T = \begin{array}{c} F = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} -0.5 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 1\times(-0.5)+6\times1 \\ 3\times0.5+4\times1 \\ 2\times1 \end{bmatrix}$$

(Nm)

$\rightarrow \tau = J^T \cdot F_0^3 = \begin{pmatrix} 5.5\ Nm \\ 5.5\ N.m \\ 2\ Nm \end{pmatrix} \begin{matrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{matrix}$

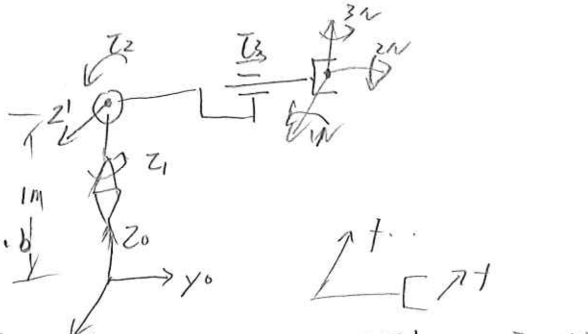
A Robots.

Static by Inspection.

If i is prismatic, then

$\tau_i = z^{i-1}\cdot f$.

If i is revolute, then



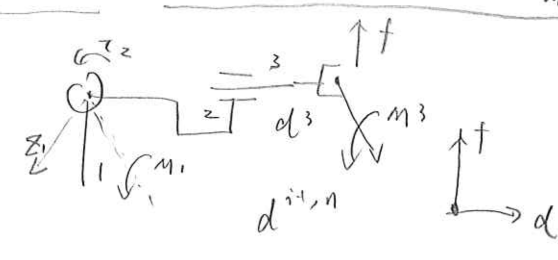
∵ $a\times b\cdot c = c\times a\cdot b$

$$\tau_i = (m^n + d^{i-n,n}\times f)\cdot z^{i-1} = m^n\cdot z^{i-1} + (z^{i-1}\times d^{i-1,n})\cdot f = [(z^{i-1}\times d^{i-1,n})^T_0 \mid (z_0^{i-1})^T]\cdot \begin{pmatrix} f_0 \\ m_0^n \end{pmatrix}$$

$$= [m^3 + 0.5\, y^0 \times (1x^2 + 2y^0 + 3z^0)]\cdot x^0$$

$$= [m^3 + (-0.5\, z^0 + 1.5\, x^0)]\cdot x^2$$

$$= m^3_{0x} + 1.5 = 4 + 1.5 = 5.5\ \text{N.m.}$$



$$\to \tau_i = [J_{vi}^T \mid J_{wi}^T]\cdot F_0^n = J^T F^n.$$

Q_1 what force does the ground (Frame 0) apply to the robot? $f_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ N

Q_2 what moment does the ground ... $m_0^0 = ?$

$$m_0^0 = m^n + d^{0n}\times f = m^n + (1z^0 + 0.5\, y^0)\times(1x^0 + 2y^0 + 3z^0) = m^n + y^0 - 2x^0 - 0.5\, z^x + 1.5\, x^2$$

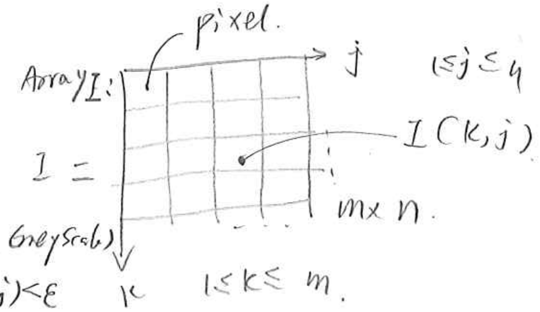
$$= m^n - 0.5\, x^0 + y^0 - 0.5\, z^0 \qquad m_0^0 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} -0.5 \\ 1 \\ -0.5 \end{pmatrix} = \begin{pmatrix} 3.5 \\ 6 \\ \underline{5.5} \end{pmatrix}$$

$$\tau_1 = m^0\cdot z^0 = 5.5\ \text{Nm}.$$

★ Robots. 3 - Image Representation & Storage

3.1. Image as array of pixels of certain value

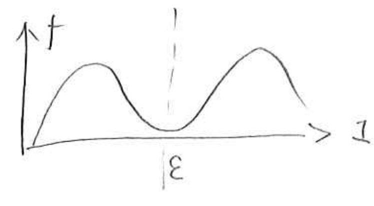
- Grey Scale & Binary Image $\begin{cases} \text{- each square is a pixel} \\ \text{- pixel coordinate } (j, k) \\ \text{- Resolution } = m \times n. \end{cases}$



3.2. Thresholding a Greyscale Image. (Binary easier than Greyscale)

Converts gray scale to Binary : $I_\varepsilon(k,j) = \begin{cases} 0 & \text{if } I(k,j) < \varepsilon \\ 1 & \text{if } I(k,j) \ge \varepsilon \end{cases}$.

eg. $I = \begin{vmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 5 & 6 & 5 & 1 \\ 2 & 6 & 6 & 6 & 1 \\ 0 & 5 & 6 & 5 & 2 \\ 1 & 1 & 2 & 2 & 3 \end{vmatrix}$, $I_4 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$



3.3. Run - length Encoding (2D → 1D). (1D less memory than 2D.)

$T' = [\,0, 0, 3, 2, 3, 2, 3, 6\,]$

4. Object Recognition

4.1. Template Matching :

Matrix I. $\quad I(k,j) = \begin{array}{c|cccc|} & & j & & n=4 \\ \hline k & 2 & 1 & 0 & 3 \\ & 0 & 0 & 5 & 0 \\ & 0 & 4 & 0 & 0 \\ & 1 & 0 & 5 & 0 \\ \hline \end{array}$ $\quad m = 4.$

