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Exercises in Numerics of Differential Equations

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Exercise 1. Consider an implicit m-stage Runge–Kutta method with Butcher tableaux and increment function

$$\begin{array}{c|c} c & A \\ \hline & b^\top \end{array} \quad \text{and} \quad \phi(t,y,h) = \sum_{j=1}^{m} b_j k_j, \tag{1}$$

respectively. Let further be $f(t,y)$ Lipschitz continuous in the second argument and $H > 0$ sufficiently small. Under these assumptions, show that the RK-method is stable, i.e., there exists a constant $C > 0$ such that for all $t \in [0,T]$, $h \in (0,H)$ and $y, \tilde{y} \in \mathbb{R}^n$ there holds

$$\|\phi(t,y,h) - \phi(t,\tilde{y},h)\| \le C\|y - \tilde{y}\|.$$

Exercise 2. Consider a Runge–Kutta method of consistency order $p \ge 1$ on a mesh $\Delta = (t_0, \dots, t_N)$. Under the assumptions of the convergence theorem (in particular, f is Lipschitz continuous in the second argument), we obtain a vector of approximations $y_\ell \approx y(t_\ell)$ satisfying

$$\max_{\ell=0,\dots,N} \|y(t_\ell) - y_\ell\| = \mathcal{O}(h_\Delta^p).$$

a) Let y_h be the interpolating linear spline (i.e., y_h is a polynomial of degree 1 for every interval $[t_\ell, t_{\ell+1}]$) given by $y_h(t_\ell) = y_\ell$ for all $\ell = 0, \dots, N$. Show that

$$\|y - y_h\|_\infty = \mathcal{O}\big(h_\Delta^{\min\{2,p\}}\big).$$

b) Let y_h be the interpolating cubic spline (i.e., y_h is a polynomial of degree 3 for every interval $[t_\ell, t_{\ell+1}]$) given by piecewise Hermite-Interpolation

$$y_h(t_\ell) = y_\ell, \quad y_h'(t_\ell) = f(t_\ell, y_\ell)$$

for all $\ell = 0, \dots, N$. Which convergence order do you expect for $\|y - y_h\|_\infty$?

Hint. Use the Δ-piecewise Lagrange, or Hermite interpolant $\tilde{y}(t)$, which approximates the exact solution $y(t)$, in a suitable manner. For b) show further, $|||q||| := |q(t_\ell)| + |q(t_{\ell+1})| + |q'(t_\ell)| + |q'(t_{\ell+1})|$ is a norm on the space of polynomials of degree ≤ 3 on the interval $[t_\ell, t_{\ell+1}]$ and equivalent to $\|\cdot\|_\infty$.

Exercise 3. Consider a linear initial value Problem

$$y'(t) = My(t), \quad y(0) = y_0 \tag{2}$$

with a matrix $M \in \mathbb{R}^{n\times n}$. Implement a general solver for this kind of problem, based on implicit Runge–Kutta methods as given in (1). To this end, write a function `linearImplicitRK` that takes as input the Matrix M, a discretization $\Delta = (t_0, \dots, t_N)$ of the interval $[0, T]$, the initial value y_0, and the Butcher tableaux of the implicit RK-method. Your function should return the corresponding vector of approximations $y_\ell \approx y(t_\ell)$.
To validate your implementation, you might want to consider $M = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, $y_0 = (d_1, \dots, d_n)^\top$, and $y(t) = (d_1 \exp(\lambda_1 t), \dots, d_n \exp(\lambda_n t))^\top$.

Hint. To get an explicit representation of the stages k_i, write them as $K := (k_1^\top, \dots, k_m^\top)^\top \in \mathbb{R}^{nm}$. Now formulate the implicit formula for the stages,

$$k_i = f\Big(t + c_i h_\ell, y_\ell + h_\ell \sum_{j=1}^{m} A_{ij} k_j\Big),$$

as implicit equation for the vector K using matrix-vector multiplication with suitable matrices in $\mathbb{R}^{nm\times nm}$.

Exercise 4. Implement the embedded Runge–Kutta method of Bogacki and Shampine. This is a scheme for adaptive time-stepping as given in the lecture (see lecture notes chapter 2, page 20) with two Runge–Kutta methods of order 2 and 3, respectively. They have the Butcher tableaux

$$\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
3/4 & 0 & 3/4 & & \\
1 & 2/9 & 1/3 & 4/9 & \\
\hline
 & 2/9 & 1/3 & 4/9 & 0 \\
 & 7/24 & 1/4 & 1/3 & 1/8
\end{array}$$

where the first b-row gives the method of order 3 and the second the method of order 2.
With this method solve the initial value problem

$$y'(t) = -200ty^2(t), \quad y(0) = 1, \quad y(t) = \frac{1}{1 + 100t^2}.$$

For different tolerances τ, plot the solution and the vector of used step-sizes. Finally, plot the error at $t = 1$ over the tolerance τ.