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Summer Term 2019

Exercises in Numerics of Differential Equations

20^{th} / 22^{nd} March 2019

Exercise 1. Consider the initial value problem

$$y'(t) = \lambda y(t), \quad t \in \mathbb{R}_{\geq 0}, \quad y(0) = 1. \tag{1}$$

a) Solve problem (1) using the Picard iterates from the proof of the Picard-Lindelöff–Theorem.

b) Let $\lambda < 0$ and y be the exact solution of (1). Let further $h > 0$ be a constant step-size and $t_\ell := \ell h$ for $\ell \in \mathbb{N}_0$. Compute the approximations y_ℓ^e and y_ℓ^i to $y(t_\ell)$ from the explicit and implicit Euler method, respectively. Investigate the behaviour of y_ℓ^e and y_ℓ^i for $\ell \to \infty$ and compare this to the exact solution $y(t_\ell)$.

c) Solve problem (1) approximatively on the interval $[0, 1]$ by implementing the explicit Euler method in a programming language of your choice. Use an equidistant mesh of $[0, 1]$. Investigate the error to the exact solution at time $t = 1$ dependent on the number of time steps.

Exercise 2. Let $\Delta = \{a = x_0 < \cdots < x_n = b\}$ be a mesh with (local) step-size $h_\ell = x_{\ell+1} - x_\ell$. For the numerical solution of the initial value problem

$$y(a) = y_0, \quad y' = f(x, y) \quad \text{in } [a, b] \tag{2}$$

an *implicit one-step method* shall be employed, i.e.

$$y_{\ell+1} = y_\ell + h_\ell \phi(x_\ell, y_\ell, y_{\ell+1}, h_\ell) \tag{3}$$

for $\ell = 0, \ldots, n-1$. Compared to explicit methods, the function ϕ additionally depends on $y_{\ell+1}$. We suppose that ϕ is Lipschitz-continuous in the 3^{rd} argument. Show that for sufficiently small h_ℓ there exists a unique solution $y_{\ell+1}$ to the implicit equation (3).

Exercise 3. An implicit one-step method has consistency order $p \geq 1$, if for all $f \in C^p([a, b] \times \mathbb{R}^n, \mathbb{R}^n)$ and hence $y \in C^{p+1}([a, b], \mathbb{R}^n)$ for the solution of (2), it holds that

$$\|y(x + h) - \{y(x) + h\phi(x, y(x), y(x + h), h)\}\| = \mathcal{O}(h^{p+1}). \tag{4}$$

The involved constant can depend on f and y, but must not depend on x or h. We consider the *(implicit) trapezoidal rule* (3) with

$$\phi(x, y, z, h) = \frac{1}{2}\big(f(x, y) + f(x + h, z)\big),$$

where $f(x, y)$ is Lipschitz-continuous in y. According to Exercise 2, this method is well-defined for sufficiently small step-sizes h_ℓ. Show that the trapezoidal rule has consistency order 2.

Exercise 4. Consider an implicit one-step method as given in (3). Let this method be stable, i.e., there exists a constant $L > 0$ such that

$$\|\phi(t, y, z, h) - \phi(t, \tilde{y}, \tilde{z}, h)\| \leq L\big(\|y - \tilde{y}\| + \|z - \tilde{z}\|\big) \quad \text{for all } t \in [t_0, T], h > 0, \text{ and } y, \tilde{y}, z, \tilde{z} \in \mathbb{R}^n.$$

Furthermore, let the method have consistency order $p \geq 1$ and let $y \in C^{p+1}([t_0, T], \mathbb{R}^n)$ be the exact solution to (2). Under these assumptions, show that there exists $C > 0$ such that

$$\max_{\ell=1,\dots N} \|y_\ell - y(t_\ell)\| \leq C h_\Delta^p,$$

where $h_\Delta = \max_{\ell=1,\dots N} h_\ell$.

***Note**: Please prepare the exercises before the exercise class. You may be chosen to present any exercise you have prepared (you can state which of the exercises you prepared at the beginning of class). In case of programming exercises, make sure that either you or one of your colleagues brings a laptop on which you can present them.*