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Exercises in Numerics of Differential Equations

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Exercise 1. On the time-interval $[t_0, T]$ consider a general m-step method of the form

$$\sum_{j=0}^{m} \alpha_j y_{\ell+j} = h\Phi(t_\ell, y_\ell, \dots, y_{\ell+m}, h). \tag{1}$$

Suppose that Φ is Lipschitz continuous in y, i.e. for all $t_\ell \in [t_0, T], h > 0$ and $y_{\ell+j}, \widetilde{y}_{\ell+j} \in \mathbb{R}^n$ it holds that

$$\|\Phi(t_\ell, y_\ell, \dots, y_{\ell+m}, h) - \Phi(t_\ell, \widetilde{y}_\ell, \dots, \widetilde{y}_{\ell+m}, h)\| \le L \sum_{j=0}^{m} \|y_{\ell+j} - \widetilde{y}_{\ell+j}\|.$$

Show that for sufficiently small $h > 0$ and arbitrary $y_\ell, \dots, y_{\ell+m-1} \in \mathbb{R}^n$, the equation (1) has a unique solution $y_{\ell+m} \in \mathbb{R}^n$.

Exercise 2. For a general m-step method as given in (1) we define the polynomial

$$q(\lambda) := \sum_{j=0}^{m} \alpha_j \lambda^j.$$

Show that the Adams–Bashforth and Adams–Moulton methods from the lecture satisfy the root condition from the convergence theorem, i.e., all solutions λ_i of $q(\lambda) = 0$ satisfy $|\lambda_i| \le 1$ and all λ_i with $|\lambda_i| = 1$ are simple roots.
Furthermore, show that $\lambda = 1$ is a root of $q(\lambda)$, if the m-step method has consistency order $p \ge 1$.

Exercise 3. Consider the so-called *backwards differentiation formulas (BDF-methods)* for the approximate solution of $y'(t) = f(t, y(t))$: Let $m \in \mathbb{N}$. Suppose we already have computed approximations $y_\ell \approx y(t_\ell)$ at points t_ℓ for $\ell = 0, \dots, m-1$. We construct a polynomial $q(t) \in \mathbb{P}_m$ approximating the exact solution by Lagrange interpolation in the $m+1$ points (t_j, y_j) for $j = 0, \dots, m$, i.e.

$$q(t) = \sum_{j=0}^{m} y_j L_j(t),$$

where the L_j are the Lagrange basis functions. The value y_m is the sought approximation. To find this value, we additionally ask for

$$q'(t_m) = f(t_m, y_m).$$

Show that this defines a linear m-step method is given, i.e., provide formulas for the coefficients α_j and the incremental function Φ. Compute the coefficients of the BDF-methods for $m = 1, 2, 3$.

Exercise 4. Consider a general explicit linear two-step method of the form

$$y_{\ell+2} + \alpha_1 y_{\ell+1} + \alpha_0 y_\ell = h\big[\beta_1 f(t_{\ell+1}, y_{\ell+1}) + \beta_0 f(t_\ell, y_\ell)\big]. \tag{2}$$

Choose the parameters α_0, α_1 and β_0, β_1 such that the consistency order of (2) is maximal. Implement method (2) to approximate the solution of

$$y' = -y \quad \text{on } [0,1], \quad y(0) = 1,$$

where you can set $y_1 = y(h) = \exp(-h)$. Vary the step-size $h = 2^{-1}, 2^{-2}, \ldots$. Does this method converge?