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Exercises in Numerics of Differential Equations

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Exercise 1. Consider an m-stage Runge–Kutta method with Butcher tableau $\begin{array}{c|c} c & A \\ \hline & b^\top \end{array}$. In the lecture, we have formulated the integrator via increments k_j, i.e.,

$$y_{\ell+1} = y_\ell + h \sum_{j=1}^{m} b_j k_j, \quad \text{where} \quad k_i = f\Big(t_\ell + c_i h\,,\, y_\ell + h \sum_{j=1}^{m} A_{ij} k_j\Big) \quad \text{for all } i = 1, \dots m.$$

Equivalently, one can formulate Runge–Kutta methods with stages Y_j, i.e.,

$$y_{\ell+1} = y_\ell + h \sum_{j=1}^{m} b_j f(t_\ell + c_i h, Y_j), \quad \text{where} \quad Y_i = y_\ell + h \sum_{j=1}^{m} A_{ij} f(t_\ell + c_i h, Y_j). \tag{1}$$

Show that both approaches lead to the same method (under the usual assumptions on f).

Exercise 2. Consider an m-stage Runge–Kutta method with Butcher tableau $\begin{array}{c|c} c & A \\ \hline & b^\top \end{array}$ such that the coefficients satisfy

$$b_i A_{ij} + b_j Aji = b_i b_j \qquad \text{for all } 1 \le i, j \le m.$$

Show that the integrator preserves quadratic invariants of an autonomous ODE.

Hint. It suffices to consider invariants of the form $I(y) = y^\top C y$. Furthermore, use the form (1) for the RK-method.

Exercise 3. Consider the Hamiltonian $H(q,p) = \frac{1}{2}q^2 + \frac{1}{2}p^2$ and the corresponding system

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} \partial_p H(q,p) \\ -\partial_q H(q,p) \end{pmatrix} \text{ in } [0,T], \qquad \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2}$$

Argue that the solution is unique and compute the exact solution of (2). Moreover, consider the explicit Euler method, the implicit Euler method, and the implicit midpoint method, i.e.,

$$y_{\ell+1} = y_\ell + h f(y_\ell), \qquad y_{\ell+1} = y_\ell + h f(y_{\ell+1}), \qquad y_{\ell+1} = y_\ell + h f\Big(\frac{y_\ell + y_{\ell+1}}{2}\Big). \tag{3}$$

Show that the first two methods do not conserve the discrete energy, whereas the latter does. To this end, show that for the discrete energy there holds

$$H(q_\ell, p_\ell) - H(q_0, p_0) \begin{cases} \ge 0 & \text{for the explicit Euler method,} \\ \le 0 & \text{for the implicit Euler method,} \\ = 0 & \text{for the implicit midpoint method.} \end{cases}$$

Hint. For $a, b \in \mathbb{R}$, use the identity

$$\frac{1}{2}(a^2 - b^2) - \frac{1}{2}(a-b)^2 = (a-b)b.$$

Exercise 4. Use the methods from (3) to solve the system (2) numerically. For $h = 0.01$ and $T = 10$, plot the values q_ℓ, p_ℓ on the q-p plane. What do you expect? What do you observe? Furthermore, for varying step-sizes h, plot the energy differences and errors

$$\big|H(q_\ell, p_\ell) - H(q(T), p(T))\big|, \qquad \big\|(q_\ell, p_\ell)^\top - (q(T), p(T))^\top\big\|$$

at time $T = 10$ over the step-size h. What do you expect? What do you observe?