Template i: $\quad T_i(ab) = \begin{array}{c|ccc|c} & & b & & n_i \\ \hline a & 0 & 4 & 0 & \\ & 3 & 0 & 5 & \\ & 0 & 4 & 0 & \\ \hline m_i & & & & \end{array}$

Error function :

$$p_i(x,y) = \sum_{a=1}^{m_i} \sum_{b=1}^{n_i} | I(x+a, y+b) - I_i(a,b) | \rightarrow \text{not Robust for Rotating, Scaling etc.}$$

eg. $p_i(0,0) = |2-0| + |1-4| + |0-0| + |0-3| + |0-0| + |5-5| + |0-0| + |4-4| + |0-0| = 8$ (with 2, 3, 3 written above the terms $|2-0|$, $|1-4|$, $|0-3|$)

Grey Scale Image → Robot Vision.

eg $I = \begin{array}{c|cccc} & & & & \rightarrow j \\ \hline & 1 & 1 & 2 & 1 \\ & 2 & 6 & 7 & 0 \\ k & 1 & 7 & 5 & 1 \\ \downarrow & 0 & 1 & 1 & 2 \end{array}$

eg: $\begin{array}{c|c|c} & \multicolumn{2}{c}{j \rightarrow} \\ \hline K & K, j & K, j+1 \\ \hline \downarrow & K+1, j & K+1, j+1 \end{array} \rightarrow \begin{cases} \dfrac{\partial I}{\partial j}(k,j) = I(k, j+1) - I(k,j) \\ \dfrac{\partial I}{\partial k}(k,j) = I(k+1, j) - I(k,j). \end{cases}$

2 - Edge Detection   For a neighbourhood of 4 pixels: $\begin{array}{c|c|c} & \multicolumn{2}{c}{\rightarrow j} \\ k & 1 & 2 \\ \hline \downarrow & 3 & 4 \end{array}$

Gradient $= \left(\dfrac{\partial I}{\partial k}, \dfrac{\partial I}{\partial j} \right) \rightarrow$ Magnitude of Gradient $= G(k,j) = \left| \dfrac{\partial I}{\partial k}(k,j) \right| + \left| \dfrac{\partial I}{\partial j}(k,j) \right|$

A Robots

eg. $G = \begin{vmatrix} 1 & 6 & 6 \\ 5 & 1 & 9 \\ 7 & 8 & 8 \end{vmatrix}$ $\quad G\varphi = \begin{array}{c} \text{start} \\ \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} \end{array} \xrightarrow{1D} c = (0, 6, 6, 4, 4, 2)$

$\rightarrow$ curve fragment $f = (6, 4, 4)$ by 1D Template matching.

3. Chain coding. (of Binary curves) eg. chain-coding template of pixel P.

$$\rightarrow \begin{vmatrix} 3 & 2 & 1 \\ 4 & P & 0 \\ 5 & 6 & 7 \end{vmatrix}$$

Direction

4.4 Area Descriptions (Binary Image).

Moments: $m_{ab} = \sum_{R} k^a j^b \quad a \geq 0,\ b \geq 0.$

where R is the region of object pixels (1-pixel).

order of m_{ab} is $a+b$.

$$m_{00} = \sum_{R} k^0 j^0 = \sum_{R} 1 = A$$

Centroid of the object is (k_c, j_c).

$$k_c = \frac{m_{10}}{m_{00}} = \frac{\sum_R k}{A}$$

$$j_c = \frac{m_{01}}{m_{00}} =$$

eg

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \ (j) \\ \hline k & 1 & 1 & 0 & 0 \\ & 0 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 0 \end{array}$$

Region (R)

$\rightarrow m_{01} = \sum_{R} j = 1 + 2 + 2 + 3 + 3 + 4 = 15$

$m_{00} = \sum_{R} 1 = A = 6$

$m_{10} = \sum_{R} k = 1 + 1 + 2 + 2 + 3 + 3 = 12$

$\rightarrow$ $j_c = \frac{m_{01}}{m_{00}} = \frac{15}{6} = 2.5$

$k_c = \frac{m_{10}}{m_{00}} = \frac{12}{6} = 2.$

$\rightarrow (k_c, j_c) = (2.5, 2)$

Central Moments $\mu_{ab} = \sum_{R} (k - k_c)^a (j - j_c)^b.$

$\mu_{00} = A$

$\mu_{10} = \sum_{R} (k - k_c) = 0$

$\mu_{20} = \sum_{R} (k - k_c)^2 = 4$

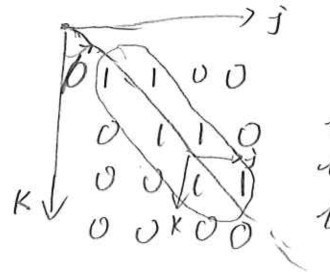
Robots

Principle Angle (of Region R)

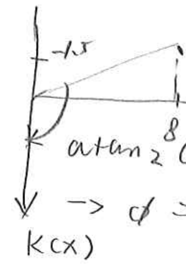
$\phi = \frac{1}{2}\arctan(2u_{11}, u_{20} - u_{02})$

$= \frac{1}{2}\,\mathrm{atan2}(8, -1.5)$

$= 50.3°$



$u_{11} = 4$
$u_{20} = 4$
$u_{02} = 5.5$

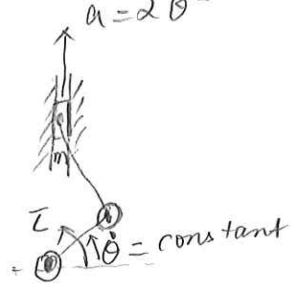


$\mathrm{atan2}(8, -1.5) = 100.6°$

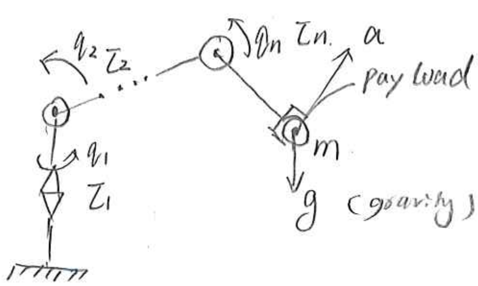
$\to \phi = \frac{1}{2} \times 100.6° = 50.3°$

