Sheet 2

Discussion of the sheet: Th., 21.03.2019

1. Let $A, N \in \mathbb{R}^{n\times n}$ be SPD and consider the iteration $x_{k+1} = x_k + N(b - A\,x_k)$. Let $M = I - W^{-1}A \quad (W = N^{-1})$ be the iteration matrix. Assume that W is SPD and even satisfies

$$2\,W > A > 0. \tag{1}$$

a) (Auxiliary identity) Show: The adjoint M^A of a matrix M with respect to the $(\cdot,\cdot)_A$ inner product, i.e.,

$$(Mx, x)_A := (AMx, x)_2 = (Ax, M^A x)_2 = (x, M^A x)_A \tag{2}$$

is given by

$$M^A = A^{-1}M^T A. \tag{3}$$

(Note that M is A-selfadjoint iff $M^T A = A\,M \iff A^{-\frac{1}{2}}M^T A^{\frac{1}{2}} = A^{\frac{1}{2}}\,MA^{-\frac{1}{2}}$, i.e., if $\hat{M} = A^{\frac{1}{2}}\,MA^{-\frac{1}{2}}$ is symmetric.)

b) Show: $\rho(M) = \|M\|_A < 1$, where $\|\cdot\|_A$ is the norm induced by the vector norm $\|x\|_A := (x,x)_A$.

c) Show: If for some $0 < \lambda \leq \Lambda$ there holds

$$0 < \lambda\,W \leq A \leq \Lambda\,W$$

then $\sigma(M) \subset [1-\Lambda, 1-\lambda]$, and thus,

$$\rho(M) \leq \max\{|1-\lambda|, |\Lambda - 1|\}.$$

2. Let $A \in \mathbb{R}^{n\times n}$ be SPD.

a) Let $\omega \in (0,1]$. Use exercise 1 to show that the damped SSOR method (equation (5.14b) in the lecture notes) converges, i.e., show $\rho(M_\omega^{SSOR}) < 1$.

b) Show that exercise 1a) can be weakened in the following way: We do not require that N, W are symmetric, but we only assume that W is positive definite, i.e., $W + W^T > 0$, and replace condition (1) by

$$W + W^T > A > 0.$$

Then, $\rho(M) \leq \|M\|_A < 1$.
Hint: Use (3) and express $N + N^T$ by means of $W + W^T$.

3. Let A be SPD and denote by $M_\omega^{SOR} = I - \omega\,(D + \omega L)^{-1}A$ the iteration matrix of the (forward) SOR method and by $\bar{M}_\omega^{SOR} = I - \omega\,(D + \omega L^T)^{-1}A$ the iteration matrix of the backward SOR method. The iteration matrix $M_\omega^{SSOR} = \bar{M}_\omega^{SOR}\,M_\omega^{SOR}$ of the damped SSOR method is given by

$$M_\omega^{SSOR} = I - \omega(2-\omega)(D + \omega L^T)^{-1} D\,(D + \omega L)^{-1} A$$

a) Show: $\bar{M}_\omega^{SOR}$ is the adjoint of M_ω^{SOR} with respect to the $(\cdot,\cdot)_A$ inner product defined in (2), i.e., $\bar{M}_\omega^{SOR} = (M_\omega^{SOR})^A$. Conclude that $\sigma(M_\omega^{SSOR}) \subset \mathbb{R}_0^+$, i.e., the spectrum is non-negative.

b) Using b), show: $\|M_\omega^{SSOR}\|_A = \|M_\omega^{SOR}\|_A^2$.