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Exercises in Numerics of Differential Equations

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Exercise 1. Consider an explicit s-step RK-method

$$\begin{array}{c|c} c & A \\ \hline & b^\top \end{array} \quad \text{with} \quad \sum_{j=1}^{i-1} a_{ij} = c_i \quad \text{for all } i = 1, \dots, s. \tag{1}$$

Let $k_i = k_i(t, y(t), h)$ be the i-th increment of the RK-method. Suppose that $f(t, y)$ is C^1 and Lipschitz continuous in y. Show that $y'(t + c_i h) - k_i = \mathcal{O}(h^2)$.

Exercise 2. Consider an explicit s-step RK-method with Butcher tableaux (1).

a) Assume that the vector b satisfies

$$\sum_{j=1}^{s-1} b_j = 1.$$

Show that the RK-method has at least consistency order $p = 1$.

b) Additionally assume that the matrix A and the vectors b, c satisfy

$$\sum_{j=1}^{s-1} b_j c_j = \frac{1}{2} \qquad \text{and} \qquad \sum_{j=1}^{i-1} a_{ij} = c_i \text{ for all } i = 1, \dots, s.$$

Show that the RK-method has at least consistency order $p = 2$.

Exercise 3. Let $y\colon [0, T] \to \mathbb{R}^n$ be the solution of the initial value problem

$$y'(t) = f(t, y), \quad y(0) = y_0.$$

Implement a general solver for this kind of problem, based on explicit Runge–Kutta methods. To this end, write a function `explicitRK` that takes as input the function f, a discretization $\Delta = (t_0, \dots, t_N)$ of the interval $[0, T]$, the initial value y_0, and the Butcher tableaux of the explicit RK-method. Your function should return the corresponding vector of approximations $y_\ell \approx y(t_\ell)$.

To validate your implementation, you might want to consider $f(t, y) = y$, $y_0 = 1$, and $y(t) = \exp(t)$, as well as $f(t, y) = (s + 1)t^s$, $y_0 = 0$, and $y(t) = t^{s+1}$ with $s \geq 0$.

Test your implementation with the Butcher tableaux of the classical RK4-method from the lecture on the following predator-prey problem:

$$\begin{aligned} y'(t) &= a_{11} y(t) - a_{12} y(t) z(t), \qquad & y(0) &= y_0, \\ z'(t) &= a_{21} y(t) z(t) - a_{22} z(t), \qquad & z(0) &= z_0. \end{aligned}$$

Use $a_{11} = 2$, $a_{22} = 1$, and $a_{12} = a_{21} = 0.01$, as well as $y_0 = 300$ and $z_0 = 150$. Use a uniform mesh with step-size $h = 0.01$ on $[0, 100]$. Vary the model parameters, as well as the step-size and observe the effects on the numerical results.

Exercise 4. Use the classical RK4-method from the lecture for solving

$$y'(t) = f(t) \text{ in } [0,1], \quad y(0) = 0, \qquad f(t) = \begin{cases} 0 & \text{for } t \le 1/2, \\ t - 1/2 & \text{for } t > 1/2. \end{cases}$$

Obviously, this problem has the exact solution

$$y(t) = \begin{cases} 0 & \text{for } t \le 1/2, \\ \frac{1}{2}(t - 1/2)^2 & \text{for } t > 1/2. \end{cases}$$

Plot the error of the approximate solution at $t = 1$ for meshes of $[0,1]$ with N equidistant nodes. What do you observe for $N = 2^n + 1$ with $n = 1, 2, \ldots$? What do you observe for $N = 2^n$ with $n = 1, 2, \ldots$? How can you explain these observations analytically?

Hint. You only need to analytically compute the error for one time step. Which one?