Sheet 7

Discussion of the sheet: Th., 23.05.2019

1. (Excercise 11.1) Show that Algorithm 11.1 (Lanczos biorthogonalization) generates bi-orthogonal sequences $(v_1, \dots, v_m)$ and $(w_1, \dots, w_m)$. In particular, we have for $j = 1, \dots, m$:

$$0 = (v_{j+1}, w_1) = \cdots = (v_{j+1}, w_j), \qquad \text{and} \qquad 0 = (w_{j+1}, v_1) = \cdots = (w_{j+1}, v_j).$$

(Hint: Proof by induction, exploiting the three term recurrences for v_j and w_j.)

2. We compare the different Krylov space methods.

a) Implement the steepest descent method (Algorithm 7.1).

b) Matlab provides efficient implementations of the CG and GMRES method (see *help pcg* and *help gmres*). Compare the convergence rates for our 2D-model problem (in MATLAB provided by gallery(poisson,N)) for the following methods: steepest descent, CG and GMRES. Plot the size of the residual in the 2-norm. Which method performs best?

c) For moderate N, we can compute the exact solution x_* via $x_* = A \setminus b$. Compare the true error in the energy norm for CG and GMRES for a fixed number of iterations. Which method is better in this metric? Why?

3. For $n = 1000$, we look at the $n \times n$ matrix given by

$$A := \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{1}$$

a) Apply the GMRES method to solve $Ax = b$ and plot the convergence of the residual. What do you observe?

b) How does the computational cost for reaching good accuracy (say $\|Ax - b\|_2 < 10^{-5}$) compare to a direct solver?

4. We look at the condition number / speed of convergence for the 1D model problem from Sheet 3, Exercise 3 b).

a) What do you expect for the condition number of A? Derive a bound of the form $\kappa(A) \leq Cn^\alpha$ for some parameter $\alpha \in \mathbb{R}$ and constant $C > 0$ independent of n.
(Hint: For small x, one can approximate $\cos(x) \approx 1 + \frac{x^2}{2}$)

b) Apply the CG method to this matrix with the right-hand side $b := (1, \dots, 1)^T$. How fast does the method converge? Does it match the prediction of Theorem 8.2?

c) Plot the convergence of the relative residuals $\|b - Ax_m\|/\|b\|$ for different choices of n, e.g. $n = 1024, 2048, 4096, 8192$. How does the number of iterations needed to reach a certain accuracy change (for example to get a relative error of 10%) as you increase n?