Robot Dynamics

Find: $\tau(q, \dot{q}, \ddot{q})$ (pose, velocity, joint Rate)



Torque = Force × Length = $F_i \cdot l_i = \tau_i$



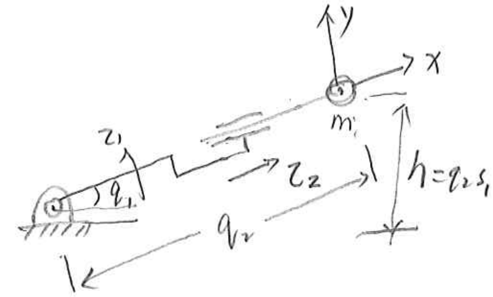
(6 degrees of Freedom)

Lagrangion Approach

$\to$ Lagrangian: $L(q, \dot{q}) \equiv K(q, \dot{q}) - V(q)$ (Kinetic Energy (KE), potential Energy (PE))

$\to \tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$

(Torque)



eg. $V(q) = mgh = mg s_1 q_2$ $\quad (s_1 = \sin q_1)$

$K(q, \dot{q}) = \frac{1}{2} m |V|^2$ (Energy takes for F to move m for static to velocity V.)

$= \frac{1}{2} m (\dot{q}_2^2 + q_2^2 \dot{q}_1^2)$

Note: $E = F \cdot L = m \cdot a \cdot L = m \cdot \frac{V}{t} \cdot (\frac{1}{2} V t)$

$= \frac{1}{2} m V^2$

or $\int m V dt = \frac{1}{2} m V^2$

$|V|^2 = |V_x|^2 + |V_y|^2 = \dot{q}_2^2 + q_2^2 \dot{q}_1^2$

$\to L = \frac{1}{2} m (\dot{q}_2^2 + q_2^2 \dot{q}_1^2) - mg s_1 q_2$ $\quad \left(\frac{d(q_2^2 \dot{q}_1^2)}{d\dot{q}_1} = 2 q_2^2 \cdot \dot{q}_1\right)$

$+ 2 q_2 \dot{q}_2 \cdot \dot{q}_1$

$\to \tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \frac{d}{dt}(m q_2^2 \cdot \dot{q}_1) + mg c_1 q_2$

$= m(\ddot{q}_1 q_2^2 + 2 q_2 \dot{q}_1 \dot{q}_2) + mg c_1 q_2$

Robots Dynamics

$$\mathcal{L} = K - V = \frac{1}{2} m \left(\dot q_2^2 + (q_2 \dot q_1)^2 \right) - m g q_2 s_1$$

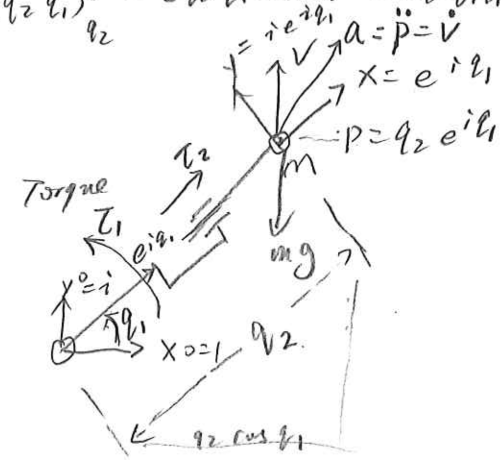
$$(q_2 \dot q_1)^2{}'_{q_2} = 2 q_2 \dot q_1 \cdot (q_2 \dot q_1)' = 2 q_2 \dot q_1 \dot q_1$$

$$(\dot q_2^2)'_{\dot q_2} = 2 \dot q_2 \qquad (\dot q_2)'_t = \ddot q_2$$

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1}$$

$$= \frac{d}{dt}(m q_2^2 \dot q_1) + m g q_2 \cos q_1$$

$$= m q_2^2 \cdot \ddot q_1 + 2 m q_2 \dot q_2 \dot q_1 + m g q_2 \cos q_1$$

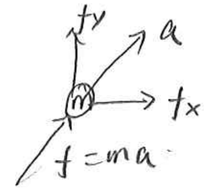


$$\tau_2 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q_2}\right) - \frac{\partial \mathcal{L}}{\partial q_2}$$

$$= m \ddot q_2 - m q_2 \dot q_1^2 + m g \sin q_1$$

$$P = q_2 \cos q_1 + i q_2 \sin q_1$$

position of the mass : $p = q_2 e^{iq_1}$



Velocity $v = \dot p = \dot q_2 e^{iq_1} + q_2 \dot q_1 i e^{iq_1}$

$$= \dot q_2 x + q_2 \dot q_1 y$$

Acceleration $a = \dot v = (\ddot q_2 + (\dot q_2 \dot q_1 + q_2 \ddot q_1) i) e^{iq_1} + (\dot q_2 + q_2 \dot q_1 i) i \dot q_1 e^{iq_1}$

$$= (\ddot q_2 - q_2 \dot q_1^2) e^{iq_1} + (q_2 \ddot q_1 + 2 \dot q_1 \dot q_2) i e^{iq_1}$$

