Sheet 4

Discussion of the sheet: Th., 11.04.2019

1. Let $A \in \mathbb{R}^{n\times n}$ be sparse, i.e., the number non-zero elements in A should be bounded by $C_A n$, where $C_A > 0$ is a constant. In this exercise, we want to give bounds for the computational effort (i.e., the number of arithmetic operations needed) of one step of the Jacobi, Gauß-Seidel and SOR-method. Assume that Nb has been precomputed and therefore does not influence the effort in each step.

a) Show that, for the Jacobi and Gauß-Seidel method, the effort can be bounded by

$$\text{effort(Jacobi)} \leq 2(C_A - 1)n, \qquad \text{effort(GS)} \leq 2(C_A - 1)n.$$

b) What is the minimal effort for the SOR-method?

2. We are now introducing the so called **ADI-method** (Alternating direction implicit method) Let S, T be SPD and $A = S + T$ as well as $\omega > 0$. Since solving $Ax = b$ is equivalent to solving

$$\omega x + Sx = b + (\omega - T)x \qquad \text{as well as} \qquad \omega x + Tx = b + (\omega - S)x,$$

we may also study the following iteration method

$$(\omega + S)x_{k+1/2} = b + (\omega - T)x_k, \qquad (\omega + T)x_{k+1} = b + (\omega - S)x_{k+1/2}.$$

a) Provide the iteration matrix M^{ADI} of the method.

b) Let C be SPD. Show that

$$\|(\omega - C)(\omega + C)^{-1}\|_2 = \max_{\lambda\in\sigma(C)} \left|\frac{\omega - \lambda}{\omega + \lambda}\right| < 1.$$

c) Compute the value $\omega_{opt} > 0$ in dependence of $\lambda_{min}(C)$, $\lambda_{max}(C)$ that minimizes $\omega \mapsto \|(\omega - C)(\omega + C)^{-1}\|_2$. What is the value of this minimum?

d) Show that

$$\begin{aligned}\rho((\omega + T)^{-1}(\omega - S)(\omega + S)^{-1}(\omega - T)) &= \rho((\omega - T)(\omega + T)^{-1}(\omega - S)(\omega + S)^{-1}) \\ &\leq \|(\omega - T)(\omega + T)^{-1}\|_2 \|(\omega - S)(\omega + S)^{-1}\|_2,\end{aligned}$$

and use that to conclude that the ADI-method converges for any $\omega > 0$.

3. (ADI for the model problem) The discretization of our 2D-model problem with zero boundary conditions (example 2.2 in the lecture notes) leads to the equations

$$\frac{2u_{ij} - u_{i+1,j} - u_{i-1,j}}{h^2} + \frac{2u_{ij} - u_{i,j+1} - u_{i,j-1}}{h^2} = f_{ij}, \qquad i, j = 1, \ldots, n.$$

With lexicographic ordering this leads to a SPD matrix $A \in \mathbb{R}^{N\times N}$ with $N = n^2$ (compare figure 2.1. in the lecture notes)

We define the splitting $A = S + T$, where S and T are matrices corresponding to

$$\frac{2u_{ij} - u_{i+1,j} - u_{i-1,j}}{h^2} \text{ and } \frac{2u_{ij} - u_{i,j+1} - u_{i,j-1}}{h^2}.$$

The ADI-method is then defined as in exercise 2 for the splitting $A = S + T$.

a) Verify that the matrix S is a block-diagonal matrix $S = \mathrm{diag}(S_{11}, \dots, S_{nn})$ with identical blocks S_{ii}. These matrices are (up to scaling with h) are the same as for the 1D- model problem (example 2.1. in the lecture notes in the x-variable). Therefore, we have

$$S_{ii} = h^{-2}\texttt{diag}([-e, 2 * e, -e], [-1, 0, 1]) \quad \text{with} \quad e = (1, 1, \dots, 1)^\top \in \mathbb{R}^n.$$

Notice that the matrix T corresponds (up to scaling with h) to the 1D-model problem in the y Variable. Moreover, T can be written as $T = P^\top SP$ with a permutation P.

b) Compute the damping parameter ω_{opt} such that $\rho(S)$ is minimal. Estimate $\rho(M^{ADI})$ in dependence of h, if ω_{opt} is employed. Is the ADI-method better than the Jacobi-method?

c) Use (a) to implement the ADI-method for the 2D-model problem. Test your implementation for $f = 0$ and $x_0 = \texttt{ones}(N, 1)$ and plot the error over number of iterations semilogarithmically. Use ω_{opt} from (b) and different choices of h.