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Exercises in Numerics of Differential Equations

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Exercise 1. Consider the midpoint rule

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array}$$

Compute its stability function. How does the stability region look like? Is the midpoint rule A-stable? Is it L-stable?

Exercise 2. Write a program that plots the stability region of an RK-method on a part of the complex plane. The input should be the Butcher tableau of the RK-method and a rectangular region of the complex plane. The plot should indicate where the method is stable (e.g. by colouring the region of stability and instability in different colours).

Hint. This is not exact science, be creative.

Exercise 3. Consider an implicit m-stage RK-method with Butcher tableau $\begin{array}{c|c} c & A \\ \hline & b^\top \end{array}$ and a problem with dimension $n = 1$. Instead of solving the (implicit) equation for the vector of stages $k \in \mathbb{R}^m$ exactly, we employ m steps of the Banach fixpoint iteration to obtain approximate stages. We set $k^{(0)} := f(t_\ell, y_\ell)(1, \ldots, 1)^\top \in \mathbb{R}^m$, define $k^{(s)}$ for $s = 0, \ldots, m$ as the s-th fixpoint iterate and set $\widetilde{k} := k^{(m)}$. This gives rise to a one-step method

$$y_{\ell+1} = y_\ell + h \sum_{j=1}^{m} b_j \widetilde{k}_j.$$

Compute the stability function of this method. Is this method A-stable?

Exercise 4. Consider an m-stage collocation method. Define the polynomial

$$M(x) := \frac{1}{m!} \prod_{i=1}^{m} (x - c_i).$$

Show that the stability function $R(z)$ with $z = \lambda h$ for the collocation method is the rational polynomial $R(z) = P(z)/Q(z)$, where, $P, Q \in \mathbb{P}_m$ are given by

$$\begin{aligned} P(z) &= M^{(m)}(1) + M^{(m-1)}(1)z + \ldots + M(1)z^m, \\ Q(z) &= M^{(m)}(0) + M^{(m-1)}(0)z + \ldots + M(0)z^m. \end{aligned}$$

Use this explicit representation of $R(z)$ to show that Gauss-methods are not L-stable.

Hint. In order to obtain the representation for $R(z)$, consider the usual model problem and $h = 1$ (which implies $z = \lambda$). From the definition of the collocation polynomial $q \in \mathbb{P}_m$ infer that

$$q'(x) - zq(x) = KM(x) \tag{1}$$

for a constant $K \neq 0$. Differentiate equation (1) $s = 0, \ldots, m$ times to obtain an expression for $q(x)$. Finally, there holds $R(z) = q(1)/q(0)$ (why?).