$$= (\ddot q_2 - q_2 \dot q_1^2) x + (q_2 \ddot q_1 + 2 \dot q_1 \dot q_2) y$$

$$= a_x \cdot x + a_y \cdot y$$

Moment of Inertia $\rightarrow I$

Force: $f = ma = m a_x \cdot x + m a_y \cdot y = f_x \cdot x + f_y \cdot y$

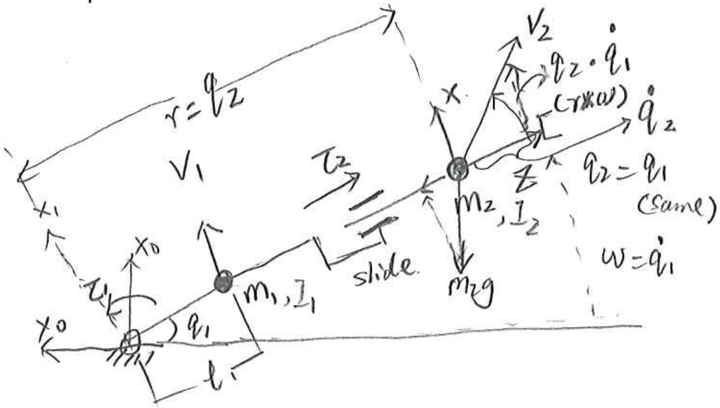
Torque: $\tau_1 = f_y q_2 = m(q_2^2 \ddot q_1 + 2 q_2 \dot q_1 \dot q_2) + I \ddot q_1$

$$\tau_2 = f_x = m a_x = m \ddot q_2 - m q_2 \dot q_1^2$$

☆ Robots

Dynamic Example:

Find $\tau(q, \dot{q}, \ddot{q})$ Let I_1 and I_2 be the moments of inertia at links 1 & 2 about their own mass centers.



$\mathcal{L} = K - V = (K_1 + K_2) - (V_1 + V_2)$

$K = K_1 + K_2$

$$K_1 = \frac{1}{2} m_1 |V_1|^2 + \frac{1}{2} I_1 \dot{q}_1^2$$
$$= \frac{1}{2} m_1 (l_1 \dot{q}_1)^2 + \frac{1}{2} I_1 \dot{q}_1^2$$
$$= \frac{1}{2} (m_1 l_1^2 + I_1) \dot{q}_1^2$$
$$= \frac{1}{2} I_{1,0} \dot{q}_1^2$$

$$K_2 = \frac{1}{2} m_2 |V_2|^2 + \frac{1}{2} I_2 \dot{q}_1^2$$
$$= \frac{1}{2} m_2 (\dot{q}_2^2 + q_2^2 \cdot \dot{q}_1^2) + \frac{1}{2} I_2 \dot{q}_1^2$$

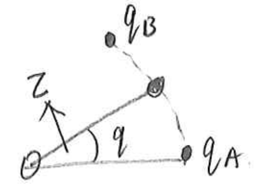
$$V = V_1 + V_2 = m_1 g h_1 + m_2 g h_2$$
$$= m_1 g l_1 S_1 + m_2 g q_2 S_1 = (m_1 g l_1 + m_2 g q_2) S_1$$

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1} = (m_1 l_1^2 + I_1 + m_2 q_2^2 + I_2) \ddot{q}_1 + 2 m_2 q_2 \dot{q}_1 \dot{q}_2 + (m_1 g l_1 + m_2 g q_2) C_1$$
$$= I \ddot{q}_1 + \cdots$$

$$\tau_2 = m_2 \ddot{q}_2 - \underbrace{m_2 q_2 \dot{q}_1^2}_{\text{Centrifugal force } (I \cdot \omega^2)} + m_2 g \sin q_1$$

☆ Robots

Trajectory (Path) Planning { Path in Time (A). / Path in space (B)



A) Path in Time.

problem: move smoothly from $q = q_A$ to $q = q_B$ in Time t_B

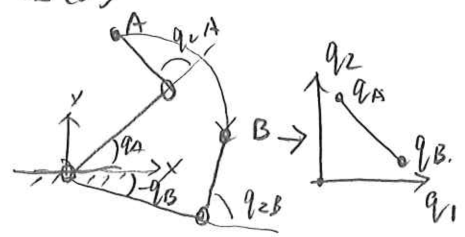
find $q(t)$ such that: $q(0) = q_A$, $q(t_B) = q_B$ (1)

$\dot{q}(0) = \dot{q}(t_B) = 0$ (2).

$\ddot{q}(t) \le a$ for all $t \in [0, t_B]$ (3)

Motor Torque:

$$\tau = (ml^2 + I)\ddot{q}$$

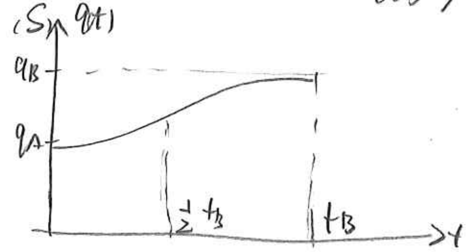


Q problem: Joint-Trajectory planning Move from $\bar{q}_A$ to $\bar{q}_B$:

with $\dot{q}(0) = 0 = \dot{q}(t_B)$

Solution: Cubic splines.

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 = q_A + (q_B - q_A)\left(3\left(\frac{t}{t_B}\right)^2 - 2\left(\frac{t}{t_B}\right)^3\right)$$



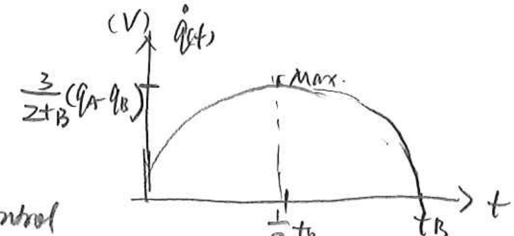
$$\rightarrow \dot{q}(t) = \frac{6}{t_B}(q_B - q_A)\left[\frac{t}{t_B} - \left(\frac{t}{t_B}\right)^2\right]$$

$$\rightarrow \ddot{q}(t) = \frac{6}{t_B^2}(q_B - q_A)\left[1 - 2\left(\frac{t}{t_B}\right)\right]$$

→ Acceleration is a straight line.

