Sheet 6

Discussion of the sheet: Th., 16.05.2019

1. (Excercise 10.1)

a) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix, and $x_0 \in \mathbb{R}^n$ an arbitrary starting value. Show that the approximation x_n obtained by the GMRES minimization criterion

$$\|b - A\,x_n\|_2 = \min_{x\in x_0+\mathcal{K}_n} \|b - A\,x\|_2$$

is the exact solution x_* of $A\,x = b$.

b) Assume additionally that for some $m \le n$ there holds $\mathcal{K}_m = \mathcal{K}_n$. Show that then already

$$x_m = x_{m+1} = \cdots = x_*$$

Hint: Show that $\mathcal{K}_k = \mathcal{K}_m = \mathcal{K}_n$ also for all $k > m$.

2. Let A be of the form

$$A = \begin{pmatrix} I_{d\times d} & Y_1 & & & & \\ & I_{d\times d} & Y_2 & & & \\ & & \ddots & \ddots & & \\ & & & I_{d\times d} & Y_{k-1} & \\ & & & & I_{d\times d} & Y_k \\ & & & & & I_{d\times d} \end{pmatrix}$$

with sub-matrices $Y_1, \dots Y_k \in \mathbb{R}^{d\times d}$, $d \in \mathbb{N}$ and $I_{d\times d}$ is the $d \times d$ identity matrix.

Show that $(I - A)^k = 0$. How many iterations does the GMRES method take (at most) to converge?

3. Let A be a positve definite (not necessarily symmetric) matrix, i.e. assume that there exists $\gamma > 0$ such that $(A\,x, x) \ge \gamma\|x\|_2^2$ for all $x \in \mathbb{R}^n$. Show that the 'restarted' GMRES(m) converges for any $m \ge 1$.

(Hint: see equation (10.15) from the lecture/lecture notes)

4. Preconditioning means that one applies an iterative scheme to the system $W^{-1}A\,x = W^{-1}\,b$ (details will be given in the lecture), where $W \approx A$ is chosen such that W^{-1} can be computed cheaply.

Show, if A, W are SPD and

$$a\,W \le A \le b\,W,$$

then

$$\kappa_\sigma(W^{-1}A) \le \frac{b}{a}$$

where $\kappa_\sigma(M) = \lambda_{max}(M)/\lambda_{min}(M)$ is the spectral conditioning of a matrix M with positive spectrum.

Also show that

$$\kappa_\sigma(W^{-1}A) = \kappa_A(W^{-1}A).$$

Here $\kappa_A(M) := \|M\|_A\|M^{-1}\|_A$. (Hint: see also Excercise 1 of Sheet 2)