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Exercises in Numerics of Differential Equations

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Exercise 1. Consider quadrature rules of Newton–Côtes type. For open and closed Newton–Côtes formulae, the quadrature points in the interval $[0, 1]$ are

$$x_\ell = \frac{k}{n} \quad \text{and} \quad x_\ell = \frac{k+1}{n+2} \quad \text{for } k = 0, \dots, n,$$

respectively. Compute the Butcher tableaux of the 3-stage Runge–Kutta methods, where the collocation points are given by the quadrature points of the open and closed Netwon–Côtes formula (for suitable n).

Exercise 2. Consider an m-stage RK-method which is generated by collocation. Let the collocation points $c_1, \dots, c_m$ be pairwise distinct. Show that the Matrix A from the Butcher tableau is invertible if and only if the collocation points are non-vanishing, i.e., $\prod_{i=1}^m c_i \neq 0$.

Hint. Use the result of plugging $q(t) = t^{k-1}$ for $k = 1, \dots, m$ into the formula

$$\sum_{j=1}^m a_{ij} q(c_j) = \int_0^{c_i} q(t) \, \mathrm{d}t \quad \text{for all} \quad i = 1, \dots, m,$$

which was proved in the lecture. Recall the regularity of the Vandermonde matrix $V = \big(c_i^{j-1}\big)_{i,j=1}^s$.

Exercise 3. Consider a differential equation $y'(t) = f(t, y(t))$ which is non-expansive, i.e., there exists a constant $L^- \leq 0$ such that

$$\big(z - \widetilde{z}\big) \cdot \big(f(t, z) - f(t, \widetilde{z})\big) \leq L^- \|z - \widetilde{z}\| \quad \text{for all} \quad t \in [t_0, T], z, \widetilde{z} \in \mathbb{R}^n.$$

Let further be y_ℓ and $\widetilde{y}_\ell$ be the vectors of approximations obtained by an m-stage RK-method which was generated by collocation from Gauss quadrature for initial values y_0 and $\widetilde{y}_0$, respectively. Show that

$$\|y_1 - \widetilde{y}_1\| \leq \|y_0 - \widetilde{y}_0\|$$

Hint. Let $q, \widetilde{q}$ be the collocation polynomials for the initial values $y_0, \widetilde{y}_0$, respectively. For collocation methods, we know that $y_1 = q(t_0 + h)$ and $\widetilde{y}_1 = \widetilde{q}(t_0 + h)$. Let $u(\theta) = \|q(t_0 + \theta h) - \widetilde{q}(t_0 + \theta h)\|^2$ and show that

$$\|y_1 - \widetilde{y}_1\|^2 = \|y_0 - \widetilde{y}_0\|^2 + \int_0^1 u'(\theta) \, \mathrm{d}\theta.$$

Use Gauss quadrature for the integral and apply the non-expansiveness of the differential equation.

Exercise 4. Consider a linear (scalar) initial value problem

$$y'(t) = M(t)y(t), \quad y(0) = y_0$$

with a time-dependent scalar function $M(t)$. Implement a general solver for this kind of problem, based on collocation methods. To this end, write a function `linearCollocation` that takes as input the function $t \mapsto M(t)$, a discretization $\Delta = (t_0, \dots, t_N)$ of the interval $[0, T]$, the initial value y_0, and the pairwise distinct collocation points $c_1, \dots, c_m \in [0, 1]$. Your function should return the corresponding vector of approximations $y_\ell \approx y(t_\ell)$.

One way to solve this exercise would be to generate the Butcher tableau from the collocation points, as was done in Exercise 1. You should, however, compute the solution from the collocation conditions directly. To this end, make the ansatz

$$q(t) = \sum_{j=0}^{m} w_j t^j \quad \text{with weights} \quad w_j \in \mathbb{R}.$$

Set $W := (w_0, \dots, w_m)^\top \in \mathbb{R}^m$ and use the collocation conditions (with $c_0 = 0$ and $0^0 = 1$) to derive a linear system of equations for W.

To validate your implementation, you might want to consider $M(t) = nt^{n-1}$, $y_0 \in \mathbb{R}$, and $y(t) = y_0 \exp(t^n)$.