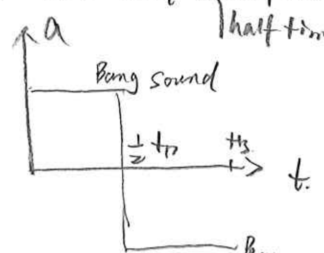
If To Get Robots most fast

velocity:

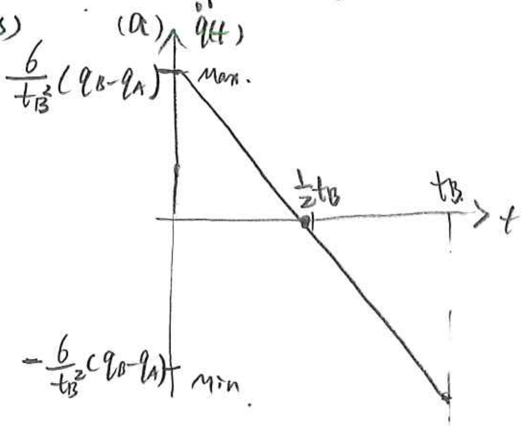


→ Min time t → { half time Max a / half time Min a. → Bang-Bang Control (No Brakes)

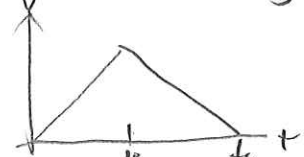
pot: →



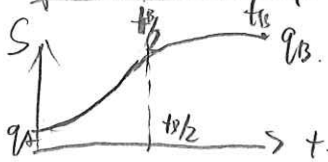
Accelation:



→



→

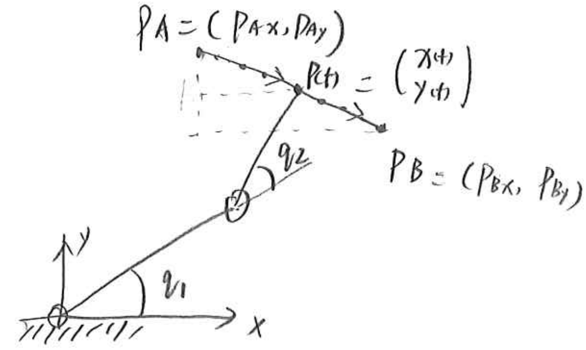


parabolic

q_B. (2 段曲线 部分)

☆ Robots

Straight line trajectory.



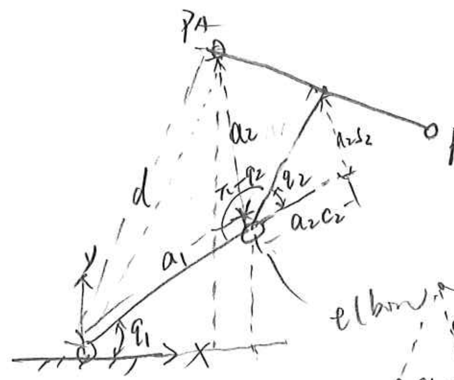
Given $P_A = \begin{pmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{pmatrix}$ $\quad P_B = \begin{pmatrix} P_{Bx} \\ P_{By} \\ P_{Bz} \end{pmatrix}$

Find $P(t) = P_A + (P_B - P_A)\left[3\left(\frac{t}{t_B}\right)^2 - 2\left(\frac{t}{t_B}\right)^3\right]$

$$\rightarrow \begin{cases} x(t) = P_{Ax} + (P_{Bx} - P_{Ax})\, f(t) & \cdots\cdots (1) \\ y(t) = P_{Ay} + (P_{By} - P_{Ay})\, f(t) & \cdots\cdots (2) \end{cases}$$

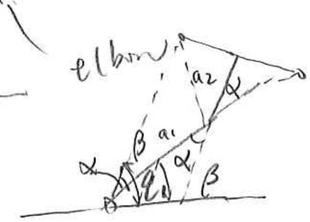
$(2) \rightarrow f(t) = \dfrac{y(t) - P_{Ay}}{P_{By} - P_{Ay}} \quad \cdots (3)$

put (3) in (1) $\rightarrow x(t) = P_{Ax} + \left(\dfrac{P_{Bx} - P_{Ax}}{P_{By} - P_{Ay}}\right)(y(t) - P_{Ay})$



Inverse Kinematics:

Given $p = \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow$ Find $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$



$d^2 = x^2 + y^2$

$d^2 = a_1^2 + a_2^2 - 2a_1 a_2 \cos(\pi - q_2)$

$= a_1^2 + a_2^2 + 2a_1 a_2 C_2$

$\rightarrow \cos q_2 = \dfrac{d^2 - a_1^2 - a_2^2}{2a_1 a_2} \rightarrow q_2 = \pm \arccos\left(\dfrac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1 a_2}\right)$

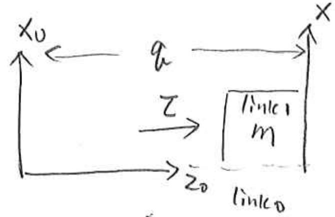
elbow up / down.

$\therefore \alpha = \arctan(a_2 S_2,\ a_1 + a_2 C_2) \Rightarrow q_1 = \beta - \alpha = \text{atan2}(y, x) - \alpha$

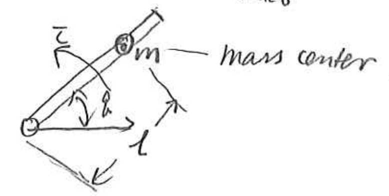
Robot

— Control.

