Sheet 1

Discussion of the sheet: Th., 14.03.2019

1. Define the column based sparse matrix format *Compressed Sparse Column* (CSC) analogously to the lecture. Formulate an algorithm that realizes the matrix-vector-multiplication of a sparse matrix $A \in \mathbb{R}^{n \times n}$ stored in the CSC-Format with a vector $x \in \mathbb{R}^n$.

2. Let $A \in \mathbb{R}^{n \times n}$ SPD and C be its Cholesky-factor, i.e., $A = CC^T$. Show that $\mathrm{Env}(C) \subset \mathrm{Env}(A)$.

3. Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times m}$. Show for the spectra of the matrices AB and BA the following relation:

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}.$$

Show that this relation implies $\rho(AB) = \rho(BA)$.

4. (Formula of Sherman-Morrison-Woodbury: low-rank perturbations of invertible matrices) Let $A \in \mathbb{R}^{n \times n}$ be invertible and u, $v \in \mathbb{R}^n$.

a) If $v^\top A^{-1} u \neq -1$, show that

$$(A + uv^\top)^{-1} = A^{-1} - \frac{1}{1 + v^\top A^{-1} u} A^{-1} u v^\top A^{-1}.$$

b) If $v^\top A^{-1} u = -1$, show that $A + uv^\top$ is not invertible (*hint:* find a vector $z \neq 0$ with $(A + uv^\top)z = 0$).

c) The formula for M^{-1}, where $M = A + uv^\top$, can also be obtained as follows: Make the ansatz $M^{-1} = A^{-1} + \alpha A^{-1} u v^\top A^{-1}$ and compute the correct α.

Let M be a rank-r-perturbation of A, i.e.,

$$M = A + \sum_{i=1}^{r} u_i v_i^\top$$

with vectors u_i, $v_i \in \mathbb{R}^n$. For M^{-1} make the ansatz

$$M^{-1} = A^{-1} + \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_{ij} A^{-1} u_i v_j^\top A^{-1}$$

and provide a linear system of equations for the coefficients α_{ij}.