Sheet 3

Discussion of the sheet: Th., 28.03.2019

1. Consider the damped Richardson method $x_{k+1} := x_k - \omega(Ax_k - b)$ with *complex* damping parameter $\omega \in \mathbb{C}$.

a) Show that: $\sigma(M^{Rich}) = \{1 - \omega\lambda \,:\, \lambda \in \sigma(A)\}$.

b) Show that: the damped Richardson method converges for all damping parameters $\omega \in \mathbb{C}\backslash\{0\}$ with the following property: the open disc $B_{1/|\omega|}(1/\omega) \subset \mathbb{C}$ around $1/\omega$ of radius $1/|\omega|$ contains the spectrum $\sigma(A)$.

c) Let $\sigma(A) \subset \{z \in \mathbb{C} \,|\, \mathrm{Re} z > 0\}$. Show that: the Richardson iteration converges for sufficiently small *real* damping parameters $\omega \in \mathbb{R}$.

d) Let $0 < \lambda_{\min}$ be the smallest eigenvalue of A and $\lambda_{\max}$ be the largest eigenvalue of A. Show that the optimal damping parameter, i.e., the parameter ω that minimizes $\rho(M_\omega^{Rich})$ is given by $\omega_{opt} = \frac{2}{\lambda_{min}+\lambda_{max}}$.

2. Let $A \in \mathbb{R}^{n\times n}$.

a) Give an example of a matrix A, for which no damping parameter $\omega \in \mathbb{C}$ exists, such that the Richardson iteration converges.

b) Show that: If A has at least one positive and one negative eigenvalue, than the Richardson method diverges for any damping parameter $\omega \in \mathbb{C}$.

3. Consider the $n \times n$ tridiagonal matrix

$$A = \begin{pmatrix} \alpha & -1 & & & \\ -1 & \alpha & -1 & & \\ & -1 & \alpha & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & \alpha \end{pmatrix}, \tag{1}$$

with a parameter $\alpha \in \mathbb{R}$.

a) Verify that the eigenvalues and eigenvectors of A are given by

$$\lambda_j = \alpha - 2\cos\left(\frac{\pi}{n+1}j\right)$$

$$v_j = \left(\sin\left(\frac{\pi}{n+1}j\right), \sin\left(2\frac{\pi}{n+1}j\right), \ldots, \sin\left(n\frac{\pi}{n+1}j\right)\right)^T.$$

b) Let $\alpha = 2$ (as in model problem 1). Will the Jacobi and Gauß-Seidel iteration converge for this matrix? If it does, what is the corresponding convergence factor?

For which $\omega \in \mathbb{R}$ will the SOR method converge?

4. The aim of this exercise is to implement and test the Jacobi, Gauß-Seidel and SOR-method (preferably in MATLAB) to solve the system

$$Ax = b \tag{2}$$

with a given starting vector $x_0 \in \mathbb{R}^n$.

a) Write functions `jacobi(A,b,x0,`$k_{\max}$`)`, `gaussseidel(A,b,x0,`$k_{\max}$`)`, `SOR(A,b,x0,`$k_{\max}$`,`ω`)` that realize $k_{\max}$- number of steps of the Jacobi, Gauß-Seidel and SOR-method (with parameter ω), i.e., they should return a vector $x_{k_{\max}}$ that approximates the exact solution x of (2).

b) Compare the implemented methods (and choose different parameters ω for the SOR-method). For comparison, use the matrix A of (1) or matrices for our 2D-model problem (in MATLAB provided by `gallery('poisson',N)` with N^2 unknowns) and plot the errors of each method over the number of iteration steps (use a double logarthmic plot `loglog`). What do you observe?