Consider 1- joint robot :



a) presmatic → equation of motion $\tau = m\ddot{q}$ ($F = ma$)



b) revolute: $\tau = (\underbrace{I}_{\text{moment of Inertia}} + ml^2)\,\ddot{q}$ (1)

Want q to follow $q_d(t)$ with small error $e = q_d - q$ (2)

$\ddot{q}_d \to \ddot{q}$

(2) → (1) : $\underbrace{m\ddot{e}}_{\text{error}} + \tau = m\ddot{q}_d$ ---- (3)

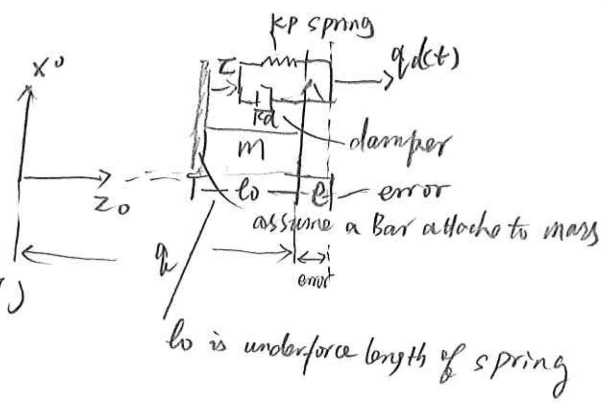
— open-loop Control: $\tau = m\ddot{q}_d$ ------ (4)

(4) → (3) $m\ddot{e} + m\ddot{q}_d = m\ddot{q}_d \to m\ddot{e} = 0 \xrightarrow{\text{Integrate}} \dot{e}(t) = \dot{e}(0) \xrightarrow{\text{Integrate}} e(t) = \dot{e}(0)t + e(0)$

Fr. equation point: If $e = 0 \to \dot{e} = 0$

error grows (unstable).

— Feed back control. $\tau = k_d\dot{e} + k_p e$ --- (5)



(5) → (3) → $m\ddot{e} + k_d\dot{e} + k_p e = m\ddot{q}_d$ ---(6)

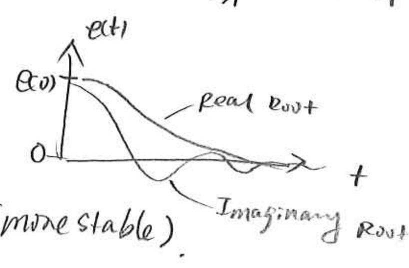
Stability of (6) Set $\ddot{q}_d = 0 \to m\ddot{e} + k_d\dot{e} + k_p e = 0$ (7)

→ Solution: $e(t) = C e^{\lambda t}$ → put into (7)

→ $mC\lambda^2 e^{\lambda t} + k_d C\lambda e^{\lambda t} + k_p C e^{\lambda t} = 0$.

→ $m\lambda^2 + k_d\lambda + k_p = 0 \xrightarrow{\text{2 Roots}} \lambda = \lambda_1, \lambda_2 = \frac{-k_d}{2m} \pm \frac{1}{2m}\sqrt{k_d^2 - 4mk_p}$

⇒ $e(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$



(5) modified $\tau = \underbrace{k_d\dot{e}}_{\text{derivative}} + \underbrace{k_p e}_{\text{proportional}} + \underbrace{K_I \int_0^t e(\tau)\,d\tau}_{\text{Integrate}}$ (more stable).

(PID)

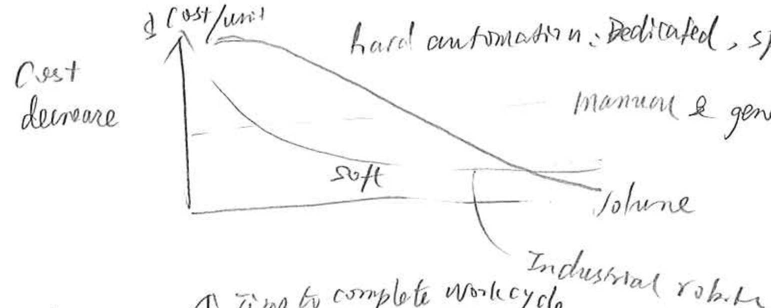
Robots

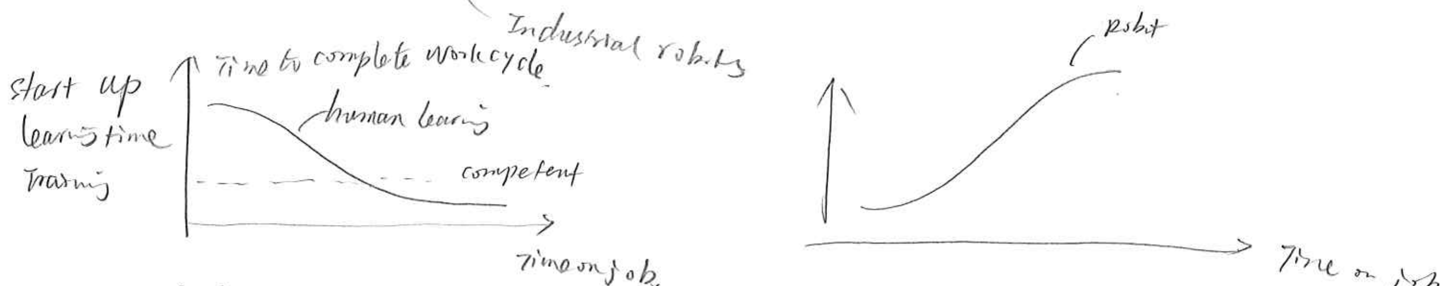
Application:
- Remote Surgery (medical) on space ships. power/Force } scale up/down → better than man. Vision(camera) Function.
- Wireless Drone → Surveillance.
- Artifical wrist → air typping
- AI.

