Sheet 8

Discussion of the sheet: Th., 06.06.2019

1. (Exercise 12.1) Let $A, M \in \mathbb{R}^{n\times n}$ be SPD. Define the M-inner product $(\cdot,\cdot)_M$ by $(x,y)_M = (Mx, y) = x^{\mathrm{T}} M\, y$. Show:

a) $M^{-1}A$ is selfadjoint with respect to the inner product $(\cdot,\cdot)_M$, i.e., $(M^{-1}A\, x, y)_M = (x, M^{-1}A\, y)_M$ for all $x, y \in \mathbb{R}^n$.

b) $M^{-1}A$ is positive definite with respect to $(\cdot,\cdot)_M$, i.e., $(M^{-1}A\, x, x)_M > 0$ for all $0 \neq x \in \mathbb{R}^n$.

c) The matrix AM^{-1} is symmetric positive definite with respect to the M^{-1}-inner product $(\cdot,\cdot)_{M^{-1}}$.

2. (Exercise 12.3) Assume that the Cholesky decomposition $M = L\,L^T$ of an SPD preconditioner M is available. Consider the split-preconditioned system with $M_L = L$ and $M_R = L^T$, i.e., applying the CG algorithm to the SPD system $L^{-1}A\,L^{-T}u = L^{-1}\,b$.

Show: The iterates $x_m = L^{-T}u_m$ coincide with those of the left-preconditioned CG method, i.e., Alg. 12.2 with preconditioner M.

(Note that the same argument holds for any SPD preconditioner M implicitly specified by a regular matrix C with $M = C\,C^T$.)

3. (Exercise 12.4)

a) Show: The speed of convergence of the PCG iteration can be estimated by

$$\|e_m\|_A \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^m \|e_0\|_A, \quad \kappa = \kappa_\sigma(M^{-1}A) = \frac{\lambda_{max}}{\lambda_{min}}$$

where λ_{max} and λ_{min} are the largest and smallest eigenvalues of $M^{-1}A$, and $\kappa_\sigma(M^{-1}A) = \lambda_{max}/\lambda_{min}$ is the spectral condition number (in general, this is not identical with $\kappa_2(M^{-1}A)$).

b) Show that a characterization of λ_{min} and λ_{max} which may be easier to check is the following: λ_{min} is the largest and λ_{max} is the smallest number such that

$$\lambda_{min}\, M \leq A \leq \lambda_{max}\, M$$

is valid.

4. (Exercise 12.5) Let $X \in \mathbb{R}^{n\times n}$ and $\varphi(X) := \|AX - I\|_F^2$. Expand $\varphi(X+H)$ to conclude

$$\varphi(X+H) = \varphi(X) - 2\,(A^{\mathrm{T}}R, H)_F + \|A\,H\|_F^2 \quad \text{with the residual matrix } R = I - A\,X.$$

Note that this expansion is exact because $\varphi(X)$ is a quadratic functional.