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Exercises in Numerics of Differential Equations

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Exercise 1. An initial value problem is called *autonomous* if its right-hand side does not explicitly depend on time, i.e.,

$$Y'(t) = F(Y(t)), \quad Y(t_0) = Y_0. \tag{1}$$

Every initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0 \tag{2}$$

can be formulated as autonomous problem by means of $Y(t) := (t, y(t))^\top$, $Y_0 := (t_0, y_0)^\top$, and $F(Y) := (1, f(Y))^\top$. Let f be arbitrary and sufficiently smooth. Show that an explicit m-stage Runge–Kutta method applied to (1) gives the same result as if applied to (2) if and only if

$$c_i = \sum_{j=1}^{i-1} a_{ij}, \quad \text{for all } i = 1, \dots, m.$$

Without further computations, which collocation methods give the same result when applied to (1) as if applied to (2)?

Exercise 2. Consider an initial value problem of the form

$$y'(t) = My(t) + g(t), \quad y(t_0) = y_0, \tag{3}$$

with a function $g\colon \mathbb{R} \to \mathbb{R}^n$ and a diagonalizable matrix $A \in \mathbb{R}^{n\times n}$, i.e., there exists a regular matrix $V \in \mathbb{R}^{n\times n}$ such that $V^{-1}MV = \operatorname{diag}(\lambda_1, \dots, \lambda_n) =: \Lambda$. Furthermore, consider the Problem

$$z'(t) = \Lambda z(t) + V^{-1}g(t), \quad z(t_0) = V^{-1}y_0. \tag{4}$$

Let y_ℓ and z_ℓ be the numerical approximations obtained by applying an m-stage RK-method to (3) and (4), respectively. Show that $y_\ell = Vz_\ell$, i.e., the RK-method is invariant under linear transformations.

Hint. If you obtain some identities for the stages of the RK-method, formulate them as implicit system of extended vectors and matrices. Moreover, assume that h is sufficiently small to solve this system.

Exercise 3. For $1 \geq \varepsilon > 0$ consider the matrix

$$M_\varepsilon = -\frac{1}{\varepsilon}\begin{pmatrix} 1 & 1-\varepsilon \\ 1-\varepsilon & 1 \end{pmatrix}.$$

Compute the eigenvalues λ_1, λ_2 and corresponding normalized eigenvectors $v_1, v_2 \in \mathbb{R}^2$ (i.e., $\|v_j\| = 1$) of M_ε. Furthermore, (analytically) compute the general solution of

$$y'(t) = M_\varepsilon y(t), \quad y(0) = y_0. \tag{5}$$

How do $y^{(j)} := y \cdot v_j$ look like? What do you expect for numerical approximations to these solutions?

Exercise 4. Solve (5) numerically on $[0,1]$ by employing the RK4-method for $\varepsilon = 10^{-j}$, $j = 0, \ldots, 6$. For every value of ε plot the errors $\|y_N - y(1)\|$, $\|y_N^{(1)} - y^{(1)}(1)\|$, and $\|y_N^{(2)} - y^{(2)}(1)\|$ over step-size h on suitably scaled axes, where $y^{(j)} := y \cdot v_j$ and $y_N^{(j)} := y_N \cdot v_j$.
From these plots infer values of ε and h such that:

1. All three errors are large.
2. At least one but not all errors are large.
3. All three errors are small.

Plot the components $y_\ell^{(1)}$ and $y_\ell^{(2)}$ of the numerical solution for these values and explain your observations.