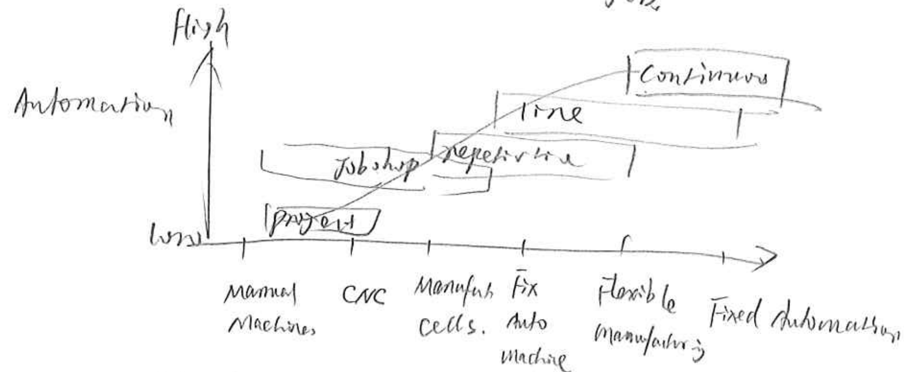
chip Gun Robot
Nortel: 15yr ago
eg 5 chips/second

Robot characteristics: Multifunctional. Reprogrammable, Repeatable, Fast & Accurate
Can Operate continuously

eg need robot to go from A → B. (or change pay load)
→ Inverse kinematic → plan a route.





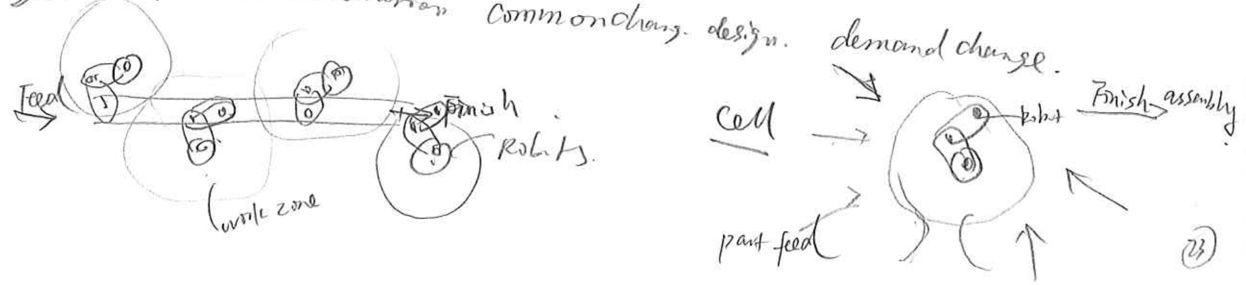


Hards vs. Soft Automation

Fixed (Hard): Few products / variation. rare design changes, Fixed demand. High Volume

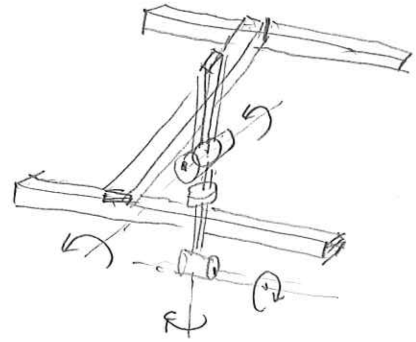
(Soft): several products & variation common change design. demand change.

Assembly line:

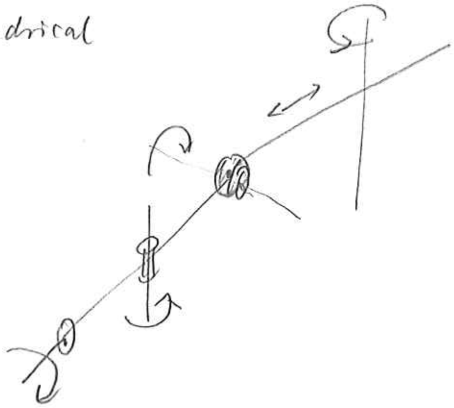


☆ Robots:

Geometry.

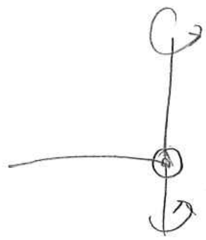


Cylindrical



sperical :

5 DOF



Jointed - spherical Geomery robot. 6 rotation DOF

power supply:

Gears + D C Motors → easy

- Hydraulic sys. (AC Motor + pumps + Hydraulic fluids (oil) + supply) → high power Ratio. → high payload
  └ out of the Robot body.
- Air driven : pneumatic Actuators & control valves
  └ cheap └ difficult

Hydraulic vs eletric Motor → advantage:

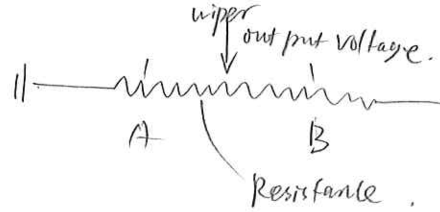
- higher payload + power
- high power/weight Ratio
- greater shock resistance.
- less backlash
- intrinsically safe
- self lubricating and cooling
- can operate in stall without damage.

Disadvantage: can leak oil, expensive, noisy, difficult control, remote power source.

A Robots.

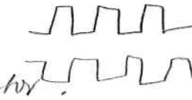
- Harmonic Drive Components. 100 : 1 Gear Ratio / outer box have one more gear than inner gear → Accuracy . eg Canadian Arm.

- potentiometer.



→ location & R.(Ω)

- position feedback.



Digital :

- Incremental Encoder. : Disk have holes, light goes through holes → light detector → Calculate Rotation speed & direction

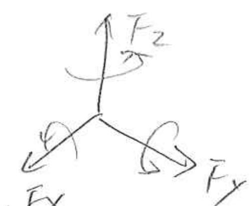
→ relative movement .

can have many segments to express different code chart (different part (joint))

Video Camera (videcon tube). VS Reverse process of TV.

- Front lighty
- Back lighty
- Structured lighting.

Force Sensing. : Sensor + tooling (Beams with strain gauges) 3 ~ 6 Axies (protected by overload pin)



compliance : (with feedback Control) Lateral, Rotational Axial — connecting links with end of pegs.

Tool :

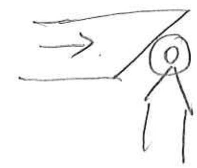
- Angular & parallel Grippers.

