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Exercises in Numerics of Differential Equations

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Exercise 1. Show that RK-methods fit into the framework of multi-step methods. Furthermore, show that they satisfy the root condition.

Exercise 2. If we drop the assumption of uniform step-size, the coefficients β_j of a linear multistep method depend additionally on the step-sizes $h_{\ell+j} = t_{\ell+j+1} - t_{\ell+j}$ for $j = 0, \dots, m-1$. Compute the coefficients of a three-step Adams–Bashforth method with non-uniform step-size:

$$y_{\ell+3} - y_{\ell+2} = \sum_{j=0}^{2} \beta_j(h_{\ell+2}, h_{\ell+1}, h_\ell) f_{\ell+j}.$$

Hint. For uniform step-size the method reads

$$y_{\ell+3} - y_{\ell+2} = \frac{h}{12}\big(23 f_{\ell+2} - 16 f_{\ell+1} + 5 f_\ell\big). \tag{1}$$

Use this to verify your results.

Exercise 3. Consider the Milne–Simpson rules from the lecture. Show that they are linear, implicit multi-step methods. Furthermore, show that they satisfy the root-condition and have consistency order $p \geq m+1$ (i.e., in case of m odd, they reach the first Dahlquist barrier).
For the Milne-Simpson rule with $m = 2$,

$$y_{\ell+2} = y_\ell + \frac{4}{3}(f_{\ell+2} + 4 f_{\ell+1} + f_\ell),$$

show that it has consistency order $p = 4$.

Exercise 4. Consider an initial value problem

$$y'(t) = f(t, y(t)) \text{ in } [0, T], \quad y(0) = y_0 \in \mathbb{R}.$$

Implement a general solver for this kind of problem, based on linear explicit m-step methods, i.e.,

$$\sum_{j=0}^{m} \alpha_j y_{\ell+j} = \sum_{j=0}^{m} \beta_j f_{\ell+j},$$

with $\alpha_m = 1$ and $\beta_m = 0$. To this end, write a function `linearExplicitLMM` that takes as input the function f, a step-size h, the end-time T, initial values $y_0, \dots, y_{m-1}$, and the coefficient vectors $(\alpha_j)_{j=0}^m$ and $(\beta_j)_{j=0}^m$ of the explicit LMM. Your function should return the corresponding vector of approximations $y_\ell \approx y(t_\ell)$.
Solve the initial value problem to the function $y(t) = \sqrt{1+t^2}$,

$$y'(t) = \frac{t}{y(t)} \text{ in } [0, 1], \quad y(0) = 1 \in \mathbb{R},$$

with the method given in (1). For the initial steps, y_1 and y_2, use

1. the exact values $y_1 = \sqrt{1+h^2}$ and $y_2 = \sqrt{1+4h^2}$,
2. the values from the first two steps of the explicit Euler method,
3. the values from the first two steps of the modified Euler method (cf. Example 2.19 in the lecture notes).

Compare your numerical results at $t = 1$ with the exact solution $y(1) = \sqrt{2}$ for different step-sizes. What rates of convergence do you expect? What rates do you get?