Sheet 5

Discussion of the sheet: Th., 02.05.2019

1. Let $A \in \mathbb{R}^{n\times n}$ be SPD, set $\phi(x) := \frac{1}{2}(Ax,x) - (b,x)$, and let x_* be the exact solution of $Ax_* = b$.

a) Show that

$$\phi(x) - \phi(x_*) = \frac{1}{2}\|x - x_*\|_A^2.$$

b) Fix $x_0 \in \mathbb{R}^n$ and let $(x_0, x_1, \ldots, x_k)$ be the sequence of CG approximations. Let $\widehat{x}_k \in \mathbb{R}^n$ be the result of a steepest-descent iteration step with search direction $0 \neq d = r_{k-1} = b - Ax_{k-1}$, starting from the CG-vector x_{k-1}. Show

$$\|x_k - x_*\|_A \leq \|\widehat{x}_k - x_*\|_A,$$

i.e. the CG-step is at least as good as the SD-step.

Conclude the local contraction property

$$\|x_k - x_*\|_A \leq q\|x_{k-1} - x_*\|_A \qquad \text{with} \qquad q := \frac{\kappa_2(A) - 1}{\kappa_2(A) + 1} < 1.$$

2. (Exercise 9.5 from the Lecture notes)
Assume that for $m = 1, 2, \ldots$, the tridiagonal matrices

$$T_m = \begin{pmatrix} \alpha_1 & \beta_2 & & & \\ \beta_2 & \alpha_2 & \beta_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{m-1} & \alpha_{m-1} & \beta_m \\ & & & \beta_m & \alpha_m \end{pmatrix} \in \mathbb{R}^{m\times m}$$

admit LU-decompositions $T_m = L_m U_m$. Of course, L_m and U_m are bidiagonal, and we adopt the notation

$$T_m = L_m U_m = \begin{pmatrix} 1 & & & & \\ \lambda_2 & 1 & & & \\ & \ddots & \ddots & & \\ & & \lambda_{m-1} & 1 & \\ & & & \lambda_m & 1 \end{pmatrix} \begin{pmatrix} \eta_1 & \omega_2 & & & \\ & \eta_2 & \omega_3 & & \\ & & \ddots & \ddots & \\ & & & \eta_{m-1} & \omega_m \\ & & & & \eta_m \end{pmatrix} \tag{1}$$

a) Verify the following recursive formulas for the values λ_m, ω_m, η_m in (1):

$$\omega_m = \beta_m, \qquad \lambda_m = \frac{\beta_m}{\eta_{m-1}}, \qquad \eta_m = \begin{cases} \alpha_1 & m = 1, \\ \alpha_m - \lambda_m \omega_m, & m > 1. \end{cases} \tag{2}$$

Conclude that the matrices L_m and U_m are recursively obtained from L_{m-1}, U_{m-1} by adding one row and column, i.e.,

$$L_m = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{L_{m-1}} & 0 \\ \hline 0^{\mathrm{T}} & \lambda_m & 1 \end{array}\right), \qquad U_m = \left(\begin{array}{c|c} U_{m-1} & \begin{matrix} 0 \\ \omega_m \end{matrix} \\ \hline 0^{\mathrm{T}} & \eta_m \end{array}\right) \tag{3}$$

b) Given the factors $L, U \in \mathbb{R}^{m\times m}$ of the LU-decomposition of a matrix $T \in \mathbb{R}^{m\times m}$ in the form (3),

$$L = \left(\begin{array}{c|c} L' & 0 \\ \hline \ell^{\mathrm{T}} & 1 \end{array}\right), \qquad U = \left(\begin{array}{c|c} U' & u \\ \hline 0^{\mathrm{T}} & \eta \end{array}\right)$$

with $L', U' \in \mathbb{R}^{(m-1)\times(m-1)}$ and $\ell, u \in \mathbb{R}^{m-1}$, conclude that L^{-1} and U^{-1} can be written as

$$L^{-1} = \left(\begin{array}{c|c} L'^{-1} & 0 \\ \hline -\ell^{\mathrm{T}} L'^{-1} & 1 \end{array}\right), \qquad U^{-1} = \left(\begin{array}{c|c} U'^{-1} & -\frac{1}{\eta}\, U'^{-1} u \\ \hline 0^{\mathrm{T}} & \frac{1}{\eta} \end{array}\right)$$

This means that after adding one row and column to the tridiagonal matrix T to obtain the new tridiagonal matrix T', the triangular inverses L'^{-1}, U'^{-1} can be also obtained from L^{-1} and U^{-1} by simply adding one row and column.

3. We consider the *minimal residual iteration* for solving $Ax = b$. In this method, in order to determine the iteration x_{k+1} in each step, the residual $\Phi(x) := \|b - Ax\|_2^2$ is minimized over the one dimensional affine space $x_k + \mathrm{span}\{r_k\}$.

a) Formulate the algorithm for computing the iterations $x_1, x_2, \ldots$.

b) Assume, that A is positive definite (but not necessarily symmetric). Define

$$\mu := \frac{1}{2}\lambda_{min}(A + A^T), \qquad \sigma := \|A\|_2.$$

Show that $\mu \leq \sigma$.

c) Show that under the assumptions of b), the *minimal residual iteration* converges and the following estimate holds

$$\|r_{k+1}\|_2^2 \leq \Big(1 - \frac{\mu^2}{\sigma^2}\Big)\|r_k\|_2^2.$$