- Vacuum cups. → pick up sheets.

- pnematic Fingers

★ Robots.

limit switch. / stops — contact sensor.
(hit).



on/off.

Capacitance sensor. - Detect eletrostati field

Inductive sensor. : Eddy currents (in object sensed)

magnetic field

optical sensor

☆ Robots Review

eg. 6 joints Robot

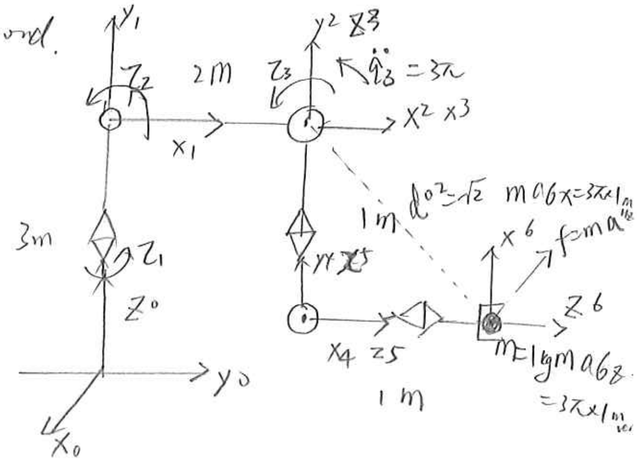
Robot moves from the pose shown

index # 1 2 3 4 5 6.

to $q = (\frac{\pi}{4} \quad 0, \frac{\pi}{2} \quad 0 \quad \frac{\pi}{2} \quad 0)$ radians in 1 second.

starting and ending with $\dot{q}(0) = \dot{q}(0) = 0$.

$q(t)$ is cubic.



Q: $\dot{q}_3(\frac{1}{2}) = ?$ $\quad q_A = 0, \quad q_B = \frac{\pi}{2}, \quad t_B = 1$

$q_3(t) = q_A + (q_B - q_A)\left[3\left(\frac{t}{t_B}\right)^2 - 2\left(\frac{t}{t_B}\right)^3\right]$

$= 0 + \frac{\pi}{2} \times (3t^2 - 2t^3)$

$\rightarrow \dot{q}_3(t) = \frac{6\pi}{2}(t - t^2)$ & $\ddot{q}_3(t) = 3\pi(1 - 2t)$

$\therefore \dot{q}_3(\frac{1}{2}) = \frac{6\pi}{2}\left[\frac{1}{2} - (\frac{1}{2})^2\right] = \frac{3\pi}{4}$ rad/s. $\quad$, $\quad \ddot{q}_3(0) = 3\pi \frac{rad}{s}$.

Torque :

$\tau_3(0) = ?$ $\quad (F \times L = M)$ Force length moment

$(\vec{\tau}_3 = d_{02} \times f. \rightarrow f = \frac{6\pi \cdot Nm}{\sqrt{2}\, m} = 3\sqrt{2}\, N.)$

$a_{6x} = 3\pi \frac{rad}{s} \times 1\, m_{Hz} = 3\pi\, m/s^2$

$a_{6z} = 3\pi \frac{rad}{s} \times 1\, m_{ver.} = 3\pi\, m/s^2$

$\tau_3 = m(a_{6x} \times 1m + a_{6z} \times 1m) = 1kg \times 6\pi\, m/s^2 = 6\pi\, N \cdot m.$

$\tau_2(0) = m \times a_{6x}\,(1m + 2m)_{Hz} + m\, a_{6z} \times 1m_{ver.}$

$= 1kg \cdot 3\pi \cdot 3m + 1kg \times 3\pi \times 1m = 12\pi\, N \cdot m.$

$\tau_1(1) = -\, m\, \underline{a_{6y}}(1) \times (2m + 1m)$

$q_1(0) = \frac{\pi}{2} \qquad q_3 = 0.$

★ Robots.

- Review

Q If $H_0^2 = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ & $H_1^2 = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\xrightarrow{\text{Find}}$ $H_0^1 = ?$

R_1^2, d_1^{12}

$$H_0^1 = H_0^2 \cdot H_2^1 = H_0^2 \cdot [H_1^2]^{-1}, \quad d_2^{21} = -d_2^{12} = -R_2^1 d_1^{12} = -(R_1^2)^T d_1^{12} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$$

$$\rightarrow H_2^1 = \left[\begin{array}{c|c} \underset{3\times 3}{(R_1^2)^T} & d_2^{21} \\ \hline 0 \;\; 0 \;\; 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$(R_1^2)^T$, d_2^{21}

Q The length of the run length code of B is: <u>8</u>

$$B = \begin{array}{c} \overset{k \downarrow \;\rightarrow j}{} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad I = (0, 2, 4 \cdots$$

μ_{11} — kj — $112.5°$ — $-67.5°$ — μ_{02}

Q which of the following curve fragments (Chain code) appear as a chain of 1 pixels:

$C = (6, 0, \boxed{0, 2, 2, 0}, 0\; 6)$ $\qquad 0, 2, 2, 0 \rightarrow$

Q $\quad m_{00} = \sum_R k^0 j^0 = A = 9$

$k_c = 2$

$j_c = 3$

$\underline{\mu_{02}} = \sum_R (k - k_c)^0 (j - j_c)^2 = 18.$ $\qquad \Big| \quad \mu_{20} = \sum_R (k - k_c)^2 (j - j_c)^0 = 6$

$\underline{\mu_{11}} = \sum_R (k - k_c)^1 (j - j_c)^1 = -6$

$\phi = \frac{1}{2} \operatorname{atan2}(2\mu_{11}, \mu_{20} - \mu_{02})$

$= \frac{1}{2} \operatorname{atan2}(-12, -12). = \frac{1}{2}(-135°) = -67.5$

or $\phi = \frac{1}{2}(225) = 112.5°$.